Absolutely Continuous Spectrum of Fourth Order Difference Operators With Unbounded Coefficients on the Hilbert Space $\ell^2(\mathbb{N})$

by

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Declaration

This thesis is my own work and has not been presented for a degree award in any other institution.

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This thesis has been submitted for examination with our approval as the university supervisors.

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DEDICATION

A special dedication to my beloved wife Lilian, children Brian,Bridget and Nicole for bearing with me through the difficult times of the economy. Not forgetting my Father Hezekiah and Mother Teresa,without whose effort,my life story would have been different.Glory be to our Almighty God.

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Abstract

Sturm-Liouville operators and Jacobi matrices have so far been developed in parallel for many years. A result in one field usually leads to a result in the other. However not much in terms of spectral theory has been done in the discrete setting compared to the continuous version especially in higher order operators. Thus, we have investigated the defficiency indices of fourth order difference operator generated by a fourth order difference equation and located the absolutely continuous spectrum of its self-adjoint extension as well as the spectral multiplicity using the M-matrix. The results are useful to mathematicians and can be applied in quantum mechanics to calculate time dilation and length contraction as used in Lorentz-Fotzgeralds transformations. The study has been carried out through asymptotic summation as outlined in Levinson Benzaid Lutz-theorem. This involved: reduction of a fourth order difference equation into first order, computation of the eigenvalues, proof of uniform dichotomy condition, calculating the deficiency indices and locating absolutely continuous spectrum. In this case we have found the absolutely continuous spectrum to be the whole set of real numbers of spectral multiplicity one.

Contents

List of Figures

Index of Notations

Chapter 1

Introduction

1.1 Background

Sturm-Liouville operators and Jacobi matrices have been developed in parallel in recent years. Actually, Sturm-Liouville equations and their discrete counterparts, Jacobi matrices are analysed using similar and related methods. Therefore, there is no doubt that the theory of Jacobi matrices is far much developed. This shows that the theory of difference equations have surely grown.

In this study, we have investigated the absolutely continuous spectrum of a fourth order self-adjoint extension operator of minimal operator generated by difference equation;

$$\begin{aligned}\mathcal{L}y(t) &= w^{-1}(t)\triangle^4 y(t-2) - i\{\triangle(q(t)\triangle^2 y(t-2)) + \\ &\quad \triangle^2(q(t)\triangle y(t-1)\} - \triangle(p(t)\triangle y(t-1)) + \\ &\quad i\{r(t)\triangle y(t-1) + \triangle(r(t)y(t)\} + m(t)y(t), \qquad (1.1)\end{aligned}$$

defined on a weighted Hilbert space $\ell^2_w(\mathbb{N})$ with the weight function $w(t) > 0$, $t \in \mathbb{N}$ where p(t), q(t), r(t) and m(t) are real-valued functions.Here the equation is in the form that makes it symmetric and also of order 4. In this case the coefficients are allowed to be unbounded. Δ is a forward difference operator such that $\Delta f(t) = f(t+1) - f(t)$; for $t \in \mathbb{N}$.The method applied is asymptotic summation as outlined in Levinson-Benzaid -Lutz theorem [8] and whose spectral parameter uniform version is given in [1,4,5]. For simplicity in computation and analysis ,we have assumed that $w(t) = 1$ unless otherwise stated. For the spectral analysis we have solved the equation L$y(t) = zy(t)$ where L is the difference operator generated by (1.1) and z is the spectral parameter,$z \in \mathbb{C}$. We have applied the M-matrix theory as developed in Hinton and Shaw [14] in order to compute the spectral multiplicity and the location of the absolutely continuous spectrum of self-adjoint extension operator. These results has been an extension of some known spectral results of fourth order differential operators to difference setting. Similarly,they have extended results found in Jacobi matrices [10] .In this thesis,chapter 1 is about introduction and some preliminary results including literature review,objectives,methodology and basic definitions.In chapter 2,we have given the results on the computation of the eigenvalues,dichotomy conditions and some results on singular continuous spectrum.Chapter 3 contains the main results in deficiency indices,absolutely continuous spectrum and the spectral multiplicity.Finally,we have summarized our results in chapter 4 and also highlighted areas of further research.

1.2 Basic Concepts

Definition 1.2.1

An operator T defined on Hilbert space H is said to be symmetric if it is densely defined and $D(T) \subset D(T^*)$ where T^* is a Hilbert adjoint of T

Definition 1.2.2

An operator T on a Hilbert space H is said to be self-adjoint if $T = T^*$ so that self adjoint operators are symmetric by definition.

Definition 1.2.3

Let T be an operator defined on Hilbert space H . A number λ is said to be in the spectrum of T if the operator $T - \lambda I$ is not invertible. The spectrum of T is denoted by $\sigma(T)$ and is defined by

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}$$

. In addition, the complement of the spectrum,$\mathbb{C} \backslash \sigma(T)$ is called the resolvent operator T and is denoted by $\rho(T)$ that is,

$$\rho(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is invertible}\}$$

. In this, one says $R_\lambda(T) = (T - \lambda I)^{-1}$ is the resolvent operator of T. Here

$$\sigma(T) \cap \rho(T) = \phi$$

We note that an operator $T - \lambda I$ fails to be invertible if it is neither one-to-one nor onto.

Definition 1.2.4

If the operator is not one-to -one, it implies that λ is an eigenvalue of the operator T. Thus the set of such $\lambda \in \mathbb{C}$ which makes $T - \lambda I$ not one-to-one forms the components of the spectrum known as the point(discrete) spectrum denoted by $\sigma_\rho(T)$

Definition 1.2.5

If $T - \lambda I$ is not invertible (does not have a bounded inverse) because $T-\lambda I$ is not onto then the spectral values λ in this case form a continuous spectrum. The set of all such λ is denoted by $\sigma_c(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ does not have a bounded inverse, $T - \lambda I$ is not onto$\}$.

Definition 1.2.6

Let T be the maximal multiplication operator defined by T$u(x) = xu(x)$ on a Hilbert space H,then the spectrum of T is absolutely continuous with the D(T) consisting of all $u \in H$ with $xu(x) \in H$.

Definition 1.2.7

A mapping $\triangle$ is known as forward difference operator if for any function

$f(t), t \in \mathbb{N}$ then

$$\triangle f(t) = f(t+1) - f(t).$$

Similarly $\triangle^*$ or ∇ is backward operator if

$$\nabla f(t) = f(t) - f(t-1).$$

Definition 1.2.8

Let H be a separable Hilbert space and let T be a densely defined symmetric linear operator on H. The operator T is closed if its graph

$$\{x \oplus Tx \in H \oplus H : x \in D(T)\}$$

is closed. If $T^{'}$ is a symmetric operator on H with $D(T) \subset D(T^{'})$ and $T' - \backslash D(T) = T$ we call T' a symmetric extension of T. symmetric operators have maximal symmetric extensions and the maximal symmetric extensions are closed only if not self-adjoint. In order to convert equation (1.1), into a first order system, we define the vector valued functions $x(t)$,$u(t)$ and $y(t)$ by,

$$x(t) = (x_1(t), x_2(t))^{tr},\ u(t) = (u_1(t), u_2(t))^{tr},\ y(t) = (x(t), u(t))^{tr}$$

where the superscript tr denotes transpose and

$$\begin{aligned}
x_1(t) &= y(t-1) \\
x_2(t) &= \triangle y(t-2) \\
u_1(t) &= p(t)\triangle y(t-1) - \triangle^3 y(t-2) + i\{\triangle(q(t)\triangle y(t-1)) + \\
&\quad q(t)\triangle^2 y(t-2)\} - ir(t)y(t) \\
u_2(t) &= \triangle^2 y(t-2) - iq(t)\triangle y(t-1).
\end{aligned}$$

Now we let

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

and

$$u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$$

Therefore the discrete linear Hamiltonian system as outlined by Hinton and Shaw[14] for differential operators and discritised by Shi[20] is of the form

$$J\triangle Y(t) = [zW(t) + P(t)]R(Y)(t) \tag{1.2}$$

where $t \in \mathbb{N}$, $W(t)$ and $P(t)$ are 4 x 4 complex Hamiltonian matrices. $W(t) = \text{diag}(w(t), 0..., 0)$, $w(t)$ is a weighted function, $x(t), u(t) \in \mathbb{C}^2$, J is a symplectic matrix, that is

$$\text{J} = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix} \quad \text{and} \quad \text{P(t)} = \begin{pmatrix} -C(t) & A^*(t) \\ A(t) & B(t) \end{pmatrix}.$$

For non-zero elements of 2 x 2 matrices A,B, and C are given by

$$A_{1,2} = 1, A_{2,2} = iq, B_{2,2} = 1, C_{1,1} = m, C_{1,2} = -C_{2,1} = ir,\ and\ \ C_{2,2} = p$$

Definition 1.2.9

Let $\ell^2_w[(0,\infty)]$ be a Hilbert space with weight function $w(t)$ and define this Hilbert space using the vector valued function $x(t), u(t)$ and $Y(t)$ by

$$\ell^2_w[(0,\infty)] = \{y; y = y(t)^{\infty}_{t=0} \subset \mathbb{C} \ \text{ and} \sum_{t=0}^{\infty}(RY^*)(t)W(t)(RY)(t) < \infty\}$$

where $RY(t)$ is a partial shift operator

$$\text{Ry(t)} = \begin{pmatrix} x(t+1) \\ u(t) \end{pmatrix}$$

Like in differential operators,a regularity condition is needed for spectral analysis of higher order difference operators,that is, there exists an n_0 such that non-trivial solutions Y(t,z) of (1.1) viz (1.2) and all $z \in \mathbb{C}$,Shi[20]

$\sum_{t=0}^{n}(RY(t,z)^{*}W(t)(RY(t,z)) > 0, n \geq 0.$

The scalar product for the vector valued functions system is,

$\sum_{t=0}^{\infty}\overline{y_1}(t+1)w(t)y(t+1) = \langle y_1, y\rangle w, y, y_1 \in \ell_w^2([o,\infty)$ *see* [20].

In this case, one defines maximal difference operator L^* on $\ell_w^2(\mathbb{N})$ by $D(L^*) = \{y(t) \in \ell_w^2(\mathbb{N})$:there exists $f(t) \in \ell_w^2(\mathbb{N})$ such that

$J\Delta Y(t) - P(t)RY(t) = W(t)f(t), t \in \mathbb{N}.$

This implies that for y(t),$f(t) \in \ell_w^2(\mathbb{N})$,then

$L^*y(t) = w(t)f(t).$

The restriction of L^* by boundary conditions at 0 and all $t \geq n+1$ for some $n \in \mathbb{N}$ results into a pre-minimal difference operator defined by $D(L') = \{y(t) \in D(L^*)$: there exists $n \in \mathbb{N}$ such that $y(0) = y(t) = 0$, for all $t \geq n+1\}$.Thus for $y(t) = D(L^{'})$ then

$L^*y(t) = L^{'}y(t).$

The closure of pre-minimal operator $L^{'}, \bar{L}'$ is defined as the minimal difference operator. This means that a minimal operator is a restriction of maximal operator L^*. We shall denote by L and L^* minimal and maximal operators respectively. It follows that L and L^* are symmetric, $L \subset L^*$ and $L = L^{**}$ as required.

In order to compute deficiency indices of L,we have used $a > 2$ as the left-end point in order for L^* to be densely defined .The result can be extrapolated to the set $\mathbb{N}$ using Remlings results [19] since deficiency indices of an operator are invariant of left-end points

Definition 1.2.10

The deficiency indices of the operator L is the pair (N_-, N_+) defined by, $\dim N(L^* \pm iI)$ and denoted by N_- and N_+, for $\dim N(L^* - i)$ and $\dim N(L^* + i)$ respectively. Here $N(L^* \pm i)$ is the null space of $L^* \pm i$. Thus

$2 \leq N_-, N_+ \leq 4$

and if $N_- = N_+$ then, there exists a symmetric self-adjoint extension H of an operator L.

Definition 1.2.11

Let

$$Y_\alpha(., z) = \begin{pmatrix} U_\alpha(., z) \\ V_\alpha(., z) \end{pmatrix}$$

be the fundamental matrix of

$$J\triangle \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} 0 & -\triangle \\ \triangle & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} -C(t) + zW & A^*(t) \\ A(t) & B(t) \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \tag{1.3}$$

with initial values of

$$Y_\alpha(a, z) = \begin{bmatrix} \alpha_1^* & -\alpha_2^* \\ \alpha_2^* & \alpha_1^* \end{bmatrix},$$

where

$$\alpha_1, \alpha_2$$

satisfy

$$\alpha_1\alpha_1^* + \alpha_2\alpha_2^* = I_2, \alpha_1\alpha_2^* - \alpha_2\alpha_1^* = 0_2 \ and \ \alpha_1\alpha_2 Y(a) = 0 \tag{1.4}$$

α_1 and α_2 are 2 x 2 matrices, that is $\alpha = (\alpha_1, \alpha_2) \in \mathbb{C}^{2\,\mathrm{X}2}$.

U_α, V_α are 4 x 2 complex-valued matrices whose every column solves $Ly = zy$ and that $V_\alpha(,..z)$ satisfy self-adjoint boundary conditions at a. Thus,columns of $Y_\alpha(,...z)$ span the 4-dimensional vector space of solutions of (1.3). Therefore in the limit point case with $Imz > 0$ one has a matrix $M \in \mathbb{C}^{2\,\mathrm{X}\,2}$ such that

$$\mathcal{X}_\alpha(t,z) = Y_\alpha(a,z) = \begin{bmatrix} I_2 \\ M(z) \end{bmatrix} = U_\alpha(t,z) + V_\alpha(t,z)M(z)$$

where$\mathcal{X}_\alpha(t,z)$ satisfy the boundary conditions of (1.4).It has been shown in [20] that if L is limit point as $t \to \infty$,then one can construct the M-matrix $M(z)$ for the Hamiltonian restriction to $[a,\infty)$ with Dirichlet boundary conditions.To do this ,let
$\begin{pmatrix} W_1(a,z) \\ W_2(a,z) \end{pmatrix}$ be a system of 2 square summable solutions for $Imz > 0$.
Then from the theory of Hinton and Shaw [14], it follows that this solutions also arise from $Y_a(t,z)\begin{pmatrix} I_n \\ M(z) \end{pmatrix}$,where $Y_a(t,z)$ is the fundamental solution of the system satisfying the appropriate boundary conditions at a.

Definition 1.2.12

If L is limit point, then L has self-adjoint extensions.With a=0,the self-adjoint extension H of L are precisely defined by

$$D(H) = y \in D(L^*) : (\alpha_1, \alpha_2)y(0) = 0, L^*y = Hy$$

and $L \subset H = H^* \subset L^*$.

1.3 Literature Review

The spectral analysis and deficiency indices of Sturm-Liouville operators have been generating a lot of interest in the field of mathematical research. Sturm-Liouville equations and their discrete counterparts, Jacobi matrices are analysed in related and almost in a similar way. It is a known fact that the spectral theory of Sturm- Lioville operators and Jacobi matrices are developed in parallel.
The accelerated growth of the theory of difference equations has played an important role in the applicable analysis and in mathematical research as a whole. The difference equations appeared earlier than differential equations and played an important role in the development of the latter.

The qualitative study of the solutions of difference systems is periodic and one can easily include method of variation of constants, the concept of exact and adjoint equations and Lagranges' and Green's identities into this analysis. The method of generating functions, a very important technique for obtaining the closed form solutions of higher order difference equations will follow immediately.

Currently,there are research papers that have developed and expanded the M-function theory for difference systems. These papers include Fischer and Remling[12], Clark and Gesztesy [9], Behncke and Nyamwala [4,5], Behncke [7], Shi [20].For example,Remling made an attempt to establish asymptotic integration as a valuable tool in spectral analysis in conjuction with the theory of the M-matrix.Remling could prove some results on the spectral theory of fourth order operators,though unbounded middle terms formed an obstacle.Behncke,Hinton and Remling finally developed the spectral theory for higher even order operators with bounded coefficients satisfying some regularity conditions.Due to this and other results on asymptotic integration by Behncke and Hinton,it was clear that one obstacle to analysis of the absolutely continuous spectrum of operators with unbounded coefficients, is the understanding of the zero's of polynomials, here the Fourier polynomials. This is experienced especially when proving some results on the spectral theory of fourth order operators, where the middle terms form an obstacle even though unbounded. The theory of M-functions as developed in these papers are equivalent but the approach in [20] has been relevant in this study because the results are closer to the traditional approach of Hinton and Shaw [14]. Actually, the analysis has been parallel to that of Shi [20].

Even though attempts have been made to compute deficiency indices and the location of absolutely continuous spectrum of unbounded operators, much has not been done for discrete operators except for papers by Behncke and Nyamwala [4,5] and that of Agure, Ambogo and Nyamwala [1] where the coefficients that were taken to be unbounded were the even order coefficients.

We have investigated the absolutely continuous spectrum of fourth order difference operator generated by (1.1) when odd order coefficients are unbounded. This has been done using asymptotic summation. Asymptotic summation is based on the discretized version of Levinson's theorem which appeared in the Benzaid-Lutz paper [8] and the result which is z-uniform is stated here below,since the assumptions in the Theorem have been used in chapter 3 to prove our main result in that chapter.

Theorem 1.3.1

Let $\Lambda(t,z) = diag\{\lambda_1(t,z), ...\lambda_{2n}(t,z)\}$ *for* $t \geq a$ *assume*

(i) $\lambda_i(t,z) \neq 0$ *for all* $1 \leq i \leq 2n$ *and* $t \geq a$

(ii)$R(t,z)$ *is a 2n x 2n matrix defined for all* $t \geq a$ *satisfying*

$$\sum_{t=0}^{\infty} \mid \frac{1}{\lambda_i(t,z)} \mid \|R(t,z)\| \leq \infty$$

for all $i = 1, 2...2n$

(iii) $\Lambda(t,z)$ *satisfy the following uniform dichotomy condition for any pair of indices* i *and* j *such that* $i \neq j$*, assume their exist a* δ *with* $0 < \delta < 1$ *such that* $|\lambda_i(t,z)| \geq \delta$ *for all* $t \geq a$*.Then either* $\mid \frac{\lambda_i(t,z)}{\lambda_j(t,z)} \mid \geq 1$ *or* $\mid \frac{\lambda_i(t,z)}{\lambda_j(t,z)} \mid \leq 1$ *for large* t*. Here the linear system*

$$X(t+1,z) = [\Lambda(t,z) + R(t,z)]X(t,z)$$

has the fundamental matrix satisfying

$$X(t,z) = [I + o(1)] \prod_{l=a}^{t-1} \Lambda(l,z) \ \ as \ \ t \to \infty.$$

1.4 Statement of the problem

The theory of difference operators occupies a central position in analysis.This is because of its continuous growth and infact it will continue playing an important role in mathematics as a whole. In this study, we have investigated the deficiency indices of minimal difference operator generated by equation (1.1) and also investigated the absolutely continuous spectrum of fourth order self-adjoint extension operator of minimal operator when the coefficients are unbounded. We have used asymptotic summation approach as outlined in Levinson-Benzaid-Lutz theorem.

1.5 Objectives of the study

The objectives of the study were:

- To compute deficiency indices of fourth order minimal difference operator when the coefficients are unbounded.
- To locate the absolutely continuous spectrum of self adjoint extension operator of the minimal difference operator generated by (1.1).
- To compute the spectral multiplicity of the absolutely continuous spectrum using the M-matrix.

1.6 Significance of the study

Our study on absolutely continuous spectrum of fourth order difference operators with unbounded coefficients on Hilbert space using asymptotic summation have contributed more knowledge to the existing results in this field. The study have extended the existing knowledge of computation of deficiency indices and spectral theory. In particular, location of absolutely continuous spectrum together with its spectral multiplicity using M-matrix have also been very useful to quantum physicists.

We believe that this study have also provided some solutions to unanswered open questions on absolutely continuous spectrum of fourth order difference operators with unbounded coefficients on Hilbert space, for example the fourth order square well problem with unbounded coefficients.

1.7 Research Methodology

We started by reducing the fourth order difference system to a first order difference equation using quasi-difference. We have also gone a head and computed the eigenvalues of the minimal operator. We have also established the dichotomy condition uniformly in the spectral parameter. Again we have calculated or approximated the eigenfunction of the difference operator using Levison-Benzaid Lutz theorem. We have also calculated deficiency indices and found that they are equal and hence we have defined the self adjoint extension operator. Finally, we have located the absolutely continuous spectrum together with its spectral multiplicity using the M-matrix.

Chapter 2

Difference Operators

2.1 Hamiltonian System

The known origin of discrete Hamiltonian systems is from discrete processes acting according to the principle of Hamiltonian systems, that is, discrete control problems and discrete physical problems. It also originated from the discretisation of continuous Hamiltonian system.
In order to define discrete Hamiltonian system of (1.1),we need to introduce quasi-differences as explained in [20] and [4,5].Thus we define the vector valued functions $x(t), u(t)$ and $Y(t)$ as in chapter one. Therefore we introduce the spectral parameter $z, z \in \mathbb{C}$ and solve the equation

$$Ly(t) = zy(t) \tag{2.1}$$

In such a case,the Hamiltonian system (1.2) which can be rewritten as
$J\triangle \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} 0 & -\triangle \\ \triangle & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} -C(t) + zW & A^*(t) \\ A(t) & B(t) \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}$
with A,B and C given by

$$A = \begin{bmatrix} 0 & 1 \\ 0 & iq \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} m-z & ir \\ -ir & p \end{bmatrix}$$

In order to ensure the existence, uniqueness and continuity of the solutions of initial value problem of (2.1),we need that $I_2 - A$ is invertible in $\mathbb{N}$, but this is always true for the fourth order case as long as $a > 2$. Let $(I_2 - A)^{-1}$ be denoted by E, then in line with the analysis of shi[20], (2.1) then has a first order system of the form

$$Y((t+1), z) = S(t, z)Y(t, z) \tag{2.2}$$

where

$$S(t, z) = \begin{bmatrix} E & EB \\ CE & I - A^* + CEB \end{bmatrix}$$

The 2x2 block matrices are then obtained from;

$$E = \begin{bmatrix} 1 & \frac{1}{1-iq} \\ 0 & \frac{1}{1-iq} \end{bmatrix}, \quad EB = \begin{bmatrix} 0 & \frac{1}{1-iq} \\ 0 & \frac{1}{1-iq} \end{bmatrix}$$

$$CE = \begin{bmatrix} m-z & \frac{m-z+ir}{1-iq} \\ ir & \frac{p-ir}{1-iq} \end{bmatrix}, \quad CEB = \begin{bmatrix} 0 & \frac{m-z+ir}{1-iq} \\ 0 & \frac{p-ir}{1-iq} \end{bmatrix}$$

Hence (2.2) becomes,

$$\begin{bmatrix} x(t+1,z) \\ u(t+1,z) \end{bmatrix} = S(t,z) \begin{bmatrix} x(t,z) \\ u(t,z) \end{bmatrix} \tag{2.3}$$

where $S(t,z)$ is a 4 x 4 transfer matrix given by;

$$\begin{bmatrix} 1 & \frac{1}{1-iq} & 0 & \frac{1}{1-iq} \\ 0 & \frac{1}{1-iq} & 0 & 1-iq \\ m-z & \frac{m-z+ir}{1-iq} & 1 & \frac{m-z+ir}{1-iq} \\ -ir & \frac{p-ir}{1-iq} & -1 & \frac{1+q^2+P-ir}{1-iq} \end{bmatrix}$$

The system (2.2) is now solved using asymptotic summation. The spectral multiplicity is computed via M-matrix.

2.2 Asymptotic Summation

We stated earlier that Levinson-Benzaid -Lutz theorem is useful in asymptotic summation. In this case the results is the extension of Levinson's theorem from differential calculus to difference setting .This result first appeared in the paper of Benzaid and Lutz [8] and has been extended by many authors, namely, Behncke and Nyamwala[4,5] as well as Shi[20]. Thus asymptotic summation is based on Levinson-Benzaid-Lutz theorem . The statement of this theorem implies that we solve for the eigenvalues of the matrix $S(t,z)$. In such a case ,we determine the characteristic polynomial $\det(S(t,z) - \lambda I_4)$ which gives;

$$\mathcal{P}(t,\lambda,z) =$$
$$(1-\lambda)^2[\tfrac{1}{(1-iq)^2} + \tfrac{q^2}{(1-iq)^2} - \tfrac{2\lambda}{1-iq} - \tfrac{\lambda q^2}{1-iq} - \tfrac{\lambda p}{1-iq} + \tfrac{\lambda ir}{1-iq} + \lambda^2] - \tfrac{2ir\lambda(1-\lambda)}{1-iq} + \tfrac{\lambda^2 m}{1-iq}.$$

Thus multiplying $\mathcal{P}(t,\lambda,z)$ by $\frac{1-iq}{\lambda^2}$ so that if λ is a root, then $\overline{\lambda}^{-1}$ is also a root, we obtain;

$$F(t,\lambda,z) = [(1-\lambda^{-1})^2(1-\lambda)^2 + p(1-\lambda^{-1})(1-\lambda) + (m-z)] + [q(1-\lambda^{-1})(1-\lambda)(i\lambda + (i\lambda)^{-1}) + r(i\lambda + (i\lambda)^{-1})].$$

In order to have a polynomial of real coefficients, we apply a transformation $\lambda = \frac{is+1}{is-1}$ that maps upper half plane into the interior of a circle, such that

$$(1-\lambda^{-1})^2(1-\lambda)^2 = \frac{16}{(s^2+1)^2}$$

$$(1-\lambda)(1-\lambda^{-1}) = \frac{4}{(s^2+1)}$$

$$i\lambda + (i\lambda)^{-1} = \frac{4s}{s^2+1}$$

and

$$Q_0(s,t,z) = \frac{16}{(s^2+1)^2} + \frac{4p}{s^2+1} + (m-z) + \frac{4q}{s^2+1}(\frac{4s}{s^2+1}) + \frac{4rs}{s^2+1}$$

The terms in the denominator can be eliminated by multiplying through by $(s^2+1)^2$ so that we have

$$Q(s,t,z) = (s^2+1)^2 Q_0(s,t,z)$$

and is given by

$$Q(s,t,z) = ms^4 + 4rs^3 + (4p+2m)s^2 + (16q+4r)s + (16+4p+m). \quad (2.4)$$

Since the transformation of (2.1) into Levinson-Benzaid -Lutz form by asymptotic summation involves diagonalisation, we need that the eigenvalues of $S(t,z)$ be distinct. By considering the resultant or the discriminant of $\mathcal{P}(\lambda,t,z)$ and $\partial_\lambda \rho(\lambda,t,z)$ one can show just like in [3], that there are only finitely many spectral values z for which $\mathcal{P}(\lambda,t,z)$ has multiple roots. Let $\omega_1 < \omega_2 < ... < \omega_k$ denote all of the real spectral values z leading to multiple roots. Following [3], the analysis will be restricted to small complex neighborhoods of $z_0 \in (\omega_i, \omega_{i+1}), \quad \mathrm{i} = 0, ..., k$ where $\omega_0 = -\infty$ and $\omega_{k+1} = \infty$. For a given $z_0 \in (\omega_i, \omega_{i+1})$, one can now choose $\epsilon > 0$ and $a > 0$ so that $\mathcal{P}(\lambda,t,z) = 0$ has no multiple roots for any z

$$z \in \mathcal{K}_\epsilon(z_0) = \{z||z-z_0| \leq \epsilon, \quad Imz \geq 0\}$$

and $t \geq a$.This is possible because the roots of $\mathcal{P}(\lambda,t,z)$ depend analytically on the coefficients. Throughout the study, it may be necessary to adjust a and ϵ repeatedly.This will be done without mentioning.

2.3 Bounded Coefficient

Definition 2.3.1

The coefficients $q(t), r(t), p(t)$ and $m(t)$ are said to be almost constant coefficients if there exists constants c_q, c_r, c_p and c_m such that

$$q(t) \to c_q, r(t) \to c_r, p(t) \to c_p, \ and \ m(t) \to c_m \ \ as \ \ t \to \infty. \tag{2.5}$$

In this case,the coefficients $q(t), r(t), p(t)$ and $m(t)$ are bounded. With this assumption, we have the following theorem, which proves that in the case of bounded coefficients, then there exists an interval in which the singular continuous spectrum of H is absent.

Theorem 2.3.2

Let H be self-adjoint extension operator of the minimal difference operator generated by (1.1). Assume the coefficients are almost constant, then

$$\sigma_{sc}(H) \cap (\underline{m}, \bar{m}) = \phi.$$

Here,

$$\underline{m} = \liminf m(t) \qquad and \qquad \bar{m} = \limsup m(t)$$

Proof. The proof is analysed both for accumulation of eigenvalues and boundedness of the M-matrix. $\sigma_{sc}(H)$ cannot lie within the interval $(\underline{m}, \bar{m})$ since if X is an open subset of $\mathbb{C}$ such that $(\underline{m}, \bar{m}) \subset X$, then we may assume that for $z \in X$,the solutions $y_j(t, z)$, j = 1,..., 4, of $(L - z)y$ analytically depend on z such that for $z \in X$ with $Imz > 0$, then def$L = (2, 2)$ and the point spectrum has no accumulation point within$(\underline{m}, \bar{m})$.The solutions $y_j(t, z)$, form the fundamental system $Y(t, z)$

of (1.1) since otherwise there would exist a solution which is in the domain of self-adjoint extension operator, implying that z is an eigenvalue. By analyticity, it follows that $y_j(t,z)$, j=1, ...,4, form a fundamental system for all $z \in X$, with possible exception for at most countably many points which cannot accumulate in X.

Finally we show that the solutions that lose their square summability as $Imz \to 0^+$ cannot contribute to singular continuous spectrum. For $z \in \mathbb{R}$, those eigenvalues ,λ, with $|\lambda| < 1$ and $|\lambda| > 1$ will lead to eigenfunctions which are z-uniformly square summable and z-uniformly non-square summable respectively and hence discrete spectrum at most. But if $Imz > 0, z \in \mathbb{C}$, then as $Imz \to 0^+$, some of the eigenfunctions from eigenvalues $\lambda, |\lambda| = 1$, lose their square summability, but since the domain of H is defined by only those eigenfunctions that are z-uniformly square summable, we need to show that $ImM(z)$ exists finitely and is bounded.

Now let $F(.,z)$ be 2 by 4 system of square summable solutions which satisfy α-boundary conditions at 0 and define the M-matrix $M(z)$ (see C.Remling[17])

$$< F(.,z), F(.,z^{'}) > (\bar{z} - z') = M^*(z) - M(z'), \tag{2.6}$$

whose discrete version is given in [20]. Then for $z = z + i\eta, z_0 \in \mathbb{R}$ we have

$$ImM(z_0) = \lim_{\eta \to 0^+} \eta < F(.,z_0 + i\eta); F(.,z_0 + i\eta) > .$$

Assume the α-boundary conditions does not give rise to a bound state since otherwise $ImM(z_0)$ will exist boundedly, then the above limit exist finitely. To see this we use the two eigensolutions given by the eigenvalues λ such that $|\lambda| < 1$ even as $Imz \to 0^+$. In this case,

$$y_j(t,z) \simeq c_{jk}\lambda^t$$

and

$$y_i \simeq c_{ik}\lambda^t,$$

where c_{ik} and c_{jk} appropriate eigenvectors and are bounded. Then by Cauchy-Swartz inequality, we obtain

$$\begin{aligned} ImM(z_0) &= \lim_{\eta\to 0^+} \eta| < y_i(t,\eta), y_j(t,\eta) > | \leq \\ & \lim_{\eta\to 0^+} \eta(\sum_{k=1}^{4} |c_{ik}|^2 |\lambda_i^t(\eta)|^2)^{\frac{1}{2}} (\sum_{k=1}^{4} |c_{jk}|^2 |\lambda_j^t(\eta)|^2)^{\frac{1}{2}}. \end{aligned}$$

The term on the right hand side is bounded absolutely as $t \to \infty$ since $|\lambda_j(\eta)|, |\lambda_i(\eta)| < 1$. Consequently, $ImM(z_0)$ is non-trivial. This shows that the spectrum of H has no singular continuous part. □

The interval $(\underline{m}, \overline{m})$ is not necessarily empty since if we assume that $m(t) = sin\frac{(t+1)}{2}\Pi$,then $\underline{m} = -1$ and $\overline{m} = 1$, hence $(\underline{m}, \overline{m}) = (-1, 1)$ yet $m(t)$ is bounded for all $t \in \mathbb{N}$.

Suppose that m, p are bounded with $q = r = 0$, then (2.4) becomes a biquadratic whose zeros can be solved explicitly. Therefore under various asymptotic conditions we obtain the following result which has been proved in Agure, Ambogo and Nyamwala [1] and we provide proof for completeness.

Theorem 2.3.3

Let m and p be bounded and suppose all the necessary and sufficient conditions for asymptotic summation are satisfied, then

(i) If $q = r = 0$ and $p^2 < 4(m - z)$ then $def L = (2, 2)$ and $\sigma(H)$ is pure discrete.

(ii) Assume all the coefficients are almost constant and that the limiting characteristic polynomial has $2l$ eigenvalues of absolute value one $(0 \leq l \leq 2)$, then the self-adjoint extension operator H has no singular continuous spectrum and $\sigma_{ac}(H)$ agrees with that of the constant coefficient limiting operator and has spectral multiplicity of l.

Proof. (i) Assune that $r(t) = q(t) = 0$ for all $t \in \mathbb{N}$ and the other coefficients bounded then the polynomial is a well known biquadratic polynomial that can be solved explicitly. Thus if $p^2 < 4(m - z)$, the discriminant of the polynomial is less than zero and hence the roots have non-zero imaginary parts. These roots are in complex conjugate pairs. Assume these roots are of the form $\alpha_j \pm \beta_j, \quad j = 1, 2$. Using analysis given in [4], the two roots with $\beta_j > 0$ will lead to eigensolutions that are z-uniformly square summable while the two roots with $\beta_j < 0$ will lead to z-uniformly non-square summable eigensolutions. Thus def $L = (2, 2)$ and the spectrum is discrete at most.

(ii) If the coefficients are almost constant, then those roots λ of

$$\mathcal{P}(\lambda, t, z) \quad \text{such that} \quad |\lambda| > 1$$

lead to solutions that are z-uniformly non-square summable while the roots $|\lambda^{-1}| < 1$ lead to z-uniformly square summable solutions. Therefore, it is the roots λ such that $|\lambda| = 1$, that lead to eigenfunctions of which half of their number lose their square summability as $Imz \to 0^+$.The eigenfunctions that lose their square summability as $Imz \to 0^+$ contribute to absolutely continuous spectrum.

Invoking the results of [3], the absolutely continuous spectrum of H coincides with that of the constant coefficients limiting operator and of spectral multiplicity equal to the number of eigenfunctions that lose their square summability as $Imz \to 0^+$.

□

The following example confirms the results of the above two theorems. Before we give the example, we state the following lemma which is from classical linear algebra.

Lemma 2.3.4

If λ and $\bar{\lambda}^{-1}$ are roots of the characteristic polynomial $\rho(\lambda, t, z)$ and assume that $\zeta = \lambda + \bar{\lambda}^{-1}$, then $|\zeta| \leq 2$ is only possible if ζ is real otherwise $|\zeta| > 2$.

Lemma 2.3.4 implies that we can obtain$|\lambda| \simeq |\bar{\lambda}^{-1}| \simeq 1$ only if $\lambda + \bar{\lambda}^{-1}$ is real otherwise we will have $|\lambda| > 1$ and $|\bar{\lambda}^{-1}| < 1$.

Example 2.3.5

Let L be a fourth order difference operator generated by a difference equation of the form

$$\triangle^4 y(t-2) - \triangle\{(c_p + t^{\beta_p})\triangle y(t-1)\} + (c_m + t^{\beta_m})y(t) = zy(t)$$

where $\beta_m, \ \beta_p < 0$ and $c_p, c_m > 0$ are constants.

Then one can easily convert the above difference equation into its first order system using quasi-differences. Here as $t \to \infty$, then

$$c_p + t^{\beta_p} \to c_p, \quad \text{while} \quad c_m + t^{\beta_m} \to c_m.$$

The characteristic polynomial $\mathcal{P}(\lambda, t, z)$ multiplied by λ^{-2} becomes

$$(1-\lambda)^2(1-\lambda^{-1})^2 + (c_p + t^{\beta_p})(1-\lambda)(1-\lambda^{-1}) + c_m + t^{\beta_m} - z = 0$$

Now let $\lambda + \lambda^{-1} = \zeta$ so that we have

$$(2-\zeta)^2 + (2-\zeta)(c_p + t^{\beta_p}) + (c_m + t^{\beta_m} - z) = 0$$

Solving for ζ by absorbing $t^{\beta_p}, t^{\beta_m} - z$ into c_p and c_m respectively we get

$$\begin{aligned} \zeta_+ &= 2 + \frac{c_p}{2} + \{\frac{c_p^2}{4} - c_m\}^{\frac{1}{2}} \\ \zeta_- &= 2 + \frac{c_p}{2} - \{\frac{c_p^2}{4} - c_m\}^{\frac{1}{2}}. \end{aligned}$$

Thus we have two broad cases to consider.

(a)$\beta_p < \beta_m < 0$. Then ζ_+ and ζ_- will be in complex conjugate pairs with non-zero imaginary parts. Applying the results of Lemma 2.3.4 ,both ζ_+ and ζ_- have absolute value greater than 2, hence each contribute (1,1) to deficiency index and the eigensolutions that are square summable are z-uniformly square summable. Hence $defL = (2,2)$ and $\sigma(H)$ is discrete at most.

(b) $\beta_m < \beta_p < 0$. This can be split into three cases as follows

(i) $|\zeta_-| \leq 2, |\zeta_+| > 2$ then the expansion of

$$\zeta_- = 2 + \frac{c_p}{2} - \{\frac{c_p^2}{4} - c_m\}^{\frac{1}{2}} \approx 2 + \frac{c_p}{2} - \frac{c_p}{2}\{1 - \frac{2c_m}{c_p^2} + ..\}.$$

Thus after two diagonalisations, we need that the correction term be summable. The term affected by this, is that associated to the spectral parameter z which is $\frac{c_m}{c_p^2}$.Hence

$$\triangle^2(\frac{c_m}{c_p^2}) \approx \triangle^2(t^{\beta_m - 2\beta_p}) \approx O(t^{\beta_m - 2\beta_p - 2}).$$

Therefore if $\beta_m - 2\beta_p < 1$ then $defL = (3,3)$ and $\sigma(H)$ is pure discrete. But if $\beta_m - 2\beta_p > 1$ then def$L = (2,2)$ and $\sigma_{ac}(H) \subset [c_m, 16 + 4c_p + c_m]$ and has a spectral multiplicity of 1

(ii) If we assume $|\zeta_+| \leq 2$ and $|\zeta_-| > 2$, we obtain similar results as in (i) above

(iii) Suppose $|\zeta_+|$,$|\zeta_-| \leq 2$, this is possible since c_p and c_m can be chosen appropriately. Then if $\beta_m - 2\beta_p < 1$,$defL = (4,4)$ and $\sigma(H)$ is pure discrete while if $\beta_m - 2\beta_p > 1$ then $defL = (2,2)$ and $\sigma(H) \subset [c_m, 16 + 4c_p + c_m]$ of spectral multiplicity 2.

2.4 Unbounded Coefficients

In this section, we compute the approximations of the eigenvalues of the characteristic polynomial $\mathcal{P}(\lambda, t, z)$ when $r(t)$ is unbounded as $t \to \infty$.In particular,we assume that

$$p(t), q(t), m(t) = o(r(t)), and\ r(t) \to \infty\ as\ t \to \infty, \tag{2.7}$$

that is $p(t), q(t)$ *and* $m(t)$ are bounded for all $t \in \mathbb{N}$ while $r(t)$ is unbounded.

Lemma 2.4.1

Suppose (2.7) is satisfied ,then the roots of the polynomial (2.4) can be approximated from the equations

i)$ms^4 + 4rs^3 + R_1(s) = 0$ *where*

$R_1(s) = (4p + 2m)s^2 + (16q + 4r)s + (16 + 4p + m)$

ii)$4rs^3 + (4p + 2m)s^2 + (16q + 4r)s + R_2(s) = 0$ *where*

$R_2(s) = ms^4 + (16 + 4p + m)$

iii)$(16q + 4r)s + (16 + 4p + m) + R_3(s) = 0$ *here* $R_3(s) = ms^4 + 4rs^3 + (4p + 2m)s^2$ *where*

$$R_1(s)s_1^{-3}m^{-1}, \frac{1}{4}R_2(s)r^{-1}s_{\pm}^{-1},$$

and $R_3(s)(16q+4r)^{-1}$ *tend to zero as* $t \to \infty$.

Then,

$$|s_1| \approx |\frac{4r}{m}|, |s_{2\pm}| \approx |\frac{4r}{16+4r}|, |s_3| \approx |\frac{16_q+4r}{16+4p+m}|$$

Proof. It suffices to show that

$$R_1(s)s_1^{-3}m^{-1},$$

$$\frac{1}{4}R_2(s)r^{-1}s_{\pm}^{-1}$$

and

$$R_3(s)(16q+4r)^{-1}$$

tend to zero as $t \to \infty$, that is, they are $o(1)$.

Now we show that $R_1(s)s_1^{-3}m^{-1}$ tend to zero as $t \to \infty$.

$$\begin{aligned}\frac{|m|^{-1}R_1(s)}{|s_1^3|} &= |m|^{-1}[|4p+2m|s_1^{-1}+|16q+4r|s_{lg}^{-2}+|16+4p+m|s_1^{-3}] \\ &\leq \frac{|4p+2m|}{4r}+\frac{|16q+4r|m||}{|4r|^2}+\frac{|16+4p+m||m|^2}{|4r|^3}\end{aligned}$$

each term goes to zero as $t \to \infty$ since $r(t) \to \infty$ too.

$$\frac{|m|^{-1}|R_1(s)|}{s_1^3} \to 0$$

Also we show that $4R_2(s)r^{-1}s_{2\pm}^{-1}$ tend to zero as $t \to \infty$

$\frac{R_2(s)}{4rs} = \frac{ms^3}{4r} + \frac{16+4p+m}{4rs}$, For s_{2+} as $r(t) \to \infty, \frac{R_2(s)}{4rs} \to 0$ *since* $\frac{R_2(s)}{4rs} \approx \frac{m\{(\frac{-2p+m}{4r})+i(1+\frac{2q}{r})\}^3}{4r}$

$+ \frac{16+4p+m}{4r\{\frac{-2p+m}{4r}+i(1+\frac{2q}{r})\}}$

For s_{2-} as $r \to \infty, \frac{R_2(s)}{4rs} \to 0$

$\frac{R_2(s)}{4rs} \approx \frac{m\{(\frac{-2p+m}{4r})-i(1+\frac{2q}{r})\}^3}{4r} + \frac{16+4p+m}{4r\{\frac{-2p+m}{4r}-i(1+\frac{2q}{r})\}}$.

Finally we show that $R(s)(16q+4r)^{-1}$ tend to zero as $t \to \infty$ that is

$$|\frac{R_3(s_3)}{16q+4r}| = |m||\frac{16+4p+m}{16+4r}|^4 + 4|r||\frac{16+4p+m}{16q+4r}|^3 +$$

$$|4p+2m||\frac{16+4p+m}{16q+4r}|^2$$

as $r(t) \to \infty$ therefore; $\frac{|R_3(s_3)|}{|16q+4r|} \to o(1)$ □

Therefore from the above results, the λ roots can be evaluated using backward substitution. From Lemma 2.4.1

$s_1 \approx \frac{-4r}{m}$ and backward substitution leads to

$$\lambda_1 = \frac{i(\frac{-4r}{m})+1}{i(i(\frac{-4r}{m})-1)} = \frac{-4ir+m}{-4ir-m}.$$

By rationalizing the denominator we get

$$\lambda_1 = \frac{-m+4ir}{m+4ir}$$

$$\approx (m^2+16r^2)^{-1}\{m^2+8irm+16r^2\}$$

$$\approx \quad \frac{1}{16r^2}\Big\{1+\frac{m^2}{16r^2}\Big\}\{16r^2-m^2+8irm\}$$

By use of binomial theorem,

$$\begin{aligned}
&\approx \quad \frac{1}{16r^2}\Big\{1-\frac{m^2}{16r^2+\frac{m^4}{265r^4}+...}\Big\}\{16r2-m^2+8irm\}\\
&\approx \quad \frac{1}{16r^2}\Big\{16r^2+8irm-2m^2+\frac{m^4}{16r^2}-\frac{irm^3}{2r^2}\Big\}\\
&\approx \quad 1+\frac{im}{2r}+O(r^{-2}).
\end{aligned}$$

Thus $|\lambda_1| \approx 1$.

The other roots can be approximated using the relation,

$$4sr^3+(4p+2m)s^2+(16q+4r)s=0$$

$$s_{2\pm} \approx -\tfrac{(4p+m)}{8r} \pm \Big\{\tfrac{(4p+m)^2-16r(16q+4r)}{8r}\Big\}^{\frac{1}{2}}$$

$$s_{2\pm} \approx -\tfrac{(4p+m)}{8r} \pm \Big\{\tfrac{(16p^2+8pm+m^2-256rq-64r^2)}{16r^2}\Big\}^{\frac{1}{2}}$$

$$s_{2\pm} \approx -\tfrac{(4p+m)}{8r} \pm \Big\{-1-\tfrac{4q}{r}0(r^{-2})\Big\}^{\frac{1}{2}}$$

$$s_{2\pm} \approx -\tfrac{(4p+m)}{8r} \pm i\Big\{1+\tfrac{4q}{r}\Big\}^{\frac{1}{2}}$$

$$s_{2+} \approx \tfrac{-(4p+m)}{8r} + i\Big\{1+\tfrac{2q}{r}\Big\}$$

$$s_{2-} \approx \tfrac{-(4p+m)}{8r} - i\Big\{1+\tfrac{2q}{r}\Big\}$$

$$\lambda_2 \approx= \frac{is_{2_+}+1}{is_{2_+}-1}$$

$$= \frac{-1-\frac{2q}{r}-i(\frac{4p+m}{8r})+1}{-1-\frac{2q}{r}-i(\frac{4p8r+m}{8r})-1}$$

$$\approx \frac{-\frac{2q}{r}-i(\frac{4p+m}{8r})}{-2-\frac{2q}{r}-i(\frac{4p+m}{8r})}$$

$$\approx \Big\{-\frac{2q}{r}-i(\frac{4p+m}{8r})\Big\}\Big\{1+\frac{q}{r}+i(\frac{4p+m}{8r})\Big\}^{-1}$$

$$\approx -\frac{1}{2}\Big\{-\frac{2q}{r}-i(\frac{4p+m}{8r})\Big\}-\frac{1}{2}\Big\{1+\frac{2q}{r}+i(\frac{4p+m}{8r})\Big\}^{-1}$$

$$\lambda_2 \approx \frac{q}{r}+i\frac{(4p+m)}{16r}+0(r^{-2})$$

$$\Rightarrow |\lambda_2| \approx O(|\frac{q}{r}|)$$

$$\lambda_3 = \frac{is_{2_-}+1}{is_{2_-}-1}$$

$$= \frac{(1+\frac{2q}{r})-i(\frac{4p+m}{8r})+1}{(1+\frac{2q}{r})-i(\frac{4p8r+m}{8r})-1}$$

$$\approx \frac{2+\frac{2q}{r}-i(\frac{4p+m}{8r})}{\frac{2q}{r}-i(\frac{4p+m}{8r})}$$

$$\approx \frac{r}{2q}\Big\{2+\frac{2q}{r}-i(\frac{4p+m}{8r})\Big\}\Big\{1-i(\frac{4p+m}{16q})\Big\}^{-1}$$

$$\lambda_3 \approx \frac{r}{q}+i\frac{(4p+m)r}{16q^2}+1...$$

$$\lambda_3 \approx \frac{r}{q}+i\frac{(4p+m)r}{16q^2}+....$$

Thus $|\lambda_3| \approx O(\frac{r}{q})$.

Finally the fourth $\lambda - root$ is obtained from $s_4 \approx \frac{-(16+4p+m)}{16+4r}$,whose expansion leads to

$$s_4 \approx -(16 + 4p + m)(16q + 4r)^{-1}$$

$$s_4 \approx -\frac{1}{r}(4 + p + m) + 0(r^{-2})$$

Thus $\lambda_4 \approx -1 + \frac{2i}{r}(4 + p + m) + 0(r^{-2})$, as $t \to \infty, |\lambda_4| \approx 1$.

2.5 Dichotomy Condition

Once we have known the approximate values for the roots of the Fourier polynomial $\mathcal{P}(t, \lambda, z)$, one has enough ingredients to establish the uniform dichotomy condition for the eigenvalues of the difference operator. The dichotomy condition is only needed for λ_1 and λ_4 since as $t \to \infty$

$$|\lambda_1| \approx |\lambda_4| \approx 1.$$

The result below which is in Nyamwala [16] simplifies the proof for dichotomy condition and will just be stated without proof since the proof can be obtained in the said reference.

Theorem 2.5.1

Let

$$u(t + 1) = [\Lambda(t) + R(t)]u(t), t \geq 0 \tag{2.8}$$

$$\Lambda(t) = diag(\lambda_1(t,z)......\lambda_{2n}(t,z))$$

be asymptotically constant difference equation such that,

$$\sum_{t=t_0}^{t-1} \|R(t)\| |\lambda_i^{-1}(t,z)| < \infty.$$

Assume eigenvalues $\lambda_i(t,z)$ *for* $i = 1,..,2n$ *satisfy*

$$\lambda_{i,0} + \lambda_{i,1} + \lambda_{i,2}$$

with $\lambda_{i,0}$ *constant,* $\lambda_{i,1}(t,z) \to 0$ *as* $t \to \infty$, $\lambda_{i,2}$ *is conditionally summable and* $\lambda_{i,0}$ *is conditionally distinct. Let* $h(t) > 0$ *be non-summable,monotonic function in* $\mathbb{N}$ *and assume the eigenvalues* $\lambda_i(t,z)$ *can be assorted into classes* $c_1,..,c_n$ *so that if* $\lambda_i(t,z), \lambda_j(t,z) \in c_k$ *then*

$$\Big(\frac{|\lambda_i(t,z)|}{|\lambda_j(t,z)| - 1}\Big) = o(h(t)).$$

If

$$\lambda_i(t,z) \in c_k, \quad \lambda_j(t,z) \in c_l, \quad k \neq l$$

then either

$$\frac{|\lambda_i(t,z)|}{|\lambda_j(t,z)|} \leq 1 - h(t)$$

or

$$\frac{|\lambda_i(t,z)|}{|\lambda_j(t,z)|} \geq 1 + h(t).$$

For each $\lambda(t,z)$ *write now* $|\lambda(t,z)| = 1 + \mu(t)$ *with* $\mu_+ = \max \ (0,\mu)$ *and*

$\mu_- = \min(0, \mu)$ *and define for each class* k

$$a_k(t) = \max_{\lambda \in c_k} \mu(t)_+ \ \text{ and } \ b_k(t) = \max_{\lambda \in c_k} \mu(t)$$

.

Then associated to each c_k *there are* $|c_k|$, *is the number of elements in the* k^{th} *class, solutions* $u(t)$ *satisfying*

$$K_1 \prod_{t=t_0}^{t-1} (1 - b_k(t)) \leq \| u(t) \| \leq K_2 \prod_{t=t_0}^{t-1} (1 + a_k(t))$$

The conditionally summable terms can be removed by a simple transformation $\prod_{t=t_0}^{t-1} \Lambda_{i,2}(s)$. The rest of the proof will follow by iteration and is identical to the proof of Theorem 5.1 in [6].

Remark 2.5.2

The theorem implies that the uniform dichotomy condition is proved only for eigenvalues $|\lambda| \approx 1$ since those eigenvalues with $|\lambda| < 1$ and $|\lambda| > 1$ will lead to z-uniformly square summable and non-square summable eigensolutions respectively.

Theorem 2.5.3

Let $z \in \kappa_\epsilon(z_o)$ *so that the* $\lambda-$ *roots of* $\mathcal{P}(\lambda, t, z)$ *are distinct. If* $z = z_0 + i\eta$ *such that* $\eta > 0$, *that is,*$Imz > 0$ *then* λ_i, *where i=1,2,3 and 4 satisfy* z*-uniform dichotomy condition.*

Proof. By application of Theorem 2.5.1, we need to show the z-uniform dichotomy condition only for λ_1 and λ_4 since λ_2 and λ_3 will lead to z-uniformly square summable and z-uniformly non-square summable eigensolutions irrespective of the dichotomy condition.

Now choose $z \in K_\epsilon(z_0)$ such that $z = z_0 + i\eta$,with $0 < \eta \leq \epsilon$,then by rewriting λ_1 and λ_4 to $O(r^{-2})$ we obtain

$$\lambda_1 \approx 1 + \frac{i(m-z)}{r} \approx (1 + \frac{\eta}{r}) + \frac{i(m-z_0)}{2r}$$

and thus $|\lambda_1| > 1$ off-the real axis while

$$\lambda_4 \approx -1 + \frac{2i(m-z)}{r} \approx (-1 + \frac{\eta}{r}) + \frac{2i(m-z_0)}{r}$$

and therefore $|\lambda_4| < 1$ off- the real axis. This is the required z-uniform dichotomy condition. $\square$

2.6 Diagonalisation

In order to convert the first order system into its Levinson-Benzaid-Lutz form,we need to diagonalise the system.This requires that we compute the eigenvectors corresponding to the eigenvalues $\lambda_j(t,z), \quad j=1,2,3,4$. Using the approach in [1,4,5,16] the components of eigenvectors can be obtained directly from the quasi-differences. In such a case replace $\triangle$ by $(\lambda-1)$ and $y(t+k)$ by λ^k. Thus we have

$$\begin{aligned}
v_1 &= \lambda_j^{-1} \\
v_2 &= (\lambda_j-1)\lambda_j^{-2} \\
v_3 &= p(\lambda_j-1)\lambda_j^{-1}-(\lambda_j-1)^3\lambda_j^{-2}+i\{(\lambda_j-1)^2q\lambda_j^{-1} \\
&\quad +q(\lambda_j-1)^2\lambda_j^{-2}\}-ir \\
v_4 &= (\lambda_j-1)^2\lambda_j^{-2}-iq(\lambda_j-1)\lambda_j^{-1}, \quad j=1,2,3,4
\end{aligned}$$

For simplicity if we compute the leading term only, the terms of diagonolising matrix can be approximated by

$$\mathrm{T(t,z)} = \begin{bmatrix} 1 & \frac{r}{q} & \frac{q}{r} & -1 \\ \frac{im}{2r} & \frac{-r^2}{q^2} & \frac{q}{r} & -2 \\ -ir & \frac{ir^2}{q} & \frac{-r}{q} & -ir \\ \frac{qm}{2r} & \frac{r^2}{q^2} & -iq & -2iq \end{bmatrix}$$

with $\det T(t,z) = O(\frac{r^3}{q})$. Hence $T^{-1}(t,z)$ can be approximated by

$$\mathrm{T}^{-1}(t,z) = \frac{-q}{3r^3}\begin{bmatrix} \frac{2ir^3}{q^2} - \frac{r^3}{q} - \frac{2r^3}{q^3} & \frac{r^3}{q^3} & \frac{-ir^2}{q} & \frac{r^3}{q^3} \\ -2rq & 2rq - 2ir & 2iq & \frac{-2r}{q} \\ \frac{2r^3}{q} + \frac{2ir^3}{q^2} & \frac{-2ir^3}{q^2} & \frac{-2ir^2}{q}\frac{2r^2}{q^2} & \frac{2ir^3}{q^2} \\ \frac{-2r^3}{q} & \frac{r^3}{q^3} & \frac{-ir^2}{q} & \frac{r^3}{q^3} \end{bmatrix}$$

Using the transformation $y(t,z) = T(t,z)v(t,z)$ and applying this to (2.3) we obtain

$$\begin{aligned} v(t+1,z) &= T^{-1}(t+1,z)S(t,z)T(t,z)v(t,z) = T^{-1}(t+1,z)(T(t,z) \\ &\quad -T(t+1,z))\Lambda(t,z)v(t,z) + \\ &\quad \Lambda(t,z)v(t,z) = (\Lambda(t,z) + R(t,z))v(t,z). \end{aligned} \tag{2.9}$$

Here

$$\Lambda(t,z) = T^{-1}(t+1,z)S(t,z)T(t,z)$$

and

$$R(t,z) = -T^{-1}(t+1,z)\triangle T(t,z)\Lambda(t,z) \quad \text{with} \;\; \Lambda(t,z) = \mathrm{diag}(\lambda_j(t,z)).$$

The remainder matrix $R(t,z)$ can then be computed explicitly. Note that the correction terms to the diagonals are given by $R_{jj}(t,z)$ but since $\lambda_2 \approx \frac{q}{r}, \quad \lambda_3 \approx \frac{r}{q}$, these two eigenvalues will lead to square and non-square summable solutions irrespective of the dichotomy condition and the contribution from the perturbing matrix $R(t,z)$. Critical though are the contribution from $R_{11}(t,z)$ and $R_{44}(t,z)$ to $\lambda_1(t,z)$ and $\lambda_4(t,z)$ respectively. However, these are given by

$$\begin{aligned} R_{11}(t,z) &\approx \frac{\lambda_1}{3}\Big\{\frac{i}{q^2}(\triangle(\frac{m}{r})) - \frac{1}{r})\triangle(r) + \frac{1}{q^2}\triangle(\frac{qm}{2r})\Big\} \\ R_{44}(t,z) &\approx \frac{\lambda_4}{3}\Big\{-\frac{1}{r}\triangle(r) + \frac{2i}{q^2}\triangle(q)\Big\} \end{aligned}$$

computed correct to $O(r^{-2})$. The system (2.9)however, is not yet Levinson-Benzaid-Lutz form since the coefficients were assumed to have second difference. Thus a second diagonalisation is required. This requires smoothness and decay conditions on the coefficients. These conditions are obtained from the matrix entries of $R(t,z)$ in (2.9).Therefore,one requires that

$$\frac{\triangle(f)}{f}, \triangle(f) \in \ell^2, \frac{\triangle^2(f)}{f}, (\triangle f)^2, \triangle^2(\frac{r}{f}), \triangle^2(f) \in \ell^1, f = m, p, q, r. \tag{2.10}$$

The second diagonalisation can then be carried out as explained in Behncke and Nyamwala[5]. One can diagonalise

$$y(t+1,z) = (\Lambda(t,z)+V(t,z)+\tilde{R}(t,z)y(t,z), V(t,z) \longrightarrow 0, t \geq a. \quad (2.11)$$

The expression $R(t,z)$ consists of ℓ^2 and ℓ^1 terms .One can thus split $R(t,z)$ into $V(t,z)$ and $\tilde{R}(t,z)$, where $V(t,z) \in \ell^2$ and $\tilde{R}(t,z) \in \ell^1$. A matrix $(I + B(t))$ formed with the eigenvectors of $(\Lambda + V)(t)$ can be used to diagonalise (2.9) .By adjoining the diagonals of $V(t)$ to $\Lambda(t)$ and dropping the spectral parameter z temporarily ,$B(t)$ is constructed as follows:

$$B_{ii} = 0, B_{ij} = (\lambda_j - \lambda_i)^{-1} V_{ij}, i \neq j,\, i,j = 1,...,2n, t \geq a$$

The corrections to the eigenvalues-diagonal are by $(\Lambda_2)_{ii} = (VB)_{ii}, i = 1,...,2n$.Applying the transformation $y(t+1) = (I + B(t))v(t)$ on (2.9) ,one obtains

$$v(t+1) = [(I + B(t+1))^{-1}(I + B(t))][\Lambda(t) + \Lambda_2(t)] +$$
$$(I + B(t+1))^{-1}R(t)(I + B(t))v(t).$$

After the second diagonalisation,the system is now in Levinson-Benzaid-Lutz form and applying Levinson-Benzaid-Lutz theorem, the solutions of (2.3) respectively (1.1) are in the form

$$y_j(t,z) = (\rho_j + r_{jj}(x,z)) \prod_{l=a}^{t-1} (\lambda_j(l,z)) \quad (2.12)$$

where ρ_j is the appropriate normalised eigenvectors and $r_{jj}(x,z) = o(1)$ or simply $r_{jj}(x,z) \in \ell^1$.

Chapter 3

Deficiency Indices and Spectrum

3.1 Introduction

In this chapter we have computed the deficiency index of the operator generated by (1.1) as well as the absolutely continuous spectrum of the self-adjoint extension of L. Besides, we have calculated the spectral multiplicity of this component of spectrum. A simple example of a fourth order difference operator with unbounded coefficient has been discussed too.

3.2 Spectrum of Difference operators

Once the dichotomy condition is settled, we can now apply Levinson's Theorem to obtain the eigenvalues of the matrix

$$S(t,z) = \begin{bmatrix} E & EB \\ CE & I - A^* + CEB \end{bmatrix}$$

Theorem 3.2.1

Assume that

$$p(t), q(t), m(t) = o(r(t)), \text{and } r(t) \to \infty \text{ as } t \to \infty, \tag{3.1}$$

that is $p(t), q(t)$ *and* $m(t)$ *are bounded for all* $t \in \mathbb{N}$ *while* $r(t)$ *is unbounded.Similarly,assume that*

$$\frac{\triangle(f)}{f}, \triangle(f) \in \ell^2, \frac{\triangle^2(f)}{f}, (\triangle f)^2, \triangle^2(\frac{r}{f}), \triangle^2(f) \in \ell^1, f = m, p, q, r,$$

are satisfied ,then $defL = (3,3)$ *if* $r^{-1}(t)$ *is summable and* $\sigma(H)$ *is pure discrete. Moreover, if* $r^{-1}(t)$ *is not summable, then* $defL = (2,2)$ *and* $\sigma_{ac}(H) = \mathbb{R}$ *of spectral multiplicity 1.*

PROOF. We consider the minimal difference operator generated by (1.1) defined on $\ell^2_w(\mathbb{N})$ with w(t)=1,then the difference equation (1.1) is converted into its Hamiltonian system (1.2) and first order system (2.2) using quasi-differences. Thus in order to apply Levinson-Benzaid-Lutz theorem to obtain solutions of (1.1) viz-a-viz those of (2.2), we need the eigenvalues of the matrix $S(t,z)$ which can be computed via its characteristic polynomial,that is

$$\det(S(t,z)) - \lambda I_4 = 0.$$

As explained in chapter 2, we can then choose an appropriate $z \in \mathcal{K}_\varepsilon(z_0)$ such that the characteristic polynomial has distinct roots.These roots are approximately given by

$$\lambda_1 \approx 1 + \frac{im}{2r} + O(r^{-2})$$

$$\lambda_2 \approx \frac{q}{r} + O(r^{-2})$$

$$\lambda_3 \approx \frac{r}{q}$$

$$\lambda_4 \approx -1 + \frac{2im}{r} + O(r^{-2})$$

□

Here we have absorbed z into m. The uniform dichotomy condition is required only for eigenvalues λ_1 and λ_4 since $|\lambda_1| \approx |\lambda_4| \approx 1$ as $t \to \infty$. But this follows immediately from Theorem 2.5.3. The system can now be diagonalised to convert it into Levinson-Benzaid-Lutz form. As in section 2.5 ,this diagonalisation is carried out using the matrix $T(t, z)$ which is computed from the corresponding eigenvectors. Hence applying the transformation $Y(t) = T(t)v(t)$, one obtains

$$v(t+1, z) = (\Lambda(t, z) + R(t, z))v(t, z) \tag{3.2}$$

where

$$\Lambda(t, z) = T^{-1}(t+1, z)S(t, z)T(t, z) \ \ and$$

$$R(t, z) = -T^{-1}(t, z)\triangle T((t, z))\Lambda(t, z)$$

where

$$\Lambda(t, z) = \text{diag}(\lambda_j(t, z)).$$

As a result of the assumptions (2.10), the system (3.1) is not yet in Levinson-Benzaid-Lutz form and can be diagonalised again using a matrix $(I + B(t, z))$ as explained in Section 2.6. After the second diagonalisation, the system is in Levinson-Benzaid-Lutz form and application of Theorem 1.3.1 (Levinson-Benzaid-Lutz theorem) will lead to eigensolutions of the form (2.9). Critical for square summability of the eigensolutions are the magnitude of the eigenvalues ,the correction terms $R_{jj}(t, z)$ after the first diagonalisation and finally the nature of the spectral parameter z.

The eigensolutions corresponding to $\lambda_2(t, z)$ and $\lambda_3(t, z)$ are square and non-square summable irrespective of the correction terms $R_{22}(t, z)$ and $R_{33}(t, z)$, and the nature of the spectral parameter z. For the other eigensolutions, if $z \in \mathbb{R}$, then the correction terms plays a role. Thus if $r^{-1}(t)$ is summable, then $R_{11}(t, z)$ and $R_{44}(t, z)$ are summable since they are $O(r^{-1}(t))$ and hence the eigensolutions $y_1(t, z), y_4(t, z)$ will be square summable implying that $def L = (3, 3)$ and self-adjoint extension operator H of L exists and is defined using α-boundary conditions. All the eigensolutions that are square summable are z-uniformly square summable and the spectrum of H is at most discrete. Suppose that $r^{-1}(t)$ is not summable, then the square summability of $y_1(t, z), y_4(t, z)$ will depend on the nature of z.In this case we choose $z \in \mathcal{K}\varepsilon(z_0)$ such that

$$z = z_0 + i\eta$$

for

$$\eta > 0, 0 < \eta \leq \varepsilon$$

$,z_0, \eta \in \mathbb{R}$

then $y_1(t,z)$ loses its square summability since $|\lambda_1| > 1$ for this z as shown in Theorem 2.5.3 This solution contributes to absolutely continuous spectrum thus only $y_2(t,z)$ and $y_4(t,z)$ are z-uniformly square summable. Therefore $defL = (2,2)$ and the self adjoint extension operator is defined using α-boundary condition both at $t = a$ and $t = \infty$. From the results of Naimark [15] the spectral multiplicity is equal to 1 and since z can be picked arbitrarily in $\mathbb{R}$ and because r(t) is unbounded we have $\sigma_{ac}(H) = \mathbb{R}$ of spectral multiplicity 1.Thus if $F(t,z)$ is 2x4 system of square summable solutions and we use

$$< F(t,z), F(t,z) > (z - \overline{z}) = M^*(z) - M(z),$$

then the rank of M(z) is 1 as $Imz \to 0^+$.

It remains to show that Im M(z) exist boundedly thus we have from Section 2 (page 23), that

$$ImM(z) = \lim_{\eta \to 0} \eta < y_4(t,z), y_4(t,z) > . \tag{3.3}$$

Here we have used $y_4(t,z)$ since its eigenvalue has absolute value of at most 1 and it is square summable even if $Imz \to 0^+$. The computation involving $y_2(t,z)$ is trivial therefore (3.2) leads to

$$\lim_{\eta\to 0} \eta \prod_{l=a}^{t-1} |\lambda_4(l,z)|^2 = \lim_{\eta\to 0} \eta \prod_{l=a}^{t-1} |(-1+\frac{2\eta}{r}) + \frac{2i(m-z)}{r}|.$$

Taking the natural logarithm and using Euler summation formula we have

$$\ln \prod_{l=a}^{t-1} |\lambda_4(l,z)|^2 \approx \sum_{a}^{t-1} 2 \ln |\lambda_4(l,z)| \approx \int_a^{t-1} \exp \frac{-c^2}{|r|}(l,z) dl$$

Where $c \in \mathbb{R}$. The power of the exponent is negative since $O < |-1+\frac{2\eta}{r}| < 1$ leading to a negative logarithm. Therefore as $\eta \to 0$, and $t \to \infty$, we have

$$ImM(z) = \lim_{\eta\to 0} \eta < y_4(t,z), y_4(t,z) >\approx \int_a^{t-1} \exp \frac{-c^2}{|r|}(l,z) dl$$

which goes to zero as $t \to \infty$. Thus $ImM(z)$ exists boundedly and thus spectral function is continuous.

Example 3.2.2

Let L be a fourth order difference operator generated by a difference equation of the form

$$\Delta^4 y(t-2) + i(t^\alpha \Delta y(t-1) + \Delta(t^\alpha y(t)) = zy(t)$$

where $\alpha > 0, \alpha \in \mathbb{R}$ and z is the spectral parameter.Thus we have

$$Q(s,t,z) = -zs^4 + 4t^\alpha s^3 - 2zs^2 + 4t^\alpha s + (16 - z)$$

and hence the roots λ can be approximated by

$$\lambda_1 \approx 1 - \tfrac{izt^{-\alpha}}{2} + O(t^{-2\alpha}),$$
$$\lambda_2 \approx O(t^{-\alpha}),$$
$$\lambda_3 \approx O(t^{\alpha}), \text{ and}$$
$$\lambda_4 \approx -1 - 2izt^{-\alpha} + O(t^{-2\alpha}).$$

Since the coefficients are power coefficients the dichotomy condition will be satisfied for λ_1 and λ_4.
If $0 < \alpha < \frac{1}{2}$ then def L=(3,3),$\sigma_{ac}(H) = \mathbb{R}$.
If $\alpha > \frac{1}{2}$ then def L=(2,2),$\sigma_{ac}(H,1) = \mathbb{R}$.

Chapter 4

Chapterwise Summary

4.1 Conclusion

In this research, it has been proved in chapter 2 Theorem 2.3.2 that there exists a bounded interval which is a subset of $\mathbb{R}$ with no singular continuous spectrum. Similarly under different asymptotic conditions, it was shown (see chapter 3 Theorems 2.3.3 and 3.2.1 together with examples 2.3.5 and 3.2.2) that $defL = (k, k) : 2 \leq k \leq 4$. Finally, we have shown in Theorem 3.2.1 that the absolutely continuous spectrum of H, the self-adjoint extension of L is the whole of $\mathbb{R}$ with spectral multiplicity one when $r(t) \to \infty$ as $t \to \infty$.

4.2 Recomendations

The main obstacle of the analysis of the absolutely continuous spectrum of operators with unbounded coefficients, is the understanding of the zero's of polynomials, here the Fourier polynomials. This is experienced especially when proving some results on the spectral theory of fourth order operators, where the middle terms form an obstacle even though unbounded.Therefore, to solve this, the results should be solved for power coefficients and the classes be extended by allowing much more general coefficients and by including the analysis of the absolutely continuous spectrum as well, whenever possible. This is possible through the refinement of Levinson's Theorem.

One can also investigate the absolutely continuous spectrum of $\mathcal{H}$ when all the coefficients of (1.1) are unbounded.

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