DUALITY PROPERTIES OF NON-REFLEXIVE BERGMAN SPACE OF THE UPPER HALF-PLANE

BY

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Declaration

This thesis is my own work and has not been presented for a degree award in any other institution.

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This thesis has been submitted for examination with our approval as the university supervisors.

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Dedication

To my mother Jane Adhiambo for her support, guidance and prayers.

Abstract

The study of duality properties of the spaces of analytic functions continues to attract the attention of many mathematicians. Most studies have concentrated on the reflexive Hardy and Bergman spaces both on the unit disk and the upper half-plane. For instance, Zhu, Peloso, among others have determined the duality properties of Hardy and Bergman spaces. For the non-reflexive Bergman spaces of the disk, it was proved by Axler that the dual and the predual are identified as big and little Bloch spaces respectively. For non-reflexive Bergman spaces of the upper half-plane $L_a^1(\mathbb{U}, \mu_\alpha)$, the dual is well known as the Bloch space $B_\infty(\mathbb{U}, i)$ but the predual is not known. In our study therefore, we have determined the predual of $L_a^1(\mathbb{U}, \mu_\alpha)$. We have also determined the group of weighted composition operators defined on predual space of $L_a^1(\mathbb{U}, \mu_\alpha)$ and investigated both its semigroup and spectral properties. To determine the predual space of $L_a^1(\mathbb{U}, \mu_\alpha)$, we used the Cayley transform as well as related works on the unit disk by Zhu, Peloso among others. To investigate the properties of the weighted composition groups, we employed functional analysis techniques as well as semigroup theory of linear operators to determine the infinitesimal generator of the semigroup and established the strong continuity property. Using spectral theory, we determined the resolvents of the infinitesimal generator which were obtained as integral operators. Finally, we used known theorems like the Hill-Yosida theorem and spectral mapping theorem to obtain the spectral properties of the obtained integral operators. The results obtained in this study is of great importance to the physicists where the concept of semigroup properties plays a major role in the evolution equations.

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Chapter 1

Introduction

1.1 Background of the study

The study of semigroups of composition operators on the spaces of analytic functions was first considered by Berkson and Porta [8] on Hardy spaces and later on by Siskakis on Bergman and Dirichlet spaces [33, 34]. Most of the work done on the duality properties of these spaces of analytic functions has mainly concentrated on Hardy spaces as in [26, 27] and the reflexive Bergman spaces [36] both on the unit disk and the upper half-plane. For extensive discussion of Bergman spaces and their composition operators, we refer to [14, 19] and references there in. Athanasios [25] identified the semigroups consisting of bounded composition operators on the Hardy spaces of the upper half-plane and determined their infinitesimal generators. Matache [26] later studied the composition operators on Hardy spaces of the right half-plane, boundedness and compactness of the Hardy spaces. The resolvents of the generators of strongly continuous groups of isometries on the Hardy and Bergman spaces were obtained by Bonyo in [11] using the similarity theory of semigroups and spectral theory. The resulting resolvents in [11] were given as integral operators for which the norms and spectra were obtained. The dual and predual of non-reflexive Bergman space on the unit disk are well known to be identified as the Bloch and little Bloch spaces respectively [36]. However, the predual of non-reflexive Bergman space of the upper half-plane is not known. This has motivated our first concern of this study. In our study we intend to determine the duality property of the non-reflexive Bergman space of the upper half-plane, as well as study both the semigroups and spectral properties of composition operators on it.

1.2 Basic Concepts

1.2.1 Vector spaces and Normed spaces

A *vector space* (*linear space*) over a field $\mathbb{K}$ is a nonempty set X together with two algebraic operations of addition and scalar multiplication defined on it such that for all $x, y \in X$ and $\alpha \in \mathbb{K}$, we have, $x+y \in X$ and $\alpha x \in X$. A *norm* $\|.\| : X \longrightarrow \mathbb{K}$ is a function on X such that for all $x, y \in X$, $\alpha \in \mathbb{K}$;

(i) $\|x\| = 0$ if and only if $x = 0$,

(ii) $\|\alpha x\| = |\alpha|\|x\|$,

(iii) $\|x + y\| \leq \|x\| + \|y\|$ (Triangle inequality).

A *normed space* is a vector space with a norm defined on it. A normed space X is said to be *complete* if every cauchy sequence in X converges.

A sequence $(x_n) \subseteq X$ is said to be convergent if for every $\epsilon > 0$ there exists a number M such that for every $n \geq M$, we have $\|x_n - x\| < \epsilon$ for all $x_n, x \in X$. A sequence of vectors $(x_n) \subseteq X$ in a normed space is said to be cauchy sequence if for every $\epsilon > o$, there exist a number M such that $\|x_m - x_n\| < \epsilon$ for all $m, n > M$. A *Banach space* is a complete normed space (complete in the metric defined by the norm).

An *inner product space* (or *pre-hilbert space*) is a vector space X with an inner product defined on X. Here, an inner product on X is a mapping from $X \times X$ into the scalar field $\mathbb{K}$; that is, with every pair of vectors x and y there is an associated scalar which is written as $\langle x, y\rangle$ and is called the inner product of x and y, such that for all vectors x, y, z and scalar α, we have,

(i) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$,

(ii) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$,

(iii) $\langle x, y\rangle = \overline{\langle y, x\rangle}$,

(iv) $\langle x, x\rangle \geq 0$, and

(v) $\langle x, x\rangle = 0 \Leftrightarrow x = 0$.

An inner product on X defines a norm on X given by

$$\|x\| = \sqrt{\langle x, x\rangle}.$$

A *Hilbert space* is a complete inner product space (complete in the metric defined by the inner product). Hence inner product spaces are normed spaces and Hilbert spaces are Banach spaces. We refer to [13, 28, 36] for details.

1.2.2 Dual spaces

Let X be a vector space over the field $\mathbb{F}$. A *linear functional* on X is a linear map $\phi : X \longrightarrow \mathbb{F}$. The set of all linear functionals on X is a vector space denoted by X^*, and called the dual space of X, where the vector space operations are defined pointwise, that is;
$(\phi+\psi)(x) := \phi(x)+\psi(x)$ and $(\alpha\phi)(x) := \alpha\phi(x)$ for all $\phi, \psi \in X^*$, $\alpha \in \mathbb{F}$.
An element of X^* is said to be a *bounded linear functional* on X.
Also $X^{**} = (X^*)^*$ is the bidual or second dual of the normed space. For all $x \in X$, define a linear functional $\widehat{x}$ on X^* by setting $\widehat{x}(\phi) = \phi(x)$ and $|\widehat{x}(\phi)| = |\phi(x)| \leq \|\phi\|\|x\|$ so that $\widehat{x} \in X^{**}$ with $\|\widehat{x}\| \leq \|x\|$.
A normed space X is called *reflexive* if it satisfies the following equivalent conditions;

1. The evaluation map $\phi : X \longrightarrow X^{**}$ is surjective.
2. The evaluation map $\phi : X \longrightarrow X^{**}$ is an isometric isomorphism of normed spaces.

A reflexive space X is a Banach space, since X is isometric to the Banach space X^{**}.

1.2.3 Unit disk and upper half plane

Let $\mathbb{C}$ be the complex plane. The set $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is called the *open unit disc*. Let dA denote the area measure on $\mathbb{D}$, normalized so that the area of $\mathbb{D}$ is 1. In terms of rectangular and polar coordinates, we have: $dA(z) = \frac{1}{\pi}dxdy = \frac{r}{\pi}drd\theta$, where $z = x + iy = re^{i\theta} \in \mathbb{D}$. For $\alpha \in \mathbb{R}, \alpha > -1$, we define a positive Borel measure dm_α on $\mathbb{D}$ by $dm_\alpha(z) = (1 - |z|^2)^\alpha dA(z)$. Moreover, if $\alpha = 0$, then $dm_0 = dA$. Thus we consider dm_α as a weighted measure and a generalization of dA.

On the other hand, the set $\mathbb{U} := \{\omega \in \mathbb{C} : Im(\omega) > 0\}$ denotes the upper half of the complex plane $\mathbb{C}$ with $Im(\omega)$ being the imaginary part of $\omega \in \mathbb{C}$. For $\alpha > -1$, we define a weighted measure on $\mathbb{U}$ by $d\mu_\alpha(\omega) = (Im(\omega))^\alpha dA(\omega)$ where $\omega \in \mathbb{U}$. Again it can easily be seen that $\alpha = 0$ coincides with the unweighted measure.

The function $\psi(z) = \frac{i(1+z)}{1-z}$ is referred to as the *Cayley transform* and maps the unit disc $\mathbb{D}$ conformally onto the upper half-plane $\mathbb{U}$ with the inverse $\psi^{-1}(\omega) = \frac{\omega-i}{\omega+i}$.

Let $\partial\mathbb{D}$ be the boundary of the disk $\mathbb{D}$. Then for any $z = e^{i\theta} \in \partial\mathbb{D}$ and $h > 0$, let

$$\mathbb{B}_h(z) = \{w = re^{it} \in \mathbb{D} : 1 - h \leq r < 1, |t - \theta| \leq h\}.$$

$\mathbb{B}_h(z)$ is called a Carleson square or a sector at $z \in \partial\mathbb{D}$. Given a positive Borel measure μ on $\mathbb{D}$, we say that μ is a Carleson measure if

$$\|\mu\| = \sup\left\{\frac{\mu(\mathbb{B}_h(z))}{h} : z \in \partial\mathbb{D}, h > 0\right\} < \infty.$$

For further details, see [13, 28, 36].

1.2.4 Analytic functions

Holomorphic function is a complex-valued function of one or more complex variables, that is, at every point of its domain, complex differentiable in a neighbourhood of the point. The existence of a complex derivative in a neighbourhood is a very strong condition, for it implies that any holomorphic function is actually infinitely differentiable and equal, locally, to its own Taylor series (analytic). A biholomorphism is a map which is bijective and holomorphic (then its inverse is also holomorphic). Analytic self-map is a mapping $f : \Omega \to \Omega$ of a domain Ω onto itself.
Homomorphism is a structure-preserving map between two algebraic structures of the same type (such as two groups, two rings or two vector spaces). Automorphism is simply a bijective homomorphism of an object with itself, that is, it is a way of mapping the object to itself while preserving all of its structure. The set of all automorphisms of an object forms a group. For further details, see [28, 36].

1.2.5 Analytic spaces of interest

For an open subset Ω of $\mathbb{C}$, let $\mathcal{H}(\Omega)$ denote the space of analytic functions on Ω.

(i) **Bergman and Hardy spaces**

For $1 \leq p < \infty$, $\alpha > -1$, the weighted Bergman space of the upper half-plane $\mathbb{U}$ is defined by
$L^p_a(\mathbb{U}, \mu_\alpha) := \left\{ f \in \mathcal{H}(\mathbb{U}) : \|f\|_{L^p_a(\mathbb{U},\mu_\alpha)} = \left(\int_{\mathbb{U}} |f(z)|^p d\mu_\alpha(z) \right)^{\frac{1}{p}} < \infty \right\}$.
In particular, $L^p_a(\mathbb{U}, \mu_\alpha) = L^p(\mathbb{U}, \mu_\alpha) \cap \mathcal{H}(\mathbb{U})$, where $L^p(\mathbb{U}, \mu_\alpha)$ or simply $L^p(\mu_\alpha)$ denotes the classical Lebesque spaces associated with the weighted measure $d\mu_\alpha$. It is important to note that the case $\alpha = 0$ yields the unweighted Bergman space. Also $\|.\|_{L^p_a(\mathbb{U},\mu_\alpha)}$ defines a norm on $L^p_a(\mathbb{U}, \mu_\alpha)$.
$L^p_a(\mathbb{U}, \mu_\alpha)$ is a Banach space with respect to the norm

$$\|f\|_{L^p_a(\mathbb{U},\mu_\alpha)} = \left(\int_{\mathbb{U}} |f(z)|^p d\mu_\alpha(z) \right)^{\frac{1}{p}} < \infty.$$

For $p = 2$, $L^2_a(\mathbb{U}, \mu_\alpha)$ is a Hilbert space. The growth condition for the weighted Bergman spaces is given as follows: For every $f \in L^p_a(\mathbb{U}, \mu_\alpha)$ and $\omega \in \mathbb{U}$, there exist a constant K such that,

$$|f(\omega)| \leq \frac{K\|f\|}{(Im(\omega))^\gamma},$$

where $\gamma = \frac{\alpha+2}{p}$.
While for $1 \leq p < \infty$, $\alpha > -1$, the Hardy spaces of the unit disk are defined by

$$H^p(\mathbb{D}) := \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|^p_{H^p(\mathbb{D})} = \sup_{0<r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta < \infty \right\}.$$

The Hardy spaces of the upper half plane $H^p(\mathbb{U})$ are also defined as

$$H^p(\mathbb{U}) := \left\{ f \in \mathcal{H}(\mathbb{U}) : \|f\|_{H^p(\mathbb{U})} = \sup_{y>0} \left(\int_{-\infty}^{\infty} |f(x+iy)|^p dx \right)^{\frac{1}{p}} < \infty \right\}.$$

$\|.\|_{H^p(.)}$ defines a norm on the Hardy spaces, and therefore H^p spaces are Banach spaces with respect to the norms both on the unit disk and on the upper half-plane. For $p = 2$, $H^2(\mathbb{D})$ and $H^2(\mathbb{U})$ are Hilbert spaces.

For $H^p(\mathbb{D})$, we state the following well known growth condition: For every $f \in H^p(\mathbb{D})$, there exists a positive constant C such that

$$|f(z)| \leq \frac{C\|f\|_{H^p(\mathbb{D})}}{(1-|z|)^{\frac{1}{p}}},$$

for every $z \in \mathbb{D}$. For further details, see [13, 28, 36].

(ii) **Bounded Mean Oscillation Analytic (BMOA) and Vanishing Mean Oscillation Analytic (VMOA) spaces**

BMOA is the Banach space of all analytic functions in the Hardy space $H^2(\mathbb{D})$ whose boundary values have bounded mean oscillation. We give the characterization of this space in terms of Carleson measures: A function $f \in H^2(\mathbb{D})$ belongs to BMOA if and only if there exists a constant $C > 0$ such that

$$\int_{R(I)} |f'(z)|^2 (1-|z|^2) dA(z) \leq C|I|,$$

for any arc $I \subset \partial\mathbb{D}$, where $R(I)$ is the Carleson rectangle determined by I, that is

$$R(I) \;:=\; \left\{ re^{i\theta} \in \mathbb{D} : 1 - \frac{|I|}{2\pi} < r < 1, e^{i\theta} \in I \right\}.$$

As usual, $|I|$ denotes the length of I and $dA(z)$ the normalized Lebesque measure on $I \subset \partial\mathbb{D}$. The corresponding BMOA norm is

$$\|f\|_{BMOA} \;:=\; |f(0)| + \sup_{I \subset \partial\mathbb{D}} \left(\frac{1}{|I|} \int_{R(I)} |f'(z)|^2 (1 - |z|^2) dA(z) \right)^{\frac{1}{2}}.$$

Trivially, each polynomial belongs to BMOA.
The closure of all polynomials in BMOA is denoted by VMOA. Alternatively, VMOA is the subspace of BMOA formed by those functions $f \in BMOA$ such that

$$\lim_{|I| \to 0} \frac{1}{|I|} \int_{R(I)} |f'(z)|^2 (1 - |z|^2) dA(z) \;= 0.$$

For more details, we refer to [10].

(iii) **Bloch and Little Bloch spaces**

The Bloch space of the unit disk, denoted by $B_\infty(\mathbb{D})$, is defined by $B_\infty(\mathbb{D}) := \{f \in \mathcal{H}(\mathbb{D}) : \|f\|_{B_{\infty,1}(\mathbb{D})} = \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)| < \infty\}$. The norm on $B_\infty(\mathbb{D})$ is given by $\|f\|_{B_\infty(\mathbb{D})} := |f(0)| + \|f\|_{B_{\infty,1}(\mathbb{D})}$, while $\|.\|_{B_{\infty,1}(\mathbb{D})}$ is a seminorm. Hence $B_\infty(\mathbb{D})$ is a Banach space with respect to the norm $\|.\|_{B_\infty(\mathbb{D})}$.
On the other hand, the Bloch space of the upper half plane denoted by $B_\infty(\mathbb{U})$ is defined by

$B_\infty(\mathbb{U}) := \{f \in \mathcal{H}(\mathbb{U}) : \|f\|_{B_{\infty,1}(\mathbb{U})} = \sup_{\omega\in\mathbb{U}} Im(\omega)|f'(\omega)| < \infty\}$. $B_\infty(\mathbb{U})$ is also a Banach space with respect to the norm given by $\|f\|_{B_\infty(\mathbb{U})} = |f(i)| + \|f\|_{B_{\infty,1}(\mathbb{U})}$.

The little Bloch space of the unit disk denoted by $B_{\infty,\circ}(\mathbb{D})$ is defined as

$$B_{\infty,\circ}(\mathbb{D}) := \left\{ f \in \mathcal{H}(\mathbb{D}) : \lim_{|z|\longrightarrow 1} (1-|z|^2)|f'(z)| = 0 \right\}$$

but with the same norm as $B_\infty(\mathbb{D})$ and therefore is also a Banach space. While on the upper half-plane, the little Bloch space is denoted by $B_{\infty,\circ}(\mathbb{U})$ and is defined by

$$B_{\infty,\circ}(\mathbb{U}) := \left\{ f \in \mathcal{H}(\mathbb{U}) : \lim_{Im(\omega)\longrightarrow 0} Im(\omega)|f'(\omega)| = 0 \right\}$$

but with the same norm as $B_\infty(\mathbb{U})$ and hence a Banach space.

For further details, see [13, 17, 28, 36].

(iv) **Dirichlet spaces**

For $\alpha \geq 0$, the weighted Dirichlet spaces of the disk $\mathbb{D}$, $\mathcal{D}_\alpha(\mathbb{D})$ consists of those analytic functions f on $\mathbb{D}$, $f \in \mathcal{H}(\mathbb{D})$, such that

$$\|f\|_{\mathcal{D}_\alpha(\mathbb{D})} \quad = \quad \left(|f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dm_\alpha(z) \right)^{\frac{1}{2}} < \infty.$$

$(\mathcal{D}_\alpha, \|.\|)$ is a Banach space with respect to the norm $\|.\|_{\mathcal{D}_\alpha}$.

(v) **Besov spaces**

An analytic function f is in the Besov space B_p if

$$\|f\|_{B_p} \quad := \quad \left(\int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)\right)^{p-2} dA(z) < \infty,$$

that is, the function $(1-|z|^2)f' \in L^p(\mathbb{D}, d\lambda)$, where $d\lambda(z) = (1-|z|^2)^{-2}dA(z)$.

1.2.6 Duality of Bergman Spaces

For $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha > -1$, the dual space of the Bergman space on the disk $L^p_a(\mathbb{D}, m_\alpha)$ is given by

$$(L^p_a(\mathbb{D}, m_\alpha))^* \quad \approx \quad L^q_a(\mathbb{D}, m_\alpha),$$

under the integral pairing

$$\langle g, f\rangle \quad = \quad \int_{\mathbb{D}} g(z)\overline{f(z)}dm_\alpha(z),$$

where $g \in L^p_a(\mathbb{D}, m_\alpha)$ and $f \in L^q_a(\mathbb{D}, m_\alpha)$.
Also for $1 < p < \infty$, $\frac{1}{p}+\frac{1}{q} = 1$ and $\alpha > -1$, the dual space of the Bergman space of the upper half-plane $L^p_a(\mathbb{U}, \mu_\alpha)$ is given by

$$(L^p_a(\mathbb{U}, \mu_\alpha))^* \quad \approx \quad L^q_a(\mathbb{U}, \mu_\alpha),$$

under the integral pairing

$$\langle g, f\rangle = \int_{\mathbb{U}} g(\omega)\overline{f(\omega)}d\mu_\alpha(w),$$

where $g \in L^p_a(\mathbb{U}, \mu_\alpha)$ and $f \in L^q_a(\mathbb{U}, \mu_\alpha)$.
It is important to note that for $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha > -1$, the Hardy and Bergman spaces ($H^p(.)$ and $L^p_a(.)$) are reflexive

For $p = 1$, the Bergman spaces $L^1_a(.)$ are non-reflexive and the dual and predual spaces of the Bergman space $L^1_a(\mathbb{D}, m_\alpha)$ are respectively given by

$$(L^1_a(\mathbb{D}, m_\alpha))^* \approx B_\infty(\mathbb{D}),$$

under the usual pairing

$$\langle g, f\rangle = \int_{\mathbb{D}} g(z)\overline{f(z)}dm_\alpha(z),$$

where $g \in L^1_a(\mathbb{D}, m_\alpha)$ and $f \in B_\infty(\mathbb{D})$, and

$$(B_{\infty,\circ}(\mathbb{D}))^* \approx L^1_a(\mathbb{D}, m_\alpha),$$

under the same pairing

$$\langle g, f\rangle = \int_{\mathbb{D}} g(z)\overline{f(z)}dm_\alpha(z),$$

where $g \in L^1_a(\mathbb{D}, m_\alpha)$ and $f \in B_{\infty,\circ}(\mathbb{D})$. We refer to [29, 36] for details.

1.2.7 Semigroups of Linear Operators

Let X be a Banach space. A one-parameter family $(T_t)_{t\geq 0}$ is a *semigroup* of bounded linear operators on X, if

1. $T_0 = I$ (Identity operator on X), and
2. $T_{t+s} = T_t \circ T_s$ for every $t, s \geq 0$ (semigroup property).

A semigroup $(T_t)_{t\geq 0}$ of bounded linear operators on X is strongly continuous if $\lim_{t\to 0^+} T_t x = x$ or $\lim_{t\to 0^+} \|T_t x - x\| = 0$ for all $x \in X$. The infinitesimal generator Γ of $(T_t)_{t\geq 0}$ is defined by $\Gamma x := \lim_{t\to 0^+} \frac{T_t x - x}{t} = \frac{\partial}{\partial t}(T_t x)|_{t=0}$ for each $x \in \mathrm{dom}(\Gamma)$, where the domain of Γ is given by $\mathrm{dom}(\Gamma) = \left\{x \in X : \lim_{t\to 0^+} \frac{T_t x - x}{t} \text{ exists}\right\}$.
We refer to [13, 17, 28] for more details on semigroup theory.

1.2.8 Spectra of Linear Operators

Let $(X, \|.\|)$ and $(Y, \|.\|)$ be Banach spaces over $\mathbb{C}$. The space $\mathcal{L}(X, Y) = \{T : X \to Y \text{ such that } T \text{ is linear and continuous}\}$, endowed with the operator norm $\|T\| = \sup_{\|x\|\leq 1} \|Tx\|$ is a Banach space. We write $\mathcal{L}(X, X) = \mathcal{L}(X)$. Now, T is said to be a *closed operator* if its graph
$G(T) := \{(x, Tx) \mid x \in \mathrm{dom}(T)\} \subseteq X \times Y$ is closed.

Theorem 1.2.1 (Closed graph theorem)
Let X and Y be Banach spaces. Then every closed linear mapping $T : X \to Y$ is continuous.

Let T be a closed operator on X. The *resolvent set* of T, $\rho(T)$, is given by $\rho(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is invertible}\}$ and its *spectrum* $\sigma(T) = \mathbb{C} \setminus \rho(T)$ .Therefore, $\sigma(T) \cup \rho(T) = \mathbb{C}$. The *spectral radius* of T is defined by $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$ with the relation $r(T) \leq \|T\|$. The *point spectrum* $\sigma_p(T) = \{\lambda \in \mathbb{C} : Tx = \lambda x \text{ for some } 0 \neq x \in \text{dom}(T)\}$. For $\lambda \in \rho(T)$, the operator $R(\lambda, T) := (\lambda I - T)^{-1}$ is, by the closed graph theorem, a bounded operator on X and is called the *resolvent of* T *at the point* λ or simply the *resolvent operator*. In fact, $\rho(T)$ is an open subset of $\mathbb{C}$ and $R(\lambda, T) : \rho(T) \to \mathcal{L}(X)$ is an analytic function with respect to λ. For further details, see [13, 17, 22, 28].

1.2.9 Composition operators

Let $\varphi : \mathbb{U} \to \mathbb{U}$ be a self-analytic map on $\mathbb{U}$. Then for any $f \in \mathcal{H}(\mathbb{U})$, the composition operator induced by φ, denoted by C_φ, is a mapping $C_\varphi : \mathcal{H}(\mathbb{U}) \to \mathcal{H}(\mathbb{U})$ defined by $C_\varphi(f) = f \circ \varphi$. Now, let $\text{Aut}(\mathbb{U})$ denotes the group of all automorphisms of $\mathbb{U}$. For $\varphi_t \in \text{Aut}(\mathbb{U})$, $t \geq 0$, the group of composition operators associated with φ_t are defined on $\mathcal{H}(\mathbb{U})$ by $C_{\varphi_t}(f) = f \circ \varphi_t$ for all $f \in \mathcal{H}(\mathbb{U})$. The corresponding group of weighted composition operators induced by φ_t will therefore be defined by
$T_t(f) := S_{\varphi_t} f = (\varphi_t')^\gamma f o \varphi_t$, where γ is an appropriately chosen weight. We refer to [13, 17, 28, 36] for more details.

1.3 Statement of the Problem

The study of duality properties of the spaces of analytic functions has mainly concentrated on the Hardy and reflexive Bergman spaces as covered in literature. For the non-reflexive Bergman space of the unit disk, $L^1_a(\mathbb{D}, m_\alpha)$, the dual and predual have been identified with Bloch and the little Bloch spaces respectively. Recently, the dual of the non-reflexive Bergman space of the upper half-plane $L^1_a(\mathbb{U}, \mu_\alpha)$ was determined while the predual still remain unknown. Moreover, the properties of semigroups of composition operators on these spaces have not been exhaustively studied. In this study therefore, we have determined the predual of the non-reflexive Bergman space $L^1_a(\mathbb{U}, \mu_\alpha)$ of the upper half-plane. Further, we have investigated both the semigroup as well as the spectral properties of the group of weighted composition operators defined on the predual of $L^1_a(\mathbb{U}, \mu_\alpha)$.

1.4 Objectives of the Study

The main objective of the study was to determine the duality properties of non-reflexive Bergman space of the upper half-plane $L^1_a(\mathbb{U}, \mu_\alpha)$ and investigate the properties of the groups of weighted composition operators defined on them.

The specific objectives were to:

1. Determine the predual of the non-reflexive Bergman space of the upper half-plane $L^1_a(\mathbb{U}, \mu_\alpha)$.

2. Determine the group of weighted composition operators on the predual space of non-reflexive Bergman space of the upper half-plane $L_a^1(\mathbb{U}, \mu_\alpha)$.

3. Investigate both the semigroup and spectral properties of the group of weighted composition operators defined on the predual space of the non-reflexive Bergman space $L_a^1(\mathbb{U}, \mu_\alpha)$.

1.5 Research methodology

To determine the predual of non-reflexive Bergman space of the upper half plane $L_a^1(\mathbb{U}, \mu_\alpha)$, we used the Cayley transform as well as related works on the unit disk $\mathbb{D}$ by Zhu, Peloso among others. Using the definition of weighted composition operators as well as the duality pairing for the established predual, we determined the group of weighted composition operators on the predual space of non-reflexive Bergman space of the upper half plane $L_a^1(\mathbb{U}, \mu_\alpha)$. To investigate the semigroup properties of the group of weighted composition operators, we employed theory of semigroups of the Linear operators and functional analysis where we determined infinitesimal generator of the group of weighted composition operator obtained, then established the strong continuity property. Using spectral theory, we obtained the resolvents of the infinitesimal generator which was given as integral operators. Finally, we used the known results like spectral mapping theorems for resolvents and semigroups, among other functional analysis theories to obtain the spectral properties of the infinitesimal generator as well as the resolvents. Then, we applied the

Hille-Yosida theorem to obtain the norm properties of the resulting integral operators.

1.6 Significance of the Study

The study of duality properties of the spaces of analytic functions have been investigated by many mathematicians, although on the non-reflexive Bergman spaces, it has not been fully exhausted specifically on the identification of predual space on the upper half-plane. We hope that our study will contribute richly to the existing literature as well as advance further research for the development of this area of research. Its therefore of great significance to determine the duality properties of the non-reflexive Bergman spaces of the upper half-plane, $L^1_a(\mathbb{U}, \mu_\alpha)$ and the properties of both semigroups and spectral of the group of weighted composition operators defined on the predual of $L^1_a(\mathbb{U}, \mu_\alpha)$. We hope the results obtained in this study shall be of importance to both applied mathematicians and theoretical physicists where the concept of semigroup properties plays a major role in evolution equations in Physics.

Chapter 2

Literature Review

The theory of semigroups of bounded linear operators began with the work of Hille and Yosida in [35]. On spaces of analytic functions they were first studied by Berkson and Porta in [8] on Hardy spaces and later on by Siskakis on Bergman and Dirichlet spaces [33, 34]. Zhu in his study gave broad definitions and basic facts of the spaces and for the composition semigroups on these spaces as in [36]. A lot have been done on semigroups but of recent work by Basallote and Blasco [7, 9, 10] the authors considered semigroups of composition operators on the Bloch space and weighted Banach spaces of analytic functions. Composition operators, induced by a fixed analytic self-map of the upper half-plane, acting between Hardy and Bloch-type spaces of the upper half-plane was also studied by Sharma, et al [32]. Weighted composition operators also appeared in the study of classical operators like Cesáro and Hilbert operators and they play an important role in the study of composition operators on Hardy spaces of half-plane.

For thorough discussion of Bergman spaces and their composition operators we refer the reader to the work by [14, 19]. In 1997, Valentine

Matache [26] studied the composition operator on Hardy spaces of right half plane, boundedness and compactness on the spaces. The resolvents of the generators of strongly continuous groups of isometries on the Hardy and Bergman spaces were obtained by Bonyo in [11] using the similarity theory of semigroups and spectral theory. The resulting resolvents in [11] were given as integral operators for which the norms and spectra were obtained. Recently, Ballamoole, Bonyo, L. Miller and G. Miller in [6] constructed integral operators associated with strongly continuous groups of invertible isometries on the Hardy spaces and the weighted Bergman spaces of the upper half-plane. Specifically, they obtained the spectrum and point spectrum of the generator and represented resolvents as integral operators related to the ces'aro's operators.
In [27], Matache also proved that composition operators are bounded on the Hardy space of the half plane if and only if the inducing map fixes the point at infinity and it has a finite angular derivative there. Later Elliot and Jurry's calculation strengthened the result on non-compactness of composition operator studied by Matache in [27]. Cima, Thomson and Wogen [12] characterized the closed-range composition operators on Hardy spaces, phrasing their result in terms of the boundary behavior of the inducing function . They also characterized the Fredholm composition operators as precisely the invertible ones.
MacCluer and Shapiro [25] characterized boundedness and compactness of composition operators in terms of Carleson measures. MacCluer [25] also studied the connection between angular derivative and components in the space of composition operators . Jarchow, Hunziker and Maschioni [21] studied similar problems for other situations that is in the class of

Hilbert-Schmidt Composition operators and in the topology induced by the Hilbert -Schmidt norm. In the paper by Mark and Fiona [15], conditions were established in which these semigroups can be extended in their parameter to sector given a priori and complete characterization of all composition operators acting on the Hardy space on the right half-plane. In 2013, Arvanitidis [4] identified the semigroups consisting of bounded composition operators on the Hardy spaces of the upper half-plane and finally the identification of the infinitesimal generator.
On the unit disk, Blasco, Contreras, Diaz-Madrigal, Martinez, Papadimitrakis and Siskakis in [10] studied the maximal subspace in Bounded Mean Oscillation (BMOA) where a general semigroup of analytic functions on the unit disk generates a strongly continuous semigroup of composition operators. A related necessary condition is also proved by Blasco et al [10] in the case when the semigroup has an inner Denjoy-Wolff point. As a byproduct they also provided a generalization of the theorem of Sarason. Arevalo and Oliva [3] later gave the general result on the separable spaces and used it to prove that semigroups are always strongly continuous in the Hardy and Bergman spaces. Complete continuity of weighted composition was given in [10]. Kang and Young in their paper [24] studied some properties of weighted Bergman spaces and their duality on the setting of the half-plane of the complex plane. They obtained some characterization of Compact Toeplitz Operators. Although most of the research were mainly on Hardy and Bergman spaces, Irevalo in [2] on her article studied the strongly continuous semigroups on the mixed norm spaces.
On the Bloch spaces, recently Antti Perala [30] calculated the norm of the Bergman projection from the space of essentially bounded functions to the

Bloch space. On the same space, Xi Fu and Zhang [18] in 2017 defined the Bloch-type spaces of the upper half- plane $\mathbb{U}$ and characterized them in terms of weighted Lipschitz functions. They also discussed the boundedness of a composition operator acting between two Bloch spaces. Also Allen and Colonna in their paper [1] established bounds on the norm of multiplication operators on the Bloch spaces of the unit disk via weighted composition operators. In doing so, they characterized the isometric multiplication operators to be precisely those induced by constant functions of modulus 1. They then described the spectrum of the multiplication operators in terms of the range of the symbol. Lastly, they identified the isometries and spectra of a particular class of weighted composition operators on the Bloch space. On functions in the little Bloch space and inner functions, Rohde [31] proved that analytic functions in the little Bloch space assume every value as a radial limit on a set of Hausdorff dimension one, unless they have radial limits on a set of positive measure. The analogue for the inner functions in the little Bloch space was also proven, and characterization of various classes of the Bloch functions in terms of their level sets were given.

The duality properties of Bergman spaces are well known in literature. For instance in [36], it is proved by Zhu that for $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha > -1$, the dual space of reflexive Bergman space on the unit disk $L^p_a(\mathbb{D}, m_\alpha)$ is $L^q_a(\mathbb{D}, m_\alpha)$. For the non-reflexive Bergman space on the unit disk $L^1_a(\mathbb{D}, m_\alpha)$, it is shown in [36] by Zhu that the dual and predual space are Bloch space and little Bloch space respectively. For the corresponding spaces of the upper half plane as in [5], that the dual space of reflexive Bergman space on the upper half plane of $L^p_a(\mathbb{U}, \mu_\alpha)$ is $L^q_a(\mathbb{U}, \mu_\alpha)$

for $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha > -1$. When $p = 1$, the space $L^1_a(\mathbb{U}, \mu_\alpha)$ is non-reflexive, its dual have been determined by Kang in [23] and it's not known what its predual would be. Therefore in this study we have determined the predual of the non-reflexive Bergman space of the upper half-plane. We have also investigated semigroup and spectral properties. The following known theorems will be useful in proving the results of this study:

Theorem 2.0.1 (Hille-Yosida theorem)

A linear operator Γ is the infinitesimal generator of a strongly continuous semigroup of contractions $(T_t)_{t\geq 0} \subseteq \mathcal{L}(X)$ if and only if;

1. *Γ is closed and $\overline{dom(\Gamma)} = X$, and*
2. *The resolvent set $\rho(\Gamma)$ of Γ contains $\mathbb{R}^+$ and for every $\lambda \geq 0$,*

$$\|R(\lambda, \Gamma)\| \leq \frac{1}{\lambda}.$$

In this case, if $h \in X$, then

$$R(\lambda, \Gamma)h = \int_0^\infty e^{-\lambda t} T_t h dt$$

is norm convergent.

Theorem 2.0.2 (Spectral mapping theorem for resolvents)

Let Γ be a closed operator on X and $\lambda \in \rho(\Gamma)$. Then it asserts that,

$$\begin{aligned} \sigma(R(\lambda, \Gamma)) \setminus \{0\} &= (\lambda - \sigma(\Gamma))^{-1} \\ &= \left\{ \frac{1}{\lambda - \mu} : \mu \in \sigma(\Gamma) \right\} \end{aligned}$$

for $\lambda \in \rho(\Gamma)$.

For details, see [28].

Theorem 2.0.3 (Spectral mapping theorem for semigroups)
Let $(T_t)_{t\geq 0}$ *be a strongly continuous semigroup on* X *and* Γ *be its infinitesimal generator. Then*

$$\sigma(T_t) \quad \supset \quad e^{t\sigma(\Gamma)}.$$

For the point spectrum,

$$e^{t\sigma_p(\Gamma)} \quad = \quad \sigma_p(T_t) \setminus \{0\}.$$

For details, see [28].

Theorem 2.0.4 (Hahn-Banach extension theorem)
Let X *be a normed space over a field* $\mathbb{F}$ *and let* $Y \subseteq X$ *be a linear subspace. Then for every* $\varphi \in Y^*$ *there exists some* $\phi \in X^*$ *such that* $\phi = \varphi$ *on* Y *and* $\|\phi\| = \|\varphi\|$.

Theorem 2.0.5 (Riesz representation theorem for measures)
Let the dual space of X *be identified with the space* X^* *of all finite complex weighted measure on* Ω *such that each measure* μ *in* X^* *defines a bounded linear functional* F_μ *on* X *as,*

$$F_\mu(f) \quad = \quad \int_\Omega f(x)d\mu(x), f \in X,$$

and every bounded linear functional on X *arises in the above manner. Here,* X^* *is equipped with the norm* $\|\mu\| = |\mu|(\Omega)$.

Theorem 2.0.6 (Fubini's theorem)

Let $(X_1, \mathcal{A}_1, \mu_1)$ and $(X_2, \mathcal{A}_2, \mu_2)$ be two complete finite measure spaces. Suppose f is an integrable function on $X_1 \times X_2$ and also $\mu = \mu_1 \times \mu_2$ be a product measure. Then;

$$\int_{X_1} \int_{X_2} f(x_1, x_2) d\mu_2 d\mu_1 = \int_{X_2} \int_{X_1} f(x_1, x_2) d\mu_1 d\mu_2 = \int_{X_1 \times X_2} f(x_1, x_2) d\mu.$$

For details, see [28].

Chapter 3

Duality of the Non-reflexive Bergman space

In this chapter, we establish the predual space of non-reflexive Bergman space of the upper half-plane $L^1_a(\mathbb{U}, \mu_\alpha)$. Before we give the main result of this chapter, we prove the following lemmas:

Lemma 3.0.1

Let $f \in B_\infty(\mathbb{U})$, then

$$\|f\|_{B_{\infty,1}(\mathbb{U})} = \frac{1}{2}\|f \circ \psi\|_{B_{\infty,1}(\mathbb{D})}.$$

In particular, $f \in B_\infty(\mathbb{U})$ if and only if $f \circ \psi \in B_\infty(\mathbb{D})$.

PROOF. Let f be a function in $B_\infty(\mathbb{U})$. Then by definition,

$$\|f\|_{B_{\infty,1}(\mathbb{U})} = \sup_{\omega\in\mathbb{U}} Im(w)|f'(w)| < \infty.$$

If ψ is the Cayley transform, let $\omega = \psi(z)$. Then

$$\begin{aligned} Im(\omega) &= \frac{\omega - \overline{\omega}}{2i} \\ &= \frac{\psi(z) - \overline{\psi(z)}}{2i}. \end{aligned}$$

But $\psi(z) = \frac{i(1+z)}{1-z}$ and $\overline{\psi(z)} = \frac{-i(1+\overline{z})}{1-\overline{z}}$, therefore

$$\begin{aligned} Im(\omega) &= \frac{\frac{i(1+z)}{1-z} - \frac{-i(1+\overline{z})}{1-\overline{z}}}{2i} \\ &= \frac{i(1+z)(1-\overline{z}) + i(1+\overline{z})(1-z)}{2i(1-z)(1-\overline{z})} \\ &= \frac{i(2-2\overline{z}z)}{2i(1-z)(1-\overline{z})} = \frac{1-|z|^2}{|1-z|^2}. \end{aligned} \tag{3.1}$$

Moreover $\psi'(z) = \frac{2i}{(1-z)^2}$ and therefore $|\psi'(z)| = \left|\frac{2i}{(1-z)^2}\right|$.
Now, by definition we have

$$\|f\|_{B_{\infty,1}(\mathbb{U})} = \sup_{z\in\mathbb{D}} \left(\frac{1-|z|^2}{|1-z|^2}\right) |f'(\psi(z))| < \infty.$$

Since $|1-z|^2 = \frac{2}{|\psi'(z)|}$, we obtain

$$\|f\|_{B_{\infty,1}(\mathbb{U})} = \frac{1}{2}\sup_{z\in\mathbb{D}}(1-|z|^2)|\psi'(z)||f'(\psi(z))| < \infty.$$

In this case, $\frac{d(f\circ\psi)(z)}{dz} = \frac{df}{d\psi}.\frac{d\psi}{dz} = f'(\psi(z))\psi'(z)$, therefore
$|\psi'(z)||f'(\psi(z))| = |(f\circ\psi)'(z)|$ and hence

$$\begin{aligned} \|f\|_{B_{\infty}(\mathbb{U})} &= \frac{1}{2}\sup_{z\in\mathbb{D}}(1-|z|^2)|(f\circ\psi)'(z)| < \infty \\ &= \frac{1}{2}\|f\circ\psi\|_{B_{\infty,1}(\mathbb{D})}. \end{aligned}$$

$\square$

Lemma 3.0.2

Let $f \in B_{\infty,\circ}(\mathbb{U})$*, then*

$$\|f\|_{B_{\infty,1}(\mathbb{U})} = \frac{1}{2}\|f \circ \psi\|_{B_{\infty,\circ}(\mathbb{D})}.$$

In particular, $f \in B_{\infty,1}(\mathbb{U})$ *if and only if* $f \circ \psi \in B_{\infty,\circ}(\mathbb{D})$*.*

Proof. Let f be a function on $B_{\infty,1}(\mathbb{U})$, then by definition

$$\lim_{Im(w)\to 0} Im(w)|f'(w)| = 0.$$

But we know from Equation 3.1 that $Im(w) = \frac{1-|z|^2}{|1-z|^2}$, hence

$$\begin{aligned}\|f\|_{B_{\infty,1}(\mathbb{U})} &= \lim_{|z|\to 1}\left(\frac{1-|z|^2}{|1-z|^2}\right)|f'(\psi(z))|. \\ &= \frac{1}{2}\lim_{|z|\to 1}(1-|z|^2)|\psi'(z)||f'(\psi(z))|.\end{aligned}$$

But it is clear that $|\psi'(z)||f'(\psi(z))| = |(f\circ\psi)'(z)|$, and hence

$$\|f\|_{B_{\infty,1}(\mathbb{U})} = \frac{1}{2}\lim_{|z|\to 1}(1-|z|^2)|(f\circ\psi)'(z)|.$$

Therefore,

$$\|f\|_{B_{\infty,1}(\mathbb{U})} = \frac{1}{2}\|f \circ \psi\|_{B_{\infty,\circ}(\mathbb{D})}.$$

$\square$

Lemma 3.0.3

Let $f \in L^1_a(\mathbb{U}, \mu_\alpha)$*, then*

$$\|f\|_{L^1_a(\mathbb{U},\mu_\alpha)} = \|S_\psi f\|_{L^1_a(\mathbb{D},m_\alpha)}.$$

In particular, $f \in L^1_a(\mathbb{U}, \mu_\alpha)$ *if and only if* $(\psi')^{\alpha+2} f \circ \psi \in L^1_a(\mathbb{D}, m_\alpha)$*.*

PROOF. Let f be a function in $L^1_a(\mathbb{U}, \mu_\alpha)$. Then

$$\begin{aligned}\|f\|_{L^1_a(\mathbb{U},\mu_\alpha)} &= \int_{\mathbb{U}} |f(w)| d\mu_\alpha(w) \\ &= \int_{\mathbb{U}} |f(w)| Im(w)^\alpha dA(w) < \infty,\end{aligned}$$

for $d\mu_\alpha(\omega) = (Im(\omega))^\alpha dA(\omega)$.
By change of variables, let $w = \psi(z)$, then $\psi'(z) = \frac{2i}{(1-z)^2}$, $Im(w) = \frac{1-|z|^2}{|1-z|^2}$ from Equation 3.1, and $dA(w) = |\psi'(z)|^2 dA(z)$.
Therefore,

$$\begin{aligned}\|f\|_{L^1(\mathbb{U},\mu_\alpha)} &= \int_{\mathbb{D}} |f(\psi(z))| \left(\frac{1-|z|^2}{|1-z|^2}\right)^\alpha |\psi'(z)|^2 dA(z) \\ &= \frac{1}{2^\alpha} \int_{\mathbb{D}} |f(\psi(z))||\psi'(z)|^{\alpha+2}(1-|z|^2)^\alpha dA(z) < \infty \\ &= \frac{1}{2^\alpha} \int_{\mathbb{D}} |(\psi'(z))^{\alpha+2}(f \circ \psi)(z)| dm_\alpha(z) < \infty \\ &= \frac{1}{2^\alpha} \int_{\mathbb{D}} |(\psi'(z))^{\gamma}(f \circ \psi)(z)| dm_\alpha(z) < \infty.\end{aligned}$$

Hence,

$$\|f\|_{L^1_a(\mathbb{U},\mu_\alpha)} = \frac{1}{2^\alpha} \|S_\psi f\|_{L^\infty_a(\mathbb{D},m_\alpha)},$$

where $\gamma = \alpha + 2$ since $p = 1$ and S_ψ is the weighted composition operator.

□

Lemma 3.0.4
Let $f \in L^\infty(\mathbb{U}, \mu_\alpha)$*, then*

$$\|f\|_{L^\infty(\mathbb{U},\mu_\alpha)} \quad = \quad \|C_\psi f\|_{L^\infty(\mathbb{D},m_\alpha)}.$$

In particular, $f \in L^\infty(\mathbb{U}, \mu_\alpha)$ *if and only if* $C_\psi f \in L^\infty(\mathbb{D}, m_\alpha)$.

Proof. Let f be a function on $L^\infty(\mathbb{U}, \mu_\alpha)$. Then f is essentially bounded which implies that $f \circ \psi$ is essentially bounded as well since composition by invertible maps preserves essential boundedness. Since ψ is an invertible mapping from $\mathbb{D}$ onto $\mathbb{U}$, it follows that $f \circ \psi \in L^\infty(\mathbb{D}, m_\alpha)$. The converse follows similarly. □

Remark 3.0.5
It is easy to verify that $C_{\psi^{-1}} = C_\psi^{-1}$. The Lemma 3.0.4 above therefore imply that C_ψ is an isometry up to a constant and at the same time invertible on the respective spaces with the inverse also acting on the same appropriate spaces.
More generally, let $\{V_1, V_2\} = \{\mathbb{D}, \mathbb{U}\}$, and let $LF(V_i, V_j)$ denote the collection of conformal mappings from V_i to V_j. Then $LF(V_i, V_j) = \mathrm{Aut}(V_i)$, and if $h \in LF(V_i, V_j)$, then $g \in \mathrm{Aut}(V_j) \mapsto h^{-1} \circ g \circ h \in \mathrm{Aut}(V_i)$ is an isomorphism from $\mathrm{Aut}(V_i)$ onto $\mathrm{Aut}(V_j)$. From each $g \in LF(V_i, V_j)$, we define a weighted composition operator $S_g : \mathcal{H}(V_j) \to \mathcal{H}(V_i)$, by

$$S_g f(z) \quad = \quad (g'(z))^\gamma f(g(z)), \text{ for all } z \in V_i.$$

We note that if $g \in LF(V_i, V_j)$ and $h \in LF(V_j, V_i)$, then it is clear by Chain Rule that $S_h S_g = S_{gh}$ and $S_g^{-1} = S_{g^{-1}}$.

The duality properties of Bergman spaces are well known in literature. For instance in [36], it is proved that for $1 < p < \infty$, $\frac{1}{p}+\frac{1}{q} = 1$ and $\alpha > -1$, the dual space of reflexive Bergman space on the unit disk $L_a^p(\mathbb{D}, m_\alpha)$ is $L_a^q(\mathbb{D}, m_\alpha)$. For the corresponding spaces of the upper half plane as in [5], that the dual space of reflexive Bergman space on the upper half plane of $L_a^p(\mathbb{U}, \mu_\alpha)$ is $L_a^q(\mathbb{U}, \mu_\alpha)$ for $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha > -1$. For the non-reflexive Bergman space on the unit disk when $p = 1$, it is shown in [36] that the dual and predual space of non-reflexive Bergman space on the unit disk $L_a^1(\mathbb{D}, m_\alpha)$ are Bloch space and little Bloch space respectively.

The next result is a recent one and is due to S. Kang [23]. It gives the dual of the non-reflexive Bergman space of the upper half-plane $L_a^1(\mathbb{U}, \mu_\alpha)$. But first we give the following definition:

Definition 3.0.6

Let $B_\infty(\mathbb{U}, i)$ denote the subspace of the Bloch space $B_\infty(\mathbb{U})$ consisting of functions vanishing at a point i. Therefore $B_\infty(\mathbb{U}, i)$ is defined as

$$B_\infty(\mathbb{U}, i) \quad := \quad \{f \in B_\infty(\mathbb{U}) : f(i) = 0\}.$$

Then $B_\infty(\mathbb{U}, i)$ is a Banach space with respect to the norm $\|f\|_{B_{\infty,i}} := \|f\|_{B_\infty(\mathbb{U})} = \|f\|_{B_{\infty,i(\mathbb{U})}}$.

Similarly, let $B_{\infty,\circ}(\mathbb{U}, i)$ denote the subspace of $B_{\infty,\circ}(\mathbb{U})$ consisting of functions vanishing at point i. Then

$$B_{\infty,\circ}(\mathbb{U}, i) \quad := \quad \{f \in B_{\infty,\circ}(\mathbb{U}) : f(i) = 0\},$$

with the norm $\|f\|_{B_{\infty,i}} := \|f\|_{B_{\infty(\mathbb{U})}} = \|f\|_{B_{\infty,i(\mathbb{U})}}$.

The following result is due to S. Kang [23].

Theorem 3.0.7

For any $\alpha \in \mathbb{R}$, $\alpha > -1$, we have

$$(L_a^1(\mathbb{U}, \mu_\alpha))^* \quad \approx \quad B_\infty(\mathbb{U}, i),$$

under the integral pairing

$$\langle g, f \rangle \quad = \quad \int_{\mathbb{U}} g(w)\overline{f(w)} d\mu_\alpha(w),$$

where $g \in L_a^1(\mathbb{U}, \mu_\alpha)$ and $f \in B_\infty(\mathbb{U}, i)$.

With the help of Theorem 3.0.7 above, we determine the predual space of $L_a^1(\mathbb{U}, \mu_\alpha)$, that is, a set whose dual is $L_a^1(\mathbb{U}, \mu_\alpha)$, but first we state some definitions and results.

Let $\mathbb{C}_\circ(\mathbb{D})$ be the subalgebra of $\mathbb{C}(\overline{\mathbb{D}})$ consisting of functions f with $f(z) \to 0$ as $|z| \to 1^-$, where $\mathbb{C}(\overline{\mathbb{D}})$ is the algebra of complex-valued continuous functions on $\overline{\mathbb{D}}$, the closure of $\mathbb{D}$.

We can now state the following Theorem due to K. Zhu and details can be found in [36].

Theorem 3.0.8

If $t > 0$ and $\alpha > -1$, then the integral operator $T = T_{t,\alpha}$ defined by

$$Tf(z) \;=\; (1-|z|^2)^t \int_{\mathbb{D}} \frac{f(\omega)}{(1-z\overline{\omega})^{2+t+\alpha}} dm_\alpha(\omega),$$

has the following properties:

1. *$T = (\alpha + t + 1)T^2$.*
2. *T is a bounded embedding of $B_\infty(\mathbb{D})$ into $L^\infty(\mathbb{D})$.*
3. *T is an embedding of $B_{\infty,\circ}(\mathbb{D})$ into $\mathbb{C}_\circ(\mathbb{D})$.*

Now, let $\mathbb{C}(\overline{\mathbb{U}})$ be the algebra of complex valued continuous functions on $\overline{\mathbb{U}}$, and $\mathbb{C}_\circ(\overline{\mathbb{U}})$ be the subalgebra of $\mathbb{C}(\overline{\mathbb{U}})$ consisting of functions f such that $f(\omega) \longrightarrow 0$ as $Im(\omega) \longrightarrow 0$.

Proposition 3.0.9

Let $\mathbb{C}_\circ(\overline{\mathbb{U}})$ be the subalgebra of $\mathbb{C}(\overline{\mathbb{U}})$ consisting of functions f such that $f(\omega) \longrightarrow 0$ as $Im(\omega) \longrightarrow 0$ and $\mathbb{C}_\circ(\mathbb{D})$ be the subalgebra of $\mathbb{C}(\overline{\mathbb{D}})$ consisting of functions f with $f(z) \to 0$ as $|z| \to 1^-$. Then $\mathbb{C}_\circ(\overline{\mathbb{U}}) = \{g \circ \psi^{-1} : g \in \mathbb{C}_\circ(\overline{\mathbb{D}})\}$.

Proof. Let $\mathbb{K} \subset \mathbb{U}$ be compact. Since Cayley transform $\psi : \mathbb{D} \longrightarrow \mathbb{U}$ is a continuous bijection, it follows that $\mathbb{K} \subset \mathbb{U}$ is compact if and only if $\psi^{-1}(\mathbb{K})$ is compact in $\mathbb{D}$. If $f \in \mathbb{C}_\circ(\mathbb{U})$ and $\epsilon > 0$, then there exists $\mathbb{K}$ compact in $\mathbb{U}$ such that $\sup_{\omega\in\mathbb{U}\setminus\mathbb{K}} |f(\omega)| < \epsilon$.
Now; if $g = f \circ \psi$ is continuous on $\mathbb{D}$ with $f = g \circ \psi^{-1}$, then

$$\sup_{z\in\mathbb{D}\setminus\psi^{-1}(\mathbb{K})} |g(z)| \;=\; \sup_{\omega\in\mathbb{U}\setminus\mathbb{K}} |f(\omega)| < \epsilon.$$

$\square$

Using Proposition 3.0.9 above, we obtain the following result which is the upper half-plane analogue of Theorem 3.0.8.

Proposition 3.0.10

For $t > 0$, $\alpha > -1$, let the integral operator T on $\mathcal{H}(\mathbb{D})$ be defined by

$$Tf(z) = (1-|z|^2)^t \int_{\mathbb{D}} \frac{f(\omega)}{(1-z\overline{\omega})^{2+t+\alpha}} dm_\alpha(\omega).$$

Let S be the corresponding integral operator on $\mathcal{H}(\mathbb{U})$ defined by $S := C_{\psi^{-1}} T C_\psi$. Then the following properties hold:

(a) $S = (\alpha+t+1)S^2$,

(b) *S is a bounded embedding of $B_\infty(\mathbb{U})$ into $L^\infty(\mathbb{U})$, and*

(c) *S is an embedding of $B_{\infty,\circ}(\mathbb{U})$ into $\mathbb{C}_\circ(\mathbb{U})$.*

Proof. For (a), from Theorem 3.0.8, we have,

$$\begin{aligned} S &= C_{\psi^{-1}} T C_\psi = C_{\psi^{-1}}(\alpha+t+1)T^2 C_\psi \\ &= (\alpha+t+1)C_{\psi^{-1}} T^2 C_\psi \\ &= (\alpha+t+1)S^2 \end{aligned}$$

For (b), we have

$$B_\infty(\mathbb{U}) \xrightarrow{C_\psi} B_\infty(\mathbb{D}) \xrightarrow{T} L^\infty(\mathbb{D}, m_\alpha) \xrightarrow{C_{\psi^{-1}}} L^\infty(\mathbb{U}, \mu_\alpha).$$

Now C_ψ is an isometry of $B_\infty(\mathbb{U})$ onto $B_\infty(\mathbb{D})$ (Lemma 3.0.1) up to constant, T which is a bounded embedding of $B_\infty(\mathbb{D})$ into $L^\infty(\mathbb{D})$, By Lemma 3.0.4, $C_{\psi^{-1}}$ is also an isometry of $L^\infty(\mathbb{D}, m_\alpha)$ onto $L^\infty(\mathbb{U}, \mu_\alpha)$, it therefore follows that $S = C_{\psi^{-1}}TC_\psi$ is a bounded embedding of $B_\infty(\mathbb{U})$ into $L^\infty(\mathbb{U}, \mu_\alpha)$ (Proposition 3.0.10).

For (c), we have

$$B_{\infty,\circ}(\mathbb{U}) \xrightarrow{C_\psi} B_{\infty,\circ}(\mathbb{D}) \xrightarrow{T} \mathbb{C}_\circ(\mathbb{D}) \xrightarrow{C_{\psi^{-1}}} \mathbb{C}_\circ(\mathbb{U}).$$

C_ψ is a bijection of $B_{\infty,\circ}(\mathbb{U})$ into $B_{\infty,\circ}(\mathbb{D})$ from Lemma 3.0.2, T is an embedding of $B_{\infty,\circ}(\mathbb{D})$ into $\mathbb{C}_\circ(\mathbb{D})$ (Theorem 3.0.8) and on the other hand $C_{\psi^{-1}}$ is also a bijection of $\mathbb{C}_\circ(\mathbb{D})$ into $\mathbb{C}_\circ(\mathbb{U})$. Therefore $S = C_{\psi^{-1}}TC_\psi$ is an embedding of $B_{\infty,\circ}(\mathbb{U})$ into $\mathbb{C}_\circ(\mathbb{U})$, which completes the proof. $\square$

We now establish the predual space of $L^1_a(\mathbb{U}, \mu_\alpha)$.

Theorem 3.0.11

For any $\alpha > -1$, we have;

$$(B_{\infty,\circ}(\mathbb{U}, i))^* \;\approx\; L^1_a(\mathbb{U}, \mu_\alpha),$$

under the pairing

$$\langle g, f\rangle \;=\; \int_{\mathbb{U}} g(\omega)\overline{f(\omega)}d\mu_\alpha(\omega),$$

where $g \in B_{\infty,\circ}(\mathbb{U}, i)$ and $f \in L^1_a(\mathbb{U}, \mu_\alpha)$. Here, $B_{\infty,\circ}(\mathbb{U}, i)$ is equipped with the same norm as $B_\infty(\mathbb{U}, i)$, that is, $\|f\|_{B_{\infty,i}(\mathbb{U},i)} := \|f\|_{B_{\infty(\mathbb{U})}} = \|f\|_{B_{\infty,i(\mathbb{U})}}$.

Proof. If $f \in L^1_a(\mathbb{U}, \mu_\alpha)$, then by Theorem 3.0.7,

$g \longmapsto \int_{\mathbb{U}} g(\omega)\overline{f(\omega)}d\mu_\alpha(\omega)$ defines a bounded linear functional on $B_{\infty,\circ}(\mathbb{U}, i)$.
Conversely, if F is a bounded linear functional on $B_{\infty,\circ}(\mathbb{U}, i)$, we want to show that there exists a function $f \in L^1_a(\mathbb{U}, \mu_\alpha)$ such that
$F(g) = \int_{\mathbb{U}} g(\omega)\overline{f(\omega)}d\mu_\alpha(\omega)$ for g in a dense subset of $B_{\infty,\circ}(\mathbb{U}, i)$.
Now we fix any positive parameter t and consider the embedding S of $B_{\infty,\circ}(\mathbb{U}, i)$ into $\mathbb{C}_\circ(\mathbb{U})$ as given by Proposition 3.0.10.
The space $X = S(B_{\infty,\circ}(\mathbb{U}, i))$ is a closed subspace of $\mathbb{C}_\circ(\mathbb{U})$ and $F \circ S^{-1} : X \to \mathbb{C}$ is a bounded linear functional on X since F and S^{-1} are both bounded, that is $\|F \circ S^{-1}\| \leq \|F\|\|S^{-1}\|$.
By the Hahn-Banach extension theorem (Theorem 2.0.4), $F \circ S^{-1}$ extends to a bounded linear functional on $\mathbb{C}_\circ(\mathbb{U})$. By the Riesz representation theorem Theorem 2.0.5, there exists a finite weighted measure μ_α on $\mathbb{U}$ such that $\|\mu_\alpha\| = \|F \circ S^{-1}\|$ and $F \circ S^{-1}(h) = \int_{\mathbb{U}} h(\omega)d\mu_\alpha(w)$, $h \in \mathbb{C}_\circ(\mathbb{U})$.
In particular, if g is a polynomial (polynomials are dense in $B_{\infty,\circ}(\mathbb{U}, i)$), then $F(g) = F \circ S^{-1} \circ S(g) = \int_{\mathbb{U}} Sg(\omega)d\mu_\alpha(\omega)$. Since $S := C_{\psi^{-1}}TC_\psi$ and letting the integral operator T on $\mathcal{H}(\mathbb{D})$ be defined by $Tf(z) = (1-|z|^2)^t \int_{\mathbb{D}} \frac{f(\omega)}{(1-z\overline{\omega})^{2+t+\alpha}}dm_\alpha(\omega)$ as in Proposition 3.0.10. Now let $f \in \mathcal{H}(\mathbb{U})$, then $S(f) = C_{\psi^{-1}}TC_\psi f = C_{\psi^{-1}}Tf \circ \psi$. By substituting for $T = \int_{\mathbb{D}} \frac{f(\omega)}{(1-z\overline{\omega})^{2+t+\alpha}}dm_\alpha(\omega)$, we obtain

$$\begin{aligned} C_{\psi^{-1}}Tf \circ \psi &= C_{\psi^{-1}} \int_{\mathbb{D}} \frac{f \circ \psi(\omega)dm_\alpha(\omega)}{(1-z\overline{\omega})^{2+t+\alpha}} \\ &= C_{\psi^{-1}} \int_{\mathbb{D}} \frac{f(\psi(\omega))dm_\alpha(\omega)}{(1-z\overline{\omega})^{2+t+\alpha}} \\ &= \int_{\mathbb{U}} \frac{f(\psi^{-1} \circ \psi)(w)d\mu_\alpha}{(1-z\overline{w})^{2+t+\alpha}} = \int_{\mathbb{U}} \frac{f(\omega)d\mu_\alpha(\omega)}{(1-z\overline{\omega})^{2+t+\alpha}}. \end{aligned}$$

Therefore, $F(g) = \int_{\mathbb{U}} Sg(w)d\mu_\alpha(\omega) = \int_{\mathbb{U}} \int_{\mathbb{U}} \frac{f(\omega)}{(1-z\overline{\omega})^{2+t+\alpha}} g(\omega)d\mu_\alpha(\omega)d\mu_\alpha(\omega)$.
By Fubini's theorem 2.0.6, we have $F(g) = \int_{\mathbb{U}} g(\omega)\overline{f(\omega)}d\mu_\alpha(\omega)$, where

$f = C_{\psi^{-1}} T C_\psi$ which is bounded since T is bounded. $\square$

Chapter 4

Groups of weighted composition operators

The automorphisms of the upper half plane $\mathbb{U}$ were classified into three distinct groups [6, Proposition 2.3], namely: the scaling, the translation and the rotation groups depending on the location of their fixed points. Since the induced groups of composition operators for rotation group are defined on the analytic spaces of the unit disk, we shall only consider groups of composition operators associated with the scaling and the translation groups in this chapter. In Section 4.1, we determine the group of weighted composition operator on $B_{\infty,\circ}(\mathbb{U}, i)$ and investigate both the semigroup and the spectral properties for the scaling group. In Section 4.2, we determine the group of weighted composition operator on $B_{\infty,\circ}(\mathbb{U}, i)$ and investigate both the semigroup and spectral properties for the translation group.

4.1 Scaling group

The automorphisms of this group are of the form $\varphi_t(z) = k^t z$, where $z \in \mathbb{U}$ and $k, t \in \mathbb{R}$ with $k \neq 0$. Now, without loss of generality, we consider the analytic self maps $\varphi_t : \mathbb{U} \longrightarrow \mathbb{U}$ of the form $\varphi_t(z) = e^{-t}z$ for $z \in \mathbb{U}$. The corresponding group of weighted composition operators on $L^p_a(\mathbb{U}, \mu_\alpha)$ is given by

$$T_t f(z) \;=\; e^{-t\gamma} f(e^{-t}z),$$

for all $f \in L^p_a(\mathbb{U}, \mu_\alpha)$, where $\gamma{=}\frac{\alpha+2}{p}$, $1 \leq p < \infty$ and $\alpha > -1$. For $p = 1$, $(T_t)_{t\geq 0}$ is defined on $L^1_a(\mathbb{U}, \mu_\alpha)$ with $\gamma{=}\frac{\alpha+2}{1} = \alpha + 2$.
Following Theorem 3.0.11, the predual of $L^1_a(\mathbb{U}, \mu_\alpha)$ is given by the duality relation

$$(B_{\infty,\circ}(\mathbb{U}, i))^* \;\approx\; L^1_a(\mathbb{U}, \mu_\alpha)$$

under the integral pairing $\langle g, f\rangle = \int_{\mathbb{U}} g(w)\overline{f(w)}d\mu_\alpha(w)$, where $g \in B_{\infty,\circ}(\mathbb{U}, i)$ and $f \in L^1_a(\mathbb{U}, \mu_\alpha)$. Note that $B_{\infty,\circ}(\mathbb{U}, i) \subseteq B_{\infty,\circ}(\mathbb{U})$.
Using the integral pairing above, we obtain the corresponding group of weighted composition operators on $B_{\infty,\circ}(\mathbb{U}, i)$ as follows.
Let $g \in B_{\infty,\circ}(\mathbb{U}, i)$ and $f \in L^1_a(\mathbb{U}, \mu_\alpha)$, then,

$$\begin{aligned}
\langle g, T_t f\rangle &= \int_{\mathbb{U}} g(z)\overline{e^{-t\gamma} f(e^{-t}z)}d\mu_\alpha(z) \\
&= \int_{\mathbb{U}} g(z)e^{-t\gamma}\overline{f(e^{-t}z)}d\mu_\alpha(z) \\
&= \int_{\mathbb{U}} g(z)e^{-t\gamma}\overline{f(e^{-t}z)}(Im(z))^\alpha dA(z).
\end{aligned}$$

By change of variables, let $w = e^{-t}z$, then $z = e^t w$, using the Jacobian, $dA(w) = |\varphi_t'(z)|^2 dA(z)$. Since $\varphi_t(z) = e^{-t}z$, it therefore follows that $dA(w) = e^{-2t}dA(z)$ and $Im(z) = e^t Im(w)$.
Thus,

$$\begin{aligned}
\langle g, T_t f\rangle &= \int_{\mathbb{U}} g(z)e^{-t\gamma}\overline{f(e^{-t}z)}(Im(z))^{\alpha}dA(z) \\
&= \int_{\mathbb{U}} g(e^t w)e^{-t\gamma}\overline{f(w)}e^{\alpha t}(Im(w))^{\alpha}e^{2t}dA(w) \\
&= \int_{\mathbb{U}} g(e^t w)e^{-t\gamma}e^{(\alpha+2)t}\overline{f(w)}(Im(w))^{\alpha}dA(w) \\
&= \int_{\mathbb{U}} g(e^t w)e^{-t\gamma}e^{t\gamma}\overline{f(w)}d\mu_{\alpha}(w) \\
&= \int_{\mathbb{U}} g(e^t w)\overline{f(w)}d\mu_{\alpha}(w) \\
&= \langle T_t^* g, f\rangle,
\end{aligned}$$

where $T_t^* g(w) = g(e^t w)$.
Now, $S_t := T_t^*$ is defined on $B_{\infty,\circ}(\mathbb{U}, i)$. But we see that $S_t g(i) = g(e^t i) \neq 0$ and therefore $S_t g$ does not vanish at i. This means that S_t does not map $B_{\infty,\circ}(\mathbb{U}, i)$ onto itself. We now therefore redefine S_t to act on $B_{\infty,\circ}(\mathbb{U})$ so that it maps the space onto itself. Thus, for all $t \geq 0$ and for all $g \in B_{\infty,\circ}(\mathbb{U})$, $S_t g(w) := g(e^t w)$ is a semigroup as well as a group of composition operators defined on $B_{\infty,\circ}(\mathbb{U})$. We shall carry out a complete study of both the semigroup and spectral properties of this group on $B_{\infty,\circ}(\mathbb{U})$.

4.1.1 Semigroup properties

In this section, we investigate the semigroup properties and determine the infinitesimal generator Γ of $(S_t)_{t\geq 0}$ on $B_{\infty,\circ}(\mathbb{U})$ where, $S_t g(w) = g(e^t w)$.

Proposition 4.1.1

$(S_t)_{t\in\mathbb{R}}$ *is a group on* $B_{\infty,\circ}(\mathbb{U})$.

Proof. It suffices to prove that both $(S_t)_{t\geq 0}$ and $(S_{-t})_{t\geq 0}$ are semigroups on $B_{\infty,\circ}(\mathbb{U})$.

First we show that $(S_t)_{t\geq 0}$ is a semigroup on $B_{\infty,\circ}(\mathbb{U})$. Indeed,

$S_0 g(w) = g(e^0 w) = g(w) \Rightarrow S_0 = I(\text{Identity})$.

For semigroup property, we have;

$$\begin{aligned} S_s S_t g(w) &= S_s g(e^t w) \\ &= g(e^t . e^s(w)) \\ &= g(e^{t+s}(w)) \\ &= S_{t+s} g(w), \end{aligned}$$

for all $t, s \geq 0$. Hence, $(S_t)_{t\geq 0}$ is a semigroup.

Similarly we need to show $(S_{-t})_{t\geq 0}$ is also a semigroup on $B_{\infty,\circ}(\mathbb{U})$.

From definition, $S_t g(w) = g(e^t w)$, it follows that $S_{-t} g(w) = g(e^{-t} w)$.

Thus,

$S_{-0} g(w) = g(e^{-0} w) = g(w) \Rightarrow S_0 = I(\text{Identity})$

Now for the semigroup property, we have;

$$\begin{aligned} S_{-s}(S_{-t}g(w)) &= S_{-s}S_{-t}(g(w)) \\ &= S_{-s}g(e^{-t}(w)) \\ &= S_{-s}g(e^{-t}(w)) \\ &= g(e^{-s}.e^{-t}(w)) \\ &= g(e^{-s+-t}(w)) \\ &= S_{-t+-s}g(w) = S_{-(t+s)}g(w), \end{aligned}$$

for all $t, s \geq 0$.

Therefore $(S_{-t})_{t\geq 0}$ is a semigroup on $B_{\infty,\circ}(\mathbb{U})$.

Hence $(S_t)_{t\in\mathbb{R}}$ is a group as desired. $\square$

Theorem 4.1.2

$(S_t)_{t\in\mathbb{R}}$ *is an isometry on* $B_{\infty,\circ}(\mathbb{U})$.

Proof. By the definition of isometry, we have;

$$\begin{aligned} \|S_t g\|_{B_{\infty,\circ}(\mathbb{U})} &= \sup_{w\in\mathbb{U}} Im(w)|S_t g'(w)| \\ &= \sup_{w\in\mathbb{U}} Im(w)|g'(e^t w)| \\ &= \sup_{w\in\mathbb{U}} Im(w)|g'(e^t w).e^t| \\ &= \sup_{w\in\mathbb{U}} Im(w)e^t|g'(e^t w)|. \end{aligned}$$

Now by change of variables, let $z = e^t w$ then $w = e^{-t}z$, and $Im(w) = e^{-t}Im(z)$.

Therefore,

$$\begin{aligned}\|S_t g\|_{B_{\infty,\circ}(\mathbb{U})} &= \sup_{z\in\mathbb{U}} e^{-t} Im(z) e^t |g'(z)| \\ &= \sup_{z\in\mathbb{U}} Im(z)|g'(z)| \\ &= \|g\|_{B_{\infty,\circ}(\mathbb{U})}, \text{ as desired.}\end{aligned}$$

Thus $(S_t)_{t\in\mathbb{R}}$ is an isometry. □

Theorem 4.1.3

$(S_t)_{t\in\mathbb{R}}$ *is strongly continuous on* $B_{\infty,\circ}(\mathbb{U})$.

Proof. From our calculation of weighted composition operator defined on $B_{\infty,\circ}(\mathbb{U})$, $S_t g(w) = g(e^t w)$. We now write $S_t = C_{\varphi_{-t}}$. Then $C_{\varphi_{-t}}$ is strongly continuous on $B_{\infty,\circ}(\mathbb{U})$ if and only if $(C_{\psi^{-1}\circ\varphi_{-t}\circ\psi})_{t\in\mathbb{R}}$ is strongly continuous on $B_{\infty,\circ}(\mathbb{D})$.

Now $\psi^{-1}\circ\varphi_{-t}\circ\psi(z)$, can be rewritten as $\psi^{-1}(\varphi_{-t}(\psi(z)))$. Recall that $\psi(z) = \frac{i(1+z)}{1-z}$ and therefore,

$$\psi^{-1}\circ\varphi_{-t}\circ\psi(z) = \psi^{-1}\left(\varphi_{-t}\left(\frac{i(1+z)}{1-z}\right)\right).$$

Since $\varphi_{-t}(z) = e^t(z)$, we get,

$$\psi^{-1}\circ\varphi_{-t}\circ\psi(z) = \psi^{-1}\left(e^t\left(\frac{i(1+z)}{1-z}\right)\right).$$

But $\psi^{-1}(z) = \frac{z-i}{z+i}$, and therefore we obtain,

$$\psi^{-1}\circ\varphi_{-t}\circ\psi(z) = \frac{e^t(\frac{i(1+z)}{1-z}) - i}{e^t(\frac{i(1+z)}{1-z}) + i}.$$

Simplifying the fraction, we have

$$\begin{aligned}\psi^{-1}\circ\varphi_{-t}\circ\psi(z) &= \frac{e^t(i(1+z))-i(1-z)}{e^t(i(1+z))+i(1-z)}\\ &= \frac{e^t(1+z)-1+z}{e^t+ze^t+1-z}\\ &= \frac{(1+e^t)z-1+e^t}{(-1+e^t)z+1+e^t}.\end{aligned}$$

Dividing both numerator and denominator by $1+e^t$ we get,

$$\psi^{-1}\circ\varphi_{-t}\circ\psi(z) = \frac{z-\frac{1-e^t}{1+e^t}}{1-\frac{1-e^t}{1+e^t}z}.$$

Now we let $a_t = \frac{1-e^t}{1+e^t}$ and substitute to obtain,

$$\begin{aligned}\psi^{-1}\circ\varphi_{-t}\circ\psi(z) &= \frac{z-a_t}{1-\overline{a}_t z}\\ &= h_a(z).\end{aligned}$$

Hence $\psi^{-1}\circ\varphi_{-t}\circ\psi(z) = h_a(z)$ by letting $\frac{z-a_t}{1-\overline{a}_t z} = h_a(z)$ and $z\to 0$ as $a_t\to 0$.

It therefore suffices to show that $\|C_{h_a}f-f\|_{B_{\infty,\circ}(\mathbb{D})}\to 0$ as $a\to 0$.

Using density of polynomials in $B_{\infty,\circ}(\mathbb{D})$, let $f(z)=z^n$. Then;

$$\begin{aligned}C_{h_a}z^n - z^n &= (h_a(z))^n - z^n, n\geq 1\\ (C_{h_a}f-f)'(z) &= n(h_a(z))^{n-1}h_a'(z) - nz^{n-1}\\ &= n[(h_a(z))^{n-1}h_a'(z) - z^{n-1}].\end{aligned}$$

But

$$\begin{aligned} h_a(z) &= \frac{z-a_t}{1-\overline{a}_t z}. \\ h_a'(z) &= \frac{(1-\overline{a}_t z)1-(z-a_t)(-\overline{a}_t)}{(1-\overline{a}_t z)^2} \\ &= \frac{1-\overline{a}_t z+\overline{a}_t z-a_t\overline{a}_t}{(1-\overline{a}_t z)^2} \\ &= \frac{1-a_t\overline{a}_t}{(1-\overline{a}_t z)^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} (C_{h_a}f-f)'(z) &= n\left[\frac{(h_a(z))^{n-1}(1-a_t\overline{a}_t)}{(1-\overline{a}_t z)^2}-z^{n-1}\right] \\ &= n\left[\frac{(\frac{z-a_t}{1-\overline{a}_t z})^{n-1}(1-a_t\overline{a}_t)}{(1-\overline{a}_t z)^2}-z^{n-1}\right] \\ &= n\left[\frac{(z-a_t)^{n-1}(1-a_t\overline{a}_t)}{(1-\overline{a}_t z)^{n-1}(1-\overline{a}_t z)^2}-z^{n-1}\right] \\ &= n\left[\frac{(z-a_t)^{n-1}(1-a_t\overline{a}_t)}{(1-\overline{a}_t z)^{n+1}}-z^{n-1}\right] \\ &= n\left[\frac{(z-a_t)^{n-1}(1-a_t\overline{a}_t)-z^{n-1}((1-\overline{a}_t z)^{n+1})}{(1-\overline{a}_t z)^{n+1}}\right]. \end{aligned}$$

Now,

$$\lim_{t\to 0^+}\|C_{h_a}f-f\|_{B_{\infty,\circ(\mathbb{D})}} = \lim_{t\to 0^+}\left(\sup_{z\in\mathbb{D}}(1-|z|^2)|(C_{h_a}f-f)'|(z)\right).$$

Therefore, by substituting for $(C_{h_a}f-f)'(z)$, we obtain
$\lim_{t\to 0^+}\|C_{h_a}f-f\|_{B_{\infty,\circ(\mathbb{D})}} = \lim_{t\to 0^+}\left(\sup_{z\in\mathbb{D}}(1-|z|^2)\left|n\left[\frac{(z-a_t)^{n-1}(1-a_t\overline{a}_t)-z^{n-1}((1-\overline{a}_t z)^{n+1})}{(1-\overline{a}_t z)^{n+1}}\right]\right|\right).$

Hence,

$$\begin{aligned}
\lim_{t\to 0^+}\|C_{h_a}f-f\|_{B_{\infty,\circ(\mathbb{D})}} &= \lim_{t\to 0^+}\left(\sup_{z\in\mathbb{D}}(1-|z|^2)\left|n\left[\frac{(z^{n-1})(1)-z^{n-1}(1)}{(1)^{n+1}}\right]\right|\right)\\
&= \lim_{t\to 0^+}\left(\sup_{z\in\mathbb{D}}(1-|z|^2)\left|n\left(\frac{z^{n-1}-z^{n-1}}{1}\right)\right|\right)\\
&= \lim_{t\to 0^+}\left(\sup_{z\in\mathbb{D}}(1-|z|^2)\left|n[z^{n-1}-z^{n-1}]\right|\right)\\
&= 0.
\end{aligned}$$

Hence, $(S_t)_{t\in\mathbb{R}}$ is strongly continuous on $B_{\infty,\circ}(\mathbb{U})$. □

Theorem 4.1.4

The infinitesimal generator Γ of $(S_t)_{t\geq 0}$ on $B_{\infty,\circ}(\mathbb{U})$ is given by $\Gamma g(w)=wg'(w)$ with the domain dom$(\Gamma)=\{g\in B_{\infty,\circ}(\mathbb{U}): wg'(w)\in B_{\infty,\circ}(\mathbb{U})\}$.

Proof. By definition, the infinitesimal generator denoted by Γ of $(S_t)_{t\geq 0}$ is given by;

$$\begin{aligned}
\Gamma g(w) &= \lim_{t\to 0^+}\frac{g(e^t w)-g(w)}{t}\\
&= \frac{\partial}{\partial t}g(e^t w)\Big|_{t=0}\\
&= e^t wg'(e^t w)\big|_{t=0}\\
&= wg'(w).
\end{aligned}$$

Therefore $\Gamma g(w)=wg'(w)$. This implies that dom$(\Gamma)\subseteq\{g\in B_{\infty,\circ}(\mathbb{U}): wg'(w)\in B_{\infty,\circ}(\mathbb{U})\}$.

To prove the reverse inclusion, we let $g\in B_{\infty,\circ}(\mathbb{U})$ be such that $wg'(w)\in B_{\infty,\circ}(\mathbb{U})$.

Thus for $w \in \mathbb{U}$, we have;

$$\begin{aligned} S_t g(w) - g(w) &= \int_0^t \frac{\partial}{\partial s} g(e^s w) ds \\ &= \int_0^t w g'(e^s w) ds. \end{aligned}$$

Now let $F(w) = wg'(e^s w)$, therefore $S_t g(w) - g(w) = \int_0^t F(w) ds$.
Thus $\lim_{t\to 0^+} \frac{S_t g - g}{t}$=$\lim_{t\to 0^+} \frac{1}{t} \int_0^t F(w) ds$ and strong continuity of $(S_s)_{s\geq 0}$ implies that $\frac{1}{t} \int_0^t \|S_s F - F\| ds \to 0^+$ as $t \to 0^+$. Hence $\text{dom}(\Gamma) \supseteq \{g \in B_{\infty,\circ}(\mathbb{U}) : wg'(w) \in B_{\infty,\circ}(\mathbb{U})\}$, which completes the proof. □

4.1.2 Spectral properties

Here we obtain the spectral properties of the generator Γ as well as the resulting resolvents.

Theorem 4.1.5

Let Γ be the infinitesimal generator of $(S_t)_{t\in\mathbb{R}}$ on $B_{\infty,\circ}(\mathbb{U})$. Then $\sigma_p(\Gamma) = \emptyset$ and $\sigma(\Gamma) = i\mathbb{R}$. In particular, Γ is an unbounded operator on $B_{\infty,\circ}(\mathbb{U})$.

Before we prove this theorem (Theorem 4.1.5), we first give the following Lemma:

Lemma 4.1.6

If $\nu \in \mathbb{C}$ and $c \in \mathbb{R}$, we have

1. $g(\omega) = c\omega^\nu \notin B_{\infty,\circ}(\mathbb{U})$ *for any* c,

2. $f(\omega) = (\omega - i)^\nu \in B_{\infty,\circ}(\mathbb{U})$ *if and only if* $Re(\nu) < 0$.

PROOF. From Lemma 3.0.2, we know that $g \in B_{\infty,\circ}(\mathbb{U})$ if and only if $g \circ \psi \in B_{\infty,\circ}(\mathbb{D})$. Then for $z \in \mathbb{D}$,

$$\begin{aligned}(g \circ \psi)(z) &= g(\psi(z)) = c(\psi(z))^{\nu} = c\left(\frac{i(1+z)}{1-z}\right)^{\nu} \\ &= ci(1+z)^{\nu}(1-z)^{-\nu}.\end{aligned}$$

Now, $g \circ \psi \in \mathcal{H}(\mathbb{D})$ if and only if $Re(\nu) > 0$ and $Re(-\nu) > 0$ which is not possible, and therefore $g \circ \psi \notin \mathcal{H}(\mathbb{D})$. Hence $g \notin B_{\infty,\circ}(\mathbb{U})$. This proves (1).

For (2), following [6, Lemma 3.2], for any $\nu \in \mathbb{C}, (\omega - i)^{\nu} \in \mathcal{H}(\mathbb{U})$ if and only if $Re(\nu) < 0$ since $\gamma = 0$ in this case. □

PROOF OF THEOREM 4.1.5. To obtain the point spectrum of Γ, let λ be an eigenvalue of Γ and g be the corresponding eigenvector. Then the point spectrum $\sigma_p(\Gamma)$ is given by $\sigma_p(\Gamma) = \{\lambda \in \mathbb{C} : \Gamma g = \lambda g$ for some $0 \neq x \in dom(\Gamma)\}$.

For our case $\Gamma g(w) = wg'(w)$. Therefore from $\Gamma g(w) = \lambda g(w)$ we have, $wg'(w) = \lambda g(w)$ which implies that $\frac{wg'(w)}{w} = \frac{\lambda g(w)}{w}$ by dividing both sides by w. Therefore,

$$g'(w) = \frac{\lambda g(w)}{w}.$$

Dividing both sides by $g(w)$ implies that $\frac{g'(w)}{g(w)} = \frac{\lambda}{w}$, and hence

$$\frac{dg(w)}{g(w)} = \lambda \frac{dw}{w}.$$

By integrating both sides, we obtain

$$\begin{aligned} \int \frac{dg}{g} &= \lambda \int \frac{dw}{w} \\ \Rightarrow \ln g(w) &= \lambda \ln(w) + \ln c. \\ \text{Therefore, } g(w) &= cw^{\lambda}. \end{aligned}$$

Now, $g(w) = cw^{\lambda}$ for $\lambda \in \mathbb{C}$. We now check when $g \in B_{\infty,\circ}(\mathbb{U})$.
From Lemma 3.0.2, we know that $g \in B_{\infty,\circ}(\mathbb{U})$ if and only if $g \circ \psi \in B_{\infty,\circ}(\mathbb{D})$ which is equivalent to $\sup_{z\in\mathbb{D}}(1-|z|^2)|(g\circ\psi)'(z)| < \infty$.
Then,

$$\begin{aligned} g \circ \psi &= g(\psi(z)) \\ &= c(\psi(z))^{\lambda}. \end{aligned}$$

But,

$$\psi(z) = \frac{i(1+z)}{1-z},$$

and therefore, $g \circ \psi = c\left(\frac{i(1+z)}{1-z}\right)^{\lambda}$ implying that,

$$\begin{aligned} (g\circ\psi)'(z) &= c\lambda\left(\frac{i(1+z)}{1-z}\right)^{\lambda-1} . \frac{i(1-z)+i(1+z)}{(1-z)^2}, \\ &= c\lambda\left(\frac{i(1+z)}{1-z}\right)^{\lambda-1} . \frac{i-iz+i+iz)}{(1-z)^2}, \\ &= \frac{2ic\lambda(i(1+z))^{\lambda-1}}{(1-z)^{\lambda+1}}, \\ &= \frac{2ic\lambda i^{\lambda-1}(1+z)^{\lambda-1}}{(1-z)^{\lambda+1}}, \\ &= 2ic\lambda(i(1+z))^{\lambda-1}(1-z)^{-(\lambda+1)}. \end{aligned}$$

Now for $(1-z)^{-(\lambda+1)}$,

$$\begin{aligned} Re(-(\lambda+1)) &= Re(-\lambda-1) > 0 \\ &\Leftrightarrow -Re(\lambda)-1 > 0 \\ &\Leftrightarrow -Re(\lambda) > 1 \\ &\Leftrightarrow Re(\lambda) < -1. \end{aligned}$$

And for $(1+z)^{\lambda-1}$,

$$\begin{aligned} Re(\lambda-1) &= Re(\lambda-1) > 0 \\ &\Leftrightarrow Re(\lambda) > 1. \end{aligned}$$

Then $(1+z)^{\lambda-1} \in B_{\infty,\circ}(\mathbb{D})$ if and only if $Re(\lambda) > 1$ and $(1-z)^{-(\lambda+1)} \in B_{\infty,\circ}(\mathbb{D})$ if and only if $Re(\lambda) < -1$. Hence $Re(\lambda)$ does not exist and thus there is no such λ. Therefore the point spectrum is empty, that is $\sigma_p(\Gamma) = \emptyset$.

Since each S_t is an invertible isometry, its spectrum satisfies $\sigma(S_t) \subseteq \partial\mathbb{D}$. Therefore the spectral mapping theorem [Theorem 2.0.3] for strongly continuous groups implies that $e^{t\sigma(\Gamma)} \subseteq \sigma(S_t) \subseteq \partial\mathbb{D}$. Now let $\lambda \in \sigma(\Gamma)$, then $|e^{t\lambda}| = 1 \Rightarrow t\lambda = 0 \Rightarrow Re(\lambda) = 0$. Thus $\lambda \in i\mathbb{R}$ and therefore $\sigma(\Gamma) \subseteq i\mathbb{R}$.

We now need to show that the reverse inclusion, that is, $i\mathbb{R} \subseteq \sigma(\Gamma)$ holds. Fix $\lambda \in i\mathbb{R}$ and assume $\lambda \notin \sigma(\Gamma)$ which implies that the resolvent operator $R(\lambda, \Gamma) : B_{\infty,\circ}(\mathbb{U}) \to B_{\infty,\circ}(\mathbb{U})$ is bounded. Consider the function $h(\omega) = (\omega - i)^{-(\lambda+1)}$. Then $Re(-(\lambda+1)) = -1 < 0$ and following Lemma 4.1.6, it is immediate that $h \in B_{\infty,\circ}(\mathbb{U})$. The image function $f = R(\lambda, \Gamma)h$ is equivalent to $(\lambda - \Gamma)f = h$ hence $\lambda f - \Gamma f = h$ which yields a differential equation

$$\lambda f(\omega) - \omega f'(\omega) \;\; = \;\; h(\omega),$$

Now, dividing both sides by -1 to obtain

$$\omega f'(\omega) - \lambda f(\omega) \;\; = \;\; -h(\omega),$$

therefore dividing by ω, that is

$$\frac{\omega f'(\omega)}{\omega} - \frac{\lambda f(\omega)}{\omega} \;\; = \;\; -\frac{h(\omega)}{\omega},$$

we obtain the differential equation of the form,

$$\begin{aligned} f'(w) - \frac{\lambda}{\omega} f(\omega) &= -\frac{h(\omega)}{\omega} \\ &= -\omega^{-1} h(\omega). \end{aligned}$$

Hence, we obtain

$$(\omega^{-\lambda} f(\omega))' \;\; = \;\; -\omega^{-1-\lambda} h(\omega) d\omega.$$

Now, we consider the function $h(\omega) = (\omega - i)^{-(\lambda+1)}$.
Hence, by substituting the function $h(\omega) = (\omega - i)^{-(\lambda+1)}$ we obtain

$$(\omega^{-\lambda} f(\omega))' \;\; = \;\; -\omega^{-1-\lambda} (\omega - i)^{-(\lambda+1)} d\omega.$$

Therefore, by integrating and dividing both side by $\omega^{-\lambda}$ of

$$\omega^{-\lambda} f(\omega) = -\int \omega^{-1-\lambda}(\omega - i)^{-(\lambda+1)} d\omega,$$

we obtain the general solution

$$f(\omega) = (\omega - i)^{-\lambda} + c\omega^{\lambda}.$$

which does not belong to $B_{\infty,\circ}(\mathbb{U})$ for any c, by Lemma 4.1.6. Thus $h \notin R(\lambda - \Gamma)$ and so $\sigma(\Gamma) = i\mathbb{R}$.

Since $\sigma(\Gamma) = i\mathbb{R}$ and since $r(\Gamma) \leq \|\Gamma\|$, it follows that Γ is unbounded on $B_{\infty,\circ}(\mathbb{U})$. $\square$

Theorem 4.1.7

Let Γ be the infinitesimal generator of $(S_t)_{t\in\mathbb{R}}$. Then the following holds;

1. *For $\lambda \in \rho(\Gamma)$, and $h \in B_{\infty,\circ}(\mathbb{U})$ then,*

   (i) $R(\lambda, \Gamma)h(w) = w^{\lambda} \int_w^{\infty} \frac{1}{z^{\lambda+1}} h(z)dz$, if $Re(\lambda) > 0$.

   (ii) $R(\lambda, \Gamma)h(w) = -w^{\lambda} \int_0^{w} \frac{1}{z^{\lambda+1}} h(z)dz$, if $Re(\lambda) < 0$.

2. *$\sigma(R(\lambda, \Gamma)) = \left\{ w : |w - \frac{1}{2Re(\lambda)}| = \frac{1}{2Re(\lambda)} \right\}$.*

3. *$r(R(\lambda, \Gamma)) = \|R(\lambda, \Gamma)\| = \frac{1}{|Re(\lambda)|}$.*

Proof. To prove (1), we take note that the resolvent set is given as $\rho(\Gamma) = \{\lambda \in \mathbb{C} : Re(\lambda) \neq 0\}$. We therefore consider the following cases:

Case 1: If $Re(\lambda) > 0$, then the resolvent operator is given by the Laplace transform. For every $h \in B_{\infty,\circ}(\mathbb{U})$, we have $R(\lambda, \Gamma)h = \int_0^{\infty} e^{-\lambda t} S_t h dt$ with convergence in norm. Therefore,

$$R(\lambda,\Gamma)h(\omega) = \int_0^\infty e^{-\lambda t}h(e^t w)dt.$$

By change of variables, let $z = e^t w$, then $w = e^{-t}z$, $\frac{dz}{dt} = we^t$ then $dt = \frac{dz}{we^t} = \frac{dz}{z}$. Therefore when $t = 0 \Rightarrow z = w$ and $t = \infty \Rightarrow z = \infty$ and so

$$\begin{aligned} R(\lambda,\Gamma)h(\omega) &= \int_\omega^\infty e^{-\lambda t}h(z)\frac{dz}{z} \\ &= \int_\omega^\infty \left(\frac{z}{\omega}\right)^{-\lambda}\frac{1}{z}h(z)dz \\ &= \frac{1}{\omega^{-\lambda}}\int_w^\infty z^{-(\lambda+1)}h(z)dz \\ &= \omega^\lambda\int_\omega^\infty z^{-(\lambda+1)}h(z)dz \\ &= \omega^\lambda\int_\omega^\infty \frac{1}{z^{\lambda+1}}h(z)dz. \end{aligned}$$

Case 2: If $Re(\lambda) < 0$, then

$$\begin{aligned} R(\lambda,\Gamma)h(\omega) &= -R(-\lambda,-\Gamma)h \\ &= -\int_0^\infty e^{\lambda t}h(e^{-t}w)dt. \end{aligned}$$

Then by change of variables, let $z = e^{-t}w$ then $e^t = \frac{w}{z}$, $\frac{dz}{dt} = -we^{-t}$ then $dt = \frac{-dz}{we^{-t}}$ and $dt = \frac{-dz}{z}$. Therefore $t = 0 \Rightarrow z = w$ and $t = \infty \Rightarrow z = 0$

and so;

$$\begin{aligned}
R(\lambda,\Gamma)h(\omega) &= -\int_0^\infty e^{\lambda t}h(z).-\frac{dz}{z} \\
&= -\int_0^\omega \left(\frac{w}{z}\right)^\lambda h(z).\frac{dz}{z} \\
&= -\omega^\lambda \int_0^\omega \left(\frac{1}{z}\right)^\lambda .\frac{1}{z}h(z)dz \\
&= -\omega^\lambda \int_0^\omega z^{-\lambda}.z^{-1}h(z)dz \\
&= -\omega^\lambda \int_0^\omega z^{-(\lambda+1)}h(z)dz \\
&= -\omega^\lambda \int_0^\omega \frac{1}{z^{\lambda+1}}h(z)dz.
\end{aligned}$$

To prove (2), we use the spectral mapping theorem [Theorem 2.0.2] for the resolvent operator which asserts that
$\sigma(R(\lambda,\Gamma)) = (\lambda-\sigma(\Gamma))^{-1} = \left\{\frac{1}{\lambda-\mu} : \mu\in\sigma(\Gamma) \text{ for } \lambda\in\rho(\Gamma)\right\}\setminus\{0\}$.
Therefore,

$$\begin{aligned}
\sigma(R(\lambda,\Gamma)) &= \left\{\frac{1}{\lambda-ir} : r\in\mathbb{R}\right\}\setminus\{0\} \\
&= \left\{\frac{1}{Re(\lambda)+i(Im(\lambda)-r)} : r\in\mathbb{R}\right\}\setminus\{0\}.
\end{aligned}$$

Rationalizing the denominator and simplifying we get,

$$\begin{aligned}
\sigma(R(\lambda,\Gamma)) &= \left\{\frac{1(Re(\lambda)-i(Im(\lambda)-r))}{(Re(\lambda)+i(Im(\lambda)-r))(Re(\lambda)-i(Im(\lambda)-r))} : r\in\mathbb{R}\right\} \\
&= \left\{\frac{(Re(\lambda)-i(Im(\lambda)-r))}{(Re(\lambda))^2+(Im(\lambda)-r)^2} : r\in\mathbb{R}\right\},
\end{aligned}$$

where,

$$\lambda = Re(\lambda) + iIm(\lambda).$$

Now by letting $w = \frac{(Re(\lambda) - i(Im(\lambda) - r))}{(Re(\lambda))^2 + (Im(\lambda) - r)^2}$ and subtracting $\frac{1}{2Re(\lambda)}$, therefore we obtain,

$$\begin{aligned}
w - \frac{1}{2Re(\lambda)} &= \frac{Re(\lambda) - i(Im(\lambda) - r)}{(Re(\lambda))^2 + (Im(\lambda) - r)^2} - \frac{1}{2Re(\lambda)} \\
&= \frac{2Re(\lambda)(Re(\lambda) - i(Im(\lambda) - r)) - ((Re(\lambda))^2 + (Im(\lambda) - r)^2)}{2Re(\lambda)((Re(\lambda))^2 + (Im(\lambda) - r)^2)} \\
&= \frac{2(Re(\lambda))^2 - 2iRe(\lambda)(Im(\lambda) - r)) - ((Re(\lambda))^2 - (Im(\lambda) - r)^2)}{2Re(\lambda)((Re(\lambda))^2 + (Im(\lambda) - r)^2)} \\
&= \frac{(Re(\lambda))^2 - (Im(\lambda) - r)^2 - 2iRe(\lambda)(Im(\lambda) - r)}{2Re(\lambda)((Re(\lambda))^2 + (Im(\lambda) - r)^2)}.
\end{aligned}$$

Now finding the magnitude of both sides of the equation and simplifying we get,

$$\begin{aligned}
\left| w - \frac{1}{2Re(\lambda)} \right|^2 &= \left| \frac{(Re(\lambda))^2 - (Im(\lambda) - r)^2 - 2iRe(\lambda)(Im(\lambda) - r)}{2Re(\lambda)((Re(\lambda))^2 + (Im(\lambda) - r)^2)} \right|^2 \\
&= \frac{[(Re(\lambda))^2 - (Im(\lambda) - r)^2]^2 + 4(Re(\lambda))^2(Im(\lambda) - r)^2}{4Re(\lambda)^2[(Re(\lambda))^2 + (Im(\lambda) - r)^2]^2} \\
&= \frac{1}{(2Re(\lambda))^2}. \\
\left| w - \frac{1}{2Re(\lambda)} \right|^2 &= \frac{1}{(2Re(\lambda))^2} \\
\left| w - \frac{1}{2Re(\lambda)} \right| &= \frac{1}{2Re(\lambda)}.
\end{aligned}$$

Therefore, $\sigma(R(\lambda, \Gamma)) = \left\{ w : |w - \frac{1}{2Re(\lambda)}| = \frac{1}{2Re(\lambda)} \right\}$.

For part (3), the spectral radius $r(R(\lambda,\Gamma))$ is by definition given as;

$$\begin{aligned} r(R(\lambda,\Gamma)) &= \sup\{|w| : w \in \sigma(R(\lambda,\Gamma))\} \\ &= \sup\left\{|w| : \left|w - \frac{1}{2Re(\lambda)}\right| = \frac{1}{2Re(\lambda)}\right\} \\ &= \frac{1}{|Re(\lambda)|}. \end{aligned}$$

Finally, to determine $\|R(\lambda,\Gamma)\|$, we use the Hille Yosida theorem [Theorem 2.0.1] as well as the fact that the spectral radius is always bounded by the norm.

Since $r(R(\lambda,\Gamma)) = \frac{1}{|Re(\lambda)|}$, then using the Hille Yosida theorem which asserts that for every $\lambda \geq 0$,

$$\|R(\lambda,\Gamma)\| \leq \frac{1}{|Re(\lambda)|},$$

we get,

$$\frac{1}{|Re(\lambda)|} = r(R(\lambda,\Gamma)) \leq \|R(\lambda,\Gamma)\| \leq \frac{1}{|Re(\lambda)|},$$

which implies that $\|R(\lambda,\Gamma)\| = \frac{1}{|Re(\lambda)|}$.

Therefore, $r(R(\lambda,\Gamma)) = \|R(\lambda,\Gamma)\| = \frac{1}{|Re(\lambda)|}$. □

4.2 Translation Group

In this group the automorphisms are of the form $\varphi_t(z) = z + kt$, where $z \in \mathbb{U}$ and $k, t \in \mathbb{R}$ with $k \neq 0$. Again, without loss of generality, we let $k = 1$ and consider self analytic maps of $\mathbb{U}$ of the form $\varphi_t(z) = z+t$. Then the corresponding group of composition operators defined on $L^p_a(\mathbb{U}, \mu_\alpha)$ is given by

$$T_t f(z) \;=\; f(z+t),$$

for all $f \in L^p_a(\mathbb{U}, \mu_\alpha)$ where $1 \leq p < \infty$. For $p = 1$, $(T_t)_{t\geq 0}$ is defined on $L^1_a(\mathbb{U}, \mu_\alpha)$.

Again by using the duality pairing $\langle g, f\rangle = \int_{\mathbb{U}} g(w)\overline{f(w)}d\mu_\alpha(w)$ where $f \in L^1_a(\mathbb{U}, \mu_\alpha)$ and $g \in B_{\infty,\circ}(\mathbb{U}, i)$, we obtain a group of composition operators on $B_{\infty,\circ}(\mathbb{U}, i)$ as follows:

Let $g \in B_{\infty,\circ}(\mathbb{U}, i)$, then

$$\langle g, T_t f\rangle \;=\; \int_{\mathbb{U}} g(z)\overline{f(z+t)}d\mu_\alpha(z).$$

Now by change of variables, let $w = z + t$, then $z = w - t$ and $dA(w) = \left|\frac{dw}{dz}\right|^2 dA(z) = dA(z)$.

Therefore,

$$\begin{aligned}
\langle g, T_t f\rangle &= \int_{\mathbb{U}} g(z)\overline{f(z+t)}(Im(w))^\alpha dA(z) \\
&= \int_{\mathbb{U}} g(w-t)\overline{f(w)}(Im(w))^\alpha dA(w) \\
&= \int_{\mathbb{U}} g(w-t)\overline{f(w)}d\mu_\alpha(w) \\
&= \langle T_t^* g, f\rangle.
\end{aligned}$$

Now, we define $S_t := T_t^*$ on $B_{\infty,\circ}(\mathbb{U}, i)$. But we see that just as in the case of the scaling group, $S_t g(i) = g(i-t)$ and therefore $S_t g$ does not vanish at i. This means that S_t does not map $B_{\infty,\circ}(\mathbb{U}, i)$ onto itself. We therefore redefine S_t to act on $B_{\infty,\circ}(\mathbb{U})$ so that it maps the space onto itself. Thus, for all $t \geq 0$ and $g \in B_{\infty,\circ}(\mathbb{U})$, $S_t g(w) = g(w-t)$ is the group of composition operators defined on $B_{\infty,\circ}(\mathbb{U})$. In the next sections, we study the semigroup properties of $(S_t)_{t\geq 0}$ on $B_{\infty,\circ}(\mathbb{U})$.

4.2.1 Semigroup properties

In this section, we prove that $(S_t)_{t\in\mathbb{R}}$ is a group, is an isometry and is strongly continuous on $B_{\infty,\circ}(\mathbb{U})$ and finally determine the infinitesimal generator Γ of $(S_t)_{t\geq 0}$ on $B_{\infty,\circ}(\mathbb{U})$.

Lemma 4.2.1

$(S_t)_{t\in\mathbb{R}}$ is a group on $B_{\infty,\circ}(\mathbb{U})$.

Proof. From definition, $S_t g(w) = g(w-t)$, therefore first we need to show that $(S_t)_{t\geq 0}$ is a group.

$S_0 g(w) = g(w-0) = g(w) \Rightarrow S_0 = I$(Identity).

And for $S_{t+s}g(w) = (S_t \circ S_s)g(w)$, we need to show that,

$$(S_t \circ S_s)g(w) \;=\; S_{t+s}g(w) = S_t(S_s g(w)).$$

We have,

$$\begin{aligned}
(S_t \circ S_s)g(w) &= S_t.S_s g(w) = S_t(g(w-s)) \\
&= g((w-s)-t) \\
&= g(w-(s+t)) \\
&= S_{s+t}g(w) = S_{t+s}g(w).
\end{aligned}$$

Therefore, $S_{t+s}g(w) = (S_t \circ S_s)g(w)$.

Hence, $(S_t)_{t \geq 0}$ is a semigroup.

Similarly, we need to show that $(S_{-t})_{t \geq 0}$ is also a group.

From definition, $S_t g(w) = g(w - t)$. Therefore $S_{-t}g(w) = g(w + t)$

Thus,

$S_{-0}g(w) = g(w - 0) = g(w) \Rightarrow S_0 = I(Identity)$.

And we need to show that,

$$(S_{-t} \circ S_{-s})g(w) = S_{-t-s}g(w) = S_{-t}(S_{-s}g(w)).$$

Therefore we have

$$\begin{aligned}
(S_{-t} \circ S_{-s})g(w) &= S_{-t}(S_{-s}g(w)) \\
&= S_{-t}(g(w + s)) \\
&= g(w + s + t) = g(w + (s + t)) \\
&= S_{-s+-t}g(w) = S_{-t+-s}g(w).
\end{aligned}$$

Therefore it is clear that $S_{-t-s}g(w) = (S_{-t} \circ S_{-s})g(w)$. Thus $(S_{-t})_{t \geq 0}$ is a semigroup.

Hence, $(S_t)_{t \in \mathbb{R}}$ is a group. □

Theorem 4.2.2

$(S_t)_{t \in \mathbb{R}}$ is an isometry on $B_{\infty,\circ}(\mathbb{U})$.

PROOF. By definition, we have;

$$\begin{aligned}
\|S_t g\|_{B_{\infty,\circ}(\mathbb{U})} &= \sup_{w\in\mathbb{U}} Im(w)|(S_t g)'(w)| \\
&= \sup_{w\in\mathbb{U}} Im(w)|(g(w-t))'| \\
&= \sup_{w\in\mathbb{U}} Im(w)|g'(w-t)|.
\end{aligned}$$

By change of variables, let $z = w - t$ then $w = z + t$ and $Im(w) = Im(z)$. Hence,

$$\begin{aligned}
\|S_t g\|_{B_{\infty,\circ}(\mathbb{U})} &= \sup_{z\in\mathbb{U}} Im(z)|g'(z)| \\
&= \|g\|_{B_{\infty,\circ}(\mathbb{U})}, \text{ as desired.}
\end{aligned}$$

This therefore means that $(S_t)_{t\in\mathbb{R}}$ is an isometry. □

Theorem 4.2.3

$(S_t)_{t\in\mathbb{R}}$ *is strongly continuous on* $B_{\infty,\circ}(\mathbb{U})$.

PROOF. We know that weighted composition operators defined on $B_{\infty,\circ}(\mathbb{U})$ is $S_t g(w) = g(w-t)$. We let $S_t = C_{\varphi_{-t}}$ which is strongly continuous on $B_{\infty,\circ}(\mathbb{U})$ if and only if $(C_{\psi^{-1}\circ\varphi_{-t}\circ\psi})_{t\in\mathbb{R}}$ is strongly continuous on $B_{\infty,\circ}(\mathbb{D})$. By computing $\psi^{-1}\circ\varphi_{-t}\circ\psi(z)$ we obtain;

$$\begin{aligned}
\psi^{-1}\circ\varphi_{-t}\circ\psi(z) &= \psi^{-1}\circ\varphi_{-t}(\psi(z)) \\
&= \psi^{-1}(\varphi_{-t}(\psi(z)))
\end{aligned}$$

By letting $\psi(z) = \frac{i(1+z)}{1-z}$ and $\varphi_{-t}(z) = z - t$ and substituting, we get

$$\begin{aligned}
\psi^{-1} \circ \varphi_{-t} \circ \psi(z) &= \psi^{-1}\left(\frac{i(1+z)}{1-z} - t\right) \\
&= \frac{\frac{i(1+z)}{1-z} - t - i}{\frac{i(1+z)}{1-z} - t + i} \\
&= \frac{\frac{i(1+z)}{1-z} - (t+i)}{\frac{i(1+z)}{1-z} + (i-t)}.
\end{aligned}$$

We simplify the fraction to obtain,

$$\begin{aligned}
\psi^{-1} \circ \varphi_{-t} \circ \psi(z) &= \frac{i(1+z) - ((t+i)(1-z))}{i(1+z) + (i-t)(1-z)} \\
&= \frac{i + iz - t + tz - i + iz}{i + iz + i - iz - t + tz} \\
&= \frac{2iz + tz - t)}{2i - t + tz)} \\
&= \frac{(2i+t)z - t}{2i - t + tz}.
\end{aligned}$$

Dividing each term of the fraction by $2i + t$, we get

$$= \frac{z - \frac{t}{2i+t}}{\frac{2i-t}{2i+t} + \frac{t}{2i+t} z}.$$

Now we let

$$\begin{aligned}
a_t &= \frac{t}{2i+t}, \\
b_t &= \frac{2i-t}{2i+t}.
\end{aligned}$$

Therefore,

$$\begin{aligned}\psi^{-1}\circ\varphi_{-t}\circ\psi(z) &= \frac{z-a_t}{b_t+a_t z}\\ &= h_a(z).\end{aligned}$$

For $\frac{z-a_t}{b_t+a_t z} = h_a(z)$ and $t\to 0$ as $a_t\to 0$ and $b_t\to 1$. It therefore suffices to show that

$\|C_{h_a}f-f\|_{B_{\infty,\circ}(\mathbb{D})}\to 0$ as $t\to 0$.

Using density of polynomial in $B_{\infty,\circ}(\mathbb{D})$, we let $f(z)=z^n$. Then;

$$C_{h_a}z^n - z^n = (h_a(z))^n - z^n, n\geq 1.$$

Therefore,

$$\begin{aligned}(C_{h_a}f-f)'(z) &= n(h_a(z))^{n-1}h_a'(z) - nz^{n-1}\\ &= n[(h_a(z))^{n-1}h_a'(z) - z^{n-1}].\end{aligned}$$

But,

$$h_a(z) = \frac{z-a_t}{b_t+a_t z}.$$

This implies that,

$$h_a'(z) = \frac{(b_t+a_t z)(1)-(z-a_t)(a_t)}{(b_t+a_t z)^2}.$$

Therefore by substituting,

$$\begin{aligned}(C_{h_a}f - f)'(z) &= n[(h_a(z))^{n-1}h'_a(z) - z^{n-1}] \\ &= n\left[\left(\frac{z-a_t}{b_t+a_tz}\right)^{n-1}\frac{(b_t+a_tz)-(z-a_t)(a_t)}{(b_t+a_tz)^2} - z^{n-1}\right] \\ &= n\left[\frac{(z-a_t)^{n-1}(b_t+a_tz)-(z-a_t)(a_t)}{(b_t+a_tz)^{n+1}} - z^{n-1}\right].\end{aligned}$$

Now,

$$\lim_{t\to 0^+}\|C_{h_a}f - f\|_{B_{\infty,\circ(\mathbb{D})}} = \lim_{t\to 0^+}\left(\sup_{z\in\mathbb{D}}(1-|z|^2)|(C_{h_a}f-f)'|(z)\right)$$

Therefore by substituting for $(C_{h_a}f - f)'(z)$, we obtain
$\lim_{t\to 0^+}\|C_{h_a}f - f\|_{B_{\infty,\circ(\mathbb{D})}} = \lim_{t\to 0^+}\left(\sup_{z\in\mathbb{D}}(1-|z|^2)\left|n\left[\frac{(z-a_t)^{n-1}(b_t+a_tz)-(z-a_t)(a_t)}{(b_t+a_tz)^{n+1}} - z^{n-1}\right]\right|\right)$
Hence, it implies that the $\lim_{t\to 0^+}\|C_{h_a}f - f\|_{B_{\infty,\circ(\mathbb{D})}} =$
$\lim_{t\to 0^+}\left(\sup_{z\in\mathbb{D}}(1-|z|^2)\left|n\left[\frac{(z-a_t)^{n-1}(b_t+a_tz)-(z-a_t)(a_t)-z^{n-1}(b_t+a_tz)^2}{(b_t+a_tz)^{n+1}}\right]\right|\right)$.
Therefore,

$$\begin{aligned}\lim_{t\to 0^+}\|C_{h_a}f - f\|_{B_{\infty,\circ(\mathbb{D})}} &= \lim_{t\to 0^+}\left(\sup_{z\in\mathbb{D}}(1-|z|^2)\left|\frac{n[z^{n-1}-0-z^{n-1}]}{1}\right|\right) \\ &= 0.\end{aligned}$$

Hence $(S_t)_{t\in\mathbb{R}}$ is strongly continuous. □

Theorem 4.2.4

The infinitesimal generator Γ of $(S_t)_{t\geq 0}$ on $B_{\infty,\circ}(\mathbb{U})$ is given by $\Gamma g(w) = -g'(w)$ with the domain dom$(\Gamma) = \{g \in B_{\infty,\circ}(\mathbb{U}) : g'(w) \in B_{\infty,\circ}(\mathbb{U})\}$.

Proof. By definition, the infinitesimal generator denoted by Γ of $g(w)$

is given by;

$$\begin{aligned}\Gamma g(w) &= \lim_{t\to 0^+}\frac{g(w-t)-g(w)}{t}\\ &= \frac{\partial}{\partial t}g(w-t)\Big|_{t=0}\\ &= -g'(w).\end{aligned}$$

Therefore, $\Gamma g(w) = -g'(w)$ with the dom$(\Gamma) = \{g \in B_{\infty,\circ}(\mathbb{U}) : g'(w) \in B_{\infty,\circ}(\mathbb{U})\}$.

This therefore implies that dom$(\Gamma) \subset \{g \in B_{\infty,\circ}(\mathbb{U}) : g'(w) \in B_{\infty,\circ}(\mathbb{U})\}$.

Now to prove the reverse inclusion, let $g \in B_{\infty,\circ}(\mathbb{U})$ such that $g'(w) \in B_{\infty,\circ}(\mathbb{U})$.

Thus for $w \in \mathbb{U}$, we have;

$$\begin{aligned}S_t g(w) - g(w) &= \int_0^t \frac{\partial}{\partial s}g(w-s)ds\\ &= \int_0^t -g'(w)ds.\end{aligned}$$

Now let $G(w) = -g'(w)$, therefore $S_t g(w) - g(w) = \int_0^t G(w)ds$. So $G(w) = -g'(w)$ is a function of $B_{\infty,\circ}(\mathbb{U})$.

Thus $\lim_{t\to 0^+}\frac{S_t g - g}{t}$=$\lim_{t\to 0^+}\frac{1}{t}\int_0^t G(w)ds$ and strong continuity of $(S_s)_{s\geq 0}$ implies that $\frac{1}{t}\int_0^t \|S_s G - G\|ds \to 0^+$ as $t \to 0^+$ hence dom$(\Gamma) \supseteq \{g \in B_{\infty,\circ}(\mathbb{U}) : -g'(w) \in B_{\infty,\circ}(\mathbb{U})\}$, which completes the proof. □

Chapter 5

Summary and Recommendations

5.1 Summary

In this thesis, we determined the predual of the non-reflexive Bergman space $L_a^1(\mathbb{U}, \mu_\alpha)$ using the Cayley transform as well as the approach used by K. Zhu on the duality of Bergman spaces of the unit disk. Specifically, we have established that the predual of $L_a^1(\mathbb{U}, \mu_\alpha)$ can be identified with $B_{\infty,\circ}(\mathbb{U}, i)$ which is the little Bloch space of the upper half-plane consisting of functions vanishing at i, see Theorem 3.0.11. We also considered the groups of composition operators induced by the group of self analytic maps of the upper half plane which are distinctively classified into scaling, translation and rotation groups. In this study we considered only the scaling and translation groups. On the scaling group, we considered the analytic self maps $\varphi_t : \mathbb{U} \longrightarrow \mathbb{U}$ of the form $\varphi_t(z) = e^{-t}z$ for $z \in \mathbb{U}$ and obtained the corresponding group of weighted composition operators on the predual of the space $L_a^1(\mathbb{U}, \mu_\alpha)$ using the duality pairing. Therefore, we obtained $S_tg(w) := g(e^tw)$ as the semigroup of composition operators defined on $B_{\infty,\circ}(\mathbb{U}, i)$. Since $(S_t)_{t\in\mathbb{R}}$ is not well defined on $B_{\infty,\circ}(\mathbb{U}, i)$, we extended the domain to $B_{\infty,\circ}(\mathbb{U})$ where $(S_t)_{t\in\mathbb{R}}$ is well defined and studied both the semigroup and spectral properties of the groups on $B_{\infty,\circ}(\mathbb{U})$. For the semigroup properties, we established the strong continuity property in Theorem 4.1.3. and also in Theorem 4.1.4, we determined the infinitesimal generator Γ of $(S_t)_{t\geq 0}$ on $B_{\infty,\circ}(\mathbb{U})$, and obtained its domain.

For the spectral properties, we obtained the spectrum and point spectrum of the infinitesimal generator Γ. In Theorem 4.1.6, we obtained the resolvent of the infinitesimal generator Γ of this group using spectral theory, which was obtained as an integral operator. Finally, using Hille-Yosida theorem, we determined the spectral radius of the infinitesimal generator as $r(R(\lambda, \Gamma)) = \frac{1}{|Re(\lambda)|}$ and the norm as $\|R(\lambda, \Gamma)\| = \frac{1}{|Re(\lambda)|}$.

On the translation group, we also considered the self analytic maps of $\mathbb{U}$ of the form $\varphi_t(z) = z + t$ and obtained a group of composition operators on $B_{\infty,\circ}(\mathbb{U}, i)$ again using the same duality pairing. In this case, we obtained $S_tg(w) = g(w - t)$ as the group of composition operators defined on $B_{\infty,\circ}(\mathbb{U}, i)$. Again, since $(S_t)_{t\in\mathbb{R}}$ is not well defined on $B_{\infty,\circ}(\mathbb{U}, i)$, we extended the domain to $B_{\infty,\circ}(\mathbb{U})$ where $(S_t)_{t\in\mathbb{R}}$ is well defined and studied both the semigroup and spectral properties of the groups on $B_{\infty,\circ}(\mathbb{U})$. For the semigroup properties, we established the strong continuity property in Theorem 4.2.3. and also determined the infinitesimal generator Γ of $(S_t)_{t\geq 0}$ on $B_{\infty,\circ}(\mathbb{U})$, where $S_tg(w) = g(w - t)$ in Theorem 4.2.4. and obtained its domain

5.2 Recommendations

From the results obtained in this study, we recommend the following for further research:

1. In this study, we considered the group of weighted composition operators corresponding to the self analytic maps defined on the scaling and translation groups on $B_{\infty,\circ}(\mathbb{U})$ and studied their semigroup and spectral properties. Therefore, we recommend an extension of the same study on the weighted composition operators corresponding to the self analytic maps defined on the rotation groups on $B_{\infty,\circ}(\mathbb{U})$.

2. From this work, we have successfully investigated the semigroup and spectral properties of weighted composition operators corresponding to the self analytic maps defined on the predual of non-reflexive Bergman spaces of the upper-half plane, but on the other spaces like Dirichlet spaces, Besov spaces and among other spaces this has not been carried out. Therefore we recommend an extension of the same investigation of semigroups and spectral properties of groups of weighted composition operators on Dirichlet spaces and Besov spaces on the upper half-plane.

3. In this study, we obtained the semigroup of composition operators defined on $B_{\infty,\circ}(\mathbb{U}, i)$ but since $(S_t)_{t\in\mathbb{R}}$ is not well defined on $B_{\infty,\circ}(\mathbb{U}, i)$, we extended the domain to $B_{\infty,\circ}(\mathbb{U})$ where $(S_t)_{t\in\mathbb{R}}$ is well defined and studied both the semigroup and spectral properties of the groups on $B_{\infty,\circ}(\mathbb{U})$. Instead of extending the domain, we recommed the use of correction factor on the obtained semigroup of

composition operators defined on $B_{\infty,\circ}(\mathbb{U}, i)$ in order to study both the semigroup and spectral properties of the groups on $B_{\infty,\circ}(\mathbb{U}, i)$.

4. From this thesis, we considered the group of weighted composition operators corresponding to the self analytic maps defined on the scaling and translation groups on $B_{\infty,\circ}(\mathbb{U})$. Therefore, we recommend an extension of the study to more general classes of the Bloch spaces, usually called, Bloch type spaces.

References

[1] R. F. Allen and F. Colonna, *Isometries and spectra of multiplication operators on the Bloch space*, Bull. Aust. Math. Soc, **79** (2009), 147-160.

[2] I. Arevalo, *A characterization of the inclusions between mixed norm spaces*, Journal of Math. Anal and Appl, **433**(2016), 1904-1905.

[3] I. Arevalo and M. Oliva, *Semigroups of weighted composition operators in the spaces of analytic functions*, Departamento de matematicas, Universidad Autonoma De Madrid-ICMAT, 28049 Madrid, Spain.

[4] A. G. Arvanitidis, *Semigroups of composition operators on Hardy spaces of the Half-Plane*. Thessaloniki, Greece, (2011).

[5] S. Axler, *Bergman spaces and their operators*, Lecture Notes at the Indiana University, Function Theoretic operators Theory conference November, 1985.

[6] S. Ballamoole, J. O. Bonyo, T. L. Miller and V. G. Miller, *Ces'aro-like operators on the Hardy and Bergman Spaces of the Half Plane*, Complex Anal. Oper. Theory, **10**, (2016), 187-203.

[7] M. Basallote, M. D. Contreras, C. Hernandez-Mancera, M. J. Martin and P. J. Paul, *Voltera Operators and Semigroups in Weighted*

Banach Spaces of Analytic functions, Collett. Math, **65**, (2014), 233-249.

[8] E. Berkson and H. Porta, *Semigroups of analytic functions and composition operators*, Michigan Math. J, **25**, (1978), 101-115.

[9] O. Blasco, M. D. Contreras, S. Diaz-Madrigal, J. Martinez, M. Papadimitrakis and A. G. Siskakis, *Semigroups of composition operators and integral operators in spaces of analytic functions*, Ann. Acad. Sci. Fenn. Math, **38**, (2013), 67-89.

[10] O. Blasco, M. D. Contreras, S. Diaz-Madrigal, J. Martinez, M. Papadimitrakis and A. G. Siskakis, *Semigroups of composition operators in BMOA and the extension of a theorem of Sarason*, Integral Equations operator theory, **61**, (2008), 45-62.

[11] J. O. Bonyo, *Spectral Analysis of certain groups of isometries on the Hardy and Bergman spaces*, Journal of Math. Analysis and Applications, **456**, (2017), 1470-1481.

[12] J. A. Cima, J. Thomson and W. Wogen, *On some properties of composition operators*, Indian Univ. Math J, **24**,(1974), 215-200.

[13] N. Dunford and J. T. Schwartz, *Linear operators part I*, Interscience publishers, New York, 1958.

[14] C. C. Cowen and B. D. MacCluer, *Composition operators on spaces of analytic functions*, Studies in advanced mathematics, (CRC Press Boca Raton FL 1995).

[15] M. Elin and F. Jacobzon, *Analyticity of Semigroups on the right half-plane*, Department of Mathematics, Ort Braude College, Karmiel 21982, Israel (2010).

[16] S. J. Elliott and A. Wynn, *Composition Operators on Weighted Bergman Spaces of a Half-Plane*, Proceedings of the Edinburg Mathematical Society, **54**, (2011), 373-379.

[17] K. -J. Engel and R. Nagel, *A Short Course on Operator Semigroups*, Universitext, Springer, New York, 2006.

[18] X. Fu and J. Zhang, *Bloch-type spaces on the upper half-plane*, Bull. Korean Math. Soc. **54**, (2017), No. 4 1337-1346.

[19] H. Hedemalm, B. Korenblum and K.Zhu, *Theory of Bergman spaces*, Graduate Text in Mathematics, Vol. **199**, (springer 2000).

[20] Z. Hu, *Extended Cesaro Operators on Bergman spaces*, Journal of Math. Analysis and Applications. **296**, (2004), 435-454.

[21] H. Hunziker, H. Jarchow and V. Maschioni, *Some topologies on the space of analytic self-maps on the unit disk*, Proc. Conf. Geometry of Banach Spaces, Strobl 1989, Camb, Univ. Press, to appear.

[22] K. B. Laursen and M. M. Neumann, *An introduction to local spectral theory*, Clarendon Press, Oxford, 2000.

[23] S. H. Kang, *Some duality of weighted Bergman Spaces of the Half-Plane*, Bull. Korean Math. Soc. **42**, (2005), 385-396.

[24] S. H. Kang and J. Y. Kim, *Toeplitz Operators on weighted analytic Bergman Spaces of the Half-Plane*, Bull. Korean Math. Soc. **37**, (2000), 437-450.

[25] B. D. MacCluer and J. H Shapiro, *Angular derivatives and compact composition operators on the Hardy and Bergman Spaces*, Canadian J. Math. **38**, (1987), 878-906.

[26] V. Matache, *Composition Operators on Hardy Spaces of Half Plane*, Proceedings of the American Mathematical Society. **127**, No.5, (1999), 1483-1491.

[27] V. Matache, *Weighted composition operators on* H^2 and applications, Complex Analysis Operat. Theory, **2**, (2008), 169-197.

[28] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences. Vol. 44, Springer-Verlag, New York, 1970.

[29] M. M. Peloso, *Classical spaces of Holomorphic functions*, Technical report, Universit di Milano, 2014.

[30] A. Perala, *Bloch space and the norm of the Bergman Projection*, Annales Academiae Scientiarum Fennilae Mathematica, Volumen **38**, 2013, 849-853.

[31] S. Rohde, *On functions in the little Bloch space and inner functions*, Transaction of the American Mathematical society, volume **348**, No. 7, Jully 1996.

[32] S. D. Sharma, A. K. Sharma and S. Ahmed, *Composition operators between Hardy and Bloch-type spaces of the upper half-plane*, Bull. Korean Math. Soc. **43**, (2007), No. 3, 475-482.

[33] A. G. Siskakis, *Semigroups of composition operators on Bergman spaces*, Bull. Austral Mathematical Society, **35**, (1987), 397-406.

[34] A. G. Siskakis, *Semigroups of composition operators on Dirichlet spaces*, Result. Math. Soc, **30**, (1996), 165-173.

[35] K. Yosida, *Functional Analysis*, Classics in Mathematics Springer-Verlag, Berlin 1995.

[36] K. Zhu, *Operator Theory in function spaces*, Marcel Dekker Inc. New York, Basel (1990).