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STEADY MHD POISEUILLE FLOW BETWEEN TWO INFINITE PARALLEL POROUS PLATES IN AN INCLINED MAGNETIC FIELD

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Abstract: In this paper, we examine the motion of a two dimensional steady flow of a viscous, electrically conducting, incompressible fluid flowing between two infinite parallel porous plates under the influence of transverse magnetic field and constant pressure gradient. The lower plate is assumed porous while the upper plate is not. The resulting coupled governing equation of motion is solved analytically by an analytical method. Analytical expression for the fluid velocity obtained is expressed in terms of Hartmann number. The effects of the magnetic inclinations to the velocity are discussed graphically. The solution of this equation is important, for example in the design of MHD power generators.

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1. Introduction

Magnetohydrodynamics (usually shortened to MHD) is the fluid mechanics of electrically conducting fluids. Some of these fluids include liquid metals(such as mercury, molten iron) and ironized gases known by Physicists as plasma, one example being the Solar atmosphere. The subject of MHD is largely perceived to have been initiated by Swedish electrical engineer Hannes Alfvén[1] in 1942. If an electrically conducting fluid is placed in a constant magnetic field, the motion of the fluid induces currents which create forces on the fluid. The production of these currents has led to the design of among other devices the MHD generators for electricity production. The equations which describe MHD flow are a combination of continuity equation and Navier-Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism. The governing equations are differential equations that have to be solved either analytically or numerically.

Shercliff[2] studied the steady motion of an electrically conducting fluid in pipes under transverse magnetic fields. Drake[3] considered flow in a channel due to periodic pressure gradient and solved the resulting equation by separation of variables method. Singh & Ram[4] considered laminar flow of an electrically conducting fluid through a channel in the presence of a transverse magnetic field under the influence of a periodic pressure gradient and solved the resulting differential equation by the method of Laplace transform. Ram *et. al.*[5] have analyzed hall effects on heat and mass transfer flow through porous media. Shimomura[6] discussed magnetohydrodynamics turbulent channel flow under a uniform transverse magnetic field. Singh[8] considered steady magnetohydrodynamic fluid flow between two parallel plates. Kazuyuki[7] discussed inertia effects in two dimensional MHD channel flow and Al-Hadhrami[9] considered flow of fluids through horizontal channels of porous materials and obtained velocity expressions in terms of the Reynolds number. Ganesh[10] studied unsteady MHD Stokes flow of a viscous fluid between two parallel porous plates. They considered fluid being withdrawn through both walls of the channel at the same rate.

In this paper we consider two dimensional Poiseuille flow of an electrically conducting fluid between two infinite parallel plates under the influence of transverse magnetic field under a constant pressure gradient and assess the effect to velocity if the lower plate is porous. The resulting differential equation is solved by an analytical method and the solution expressed in terms of Hartmann number.

2. Mathematical Formulation

The concept of magnetohydrodynamics phenomenon can simply be described as follows: Consider an electrically conducting fluid moving with velocity $\mathbf{V}$. At right angles to this flow, we apply a magnetic field, the field strength of which is represented by the vector $\mathbf{B}$. We shall assume that the fluid has attained steady state conditions i.e. flow variables are independent of the time t. This condition is purely for analytic reasons so that no macroscopic charge density is being built up at any place in the system as well as all currents are constant in time. Because of the interaction of the two fields, namely, velocity and magnetic fields, an electric field vector denoted $\mathbf{E}$ is induced at right angles to both $\mathbf{V}$ and $\mathbf{B}$. This electric field is given by

$$\mathbf{E} = \mathbf{V} \times \mathbf{B} \tag{1}$$

If we assume that the conducting fluid is isotropic in spite of the magnetic field, we can denote the electrical conductivity of the fluid by a scalar σ. By Ohm's law[11], the density of the current induced in the conducting fluid denoted $\mathbf{J}$ is given by

$$\mathbf{J} = \sigma \mathbf{E} \tag{2}$$

or

$$\mathbf{J} = \sigma(\mathbf{V} \times \mathbf{B}) \tag{3}$$

Simultaneously occurring with the induced current is the Lorentz force $\mathbf{F}$ given by

$$\mathbf{F} = \mathbf{J} \times \mathbf{B} \tag{4}$$

This force occurs because, as an electric generator, the conducting fluid cuts the lines of the magnetic field. The vector $\mathbf{F}$ is the vector cross product of both $\mathbf{J}$ and $\mathbf{B}$ and is a vector perpendicular to the plane of both $\mathbf{J}$ and $\mathbf{B}$. This induced force is is parallel to $\mathbf{V}$ but in opposite direction. Laminar flow through a channel under uniform transverse magnetic field is important because of the use of MHD generator, MHD pump and electromagnetic flow meter.

We now consider an electrically conducting, viscous, steady, incompressible fluid moving between two infinite parallel plates both kept at a constant distance $2h$ between them. Both plates of the channel are fixed with no motion. This is plane Poiseuille flow. The equations of motion are the continuity equation

$$\nabla . \mathbf{V} = 0 \tag{5}$$

and the Navier-Stokes equations

$$\rho[(\mathbf{V}.\nabla)]\mathbf{V} = \mathbf{f}_B - \nabla p + \mu\nabla^2\mathbf{V} \tag{6}$$

where ρ is the fluid density, $\mathbf{f}_B$ is body force per unit mass of the fluid, μ is the fluid viscosity and p is the pressure acting on the fluid. Let us examine unidirectional flow, in which we choose the axis of the channel formed by the two plates as the $x-$ axis and assume that flow is in this direction. If $\mathbf{V} = \overline{u}(\overline{x},\overline{y},\overline{z})\mathbf{i}+\overline{v}(\overline{x},\overline{y},\overline{z})\mathbf{j}+\overline{w}(\overline{x},\overline{y},\overline{z})\mathbf{k}$ in which $\overline{u}$, $\overline{v}$ and $\overline{w}$ are the components of the velocity in $\overline{x}-$, $\overline{y}-$ and $\overline{z}-$ directions respectively and bars denote dimensionless quantities, then this implies $\overline{v} = \overline{w} = 0$ and $\overline{u} \neq 0$. Continuity equation yields

$$\frac{\partial \overline{u}}{\partial \overline{x}} + \frac{\partial \overline{v}}{\partial \overline{y}} + \frac{\partial \overline{w}}{\partial \overline{z}} = 0$$

But

$$\frac{\partial \overline{v}}{\partial \overline{y}} + \frac{\partial \overline{w}}{\partial \overline{z}} = 0$$

so that $\frac{\partial \overline{u}}{\partial \overline{x}} = 0$ from which we infer that $\overline{u}$ is independent of $\overline{x}$. This makes the non linear term $[(\mathbf{V}.\nabla)\mathbf{V}]$ in the Navier-Stokes equations zero. We neglect body forces $\mathbf{f}_B$ which are mainly due to gravity in the Navier-Stokes equations and replace them with the Lorentz force and assuming that the flow is two dimensional i.e. that flow variables are independent of $z-$ direction, it means that the governing equations for this flow are

$$0 = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial \overline{x}} + \nu\frac{\partial^2 \overline{u}}{\partial \overline{y}^2} + \frac{F_x}{\rho} \tag{7}$$

$$0 = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial \overline{y}} \tag{8}$$

where $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity and F_x is the component of the magnetic force in the $\overline{x}-$ direction. From (8) we note that $\overline{p}$ is a function of $\overline{x}$ only. Also assuming unidirectional flow $\overline{v} = \overline{w} = 0$ and $B_x = B_z = 0$ so that $\mathbf{V} = \overline{u}\mathbf{i}$ and $\mathbf{B} = B_0\mathbf{j}$ where B_0 is the magnetic field strength component assumed to be applied to a direction perpendicular to fluid motion ($\overline{y}-$ direction). Now

$$\mathbf{F}_x = \sigma[(\overline{u}\mathbf{i} \times \mathbf{j}B_0)] \times \mathbf{j}B_0] \tag{9}$$

from which we find that

$$\frac{F_x}{\rho} = -\frac{\sigma}{\rho}B_0^2\overline{u} \tag{10}$$

using the above equation in equation (7) we find

$$0 = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial \overline{x}} + \nu \frac{\partial^2 \overline{u}}{\partial \overline{y}^2} - \frac{\sigma}{\rho} B_0^2 \overline{u} \tag{11}$$

or

$$\frac{d^2 \overline{u}}{d\overline{y}^2} - \frac{\sigma}{\mu} B_0^2 \overline{u} = \frac{1}{\mu}\frac{d\overline{p}}{d\overline{x}} \tag{12}$$

or

$$\frac{d^2 \overline{u}}{d\overline{y}^2} - \frac{\sigma}{\mu} B_0^2 \sin(\alpha) \overline{u} = \frac{1}{\mu}\frac{d\overline{p}}{d\overline{x}} \tag{13}$$

where α is the angle between $\mathbf{V}$ and $\mathbf{B}$. Equation (13) is general in the sense that the two fields can be assessed at any angle α for $0 \le \alpha \le \pi$ and is solved subject to boundary conditions $\overline{u} = 0$ when $\overline{y} = \pm 1$. Let l be the characteristic length, the dimensionless equation (13) reverts back to the non-dimensionless form if we define the dimensionless quantities as follows

$$x = \frac{\overline{x}}{l}, \quad y = \frac{\overline{y}}{l}, \quad p = \frac{\overline{p} l^2}{\rho \nu^2}, \quad u = \frac{\overline{u} l}{\nu} \tag{14}$$

We use the quantities in the above equation into (13) to find

$$\frac{d^2 u}{dy^2} - \frac{\sigma}{\mu} B_0^2 l^2 \sin^2(\alpha) u = \frac{dp}{dx} \tag{15}$$

or

$$\frac{d^2 u}{dy^2} - M^2 u - \frac{dp}{dx} = 0 \tag{16}$$

where $M = M^* \sin\alpha$ and $M^* = lB_0\sqrt{\frac{\sigma}{\mu}} = Ha$ where Ha is the Hartmannn number given by $Ha^2 = \frac{\sigma B_0^2 l^2}{\mu}$. Differentiate equation (16) with respect to x to find $\frac{d^2 P}{dx^2} = 0$ and on integrating this we find $\frac{dp}{dx} = -c$(a constant). Hence on substitution, (16) becomes

$$\frac{d^2 u}{dy^2} - M^2 u + c = 0 \tag{17}$$

whose solution under the boundary conditions $u = 0$ for $y = \pm 1$ is given by

$$\frac{u}{c} = \frac{1}{M^2}\left(1 - \frac{\cosh My}{\cosh M}\right) \tag{18}$$

3. MHD Fluid Flow Between Two Infinite Parallel Porous Plates

Suppose v_0 is the characteristic velocity moving perpendicular to the fluid flow to maintain a steady fluid flow at a constant given pressure gradient. For lower porous plate, this characteristic velocity is the one which will maintain a steady fluid flow against the suction and injection of the fluid in which it is moving perpendicular to the fluid flow. The origin is taken at the centre of the channel and the x, y coordinate axes are parallel and perpendicular to the channel walls respectively. The governing equation will be

$$v_0 \frac{du}{dy} = -\frac{1}{\rho}\frac{dp}{dx} + \frac{\mu}{\rho}\frac{d^2u}{dy^2} \tag{19}$$

where μ is the effective viscosity of the porous region. Differentiate equation (19) with respect to x to find $\frac{d^2P}{dx^2} = 0$ and on integrating this we find $\frac{dp}{dx} = -p$(a constant). Hence on substitution, (19) becomes

$$\frac{d^2u}{dy^2} - \frac{v_0}{\nu}\frac{du}{dy} + \frac{p}{\mu} = 0 \tag{20}$$

If the fluid is under the influence of inclined magnetic forces, then the above differential equation becomes

$$\frac{d^2u}{dy^2} - \frac{v_0}{\nu}\frac{du}{dy} + \frac{p}{\tilde{\mu}} - M^2u = 0 \tag{21}$$

For simplicity, let us assume that $h = 1$ and solve this equation by variation of constants method with the boundary conditions $u = 0$, $y = 1$ and $u = 0$, $y = -1$ to find the solution

$$\frac{u(y)}{c} = \frac{1}{M^2}\left\{1 + \frac{1}{\sinh 2\beta}\left[e^{\frac{A}{2}(y+1)}\sinh\beta(y-1) - e^{\frac{A}{2}(y-1)}\sinh\beta(y+1)\right]\right\} \tag{22}$$

where $c = \frac{p}{\mu}$ – constant for the fluid, $A = \frac{v_0}{\nu}$, $\beta = \sqrt{(\frac{A^2}{4} + M^2)}$, $M = M^*\sin\alpha$, and $M^* = B_0 l\sqrt{\frac{\sigma}{\mu}}$. Flow velocity for Hartmann numbers $Ha = 0.5$, $Ha = 1.0$, $Ha = 1.5$ and angle of inclinations $\alpha = 15^0$, $\alpha = 30^0$ are depicted in the figures below.

4. Conclusion

High Hartmann flow i.e. high magnetic field strength decreases the velocity.

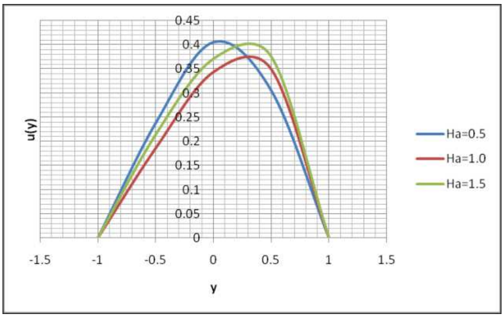

Figure 1: Velocity flow for various Hartmann flow numbers($\alpha = 15^0$)

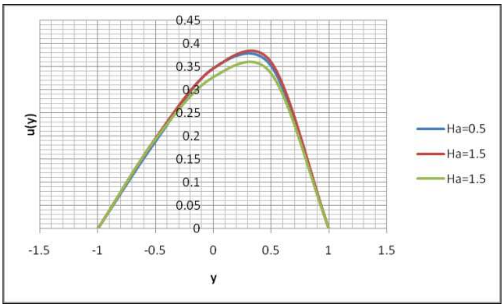

Figure 2: Velocity flow for various Hartmann flow numbers($\alpha = 30^0$)

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