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MATHEMATICAL INVESTIGATION ON THE EFFECT OF NUMBER OF BRANCHES ON WATER HAMMER

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Abstract

This paper looks at water hammer with an emphasis on the number of branches. The mathematics of water hammer are described in a transient condition. This includes the momentum and continuity equations for conduits. A numerical solution for this equation is provided via the method of characteristics. The boundary conditions at a common junction are manipulated in abid to establish the relationship between the number of branches and the magnitude of pressure developed. The relationship developed is found to reflect closely to the maximum output pressure head obtained with respect to the number of branches in a system through a case study.

1. Introduction

Water hammer is the result of an event which is associated with a rapid velocity (or pressure) change, the result of an accident or a normal

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operational matter in a pipeline system. Any flow in a conduit is a candidate of water hammer though it is commonly experienced in long conduits as explained by Massey [12]. The phenomenon is commonly associated with events like; sudden closure of water valves, in quick start up or stop of a transfer system, and in pulsating pump like the heart. The consequences of water hammer have catastrophic effects. For instance, water hammer experience can lead to system component failure. The effect of system design on water hammer is not foreign and immaterial as illustrated by Sharp and Sharp [16]. For instance, the maximum pressure head developed can be optimized by taking appropriate measures concerning system design. There is therefore, a need for conducting further research on this area. This paper therefore undertakes to present a case study which is hoped, will expose the effect of number of branches on water hammer. Due to its efficiency, high degree of accuracy and popularity, the method of characteristics is applied in order to accomplish the analysis.

2. Mathematical Formulation

The basic equations for unsteady one dimensional flow are the equation of momentum and continuity as shown below, respectively,

$$\text{Continuity: } \frac{\partial p}{\partial t} + \rho a^2 \frac{\partial v}{\partial x} = 0, \tag{1}$$

$$\text{Momentum: } \frac{\partial v}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial x} + \rho \sin\theta + \frac{fv|v|}{2D} = 0, \tag{2}$$

ρ is the density of fluid and $|v|$ is an absolute velocity, where p = pressure, v = velocity, a = wave speed, D = diameter of pipe, f = friction factor, x = x direction.

The derivation of this equations is given in Kumar [10]. The system of equations is hyperbolic. The transformation of this system to the characteristic curves leads to the direction conditions:

$$\frac{dx}{dt} = v \pm a. \tag{3}$$

Along the characteristic curves, the compatibility conditions are valid:

$$\frac{dv}{dt} + \frac{1}{\rho a}\frac{dp}{dx} + \rho \sin\theta + \frac{fv|v|}{2D} = 0, \tag{4}$$

$$\frac{dv}{dt} - \frac{1}{\rho a}\frac{dp}{dx} + \rho \sin\theta + \frac{fv|v|}{2D} = 0. \tag{5}$$

3. Finite Difference Formulation

The solution of the above equations may be visualized on an *x*-*t* plane. The conduit is divided into *N* reaches such that: $\Delta x = \frac{L}{N}$, *L* being the length of the pipe. The time increment Δt is determined by the intersection of *C* + and *C* – lines such that $\frac{\Delta x}{\Delta t} = a$ since $\nu << a$ for the *C* + characteristic line and $\frac{\Delta x}{\Delta t} = -a$ for the *C* – characteristic line (see Figure 1).

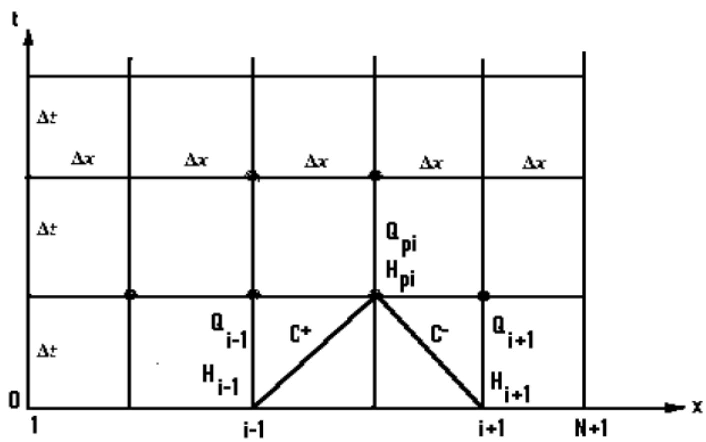

Figure 1. Rectangular grid for solution of characteristics equation.

Equations (1) and (2) are converted into difference form as shown below:

$$\frac{v_{i,j} - v_{i-1,j-1}}{\Delta t} + \frac{1}{\rho a}\frac{p_{i,j} - p_{i-1,j-1}}{\Delta x} + \rho \sin\theta + \frac{f v_{i-1,j-1}|v_{i-1,j-1}|}{2D} = 0, \tag{6}$$

$$\frac{v_{i,j} - v_{i-1,j+1}}{\Delta t} + \frac{1}{\rho a}\frac{p_{i,j} - p_{i-1,j+1}}{\Delta x} + \rho \sin\theta + \frac{f v_{i-1,j+1}|v_{i-1,j+1}|}{2D} = 0. \quad (7)$$

The nomenclature is such that $p_{i,j}$ and $v_{i,j}$ are pressure and velocity at the ith reach and jth time step, respectively, that is at (i, j) node.

Rearranging terms in equations (6) and (7), we have

$$p_{i,j} - p_{i-1,j-1} = -\rho a(v_{i,j} - v_{i-1,j-1}) - \rho g \sin\theta + \frac{\rho \Delta x f v_{i-1,j-1}|v_{i-1,j-1}|}{2D}, \quad (8)$$

$$p_{i,j} - p_{i-1,j+1} = -\rho a(v_{i,j} - v_{i-1,j+1}) - \rho g \sin\theta + \frac{\rho \Delta x f v_{i-1,j+1}|v_{i-1,j+1}|}{2D}. \quad (9)$$

For convenience, p and v are replaced with hydraulic grade line H and discharge Q. In this case, we have $v = \frac{Q}{A}$ and $p = \rho g(H - Z)$ whereby A is the cross section area of pipe and Z is the height as measured from the datum (see Figure 2).

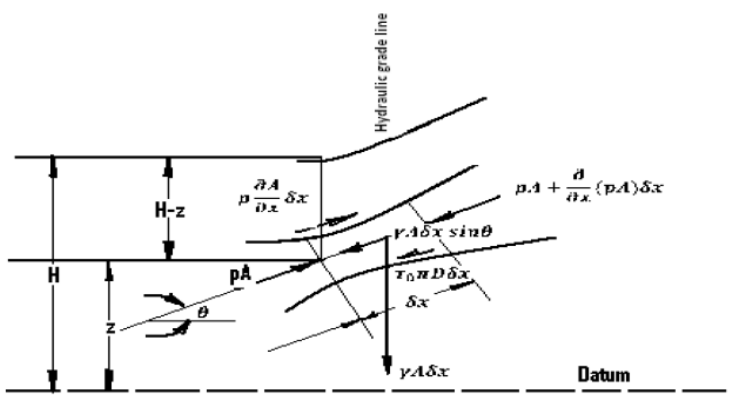

Figure 2. Free body diagram for the derivation of equations of momentum and continuity.

$$p_{i,j} = \rho g(H_{i,j} - Z_{i,j}),$$

$$p_{i-1,j-1} = \rho g(H_{i-1,j-1} - Z_{i-1,j-1}),$$

$$p_{i+1,j-1} = \rho g(H_{i+1,j-1} - Z_{i+1,j-1}).$$

It follows that

$$p_{i,j} - p_{i-1,j-1} = \rho g(H_{i,j} - H_{i-1,j-1}) - \rho g(Z_{i,j} - Z_{i-1,j-1}),$$

$$p_{i,j} - p_{i-1,j-1} = \rho g(H_{i,j} - H_{i-1,j-1}) - \rho g \Delta x \sin\theta, \tag{10}$$

$$p_{i,j} - p_{i+1,j-1} = \rho g(H_{i,j} - H_{i+1,j-1}) - \rho g(Z_{i,j} - Z_{i+1,j-1}),$$

$$p_{i,j} - p_{i+1,j-1} = \rho g(H_{i,j} - H_{i+1,j-1}) - \rho g \Delta x \sin\theta. \tag{11}$$

Comparing equations (8) and (10) and making $H_{i,j}$ the subject of formula, we have for the $C+$ characteristic

$$H_{i,j} = H_{i-1,j-1} - \frac{a}{g}\left(\frac{Q_{i,j} - Q_{i-1,j-1}}{A}\right) - \frac{\Delta x f Q_{i-1,j-1}|Q_{i-1,j-1}|}{2gDA^2}. \tag{12}$$

Comparing equations (9) and (11) and making $H_{i,j}$ the subject of formula, we have for the C-characteristic

$$H_{i,j} = H_{i+1,j-1} - \frac{a}{g}\left(\frac{Q_{i,j} - Q_{i+1,j-1}}{A}\right) - \frac{\Delta x f Q_{i+1,j-1}|Q_{i+1,j-1}|}{2gDA^2}. \tag{13}$$

Equations (12) and (13) can be simplified by writing

$$B = \frac{a}{gA}, \quad R = \frac{\Delta x f Q|Q|}{2gDA^2}.$$

Equation (12) reduces to

$$H_{i,j} = H_{i-1,j-1} - B(Q_{i,j} - Q_{i-1,j-1}) - RQ_{i-1,j-1}|Q_{i-1,j-1}|. \tag{14}$$

Equation (13) reduces to

$$H_{i,j} = H_{i+1,j-1} - B(Q_{i,j} - Q_{i+1,j-1}) - RQ_{i+1,j-1}|Q_{i+1,j-1}|. \tag{15}$$

Given that Q_{i-j}, H_{i-j}, Q_{i+j} and H_{i+j} are known at earlier time, it is possible to solve for $H_{i,j}$ and $Q_{i,j}$ at any given internal section.

Iterative computation is applied whereby it is deemed that initial conditions are known along the conduit. The computation is done along the whole conduit after which the time is increased by Δt. The process is repeated up to a specified maximum time. Parameters at the end and downstream end are found using the compatibility equations.

4. Boundary Condition at Junction

The continuity equation must be satisfied at each instant of time. This means that there is no storage. At the junction, a common hydraulic grade line elevation is assumed. A double-subscript notation is used in order to differentiate the pipes. In this case, the first subscript refers to the pipe number and the second to the pipe section number.

For the above case

$$Q_{p1,\,NS} = -\frac{H_{P1,\,NS}}{B_1} + \frac{C_{P1}}{B_1},$$

$$-Q_{p2,\,NS} = -\frac{H_{p1,\,NS}}{B_2} + \frac{C_{m2}}{B_2},$$

$$-Q_{p3,\,NS} = -\frac{H_{p1,\,NS}}{B_3} + \frac{C_{m3}}{B_3}$$

since

$$H_{p1,\,NS} = H_{p2,1} = H_{p3,\,2}.$$

Therefore, by adding the above two equations

$$\sum Q_p = 0 = -H_{p1,\,NS} \sum \frac{1}{B_i} + \frac{C_{p1}}{B_1} + \frac{C_{m2}}{B_2} + \frac{C_{m3}}{B_3},$$

$$H_{P1,\,NS} = \frac{\frac{c_{p1}}{B_1} + \frac{c_{m2}}{B_2} + \frac{c_{m3}}{B_3}}{\sum \left(\frac{1}{B_i}\right)}$$

which can be extended to n pipes as follows:

$$H_{P1,\,NS} = \frac{\frac{c_{p1}}{B_1} + \frac{c_{m2}}{B_2} + \frac{c_{m3}}{B_3} + \cdots + \frac{c_{mn-1}}{B_{n-1}} + \frac{c_{mn}}{B_n}}{\sum\left(\frac{1}{B_i}\right)}. \qquad (16)$$

5. Relationship Between Pressure Head and Number of Branches

Since equation (16) involves all the pipes it is possible to develop the relationship between pressure head and the number of pipes in a pipe network or system. In order to achieve this we need to make the assumption that all the branches are of equal diameter. It follows that

$$B_1 = B_2 = B_3 = \cdots = B_{n-1} = B_n = B$$

and

$$c_{m2} = c_{m3} = \cdots = c_{mn-1} = c_n = C \text{ and } c_{p1} = A.$$

It follows that

$$H_{p1,\,NS} = \frac{\frac{A}{B} + \frac{C}{B} + \frac{C}{B} + \cdots + \frac{C}{B} + \frac{C}{B}}{\frac{n}{B}} = \frac{\frac{1}{B}(A + (n-1)C)}{\frac{n}{B}}$$

$$= \frac{A + (n-1)C}{n} = \frac{A + (n-1)C}{n} = \frac{1}{n}(A - C) + C.$$

But $A - C = \text{constant}$.

It follows that

$$H_{P1,\,NS} = \frac{\text{constant 1}}{n} + \text{constant 2}, \qquad (17)$$

where n is the number of branches.

6. Results

Figures 3 and 4 show the graphs of maximum change in maximum change in pressure against number of branches for a case study for branches fixed and measurements taken at different points. The figures show that pressure decreases with increase in number of branches.

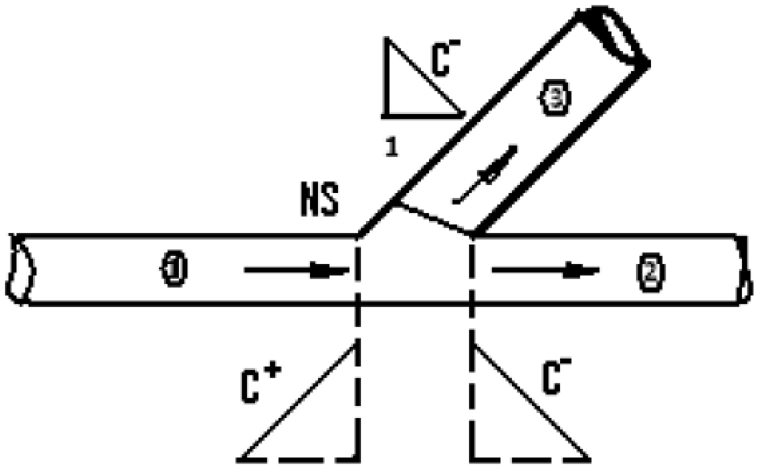

Figure 3. Pipe line junction.

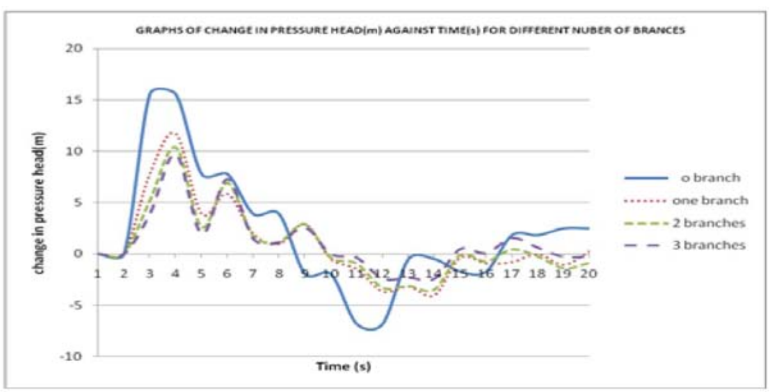

Figure 4. Graph of change in pressure against time in seconds.

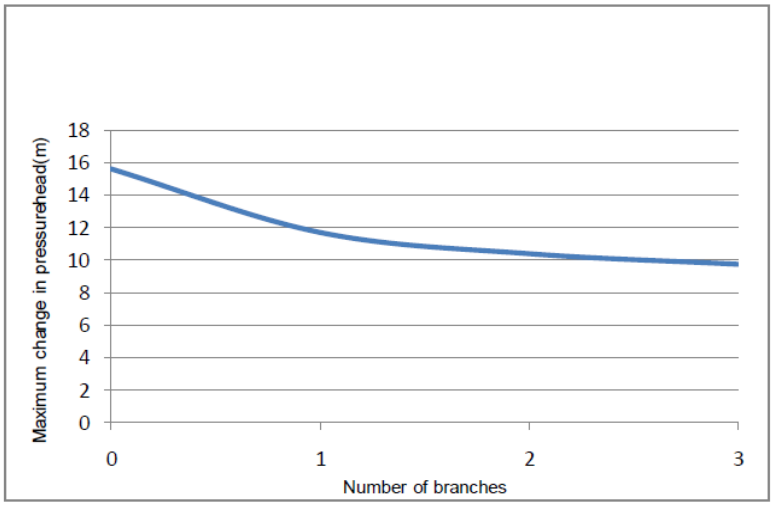

Figure 5. Graph of maximum change in pressure against number of branches.

Branches fixed at 1200m from the reservoir.

Measurement taken at 3600m from the reservoir.

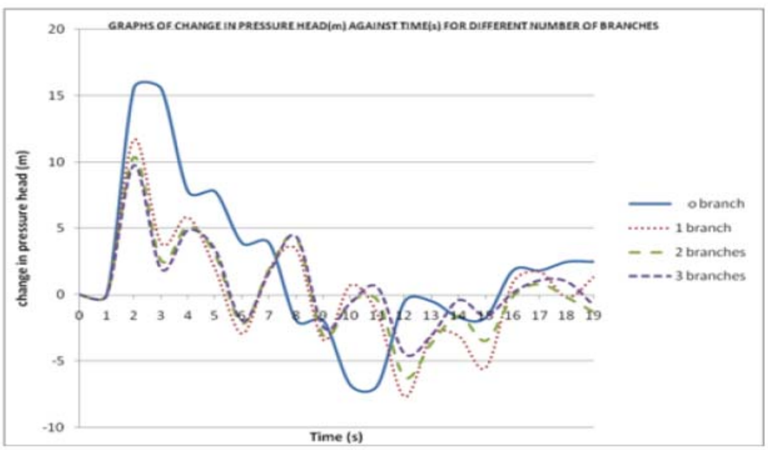

Figure 6. Graph of maximum change in pressure against time in seconds.

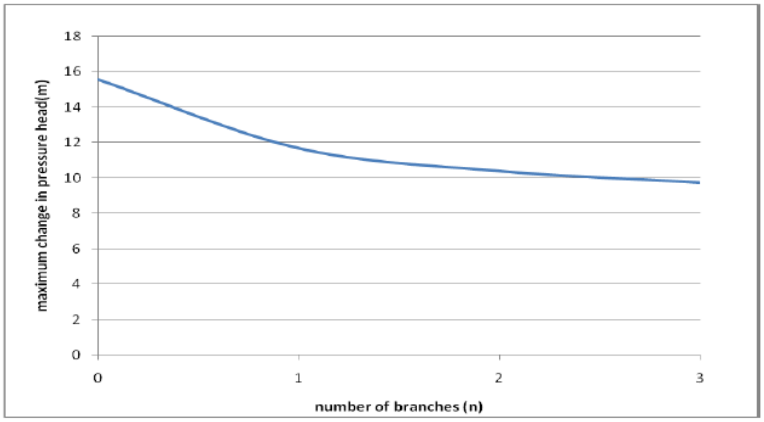

Figure 7. Graph of maximum change in pressure against number of branches.

Branches fixed at 2400m from the reservoir.

Measurement taken at 3600m from the reservoir.

7. Conclusion

Increase in branches decreases the highest pressure obtained and the output reflects equation (17).

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