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Solutions of Second-Order Partial Differential Equations in Two Independent Variables using Method of characteristics

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Abstract

In this paper we have classified second order linear PDEs into three types, hyperbolic, parabolic and elliptic. Hyperbolic equations have two distinct families of (real) characteristic curves, parabolic equations have a single family of characteristic curves, and the elliptic equations have none. All the three types of equations can be reduced to canonical forms. Hyperbolic equations reduce to a form coinciding with the wave equation in the leading terms, the parabolic equations reduce to a form modeled by the heat equation, and the Laplace's equation models the canonical form of elliptic equations. Thus, the wave, heat and Laplace's equations serve as canonical models for all second order constant coefficient PDEs.

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Introduction

The theory of partial differential equations of the second order is a great deal more complicated than that of the equations of the first order, and it is much more typical of the subject as a whole. Within the context, considerably better results can be achieved for equations of the second order in two independent variables than for equations in space of higher dimensions. Linear equations are the easiest to handle. In general, a second order linear partial differential equation is of the form, Wanjala et al [1];

$$A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} + D(x,y)u_x + E(x,y)u_y + F(x,y)u = G(x,y) \qquad (1)$$

where A, B, C, D, E, F and G are in general functions of x and y but they may be constants. The subscripts are defined as partial derivatives, that is, $u_x = \frac{\partial u}{\partial x}$.

The above equation can be written in a shorter form as;

$$A(x, y)u_{xx} + 2B(x, y)u_{xy} + C(x, y)u_{yy} = f(x, y, u, u_x, u_y)$$
$$\Rightarrow Au_{xx} + 2Bu_{xy} + Cu_{yy} = f(x, y, u, u_x, u_y)$$
$$\Rightarrow Au_{xx} + 2Bu_{xy} + Cu_{yy} = 0$$

if homogenous.

The classification of partial differential equations is suggested by the classification of the quadratic equation of conic sections in analytic geometry; The equation;

$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$,

represents hyperbola, parabola, or ellipse accordingly as B^2-4AC is positive, zero, or negative, Farlow [6].

Classifications of PDE are;

(i) Hyperbolic if $B^2-AC>O$

(ii) Parabolic if $B^2-AC=0$

(iii) Elliptic if $B^2-AC<0$

The classification of second-order equations is based upon the possibility of reducing equation (1) by coordinate transformation to canonical or standard form at a point. An equation is said to be hyperbolic, parabolic, or elliptic at a point (x_0, y_0) accordingly as;

$$B^2(x_0,y_0)-4A(x_0,y_0)C(x_0,y_0) \tag{2}$$

is positive, zero, or negative. If this is true at all points, then the equation is said to be hyperbolic, parabolic, or elliptic in a domain. In the case of two independent variables, a transformation can always be found to reduce the given equation to canonical form in a given domain. However, in the case of several independent variables, it is not, in general, possible to find such a transformation, Strauss [8].

To transform equation (1) to a canonical form we make a change of independent variables. Let the new variables be;

$$\xi = \xi(x, y),\ \eta = \eta(x, y) \tag{3}$$

Assuming that ξ and η are twice continuously differentiable and that the Jacobian;

$$J=\begin{vmatrix} \xi_x \xi_y \\ \eta_x \eta_y \end{vmatrix} \quad (4)$$

is nonzero in the region under consideration, then x and y can be determined uniquely from the system (3). Let x and y be twice continuously differentiable functions of ξ and η. Then we have

$$U_x=\frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x}+\frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}$$

$$U_y=\frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial y}+\frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}$$

$$U_{xx}=\frac{\partial u_x}{\partial \xi}\cdot\frac{\partial \xi}{\partial x}+\frac{\partial u_x}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}$$

$$U_{xy}=\frac{\partial u_x}{\partial \xi}\cdot\frac{\partial \xi}{\partial y}+\frac{\partial u_x}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}$$

$$U_{yy}=\frac{\partial u_y}{\partial \xi}\cdot\frac{\partial \xi}{\partial y}+\frac{\partial u_y}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}$$

Substituting these values in equation (1) we obtain

$$A^*(x,y)u_{xx}+2B^*(x,y)u_{xy}+C^*(x,y)u_{yy}+D^*(x,y)u_x+E^*(x,y)u_y+F^*(x,y)u=G^*(x,y)$$

where

$$A^*=A\xi_x^2+B\xi_x\xi_y+C\xi_y^2$$

$$B^*=2A\xi_x\eta_x+B(\xi_x\eta_y+\xi_y\eta_x)+2C\xi_y\eta_y$$

$$C^*=A\eta_x^2+B\eta_x\eta_y+C\eta_y^2$$

$$D^*=A\xi_{xx}+B\xi_{xy}+C\xi_{yy}+D\xi_x+E\xi_y$$

$$E^*=A\eta_{xx}+B\eta_{xy}+C\eta_{yy}+D\eta_x+E\eta_y$$

$$F^*=F$$

$$G^*=G$$

The resulting equation (5) is in the same form as the original equation (1) under the general transformation (3). The nature of the equation remains invariant under such a transformation if the Jacobian does not vanish. This can be seen from the fact that the sign of the discriminant does not alter under the transformation, that is,

$$B^{*2}-4A^{*}C^{*}=J^{2}(B^{2}-4AC) \tag{6}$$

which can be easily verified, Evans [4].

It should be noted here that the equation can be of a different type at different points of the domain. We shall assume that the equation under consideration is of the single type in a given domain. The classification of equation (1) depends on the coefficients A(x, y), B (x, y), and C (x, y) at a given point (x,y). We therefore, rewrite equation (1) as;

$$Au_{xx}+Bu_{xy}+Cu_{yy}=H(x,y,u,u_{x},u_{y}) \tag{7}$$

and equation (5) as;

$$A^{*}u_{\xi\xi}+B^{*}u_{\xi\eta}+C^{*}u_{\eta\eta}=H(\xi,\eta,u,u_{\xi},u_{\eta}) \tag{8}$$

We consider the problem of reducing equation (7) to canonical form. We suppose first that none of A, B, C, is zero. Let ξ and η be new variables such that the coefficients A and C in equation (8) vanish. Thus, from (8), we have;

$$A^{*}=A\xi_{x}^{2}+B\xi_{x}\xi_{y}+C\xi_{y}^{2}=0$$

$$C^{*}=A\eta_{x}^{2}+B\eta_{x}\eta_{y}+C\eta_{y}^{2}=0$$

These two equations are of the same type and hence we may write them in the form;

$$A\zeta_{x}^{2}+B\zeta_{x}\zeta_{y}+C\zeta_{y}^{2}=0 \tag{9}$$

in which ζ stand for either of the functions ξ or η. Dividing through by ζ_{y}^{2}, equation (9) becomes, Jost [3];

$$A\left(\frac{\zeta_{x}}{\eta_{y}}\right)^{2}+B\left(\frac{\zeta_{x}}{\eta_{y}}\right)+C=0 \tag{10}$$

Along the curve ζ = constant, we have;

$$d\zeta=\zeta_{x}dx+\zeta_{y}dy=0$$

Thus,

$$\frac{dy}{dx}=-\frac{\zeta_{x}}{\zeta_{y}} \tag{11}$$

and therefore, equation (10) may be written in the form;

$$A\left(\frac{dx}{dy}\right)^{2}+B\left(\frac{dx}{dy}\right)+C=0 \tag{12}$$

the roots of which are

$$\frac{dy}{dx}=\frac{B+\sqrt{B^2-4AC}}{2A} \quad (13)$$

$$\frac{dy}{dx}=\frac{B-\sqrt{B^2-4AC}}{2A} \quad (14)$$

These equations, which are known as the characteristic equations, are ordinary differential equations for families of curves in the xy-plane along which ξ = constant and η = constant. The integrals of equations (13) and (14) are called the characteristic curves. Since the equations are first-order ordinary differential equations, the solutions may be written as;

$\varphi_1(x,y)=c_1$

$\varphi_2(x,y)=c_2$

with c_1 and c_2 as constants.

Hence the transformations

$\xi=\varphi_1(x,y),\ \eta=\varphi_2(x,y)$

will transform equation (7) to a canonical form.

We show that the characteristic of any hyperbolic PDE can be transformed as;

$U_{\xi\eta}=0$

known as the standard form, Dennemeyer [2].

The characteristic different equation is given by;

$$\frac{dy}{dx}=\frac{B\pm\sqrt{B^2-AC}}{A}$$

Hyprbolic Equations

We can find the characteristics of $3U_{xx}+10U_{xy}+3U_{yy}=0$ and show that the standard form is $U_{\xi\eta}=0$

Given $3U_{xx}+10U_{xy}+3U_{yy}=0$ (15)

From the relation above; A = 3; B = 5; C = 3

$\Rightarrow B^2-AC=25-9=16>0.$

This is hyperbolic equation.

$$\frac{dy}{dx}=\frac{5\pm\sqrt{5^2-3.3}}{3}=\frac{5\pm\sqrt{25-9}}{3}=\frac{5\pm4}{3}$$

$\Rightarrow \frac{dy}{dx}=3; \ \frac{dy}{dx}=\frac{1}{3}$

$\Rightarrow y=3x+c_1; \ y=\frac{1}{3}x+c_2$

giving $y-3x=c_1; \ 3y-x=c_2$

$\Rightarrow y-3x=\xi; \ 3y-x=\eta$

It can be shown that the characteristics are, Haberman [5];

$\xi=y-3x; \ \eta=3y-x$ (16)

Next we now find U_{xx}, U_{xy} and U_{yy}

Under transformation (2) we have

$U_x=\frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x}+\frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}=\frac{\partial u}{\partial \xi}.(-3)+\frac{\partial u}{\partial \eta}.(-1)=-3u_\xi-u_\eta$ (17)

$U_y=\frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial y}+\frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}=\frac{\partial u}{\partial \xi}.(1)+\frac{\partial u}{\partial \eta}.(3)=u_\xi+3u_\eta$ (18)

$U_{xx}=\frac{\partial u_x}{\partial \xi}\cdot\frac{\partial \xi}{\partial x}+\frac{\partial u_x}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}$

$=\frac{\partial}{\partial \xi}(-3u_\xi-u_\eta).(-3)+\frac{\partial}{\partial \eta}(-3u_\xi-u_\eta).(-1)$

$=(-3u_{\xi\xi}-u_{\xi\eta}).(-3)+(-3u_{\xi\eta}-u_{\eta\eta}).(-1)$

$=9u_{\xi\xi}+3u_{\xi\eta}+3u_{\xi\eta}+u_{\eta\eta}$

$=9u_{\xi\xi}+6u_{\xi\eta}+u_{\eta\eta}$ (19)

$U_{xy}=\frac{\partial u_x}{\partial \xi}\cdot\frac{\partial \xi}{\partial y}+\frac{\partial u_x}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}$

$=\frac{\partial}{\partial \xi}(-3u_\xi-u_\eta).(1)+\frac{\partial}{\partial \eta}(-3u_\xi-u_\eta).(3)$

$=(-3u_{\xi\xi}-u_{\xi\eta}).(1)+(-3u_{\xi\eta}-u_{\eta\eta}).(3)$

$=-3u_{\xi\xi}-u_{\xi\eta}-9u_{\xi\eta}-3u_{\eta\eta}$

$=-3u_{\xi\xi}-10u_{\xi\eta}-3u_{\eta\eta}$ (20)

$U_{yy}=\frac{\partial u_y}{\partial \xi}\cdot\frac{\partial \xi}{\partial y}+\frac{\partial u_y}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}$

$=\frac{\partial}{\partial \xi}(u_\xi+3u_\eta).(1)+\frac{\partial}{\partial \eta}(u_\xi+3u_\eta).(3)$

$=(u_{\xi\xi}+3u_{\xi\eta}).(1)+(u_{\xi\eta}+3u_{\eta\eta}).(3)$

$=u_{\xi\xi}+3u_{\xi\eta}+3u_{\xi\eta}+9u_{\eta\eta}$

$=u_{\xi\xi}+6u_{\xi\eta}+9u_{\eta\eta}$ (21)

Substituting (19), (20), and (21) into (15) gives;

$$3(9u_{\xi\xi}+6u_{\xi\eta}+u_{\eta\eta})+10(-3u_{\xi\xi}-10u_{\xi\eta}-3u_{\eta\eta})+3(u_{\xi\xi}+6u_{\xi\eta}+9u_{\eta\eta})=0$$
$$\Rightarrow 27u_{\xi\xi}+18u_{\xi\eta}+3u_{\eta\eta}-30u_{\xi\xi}-100u_{\xi\eta}$$
$$-30u_{\eta\eta}+3u_{\xi\xi}+18u_{\xi\eta}+27u_{\eta\eta}=0$$

$64u_{\xi\eta}=0$

$\Rightarrow u_{\xi\eta}=0$

Integrating $u_{\xi\eta}$ partially w.r.t ξ, we get

$\int u_{\xi\eta}d\xi=u_\eta+f(\eta)=0$

$\Rightarrow u_\eta=f(\eta)$

Integrating u_η partially w.r.t η, we get

$\int u_\eta d\eta=f(\eta)+g(\xi)$

$\Rightarrow u(\xi\eta)=f(\eta)+g(\xi)$

Substituting the variables x,y, we have

$U(x,y)=f(3y-x)+g(y-3x)$

Parabolic Equations

We can find the characteristics of the following equation and reduce it to the following equation and reduce it to the appropriate standard form and then obtain the general solution, Boyce [7];

$U_{xx}+4U_{xy}+4U_{yy}=0$ (22)

From the relation above; A=1;B=2;C=4

$\Rightarrow B^2-AC=4-4=0.$

This is parabolic equation. In this case; $\frac{dy}{dx}=\frac{2\pm\sqrt{4-4}}{1}=2$

Integrating we have; $y-2x=c$

The characteristics are defined by $\eta=y-2x$ and $\xi=x$.

Under this transformation;

$U_x=u_\xi-2u_\eta$

$U_y=u_\eta$

$$U_{xx}=u_{\xi\xi}-4u_{\xi\eta}+4u_{\eta\eta} \tag{23}$$

$$U_{xy}=u_{\xi\eta}-2u_{\eta\eta} \tag{24}$$

$$U_{yy}=u_{\eta\eta} \tag{25}$$

Substituting equations (23), (24), and (25) into equation (22) gives

$(u_{\xi\xi}-4u_{\xi\eta}+4u_{\eta\eta})+4(u_{\xi\eta}-2u_{\eta\eta})+4u_{\eta\eta}$

Simplifying we get

$$u_{\xi\xi}=0 \tag{26}$$

Integrating $u_{\xi\xi}$ w.r.t ξ, we get;

$u_\xi=f(\eta)$

Integrating u_ξ partially w.r.t η, we get;

$u=\xi f(\eta)+g(\eta)$

Thus the required solution is given by;

$U(x,y)=\xi f(y-2x)+g(y-2x)$

Elliptic Equations

We consider the equation of the type, Boyce [7];

$$u_{xx}+u_{yy}=0 \tag{27}$$

and reduce it to the canonical form. In this case, A = 1; B = 0; C = 1

$\Rightarrow B^2-AC=0-1<0.$

This is elliptic equation.

$\Rightarrow \frac{dy}{dx} = \frac{0 \pm \sqrt{0-1}}{1} = \sqrt{-1} = \pm i$

$\Rightarrow \frac{dy}{dx} = \pm i$ or $dy = \pm i dx$

Integrating we get; $x+iy=c_1$ and $x-iy=c_2$

Hence the characteristics are given by; $\xi = x+iy$ and $\eta = x-iy$

Under this transformation;

$U_x = u_\xi + u_\eta$

$U_y = u_\eta i - u_\eta i = i(u_\eta - u_\eta)$

$$U_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \tag{28}$$

$$U_{yy} = u_{\eta\eta} + 2u_{\xi\eta} - u_{\eta\eta} \tag{29}$$

Substituting equation (28) and (29) in equation (27) gives;

$u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} - u_{\eta\eta} + 2u_{\xi\eta} - u_{\eta\eta} = 0$ or; $4u_{\xi\eta} = 0$

$\Rightarrow u_{\xi\eta} = 0$

Integrating $u_{\xi\eta}$ partially w.r.t η, we get;

$u_\xi = \varphi(\xi)$

Integrating u_ξ w.r.t ξ, we get;

$u = \varphi(\xi) + \varphi(\eta)$

Thus the required solution is given by;

$$u(x,y) = \phi(x+iy) + \phi(x-iy)$$
$$\Rightarrow u(x,y) = \phi(x+iy) + \bar{\phi}(x+iy)$$

where $\bar{\phi}$ is complex conjugate of ϕ

Conclusion

The second-order linear PDEs can be classified into three types, which are invariant under changes of variables. The types are determined by the sign of the discriminant. This exactly corresponds to the different cases for the quadratic equation satisfied by the slope of the characteristic curves. Hyperbolic equations have two distinct families of (real) characteristic curves, parabolic equations have a single family of characteristic curves, and the elliptic equations have none. All the three types of equations can be reduced to canonical forms.

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