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On the Solution of Confined Aquifer Flow Equations: Finite Difference Approximations

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Abstract

In general, groundwater flow in porous media on a microscopic level is three-dimensional. However, when considering regional flow problems i.e. groundwater flow on a macroscopic level, it is noted that because of the low ratio of aquifer thickness to horizontal length, the flow in the aquifer is practically horizontal[21]. Flow in confined aquifers is therefore described by a two-dimensional second order non-linear partial differential equation with variable coefficients on a finite domain. In this paper, we examine the case of a confined aquifer that is being pumped

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and recharged at the same time. The concepts of stability, consistency and convergence of the solution are assumed since the numerical schemes involved are known to be unconditionally stable for all finite values of mesh ratio[2]. The practical application of the results is illustrated by presenting a numerical example whose solution is compared to the measured data observed in a field site.

Mathematics Subject Classification: 65C20

Keywords: Finite difference methods, confined aquifer flow equations, Alternating Direction Implicit (ADI), nonlinear partial differential equation

1 Introduction

Flow through porous media has been studied extensively from the late 18th century. A wide range of problems in civil engineering requires the understanding of groundwater flow through porous media. These problems include determination of yield of wells situated at different locations above aquifers, determination of flow patterns under a dam in hydraulics, the study of contaminant transport in groundwater in environmental geotechnics and problems of outflow from oil bearing strata in petroleum engineering.

The finite difference method, based on the approximate substitution of derivatives by difference quotients, has been developed theoretically and practically by the work of many investigators. The results have been a number of finite difference techniques. The finite difference method obtains a finite system of linear equations from a partial differential equation by discretizing the domain and operators. For the origin and development of the various finite difference schemes, see e.g. [3, 1, 4, 5, 6, 7, 8, 9, 10, 11, 12]. Several other numerical schemes have also been used to approximate aquifer flow equations, see [13, 14, 15]. All the numerical methods used above have always resulted in a finite system of a large number of linear equations that have been more convenient for a computer solution. This has led to the development of computer codes for almost all classes of problems encountered in the management of groundwater, see e.g. [16, 17, 18, 19]. Other computer codes have also been developed lately e.g. FEMWATER and modern variations of FORTRAN. In this paper, the two - dimensional nonlinear groundwater flow equations are linearised and simplified using the ADI methods, see Jain [2]. The numerical scheme is then used to verify and validate the variation of groundwater levels in wells for Kisumu District(Kenya) hydrogeologic setting, incorporating measured data observed in a field site, see [20]. This has been achieved by generating a large matrix that is then solved using MATLAB. We are unaware of any published work on solving confined aquifer flow equations using the

ADI methods. The rest of this paper is organised as follows: In Section 2, we present groundwater flow model in two - dimensions and its discretization using the ADI methods. In Section 3, we present a numerical application of the derived ADI scheme using data obtained from wells located within Kisumu District(Kenya) hydrogeologic setting. In the conclusions, we analyze the efficiency and accuracy of the method used and briefly discuss the relevance of our results to civil engineers on the current problems in modelling and simulation of groundwater flow.

2 Finite difference (ADI) approximations to the two-dimensional confined aquifer flow equation

Consider a two-dimensional confined aquifer flow equation of the form, see Bear [21]

$$S\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(T\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(T\frac{\partial h}{\partial y}\right) - P(x,y,t) + R(x,y,t) \tag{2.1}$$

where h is the piezometric head defined as $h = \frac{p}{\rho_l g} + z$, p is the pressure and z is the positive vertical coodinate, $P(x,y,t)$ is the rate of pumping(per unit area of aquifer), $R(x,y,t)$ is the rate of recharge (per unit area of aquifer). S is the aquifer storativity defined by $S = \phi\rho_l cgb$ where ϕ is the porosity of the soil/rock surface, ρ_l is the liquid density, c is the chemical/contaminant concentration in the water, g is the gravitational acceleration and b is the thickness of the aquifer. T is the aquifer transmitivity defined by $T = kb$, in which b may vary with x and y and k is the permeability of the medium. The aquifer storativity is defined as the volume of water added to storage in a unit area of aquifer, per unit rise of piezometric head. Hence the left side of equation (2.1) expresses the volume of water added to storage in the aquifer, per unit area per unit time. Thus equation (2.1) implies that the excess of inflow over outflow of water in a unit area of an aquifer, per unit time, at a point, is equal to the rate at which water volume is being stored, where storage is due to fluid and solid matrix compressibilities. In equation (2.1), we can reasonably assume that the rate of pumping and recharge are constants, since all the wells in the considered domain have the same source of recharge. We then solve (2.1) inside the aquifer model with the following initial and boundary conditions:

(i) Specified head boundary: $h(x,y,t) = h(x,y,0) = 13m$ (lower value of the piezometric head in the considered domain at any time).

(ii) Vertical boundaries: $\frac{\partial h}{\partial x} = 0$ at $x = 0, I$; $\frac{\partial h}{\partial y} = 0$ at $y = 0, J$, indicating zero flux boundary where I and J are intervals on the $x-$ and $y-$ axes respectively.

(iii) The bottom of the flow system is an impermeable rock unit (unfractured igneous rocks). Thus, this physical boundary is also represented as a zero flux boundary.

Let $m, l,$ and n be index sets in the $x, y,$ and t directions respectively. Here, $x = m\Delta x,\ y = l\Delta y$ and $t = n\Delta t$ where $\Delta x, \Delta y$ and Δt are spacing in the $x, y,$ and t directions respectively. We may therefore write:

$$h(x,y,t) = h(m\Delta x, l\Delta y, n\Delta t) = h^n_{m,l} \tag{2.2}$$

Equation (2.1) can be linearized using the ADI methods by a procedure that we describe below, see [2]. We consider one space dimension, and then later translate the result to two- dimensional space. We then have:

$$S\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(T\frac{\partial h}{\partial x}\right) - P(x,t) + R(x,t) \tag{2.3}$$

Substitution of $T\frac{\partial h}{\partial x} = Z(x,t)$ into (2.3) and rearranging gives

$$\frac{\partial Z}{\partial x} = S\frac{\partial h}{\partial t} + P(x,t) - R(x,t) \tag{2.4}$$

which can be written as

$$\frac{\partial Z}{\partial x} = K(h; x, t) \tag{2.5}$$

On integrating (2.5) between the limits $(x_{m-\frac{1}{2}}, x_{m+\frac{1}{2}})$, we obtain the result

$$Z(x_{m+\frac{1}{2}}, t) - Z(x_{m-\frac{1}{2}}, t) = \int_{x_{m-\frac{1}{2}}}^{x_{m+\frac{1}{2}}} K(h; \lambda, t)d\lambda \tag{2.6}$$

Integrating (2.5) between the limits $(x_{m-\frac{1}{2}}, x)$ and then multiplying this with $\frac{1}{T(x,t)}$, and integrating again between (x_{m-1}, x_m), we get

$$\int_{x_{m-1}}^{x_m} \frac{Z(x,t) - Z(x_{m-\frac{1}{2}}, t)}{T(x,t)}dx = \int_{x_{m-1}}^{x_m} \frac{\int_{x_{m-\frac{1}{2}}}^{x} K(h;\lambda,t)d\lambda}{T(x,t)}dx \tag{2.7}$$

Substituting for $T\frac{\partial h}{\partial x} = Z(x,t)$ in (2.7), we obtain

$$\int_{x_{m-1}}^{x_m} \frac{\partial h}{\partial x} - \frac{Z(x_{m-\frac{1}{2}}, t)}{T(x,t)}dx = \int_{x_{m-1}}^{x_m} \frac{\int_{x_{m-\frac{1}{2}}}^{x} K(h;\lambda,t)d\lambda}{T(x,t)}dx \tag{2.8}$$

Integrate with respect to x to obtain

$$h(x_m,t)-h(x_{m-1},t)-Z(x_{m-\frac{1}{2}},t)\int_{x_{m-1}}^{x_m}\frac{dx}{T(x,t)}=\int_{x_{m-1}}^{x_m}\frac{\int_{x_{m-\frac{1}{2}}}^{x}K(h;\lambda,t)d\lambda}{T(x,t)}dx \tag{2.9}$$

Rearranging, we get

$$Z(x_{m-\frac{1}{2}},t)\int_{x_{m-1}}^{x_m}\frac{dx}{T(x,t)}=[h(x_m,t)-h(x_{m-1},t)]-\int_{x_{m-1}}^{x_m}\frac{\int_{x_{m-\frac{1}{2}}}^{x}K(h;\lambda,t)d\lambda}{T(x,t)}dx \tag{2.10}$$

Solving for $Z(x_{m-\frac{1}{2}},t)$ yields

$$Z(x_{m-\frac{1}{2}},t)=\frac{h(x_m,t)-h(x_{m-1},t)}{\int_{x_{m-1}}^{x_m}\frac{dx}{T(x,t)}}-\frac{1}{\int_{x_{m-1}}^{x_m}\frac{dx}{T(x,t)}}\int_{x_{m-1}}^{x_m}\frac{dx}{T(x,t)}\int_{x_{m-\frac{1}{2}}}^{x}K(h;\lambda,t)d\lambda \tag{2.11}$$

Similarly, we have

$$Z(x_{m+\frac{1}{2}},t)=\frac{h(x_{m+1},t)-h(x_m,t)}{\int_{x_m}^{x_{m+1}}\frac{dx}{T(x,t)}}-\frac{1}{\int_{x_m}^{x_{m+1}}\frac{dx}{T(x,t)}}\int_{x_m}^{x_{m+1}}\frac{dx}{T(x,t)}\int_{x_{m+\frac{1}{2}}}^{x}K(h;\lambda,t)d\lambda \tag{2.12}$$

Substituting (2.11) and (2.12) into (2.6) and integrating between the limits (t_n,t_{n+1}), we obtain the integral identity as follows

$$\begin{aligned}\int_{t_n}^{t_{n+1}}\frac{h(x_{m+1},t)-h(x_m,t)}{\int_{x_m}^{x_{m+1}}\frac{dx}{T(x,t)}}dt-\int_{t_n}^{t_{n+1}}\frac{h(x_m,t)-h(x_{m-1},t)}{\int_{x_{m-1}}^{x_m}\frac{dx}{T(x,t)}}dt=\\ \int_{t_n}^{t_{n+1}}dt\int_{x_{m+\frac{1}{2}}}^{x_{m+\frac{1}{2}}}K(h;\lambda,t)d\lambda+\int_{t_n}^{t_{n+1}}\frac{dt}{\int_{x_m}^{x_{m+1}}\frac{dx}{T(x,t)}}\int_{x_m}^{x_{m+1}}\frac{dx}{T(x,t)}\int_{x_{m+\frac{1}{2}}}^{x}K(h;u,t)du\\ -\int_{t_n}^{t_{n+1}}\frac{dt}{\int_{x_{m-1}}^{x_m}\frac{dx}{T(x,t)}}\int_{x_{m-1}}^{x_m}\frac{dx}{T(x,t)}\int_{x_{m-\frac{1}{2}}}^{x}K(h;\lambda,t)d\lambda\end{aligned} \tag{2.13}$$

The integrals in (2.13) may be evaluated by the quadrature formulas:

$$\begin{aligned}\int_{t_n}^{t_{n+1}}f(t)dt &\approx k[\gamma_1 f(t_n)+(1-\gamma_1)f(t_{n+1})]\\ \int_{t_n}^{t_{n+1}}f(t)g'(t)dt &\approx [\gamma_1 f(t_n)+(1-\gamma_1)f(t_{n+1})][g(t_{n+1}-g(t_n))]\end{aligned}$$

where $\gamma_1,\ 0 \le \gamma_1 \le 1$ is a parameter.

$$\int_{x_m}^{x_{m-1}} \frac{dx}{\phi(x)} \approx R\frac{1}{\phi(x_{m-\frac{1}{2}})}; \quad \int_{x_m}^{x_{m+1}} \frac{dx}{\phi(x)} \approx R\frac{1}{\phi(x_{m+\frac{1}{2}})}; \quad \int_{x_{m-\frac{1}{2}}}^{x_{m+\frac{1}{2}}} \phi(x)dx \approx a\phi(x_i);$$

$$\int_{x_{m-1}}^{x_m} \frac{dx}{\phi(x)} \int_{x_{m-\frac{1}{2}}}^{x} \psi(s)ds \approx 0; \quad \int_{x_m}^{x_{m+1}} \frac{dx}{\phi(x)} \int_{x_{m+\frac{1}{2}}}^{x} \psi(s)ds \approx 0$$

Equation (2.13) can now be simplified to

$$\begin{aligned}[](1-\gamma_1)S_m^{n+1} + \gamma_1 S_m^n]\frac{h_m^{n+1} - h_m^n}{k} - \frac{(1-\gamma_1)}{R^2}[T_{m-\frac{1}{2}}^{n+1} h_{m-1}^{n+1} - (T_{m-\frac{1}{2}}^{n+1} + T_{m+\frac{1}{2}}^{n+1})h_m^{n+1} + T_{m+1}^{n+1}h_{m+1}^{n+1}] \\ -\frac{\gamma_1}{R^2}[T_{m-\frac{1}{2}}^{n} h_{m-1}^{n} - (T_{m-\frac{1}{2}}^{n} + T_{m+\frac{1}{2}}^{n})h_m^{n} + T_{m+1}^{n}h_{m+1}^{n}] + (1-\gamma_1)P_m^{n+1}h_m^{n+1} + \gamma_1 P_m^n h_m^n \\ = (1-\gamma_1)R_m^{n+1} + \gamma_1 R_m^n\end{aligned} \tag{2.14}$$

where

$$T_{m\pm\frac{1}{2}}^n = T(x_{m\pm\frac{1}{2}}, t_n); \quad S_m^n = S(x_m, t_n); \quad P_m^n = P(x_m, t_n) \text{ and } R_m^n = R(x_m, t_n)$$

Equation (2.14) may be written as

$$\begin{aligned}[](1-\gamma_1)S_m^{n+1} + \gamma_1 S_m^n]\Delta_t h_m^{n+1} - [(1-\gamma_1)r\delta_x(T_m^{n+1}\delta_x h_m^{n+1}) + \gamma_1 r\delta_x(T_m^n \delta_x h_m^n)] \\ +(1-\gamma_1)kP_m^{n+1}h_m^{n+1} + \gamma_1 kP_m^n h_m^n = (1-\gamma_1)kR_m^{n+1} + \gamma_1 kR_m^n\end{aligned} \tag{2.15}$$

The values $\gamma_1 = 1$ and $\gamma_1 = 0$ give the ADI explicit and implicit schemes of order of accuracy $(k + n^2)$ respectively. The value $\gamma_1 = \frac{1}{2}$ gives the Crank-Nicholson scheme of order $(k^2 + n^2)$. Since ADI methods are second order accurate and are unconditionally stable and convergent for all finite values of the mesh ratio, we choose $\gamma_1 = 0$ and use the implicit scheme. We then re-write equation (2.15) as

$$S_m^{n+1}\Delta_t h_m^{n+1} - r\delta_x(T_m^{n+1}\delta_x h_m^{n+1}) + kP_m^{n+1}h_m^{n+1} = kR_m^{n+1} \tag{2.16}$$

Since the values of S, R and P have been taken as constants at particular instances, we have

$$S\Delta_t h_m^{n+1} - r\delta_x(T_m^{n+1}\delta_x h_m^{n+1}) + kPh_m^{n+1} = kR \tag{2.17}$$

Similarly, considering the y-space dimension, we write

$$S\frac{\partial h}{\partial t} = \frac{\partial}{\partial y}\left(T\frac{\partial h}{\partial y}\right) - P(y,t) + R(y,t) \tag{2.18}$$

which gives the finite difference scheme of the form

$$S\Delta_t h_l^{n+1} - r\delta_y(T_l^{n+1}\delta_y h_l^{n+1}) + kPh_l^{n+1} = kR \tag{2.19}$$

Combining equations (2.17) and (2.19) gives the required finite difference scheme approximation to equation (2.1). This is expressed as

$$S\Delta_t h_m^{n+1} - r\delta_x(T_m^{n+1}\delta_x h_m^{n+1}) + kPh_m^{n+1} = S\Delta_t h_l^{n+1} - r\delta_y(T_l^{n+1}\delta_y h_l^{n+1}) + kPh_l^{n+1} \tag{2.20}$$

and can be simplified to

$$S\Delta_t(h_m^{n+1} - h_l^{n+1}) + r[-\delta_x(T_m^{n+1}\delta_x h_m^{n+1} + \delta_y(T_l^{n+1}\delta_y h_l^{n+1}] + kP(h_m^{n+1} - h_l^{n+1}) = 0 \tag{2.21}$$

and written as

$$Sh_{i+1,j}^{n+1} - Sh_{i,j+1}^{n+1} + rT(h_{i,j-1}^{n+1} + h_{i,j+1}^{n+1} - h_{i+1,j}^{n+1} - h_{i-1}^{n+1}) + kPh_{i,j}^{n+1} = 0 \tag{2.22}$$

3 Numerical Application

Consider a schematic representation of the aquifer model with mesh spacing as shown in Figure 1. We now examine the use of equation (2.22) vis-a-

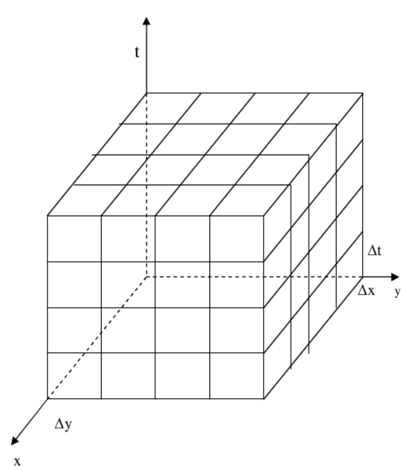

Figure 1: Discretised aquifer model

vis the nodes of Figure 1 using the implicit ADI method described so far. Equation (2.22) holds in the interior of the model of Figure 1 with the initial and boundary conditions described above. We then proceed with the solution as follows: Assume the solution is known at the time level t_n . We determine

the solution at t_{n+1} as follows:
Sub-divide the interval I on the x-axis into four equal spacing, sub-divide the interval J on the y-axis into four equal spacing and finally sub-divide the interval N on the t-axis into thirty two equal spacing. The size of the system of equations depends on the number of slices we make in the cube. We can minimize the complexity of the system without loss of generality by taking $m = 4$ i.e. $\Delta x = \Delta y = \frac{1}{4}$ on the x and y axes and use $\Delta n = \frac{1}{32}$ on the t axis as an illustration. This gives $r = \frac{1}{2}$ where $r = \frac{\Delta n}{(\Delta x)^2}$. We label all the mesh-points of the cube (Figure 1) in the aquifer model as $h_1, h_2, \ldots, h_{81}$. Apply equation (2.22) to each interior point of the model to obtain a system of 9 equations in 9 unknowns. The system of equations can be expressed in matrix vector form as:

$$\mathbf{Ah} = \mathbf{b} \tag{3.1}$$

where $\mathbf{A}$ is a $(m-1)^2$ matrix of known coefficients, $\mathbf{h}$ is the vector of unknown quantities (piezometric head at internal mesh points) and $\mathbf{b}$ is a vector of known quantities (from boundary conditions, aquifer storativity, rate of pumping and rate of recharge). We repeat the above procedure for time $t_{n+2}, t_{n+3}, \ldots$ until a stable solution is achieved. But since the implicit ADI scheme is unconditionally stable, we will consider in our scheme, only one step i.e. at time t_{n+1}. For purely illustrative purposes, we shall solve the aquifer flow equation inside the model of Figure 1 when subjected to the following boundary conditions obtained from the field data.

$$\left.\begin{array}{rcl} h(0,y,t) & = & 13m \\ h(1,y,t) & = & 50m \\ h(x,0,t) & = & 11m \\ h(x,1,t) & = & 42m \\ h(x,y,0) & = & 10m \\ h(x,y,1) & = & 50m \end{array}\right\} \tag{3.2}$$

The average values of the other parameters considered in the domain of study were recorded as follows:

$$\left.\begin{array}{rcl} S & = & 36m^3/minute \\ T & = & 1m^2/minute \\ R & = & 23m^3/minute \\ P & = & 6m^3/minute \end{array}\right\} \tag{3.3}$$

The model has nine internal points, at which the piezometric heads are to be evaluated, and may be written in vector form as: $[h^1_{11}, h^1_{22}, h^1_{23}, h^1_{24}, h^1_{32}, h^1_{33}, h^1_{34}, h^1_{42}, h^1_{44}]^T$. This can be renamed as: $[h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9]^T$. Applying the scheme, we obtain a system of equations, which we represent in matrix form as

$$\begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 35 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 35 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 35 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & -35 & 0 & 35 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 6 & 35 & 0 & -35 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 35 & 0 & 0 \\ 0 & 1 & 6 & -35 & 0 & 0 & 0 & 35 & 0 \\ 0 & 0 & -1 & 0 & 1 & 6 & -35 & 0 & 35 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \\ h_9 \end{bmatrix} = \begin{bmatrix} 251 \\ 1720 \\ 1413 \\ 1463 \\ 8 \\ -42 \\ 1691 \\ 50 \\ 0 \end{bmatrix}$$

Using MATLAB, we obtain

$$\begin{aligned} h_1 &= 41.8m & h_6 &= 29.6m \\ h_2 &= 49.1m & h_7 &= 48.3m \\ h_3 &= 40.4m & h_8 &= 34.9m \\ h_4 &= 41.8m & h_9 &= 43.5m \\ h_5 &= 32.2m \end{aligned}$$

4 Conclusion

In this paper we have presented a numerical scheme for groundwater flow in an anisotropic heterogenous fully confined aquifer based on the ADI method. Considering the results in Section 3, it can be seen that the numerical scheme has reliably validated and predicted the variations of groundwater levels for the 120 wells located within Kisumu District hydrogeologic setting. From these results of the numerical example, the finite difference approximation has yielded solutions which reliably replicate those obtained from measured field data. This is critical in planning and implementing groundwater remediation within the considered region.
We wish to point out that in differencing the groundwater flow equation, we have developed a numerical scheme that is only second order accurate. We have also considered a domain defined by the horizontal Cartesian space co-ordinates. Future research may take into consideration higher order accuracy, and a medium defined by cylindrical polar co-ordinates that describe radial groundwater flow into wells, and well pumping test analysis.

References

[1] D'Yakonov, G. E. (1963): On the application of disintegration difference operators. *Z. Vycist. Mat. Mat. Fiz*, vol 3, pp385-388

[2] Jain, M. K. (1984): Numerical Solution of Differential Equations, 2nd ed. Wiley Eastern Limited, New Delhi.

[3] Crank, J. and Nicholson, P. (1947): A practical method for the numerical evaluation of solutions of partial differential equations of the heat conduction type. *Proc. Comb. Phil. Sco.*, vol. 43, 50-57.

[4] Douglas, J Jr. and Rachford, H. H. (1956): On the numerical solution of heat conduction problems in two and three space variables. *Trans. Amer. Math. Soc.*, 82(20), PP421-439.

[5] Dufort, E. C. and Frankel, S. F. (1953): Stability conditions in the numerical treatment of parabolic differential equations. *Math. Comp.*, vol. 7, pp135-152.

[6] Larkin, B. K. (1964): Some stable explicit difference approximation to the diffusion equation. *Math. Comp.*, vol. 18, pp196-202.

[7] Lax, P. D. and Richtmyer, R. D. (1956): Survey of the stability of linear finite differential equations.*Comm. Pure Appl. Math.*,vol. 9, pp267-293.

[8] Li, Z. and Shen, Y. (1999): A numerical method for solving heat equations involving interfaces. *Electronic Journal of Differential Equations, Conf.*, vol. 3, pp100-108.

[9] Li, Z. and Mayo, A. (1993): ADI method for heat equations with discontinuities along an arbitrary interface. *Proc. Symp. Appl. Math, AMS*, vol. 48, pp311-315.

[10] Peaceman, D. W. and Rachford, H. H. (1955): The numerical solution of parabolic and elliptic equations. *J. Soc. Indust. Appli. Math.*, 3(1), pp28-41.

[11] Sarmaskii, A. A. (1964): Local one- dimensional difference schemes for multidimensional hyperbolic equations in an arbitrary region. *Zh. Vychisl. Mat. Mat. Fiz.*, 4(4), PP638-648.

[12] Saul'yen, V. K. (1964): Integration of Equations of Parabolic Type by the Method of Nets. Pergamon Press.

[13] Charvent, G. and Jaffre, J. (1986): Mathematical Models and Finite Elements for Reservoir Simulation. North Holland, Amsterdam.

[14] Herrling, B. and Heckele, A. (1986): Coupling of finite element and optimization methods for management of groundwater systems. *Advances in Water Resources*, 9(4), pp190-195.

[15] Tegnander, C. (2001): Models for groundwater flow: A numerical comparison between Richard's model and the fractional flow model. *Transport in Porous Media*, 43(2), pp213-224.

[16] McDonald, M. G. and Harbaugh, W. A. (1988): A modular three-dimensional finite-difference groundwater flow model. *Volume 6 of Techniques of Water Resources Investigation of the United States Geological Survey*, Chapter A1, Scientific Software Group.

[17] Trescott, P. C. (1975): Documentation of finite-difference model for Simulation of three-dimensional groundwater flow. *U.S Geological Survey Open-File Report*, vol. 32, pp75-438.

[18] Trescott, P. C. and Larson, S. P. (1976): Documentation of finite-difference model for Simulation of three-dimensional groundwater flow. *U.S Geological Survey Open-File Report*, vol. 21, pp75-591, Supplement to Open File Report 75-438.

[19] Trescott, P. C., Pinder, P. C. and Larson, S. P. (1976): Finite-difference model for aquifer simulation in two-dimensions with results of numerical experiments in Techniques of Water-Resources Investigation of the U.S Geological Survey. Book 7, 116.

[20] Ker, Priestman and Associates (1988): Water Resources Assessment Survey Report of Muhoroni, Nyando and Winam Divisions, Lake Basin Development Authoroty(LBDA), Nairobi.

[21] Bear, J. (1979): Hydraulics of Groundwater. McGraw-Hill, New York.

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