Applied Mathematical Sciences, Vol. 11, 2017, no. 12, 561 - 570
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ams.2017.7133

General Analytical Solution for the One Dimensional Regular Cauchy Problem of Euler-Poisson-Darboux Equation

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Abstract

A general analytical solution to the one dimensional regular Cauchy problem of Euler-Poisson-Darboux (EPD) equation has been investigated. The one dimensional EPD which is a Partial Differential Equation (PDE) with initial conditions is transformed into Ordinary Differential Equation (ODE) using Similarity Transformation. The first derivative of the ODE is eliminated by substitution technique. The coefficient of the first derivative is equated to zero and then solved. The general solution is a product of two terms. The first term is the one obtained when the first derivative is eliminated from the ODE and the second term is the complementary function (cf) obtained from the remaining part of ODE. The arbitrary constants of the cf are obtained in terms of x and t when the initial conditions are substituted in the general solution. The general solution is a solution for one dimensional regular Cauchy EPD's and degenerate EPD's, which by coincident are one dimensional wave equations.

Mathematics Subject Classification: 35Q05

Keywords: Regular Cauchy, Singular Cauchy, Similarity Transformation, Degenerate, General Analytical Solution

1 Introduction

The regular Cauchy problem of Euler-Poisson-Darboux equation appears in various areas of Mathematics and Physics, such as the theory of surfaces, the propagation of sound, etc [3]. The Cauchy problem of Euler-Poisson-Darboux equation takes the form:

$$\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} + ... + \frac{\partial^2 U}{\partial x_n^2} = \frac{\partial^2 U}{\partial t^2} + \frac{k}{t}\frac{\partial U}{\partial t}$$

$$U(x_1, ..., x_n, 0) = f(x_1, ...x_n)$$

$$\frac{\partial U}{\partial t}(x_1, ..., x_n, 0) = 0 \tag{1}$$

where $x_1, x_2, ...x_n$ are points in $\mathbf{R}^n$, k is a real parameter, t is a time parameter, f is function, U is diplacement of a wave and $\mathbf{R}^n$ is Euclidean space. The problem is called singular if $\frac{k}{t} \to \infty$ as $t \to 0$ and degenerate if $\frac{k}{t} \to 0$ as $t \to 0$. When the initial conditions in (1) are replaced by

$$U(x_1, ..., x_n, t) = f(x_1, ...x_n)$$

$$\frac{\partial U}{\partial t}(x_1, ..., x_n, t) = 0$$

the problem becomes regular Cauchy when $-\infty < \frac{k}{t} < \infty$. Davis [5] found the explicit solution for a regular Cauchy problem for the n-dimensional EPD equation. To obtain the solution , Davis extended the method of Riesz to include non self adjoint equations. Existence and uniqueness were shown. Asral [1] solved the regular Cauchy problem for the EPD equations using the method of ascent. In our earlier paper, Manyonge *et al* [6], we found the weak solution for the singular Cauchy problem of EPD equation for $n = 4$ by applying Fourier transform to form Bessel differential equation. Convolution theorem is then applied on the inverse transform of the solution of Bessel differential equation. In the present paper, we have obtained analytically, a general solution for one dimensional regular Cauchy problem of Euler -Poisson-Darboux equation . The solution also works for one dimensional EPD's that are degenerate.

2 One dimensional regular Cauchy problem of Euler-Poisson-Darboux equation

The regular Cauchy problem of Euler-Poisson-Darboux (EPD) equation for n-dimensions takes the form

$$\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} + ... + \frac{\partial^2 U}{\partial x_n^2} = \frac{\partial^2 U}{\partial t^2} + \frac{k}{t}\frac{\partial U}{\partial t} \tag{2}$$

$$U(x_1, ..., x_n, t) = f(x_1, ...x_n) \tag{3}$$

$$\frac{\partial U}{\partial t}(x_1, ..., x_n, t) = 0 \tag{4}$$

For one dimensional EPD, equations (2) to (4) become:

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 U}{\partial t^2} + \frac{k}{t}\frac{\partial U}{\partial t} \tag{5}$$

$$U(x, t) = f(x) \tag{6}$$

$$\frac{\partial U}{\partial t}(x, t) = 0 \tag{7}$$

The main task is to find a function $U(x, t)$ which satisfies equations (5) to (7) above.

3 Converting PDE of EPD into ODE by Similarity Transformation

To apply the Similarity Transformation method [2], let $U(x, t) = G(x^n t)$ so that:

$$\frac{\partial U}{\partial x} = ntx^{n-1}G'(x^n t) \tag{8}$$

$$\frac{\partial^2 U}{\partial x^2} = n(n-1)tx^{n-2}G'(x^n t) + n^2t^2x^{2(n-1)}G''(x^n t) \tag{9}$$

$$\frac{\partial U}{\partial t} = x^n G'(x^n t) \tag{10}$$

$$\frac{\partial^2 U}{\partial t^2} = x^{2n}G''(x^n t) \tag{11}$$

Substituting equations (9) to (11) in equation (5) gives

$$n(n-1)tx^{n-2}G'(x^n t) + n^2t^2x^{2n-2}G''(x^{nt}) = x^{2n}G''(x^n t) + \frac{k}{t}x^n G'(x^n t) \tag{12}$$

Let $\sigma = x^n t$ so that $t = \frac{\sigma}{x^n}$ then equation (12) becomes

$$n(n-1)\sigma x^{-2}G'(\sigma) + n^2\sigma^2 x^{-2}G''(\sigma) = x^{2n}G''(\sigma) + \frac{k}{\sigma}x^{2n}G'(\sigma)$$

The equation above is made consistent only when $-2 = 2n$ so that $n = -1$. The equation then becomes

$$G''(\sigma) + \left(\frac{2\sigma^2 - k}{\sigma^3 - \sigma}\right)G'(\sigma) = 0 \tag{13}$$

Equation (5) which is a PDE has been converted into an ODE as shown in equation (13).

4 Elimination of first derivative

The first derivative in equation (13) is removed by using the following transformation [4]: Let

$$\frac{d^2G}{d\sigma^2} + A\frac{dG}{d\sigma} + BG = C \tag{14}$$

where A, B and C are functions of σ and coefficients of G and its derivatives. Let also $G = wz$, where z is not an integral solution of complementary function (cf), then

$$\frac{dG}{d\sigma} = z\frac{dw}{d\sigma} + w\frac{dz}{d\sigma}$$

$$\frac{d^2G}{d\sigma^2} = z\frac{d^2w}{d\sigma^2} + 2\frac{dw}{d\sigma}\frac{dz}{d\sigma} + w\frac{d^2z}{d\sigma^2}$$

Putting G, $\frac{dG}{d\sigma}$ and $\frac{d^2G}{d\sigma^2}$ in equation (14) we obtain

$$\frac{d^2w}{d\sigma^2} + \left(A + \frac{2}{z}\frac{dz}{d\sigma}\right)\frac{dw}{d\sigma} + \left(\frac{d^2z}{d\sigma^2} + A\frac{dz}{d\sigma} + Bz\right)\frac{w}{z} = \frac{C}{z} \tag{15}$$

The first derivative is eliminated by equating the second bracket on the left in equation (15) to zero. z in $G = wz$ is not part of cf

$$A + \frac{2}{z}\frac{dz}{d\sigma} = 0$$

$$z = e^{-\frac{1}{2}\int Ad\sigma} \tag{16}$$

The value of first bracket on the left of equation (15) is found by using equation (16) as follows:

$$\frac{dz}{d\sigma} = -\frac{1}{2}e^{-\frac{1}{2}\int Ad\sigma} = -\frac{1}{2}Az \tag{17}$$

$$\frac{d^2z}{d\sigma^2} = -\frac{1}{2}\frac{dA}{d\sigma}z - \frac{A}{2}\frac{dz}{d\sigma} = -\frac{1}{2}\frac{dA}{d\sigma}z - \frac{A}{2}\left(-\frac{1}{2}Az\right) = -\frac{1}{2}\frac{dA}{d\sigma}z + \frac{1}{4}A^2z \quad (18)$$

Equations (17) and (18) are put in the coefficient of w in equation (15) and become

$$\frac{d^2z}{d\sigma^2}+A\frac{dz}{d\sigma}+Bz = -\frac{1}{2}\frac{dA}{d\sigma}z+\frac{1}{4}A^2z+A\left(-\frac{1}{2}Az\right)+Bz = z\left(B-\frac{1}{2}\frac{dA}{d\sigma}-\frac{1}{4}A^2\right) \quad (19)$$

When the first derivative is eliminated from equation (15), the equation becomes

$$\frac{d^2w}{d\sigma^2} + w\left(B - \frac{1}{2}\frac{dA}{d\sigma} - \frac{1}{4}A^2\right) = \frac{C}{z}$$

$$\frac{d^2w}{d\sigma^2} + w\left(B - \frac{1}{2}\frac{dA}{d\sigma} - \frac{1}{4}A^2\right) = Ce^{\frac{1}{2}\int Ad\sigma} \quad (20)$$

From equation (20), let

$$B_1 = \left(B - \frac{1}{2}\frac{dA}{d\sigma} - \frac{1}{4}A^2\right)$$

$$C_1 = Ce^{\frac{1}{2}\int Ad\sigma}$$

Putting B_1 and C_1 in equation (20) gives

$$\frac{d^2w}{d\sigma^2} + B_1w = C_1 \quad (21)$$

Since $G = wz$, w is obtained from the solution of equation (21) which is an ODE while z obtained from the integration of equation (16).

5 General solution for the one dimensional Euler-Poisson-Darboux equation

5.1 Elimination of the first derivative $\frac{dG}{d\sigma}$

Using equations (13) and (14)

$$A = \frac{2\sigma^2 - k}{\sigma^3 - \sigma}, B = 0, C = 0$$

$$B_1 = \left(B - \frac{1}{2}\frac{dA}{d\sigma} - \frac{A^2}{4}\right) \quad (22)$$

$$\frac{dA}{d\sigma} == \frac{-2\sigma^4 - 2\sigma^2 + 3\sigma^2k - k}{(\sigma^3 - \sigma)^2}$$

$$\frac{A^2}{4} = \frac{1}{4}\left(\frac{2\sigma^2 - k}{\sigma^3 - \sigma}\right)^2 = \frac{1}{4}\left(\frac{4\sigma^4 - 4\sigma^2 k + k^2}{(\sigma^3 - \sigma)^2}\right)$$

Equation (22) becomes

$$B_1 = \frac{(2\sigma^2 + k)(2 - k)}{4(\sigma^3 - \sigma)^2} \tag{23}$$

Since

$$C_1 = Ce^{\frac{1}{2}\int A d\sigma}$$

and $C = 0$ then $C_1 = 0$. Putting B_1 and C_1 in equation (21), gives

$$\frac{d^2 w}{d\sigma^2} + \left(\frac{(2\sigma^2 + k)(2 - k)}{4(\sigma^3 - \sigma)^2}\right) w = 0 \tag{24}$$

5.2 The complementary function (cf)

From equation (24), let m be its root then

$$m = \pm i\sqrt{\frac{(2\sigma^2 + k)(2 - k)}{4(\sigma^3 - \sigma)^2}}$$

The cf is therefore

$$w = M\cos\sqrt{\frac{(2\sigma^2 + k)(2 - k)}{4(\sigma^3 - \sigma)^2}}\sigma + N\sin\sqrt{\frac{(2\sigma^2 + k)(2 - k)}{4(\sigma^3 - \sigma)^2}}\sigma \tag{25}$$

Where M and N are arbitrary constants. Since $\sigma = \frac{t}{x}$

$$\sqrt{\frac{(2\sigma^2 + k)(2 - k)}{4(\sigma^3 - \sigma)^2}}\sigma = x\frac{\sqrt{(2t^2 + x^2 k)(2 - k)}}{2(t^2 - x^2)}$$

Equation (25) becomes

$$w = M\cos x\frac{\sqrt{(2t^2 + x^2 k)(2 - k)}}{2(t^2 - x^2)} + N\sin x\frac{\sqrt{(2t^2 + x^2 k)(2 - k)}}{2(t^2 - x^2)} \tag{26}$$

Equation (26) is the complementary function (cf).

5.3 Finding z which is not part of complementary function

From equation (16)

$$z = \exp -\frac{1}{2}\int \frac{2\sigma^2 - k}{\sigma^3 - \sigma} d\sigma = \frac{(\sigma^2 - 1)^{\frac{k-2}{4}}}{\sigma^{\frac{k}{2}}}$$

Again since $\sigma = \frac{t}{x}$,

$$z = \frac{(t^2 - x^2)^{\frac{k-2}{4}} x}{t^{\frac{k}{2}}} \tag{27}$$

Equation (27) gives the value of z, which is not part of complementary function

5.4 General solution for the one dimensional Euler-Poisson-Darboux equation

The general solution of the one dimensional Euler-Poisson-Daxboux equation is given by

$$U(x,t) = G(\sigma) = G\left(\frac{t}{x}\right) = wz \tag{28}$$

Putting equations (26) and (27) in equation (28) gives

$$U(x,t) = \left[M \cos x \frac{\sqrt{(2t^2 + x^2k)(2-k)}}{2(t^2 - x^2)} + N \sin x \frac{\sqrt{(2t^2 + x^2k)(2-k)}}{2(t^2 - x^2)}\right] \left(\frac{(t^2 - x^2)^{\frac{k-2}{4}} x}{t^{\frac{k}{2}}}\right) \tag{29}$$

6 General solution for the EPD in terms of x and t

To find the general solution for the EPD in terms of x and t, we put initial conditions in equation (29) to obtain values of arbitrary costants M and N in terms of x and t.

6.1 Substituting U(x,t)=f(x)

Putting $U(x,t) = f(x)$ in equation (29) we have

$$f(x) = \left[M \cos x \frac{\sqrt{(2t^2 + x^2k)(2-k)}}{2(t^2 - x^2)} + N \sin x \frac{\sqrt{(2t^2 + x^2k)(2-k)}}{2(t^2 - x^2)}\right] \left(\frac{(t^2 - x^2)^{\frac{k-2}{4}} x}{t^{\frac{k}{2}}}\right) \tag{30}$$

6.2 Substituting $\frac{\partial U}{\partial t} = 0$

Before differentiating equation (29) partially with respect to t, let $P = M \cos x \frac{\sqrt{(2t^2+x^2k)(2-k)}}{2(t^2-x^2)}$, $R = N \sin x \frac{\sqrt{(2t^2+x^2k)(2-k)}}{2(t^2-x^2)}$ and $Q = \frac{(t^2-x^2)^{\frac{k-2}{4}} x}{t^{\frac{k}{2}}}$

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial t}(PQ) + \frac{\partial}{\partial t}(RQ) = PQ' + P'Q + RQ' + R'Q = 0 \tag{31}$$

Putting P, Q, R, S and their primes in equation (31) gives

$$\left(\frac{x(t^2-x^2)^{\frac{k-6}{4}}(x^2k-2t^2)}{2t^{\frac{k+2}{2}}}\right) M\cos x\frac{\sqrt{(2t^2+x^2k)(2-k)}}{2(t^2-x^2)}$$

$$+$$

$$\left(\frac{(t^2-x^2)^{\frac{k-2}{4}}x}{t^{\frac{k}{2}}}\right)\left(\frac{xt(2-k)^{\frac{1}{2}}(t^2+x^2+x^2k)}{(2t^2+x^2k)^{\frac{1}{2}}(t^2-x^2)^2}\right) M\sin x\frac{\sqrt{(2t^2+x^2k)(2-k)}}{2(t^2-x^2)}$$

$$+$$

$$\left(\frac{x(t^2-x^2)^{\frac{k-6}{4}}(x^2k-2t^2)}{2t^{\frac{k+2}{2}}}\right) N\sin x\frac{\sqrt{(2t^2+x^2k)(2-k)}}{2(t^2-x^2)}$$

$$-$$

$$\left(\frac{(t^2-x^2)^{\frac{k-2}{4}}x}{t^{\frac{k}{2}}}\right)\left(\frac{xt(2-k)^{\frac{1}{2}}(t^2+x^2+x^2k)}{(2t^2+x^2k)^{\frac{1}{2}}(t^2-x^2)^2}\right) N\cos x\frac{\sqrt{(2t^2+x^2k)(2-k)}}{2(t^2-x^2)} = 0 \tag{32}$$

6.3 Values of arbitrary constants M and N

Arbitrary constants M and N are found by solving equations (30) and (32) simultaneously , therefore let $D = x\frac{\sqrt{(2t^2+x^2k)(2-k)}}{2(t^2-x^2)}$, $E = \frac{x(t^2-x^2)^{\frac{k-6}{4}}(x^2k-2t^2)}{2t^{\frac{k+2}{2}}}$, $F = \frac{xt(2-k)^{\frac{1}{2}}(t^2+x^2+x^2k)}{(2t^2+x^2k)^{\frac{1}{2}}(t^2-x^2)^2}$ and $H = \frac{(t^2-x^2)^{\frac{k-2}{4}}x}{t^{\frac{k}{2}}}$ .Putting D, E, F and H in equation (30) it becomes

$$M = \frac{f(x) - NHsinD}{HcosD} \tag{33}$$

Putting again D, E, F and H in equation (32) it becomes

$$MEcosD + MHFsinD - NFHcosD + NEsinD = 0 \tag{34}$$

Putting equation (33) in (34), gives

$$N = \frac{f(x)(E\cos D + HF\sin D)}{H^2F} \tag{35}$$

Putting equation (35) in equation (33) it becomes

$$M = \frac{f(x)(HF\cos D + E\sin D)}{H^2F} \tag{36}$$

6.4 Substituting M and N in the general solution

Putting equations (35) and (36) in equation (29) gives

$$U(x,t) = f(x)\frac{HF + E\sin 2D}{HF} \tag{37}$$

Writing equation (37) in terms of x and t gives

$$U(x,t) = f(x)\left[1 + \frac{(t^2 - x^2)(x^2k - 2t^2)(2t^2 + x^2k)^{\frac{1}{2}}\sin 2x\frac{\sqrt{(2t^2+x^2k)(2-k)}}{2(t^2-x^2)}}{2xt^2(2-k)^{\frac{1}{2}}(t^2 + x^2 + x^2k)}\right] \tag{38}$$

7 Results

7.1 Regular case

Equation (38) is the general analytical solution for the regular Cauchy case when $t > 0$ and k is a real parameter,

7.2 Degenerate case

When $k = 0$, equation (38) becomes

$$U(x,t) = f(x)\left[1 - \frac{t(t^2 - x^2)\sin\frac{2xt}{t^2-x^2}}{x(t^2 + x^2)}\right]$$

which is the general analytical solution to the one dimensional wave equation.

Conclusion

In this research, the general analytical solution for the one dimensional regular Cauchy EPD has been worked out . The same equation is also the general analytical solution for the one dimensional wave equation when $k = 0$.

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Received: February 8, 2017; Published: March 1, 2017