Applied Mathematical Sciences, Vol. 5, 2011, no. 9, 441 - 445

On the Generalized Reid Inequality and the Numerical Radii

J. O. Bonyo[1], D. O. Adicka[2], J. O. Agure[3]

[1,3]Department of Mathematics and Applied Statistics
Maseno University, Box 333 Maseno, Kenya

[2]School of Science and Technology
Kabianga University College, Box 1 Kabianga, Kenya

Abstract

In this paper, we extend the generalized Reid inequality to include the numerical radii for the product of two Hilbert space operators.

Mathematics Subject Classification: 47A12, 47A30, 47A63

Keywords: Generalized Reid inequality, numerical radius, spectral radius, usual operator norm

1 Introduction

Let $B(H)$ denotes the C* - algebra of all bounded linear operators on a complex Hilbert space H with inner product $\langle .,. \rangle$, here $\langle .,. \rangle$ denotes the standard inner product in $\mathbb{C}^n$. For $A \in B(H)$, the usual operator norm of an operator A, denoted by $\|A\|$, is defined by

$$\|A\| \;=\; \sup_{\|x\|=1} \|Ax\|, \text{for all } x \in H,$$

where $\|x\| = \langle x, x \rangle^{\frac{1}{2}}$.
The numerical range of A, known also as the field of values of A, is defined as the set of complex numbers given by

$$W(A) \;=\; \{\langle Ax, x \rangle : x \in H, \|x\| = 1\}.$$

[1]First author(E-mail:jobbonyo@yahoo.com)
[2]adikadaniel78@yahoo.com
[3]johnagure@maseno.ac.ke

The most important properties of the numerical range are that it is convex and its closure contains the spectrum of the operator.
A unitarily invariant norm $|||.|||$ on H is a norm on the ideal $\mathcal{C}_{|||.|||}$ of $B(H)$, satisfying $|||UAV||| = |||A|||$ for all $A \in B(H)$ and all unitary operators U and V in $B(H)$. It is called unitarily invariant norm (or invariant under similarities) if $|||UAU^*||| = |||A|||$ for all $A \in B(H)$ and all unitary operators U in $B(H)$.
The most familiar example of weakly unitarily invariant norm is the numerical radius $w(A)$ defined by

$$w(A) = \sup\{|\lambda| : \lambda \in W(A)\}.$$

It is well known that $w(.)$ defines a norm on $B(H)$ and that for every $A \in B(H)$,

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|. \tag{1}$$

Thus, the usual operator norm and the numerical radius norm are equivalent. For basic properties of the numerical radius, we refer to [5, 8, 9].
The study of numerical radii inequalities has very recently been of great interest to mathematicians, see [3], [4], [6], [7]. We join the group of these scholars as the results obtained in this paper add reasonably to the existing literature. It is important to clarify here that the breakthrough into the recent development on this area of study has been facilitated greatly by the use of classical and operator inequalities that had earlier been widely studied, see for example [2].
On the other hand, the generalized Reid inequality as proved by Halmos [8] asserts that if A and B are operators in $B(H)$ such that A is positive and AB is self-adjoint, then

$$|\langle ABx, x\rangle| \leq \gamma(B)\langle Ax, x\rangle \text{for all}\, x \in H. \tag{2}$$

Here $\gamma(B)$ denotes the spectral radius of B.
A weaker version of this inequality was proved by Reid [10] in which $\gamma(B)$ is replaced by $\|B\|$ in (2) and this gives the famous Reid inequality. This inequality has been studied by several authors, see for instance [1] or [2]. In [1], Lin presented generalizations of Reid's and Halmos inequalities via polar decomposition approach.
Kittaneh [2] later extended both the generalized Reid inequality and the generalized mixed Schwarz inequality. It is shown in [2] that for A, B in $B(H)$ such that $|A|B = B^*|A|$ and if f and g are nonnegative functions on $[0, \infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$, then

$$|\langle ABx, y\rangle| \leq \gamma(B)\|f(|A|)x\|\|g(|A^*|)y\| \text{for all}\, x,\, y \in H. \tag{3}$$

It is important to note here that $|A| = (A^*A)^{\frac{1}{2}}$ and $|A^*| = (AA^*)^{\frac{1}{2}}$.
In the next section of this paper, we extend the Kittaneh's inequality (3) to include the numerical radius for the product of two Hilbert space operators.

2 Generalized Reid Inequality and the Numerical radius

In our work, we shall need the following basic Lemmas. The first Lemma, which is called Hölder-McCarthy inequality, is a well-known result that follows from spectral theorem for positive operators and Jensen's inequality (see [2]).

Lemma 2.1. *Let $A \in B(H)$ be positive operator and let $x \in H$ be any unit vector. Then*

$$\begin{aligned}\langle Ax, x\rangle^r &\leq \langle A^r x, x\rangle \text{ for all } r \geq 1, \text{ and} \\ \langle A^r x, x\rangle &\leq \langle Ax, x\rangle^r \text{ for all } 0 < r \leq 1.\end{aligned}$$

The second Lemma also follows from spectral theorem and is due to Kittaneh [2].

Lemma 2.2. *For a self-adjoint operator A in $B(H)$,*

$$|\langle Ax, x\rangle| \leq \langle |A|x, x\rangle \text{ for all } x \in H.$$

Our first result is an extension of the generalized Reid inequality.

Theorem 2.3. *Let A and B be operators in $B(H)$ such that A is positive and AB is self-adjoint. Then for $r \geq 1$, we have*

(i) $w^r(AB) \leq \| |A|^r |B|^r \|$ *and*

(ii) $w^r(AB) \leq \gamma^r(B)\|A^r\|$.

Proof.

(i) For every unit vector $x \in H$, we have

$$\begin{aligned}|\langle ABx, x\rangle|^r &\leq \langle |AB|x, x\rangle^r \\ &\leq \langle |AB|^r x, x\rangle \\ &\leq \langle |A|^r |B|^r x, x\rangle.\end{aligned}$$

Now the result follows at once by taking the supremum over all unit vectors in H.

(ii) By the generalized Reid inequality, we have

$$\begin{aligned}|\langle ABx, x\rangle|^r &\leq \gamma^r(B)\langle Ax, x\rangle^r \\ &\leq \gamma^r(B)\langle A^r x, x\rangle.\end{aligned}$$

Hence it immediately follows that

$$w^r(AB) \leq \gamma^r(B)\|A^r\|.$$

□

Next, we extend Kittaneh's extension of the generalized Reid inequality to obtain generalized numerical radii inequalities for the product of two Hilbert space operators.

Theorem 2.4. *Let A and B be operators in $B(H)$ such that $|A|B = B^*|A|$ and let f and g be nonnegative functions on $[0,\infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0,\infty)$, then*

$$w^r(AB) \;\; \leq \;\; \gamma^r(B)\|f^{2r}(|A|)g^{2r}(|A^*|)\|^{\frac{1}{2}}, \quad r \;\geq\; 1.$$

Proof. From the inequality (3), for all $x \in H$ we have

$$\begin{aligned} |\langle ABx, x\rangle|^{2r} &\leq \gamma^{2r}(B)\|f(|A|)x\|^{2r}\|g(|A^*|)x\|^{2r} \\ &= \gamma^{2r}(B)\langle f(|A|)x, f(|A|)x\rangle^r\langle g(|A^*|)x, g(|A^*|)x\rangle^r \\ &= \gamma^{2r}(B)\langle f^2(|A|)x, x\rangle^r\langle g^2(|A^*|)x, x\rangle^r \\ &\leq \gamma^{2r}(B)\langle f^{2r}(|A|)x, x\rangle\langle g^{2r}(|A^*|)x, x\rangle \\ &\leq \gamma^{2r}(B)\langle f^{2r}(|A|)g^{2r}(|A^*|)x, x\rangle. \end{aligned}$$

That is

$$|\langle ABx, x\rangle|^{2r} \;\; \leq \;\; \gamma^{2r}(B)\langle f^{2r}(|A|)g^{2r}(|A^*|)x, x\rangle.$$

Thus by taking supremum over all unit vectors in H, we get

$$w^{2r}(AB) \;\; \leq \;\; \gamma^{2r}(B)\|f^{2r}(|A|)g^{2r}(|A^*|)\|.$$

and so

$$w^r(AB) \leq \gamma^r(B)\|f^{2r}(|A|)g^{2r}(|A^*|)\|^{\frac{1}{2}}. \tag{4}$$

□

Another extension is the following result similar to the result above.

Theorem 2.5. *Let A, B, f and g be defined as in Theorem 2.4, then for $r \geq 1$,*

$$w^r(AB) \;\; \leq \;\; \frac{1}{2}\gamma^r(B)\|f^{2r}(|A|) + g^{2r}(|A^*|)\|.$$

Proof. For all $x \in H$, we have

$$\begin{aligned} |\langle ABx, x\rangle|^{2r} &\leq \gamma^{2r}(B)\|f(|A|)x\|^{2r}\|g(|A^*|)x\|^{2r} \\ &= \gamma^{2r}(B)\langle f(|A|)x, f(|A|)x\rangle^r\langle g(|A^*|)x, g(|A^*|)x\rangle^r \\ &\leq \gamma^{2r}(B)\langle f^{2r}(|A|)x, x\rangle\langle g^{2r}(|A^*|)x, x\rangle \\ &\leq \frac{1}{4}\gamma^{2r}(B)\Big(\langle f^{2r}(|A|)x, x\rangle + \langle g^{2r}(|A^*|)x, x\rangle\Big)^2 \text{ (by the arithmetic-geometric mean inequality)} \end{aligned}$$

Thus

$$|\langle ABx, x\rangle|^r \leq \frac{1}{2}\gamma^r(B)\Big(\langle f^{2r}(|A|)x, x\rangle + \langle g^{2r}(|A^*|)x, x\rangle\Big)$$

and so taking supremum over $x \in H$ with $\|x\| = 1$, we obtain

$$w^r(AB) \leq \frac{1}{2}\gamma^r(B)\|f^{2r}(|A|) + g^{2r}(|A^*|)\| \tag{5}$$

which is the desired result. □

ACKNOWLEDGEMENTS. The author[1] would like to thank Maseno University for assistantship during the period of this research.

References

[1] C.-S. Lin, *Polar decomposition approach to Reids's inequality*, Journ. of Inequal. and Appl., **7** (2002), 779-785.

[2] F. Kittaneh, *Notes on some Inequalities for Hilbert space Operators,* Publ. Res. Inst. Math. Sci. **24** (1988), 283-293.

[3] H. Albadawi and Kh. Shebrawi, *Inequalities for powers of the numerical radii of Hilbert space Operators,* Int. Journal of Math. Analysis, **2** (2008), 527-532.

[4] Kh. Shebrawi and H. Albadawi, *Numerical Radius and Operaors Norm Inequalities,* Journ. of Inequal. and Appl., Vol. 2009 (2009), 1-11.

[5] K. E. Gustafson and D. K. M. Rao, *Numerical range: the field of values of linear operators and matrices*, Springer-Verlag, New York,(1997).

[6] M. Al-Hawari, *Sharper Inequalities for Numerical Radius for Hilbert space Operator,* Int. Journal of Math. Analysis **3** (2009), 1021-1025.

[7] M. Al-Hawari, R. Hatamleh, A. Qazza, *Sharper Inequalities for Powers of the Numerical Radius for Hilbert space Operators,* Int. Mathematical Forum **4** (2009), 1413-1417.

[8] P. R. Halmos, *A Hilbert space problem book*, Van Nostrand, Princeton, (1970).

[9] R. Bhatia, *Matrix Analysis*, Springer-Verlag, Berlin,(1997).

[10] W. Reid, *Symmetrizable completely continuous transformations in Hilbert Space*, Duke Math. Journ., **18** (1951), 41-56.

Received: November, 2009