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On Regular Prime Graphs of Solvable Groups

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Abstract

Let G be a finite solvable group and $\Delta(G)$ be the prime vertex graph on $\mathrm{c.d.}(G)$. We show that if $\Delta(G)$ is noncomplete and regular, then G is the direct product of groups with disconnected graphs of 2 vertices.

Mathematics Subject Classification: 20C15

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1 Introduction

Let G be a group and $\mathrm{Irr}(G)$ be the set of irreducible complex characters of G. The set of character degrees of the elements of $\mathrm{Irr}(G)$ is denoted by $\mathrm{c.d.}(G)$. Several graphs have been attached to the set $\mathrm{c.d.}(G)$ and whichever one decides to study is a matter or choice. The prime graph of G, $\Delta(G)$, has as its vertices the set, $\rho(G)$, which contains primes that divide some member of $\mathrm{c.d.}(G)$. There is an edge between two members $t, r \in \rho(G)$ is $rt|a$ for some $a \in \mathrm{c.d.}(G)$.

The prime vertex graphs on the character degree set of a finite group have been of interest for quite sometime now. The main objective being identifying which simple graphs can occur as a prime graph of some finite group. H. P. Tong-Viet [7] showed that the complete graph of order 4 is the only 3-regular graphs that occurs as the prime graph of any finite group.

All the simple regular graphs that can occur as prime graphs of solvable groups were later determined by C. P. M. Zuccari in [8]. He proved $\Delta(G)$ is the prime graph of G which is regular of order n, then $\Delta(G)$ is $(n-2)$-regular graphs or complete graph of degree $n-1$. He further showed that, if $F(G)$ is abelian in this case, then G is the direct product of groups having disconnected graph with two vertices.

In [6], M. Lewis and Q. Meng showed that if $\Delta(G)$ is a square, then G is the direct product of two groups having disconnected graphs.

Theorem A. *Let G be a solvable group such that $\Delta(G)$ is noncomplete and regular. Then G is the direct product of groups with disconnected prime graphs of 2 vertices.*

Notations

Let G be a finite group. $\delta(q)$ is the set of primes in $\rho(G)$ that are not adjacent to q and are different from q. $\sigma(G)$ is the set of primes p such that G has a normal nonabelian Sylow p-subgroup. Let $p \in \rho(G)$, $\mathrm{d}_G(p)$ is the degree of vertex p in $\Delta(G)$ and $\mathrm{d}_F(p)$ is the degree of vertex p in $\Delta_F(G)$. For some integer n, $\pi(n)$ denotes the set of prime divisors of n. Let $N \leq G$. $\pi(G/N)$ denotes $\pi(|G:N|)$.

Define the upper Fitting series $1 = F_0 \leq F_1 \leq \cdots \leq F_h = G$ where $F_1 = F(G)$ the Fitting subgroup of G and

$$F_i/F_{i-1} = F\left(G/F_{i-1}\right).$$

The smallest positive integer h such that $F_h = G$ is called the Fitting height of G.

2 Preliminary Results

The following results will be used in the proof of the Main Theorem. They are stated without proofs.

Lemma 2.1. [2, Theorem 6] *Suppose $N \trianglelefteq G$. If N is a Hall subgroup of G, then each invariant irreducible character of N extends to a character in G.*

Lemma 2.2. [8, Lemma 2.4] *Let G be a finite group with $F = F(G)$. If $q \in \rho(G)$, then $|\delta(q) \cap \sigma(G)| \leq 1$.*

Lemma 2.3. [8, Lemma 2.5] *Let G be a finite solvable group and $F = F(G)$. Assume that all vertices of $\Delta(G)$ are non-complete. Then one of the following holds:*

(i) The induced subgraph generated by $\pi(G/F)$, $\Delta_F(G)$, has no complete vertices.

(ii) there exists at least one vertex $q \in \pi(G/F)$ such that $d_G(q) = |\rho(G)| - 2$.

Lemma 2.4. [5, Theorem 3.2] *Let G be a solvable group of Fitting height 2 with Fitting subgroup F . Let p be a prime in $\rho(F)$ such that p is not adjacent in $\Delta(G)$ to some prime in $\rho(G) \setminus \{p\}$. Let ρ be the primes in $\rho(G) \setminus \{p\}$ that are not adjacent to p. Then G has a normal nonabelian Sylow p-subgroup P, and if H is a Hall ρ-subgroup of G, then PH is normal in G. Furthermore, the graph $\Delta(PH)$ has two connected components $\{p\}$ and ρ.*

3 Main Result

Proof of Theorem A. Suppose that $|\rho(G)| = n$. Suppose that $\Delta(G)$ is not complete. By [8, Theorem A], $\Delta(G)$ must be $(n-2)$-regular. Since $\Delta(F(G))$ is a complete subgraph of $\Delta(G)$ with vertices $\sigma(G)$. It follows that $\sigma(G)$ spans a complete subgraph of $\Delta(G)$.

G satisfies the hypothesis of Lemma 2.3. Suppose that (i) holds. Since $\mathrm{d}_G(p) = n-2$ for each p in $\rho(G)$, it follows that for each $r \in \pi(G/F)$ that $|\delta(r)| = 1$ in $\Delta_F(G)$ as well as in $\Delta(G)$. Implying that r must be adjacent to all primes in $\sigma(G)$. Since $\sigma(G)$ spans a complete subgraph, then all primes in $\sigma(G)$ will be complete. A contradiction. Hence (ii) must hold.

Let $\sigma = \sigma(G)$. Let H be the Hall σ-subgroup of G. Let $\theta = \theta_1 \times \theta_2 \times \cdots \times \theta_{|\sigma|} \in \mathrm{Irr}(H)$ with θ_i nonlinear for each i and $\theta_i \in \mathrm{Irr}(P_i)$ where each P_i is a normal nonabelian Sylow p_i-subgroup of G.

Let $T = I_G(\theta)$. Assume that θ is G-invariant. By Lemma 2.1, θ is extendible to G. By [3, Corollary 6.17], $a\theta(1) \in \mathrm{c.d.}(G)$ for each $a \in \mathrm{c.d.}(G/H)$. This means that $\Delta(G)$ will have a complete vertex, a contradiction. Hence all degrees in $\mathrm{c.d.}(G/N)$ are 1 which implies that G/N is an abelian group. Since $G/N \cong G/F \times F/N$, we have that G/F is abelian. In particular, G has Fitting height 2. Notice that if s, r are two primes in $\sigma(G)$, then $\delta(r) \neq \delta(s)$, since otherwise, $\Delta(G)$ will not be regular. This would imply that $|\sigma(G)| = |\pi(G/F)|$. Thus for each $p_i \in \sigma(G)$, there is a unique $q_i \in \pi(G/F)$ such that $\delta(p_i) = \{q_i\}$. By Lemma 2.4, if Q_i is the Sylow q_i-subgroup of G, then $P_iQ_i \trianglelefteq G$ and $\Delta(P_iQ_i)$ is has two connected components $\{p_i\}$ and $\{q_i\}$. This is true for each $1 \leq i \leq |\sigma(G)|$. Let $M_i = P_iQ_i$. Since $M_i \cap M_j = \varnothing$ and each $M_i \trianglelefteq G$. It is easy to see that $G = \prod_{i=1}^{|\sigma(G)|} M_i$.

Suppose that θ is not G-invariant. Then $T < G$. Let $\psi \in \mathrm{Irr}(T|\theta)$ so that $\psi^G = \chi$ for some $\chi \in \mathrm{Irr}(G|\theta)$. Clearly, $\chi(1) = |G : T|\psi(1)$. It is also clear that $\theta(1)|\psi(1)$. Since $(|G : T|, \theta(1)) = 1$, it follows that there is a prime $p \in \pi(G/F)$ that is adjacent to every vertex in σ. If $\Delta_F(G)$ is complete, then p will be complete in $\Delta_F(G)$ and hence complete in $\Delta(G)$. A contradiction. Hence $\Delta_F(G)$ cannot be complete.

By (ii) of Lemma 2.3, $\Delta_F(G)$ has at least one vertex of degree $n-2$. Let $t \in \pi(G/F)$ and suppose that $q \in \rho(G)$ that is not adjacent to t. Suppose that t is a complete vertex in $\Delta_F(G)$, then it must be connected to all but one prime in $\sigma(G)$. To much up for regularity of vertices in $\sigma(G)$, there must be as many complete primes in $\pi(G/F)$ as the primes in $\sigma(G)$. This clearly implies that there is no number of noncomplete primes in $\pi(G/F)$ can result into $\Delta(G)$ being regular. In other words, no primes in $\pi(G/F)$ is noncomplete. Hence $\Delta_F(G)$ is complete, a contradiction.

□

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