International Mathematical Forum, Vol. 11, 2016, no. 14, 659 - 665
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/imf.2016.51077

On Characterization of Scalar Operators via Semigroup and the Associated Generators

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Abstract

In this paper, we characterize the scalar operators by using the semigroup theory and the corresponding generators of $(\alpha, \alpha+1)$ type $\mathbb{R}$ operators. In particular, we show that if a densely defined operator H generates a contraction semigroup, then both H and H^* are scalar operators and if H admits a $\mathcal{U}$(Algebra of smooth functions) functional calculus of scalar type, then H^* also admits a $\mathcal{U}$ functional calculus of scalar type.

Keywords: Semigroup, $(\alpha, \alpha+1)$ type $\mathbb{R}$ operators

1 Introduction

A fundamental problem in spectral theory involves finding a criteria for an operator to be of scalar type. Many approaches have been used so far, for

example Kantorovitz [3], established this using the boundedness of operators with real spectrum acting on a reflexive Banach space. Let X be a Banach space and X^* denote its dual. By a linear operator A on X, with D(A)(Domain of A), we mean a function $A : D(A) \to X$ that is $\mathbb{C}$ linear. We say A is densely defined if $D(A)$ is dense subset of X with respect to norm topology. The algebra of bounded linear operators will be denoted by $L(X)$.
We suppose that H is a closed densely defined operator on a Banach space X with $\sigma(H) \subseteq \mathbb{R}$. We suppose also that the resolvent $\| (z-H)^{-1} \|$ is defined and bounded for all $z \notin \mathbb{R}$ and satisfy the hypothesis below

$$\| (z-H)^{-1} \| \leq c \mid Iz \mid^{-1} \left(\frac{\langle z \rangle}{\mid Iz \mid} \right)^{\alpha} \tag{1}$$

for some $\alpha \geq 0$ and $c > 0$ then H is of $(\alpha, \alpha+1)$ type $\mathbb{R}$.
Here $\langle z \rangle := (1+ \mid z \mid^2)^{\frac{1}{2}}$ and Iz is the imaginary part of z. The hypothesis above appears in [1] which we can state is important in application of functional Calculus of $(\alpha, \alpha+1)$ type $\mathbb{R}$ [2]. Note that if T is a linear operator in X with real spectrum, then $R_T(t) = (I - itT)^{-1}$, $t \in \mathbb{R}$ is well defined $L(X)$-valued map [5].
Consider a linear operator A on a Banach space X that generates a uniformly bounded holomorphic semigroup $\{e^{-\lambda A}\}_{Re(\lambda) \geq 0}$. This implies that in an equivalent norm; A, iA and $(-iA)$ generates a one parameter contraction group which is true if and only if A is closed and densely defined and its spectrum contained in $[0, \infty)$ [4]. The richest functional calculus occurs in the well known setting of self adjoint operators on a Hilbert space, which admits a functional calculus for any Borel measurable function on the spectrum of the operator via spectral theorem. Note that a spectral operator of scalar type on an arbitrary Banach space admits this functional calculus. We therefore wish to characterize the scalar operators by using the semi-group theory and the corresponding generators of $(\alpha, \alpha+1)$ type $\mathbb{R}$ operators. We shall also show that if H satisfy (1) and admits $\mathcal{U}$ functional calculus defined in [2], and it is of scalar type, then H^* also admits the same functional calculus and it is of scalar type for every $f \in C_c(\mathbb{R}) \subseteq \mathcal{U}$(Algebra of smooth functions).

2 Definitions and theorems

Definition 2.1 Let $A \in L(X)$, then there exist a constant $C \geq 1$ and $\gamma \geq 0$ such that

$$\| e^{tA} \| \leq Ce^{t\gamma} \tag{2}$$

for all $t \geq 0$.

Theorem 2.2 *Let H be a bounded operator with $\sigma(H) \subseteq \mathbb{R}$ and $T_t = e^{iHt}$ such that*

$$\| T_t \| \leq C(1+ | t |)^{\alpha} \tag{3}$$

where α is non-negative integer. Then H is of $(\alpha, \alpha+1)$-type $\mathbb{R}$
Proof: See[6]

The following are the immediate corollaries arising from the above theorem.
Corollary 2.3 *If $\alpha \geq 0$ is the minimal constant such that inequality (3) holds, then $\{T_t\}$ is α contractive*
Corollary 2.4 *If $\{T_t\}$ is 0 contractive and $C = 1$, then H is of $(0, 1)$-type $\mathbb{R}$ and (3) reduces to a contraction semi-group and $(0, \infty) \subseteq \rho(H)$*
Theorem 2.5 (Hille Yosida) *A closed densely defined linear operator A on a Banach space X is the infinitesimal generator of a semigroup $\{T_t\}$ if and only if there exist a constant C and γ such that for every $\lambda > \gamma$, $(\lambda I - A)$ is invertible with*

$$\| (\lambda I - A)^{-m} \| \leq C(\lambda - \gamma)^{-m} \tag{4}$$

$\forall m \in \mathbb{N}$
Corollary 2.6 *If $m = 1$ (4) reduces to*

$$\| (\lambda I - A)^{-1} \| \leq C(\lambda - \gamma)^{-1} \tag{5}$$

Theorem 2.7 (Stones theorem) *Every one parameter group of unitary transformation is of the form e^{iHt} with H self adjoint.*
Theorem 2.8 *A densely defined linear operator H acting on a reflexive Banach space X is scalar if it is of $(0, 1)$ type $\mathbb{R}$ and $\| f(H) \| \leq \| f \|_{\infty}$ for each $f \in \mathcal{U}$.*
Proof: See[1]
Theorem 2.9 (Green's theorem) *If Γ is a smooth positive Jordan system with $G := ins\Gamma, f \in C(\overline{G}) \cap C^1(G)$ and $\frac{\partial f}{\partial \overline{z}}$ is integrable over G then*

$$\int_{\Gamma} f(z)dz = 2i \int_{G} \frac{\partial f}{\partial \overline{z}} dxdy \tag{6}$$

Corollary 2.10 *Let H be an operator on X with $G \subset \rho(H)$ and $g(z) := f(z)(z - H)^{-1}$ is such that $g \in C(\overline{G}) \cap C^1(G)$ and $\frac{\partial g}{\partial \overline{z}}$ is integrable over G then*

$$\int_{G} \frac{\partial}{\partial \overline{z}} f(z)(z - H)^{-1} dxdy = \frac{1}{2i} \int_{\Gamma} f(z)(z - H)^{-1} dz \tag{7}$$

Proof: See[2]
Theorem 2.11 Let H be an operator of $(\alpha, \alpha+1)$ type $\mathbb{R}$ for some $\alpha \geq 0$. If $w \in \mathbb{C} \setminus \mathbb{R}$ and $r_w(x) := (w-x)^{-1}$ for all $x \in \mathbb{R}$ then $r_w \in \mathcal{U}$ and $r_w(H) = (w-H)^{-1}$.
Proof: See[2]

3 Main Results

Theorem 2.12: *A closed densely defined operator $H \in L(X)$ is a scalar operator if the following two conditions are satisfied.*
(i) If H generates a contraction semigroup then H is self adjoint operator
(ii) If If H is self adjoint then its contraction semigroup generates a scalar operator.

Proof: Let H be a closed densely defined operator on $L(X)$ satisfying (4), and $\sigma(H) \subseteq \mathbb{R}$. Then H generates a strongly continuous semi group given by (3). It follows from (4) that $\| (\lambda I - iH)^{-m} \| \leq C(\lambda-\gamma)^{-m}$ for all $m \in \mathbb{N}$ Since $\{T_t = e^{iHt}\}$ is 0 contractive, it follows that $\| (\lambda I - iH)^{-1} \| \leq \lambda^{-1}$. This implies that iH is the generator of the contraction semigroup. It follows from Stones theorem that H is self adjoint.
Now for $\lambda > 0$ and $x \in X$ one has;

$$(\lambda I - iH)^{-1}x = \int_0^\infty e^{-\lambda t} T_t x dt \tag{8}$$

Using standard properties of Laplace transform and semigroup property we've;
$\| R_\lambda^m x \| = \frac{1}{(m-1)!} \int_0^\infty t^{m-1} e^{-\lambda t} \| T_t x \| dt$ where $m = 1, 2, 3...$

$$= \frac{1}{(m-1)!} \int_0^\infty t^{m-1} e^{-\lambda t} \| e^{iHt} x \| dt$$

$$\leq \| x \| \frac{C}{(m-1)!} \int_0^\infty t^{m-1} e^{-\lambda t} (1+ \mid t \mid)^\alpha dt$$

$$\leq \frac{C}{(m-1)!} \int_0^\infty t^{m-1} e^{(\alpha-\lambda)t} (1+ \mid t \mid)^\alpha dt \; with \; \| x \| = 1 \; By \, Partial \, integration$$

Applying corollary (2.4) and taking $m = 1$, we have;

$$\| R_\lambda \| \leq \int_0^\infty e^{-\lambda t} dt \leq \frac{1}{\lambda}$$

This implies that iH generates a contraction semi group. Since each T_t is self adjoint, it implies that the resolvent set $(\lambda I - iH)^{-1}$ is also self adjoint and so $(\lambda I - iH)$ is also self adjoint. If $(\lambda I - iH)$ is self adjoint then H is also self adjoint and so it is of $(0,1)$-type $\mathbb{R}$ and hence by theorem (2.8) it is a scalar operator. Now since e^{iHt} is a scalar operator with bounded resolution of the identity, then it follows that the generator iH of T_t is also a scalar operator with the same bounded resolution of the identity E such that

$$iH = \int_C (i\lambda)E(d\lambda)$$

and $T_t = \int_C e^{it\lambda}E(d\lambda)$ for all $t \in \mathbb{R}$ It follows from spectral theorem that a unique projection valued measure $E(.)$ from the Borel σ-field on $\mathbb{R}$ exist such that

$$f(H) = \int_{-\infty}^{\infty} f(\lambda)dE(\lambda)$$

and this completes our proof.
We next state our second result via the following theorem.

Theorem 2.13:
If $\mathcal{H}$ is A Hilbert space and H is a bounded scalar type operator on $\mathcal{H}$, then H^* is also a scalar type operator and it admits $\mathcal{U}$ functional calculus.
Proof. Let H be a scalar type operator and $\mathcal{U}$ denote algebra of smooth functions. Also let $C_c(\mathbb{R})$ be the subalgebra of $\mathcal{U}$ generated by $\{g_\mu : g_\mu(\lambda) = (\mu-\lambda)^{-1}, \mu \notin \mathbb{R}\}$. It follows from theorem (2.11) that $g_\mu(\lambda) \in \mathcal{U}$ and $g_\mu(H) = (\mu - H)^{-1}$.
For $f \in C_c(\mathbb{R})$, and for all $z \notin \mathbb{R}$, Helffer and Sjostrand integral functional calculus [8] yields;

$$f(H) := \int_G \frac{\partial}{\partial \overline{z}}\tilde{f}(z)(z-H)^{-1}dxdy = \frac{1}{2i}\int_\Gamma \tilde{f}(z)(z-H)^{-1}dz.$$

It follows from theorem (2.8) that $\| f(H) \|$ is bounded for each $f \in C_c(\mathbb{R})$. Since $C_c(\mathbb{R})$ is dense in $\mathcal{U}$, hence the homomorphism $f \to f(H)$ extends to a continuous homomorphism $h : C_c(\mathbb{R}) \to L(X)$.
It follows that $h(g_\mu) = (\mu - H)^{-1}$ for $\mu \notin \mathbb{R}$ and Its dual

$$h(g_\mu)^* = [(\mu - H)^{-1}]^* = [(\mu - H^*)]^{-1} \tag{9}$$

Now;

$$f(H) := \int_G \frac{\partial}{\partial \overline{z}}\tilde{f}(z)(z-H)^{-1}dxdy$$

$$= \frac{1}{2i}\int_{\Gamma} f(z)[(z-H)^{-1}]dz$$

$$= \frac{1}{2i}\int_{\Gamma} f(z)[(z-H)^{-1}]^{*}dz = \frac{1}{2i}\int_{\Gamma} f(z)[(z-H)^{*}]^{-1}dz$$

and so

$$(f(H))^{*} = \frac{1}{2i}\int_{\Gamma} f(z)[(z-H)^{*}]^{-1}dz \quad \forall f \in \mathcal{U}$$

Since $D(H)$ is densed. By [7], there exist a spectral measure G of class X^* defined on the Borel sets with values in $L(X^*)$ such that

$$(f(H))^{*} = \int f(\lambda)G(d\lambda) \quad \forall f \in \mathcal{U}$$

Hence H^* is a scalar operator and this completes our proof.

Acknowledgements. The first author would like to thank University of Kabianga (Kenya) for assistantship during the period of research.

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Received: October 29, 2015; Published: July 19, 2016