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Zero Inflated Poisson Mixture Distributions

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Abstract

This paper presents a method of constructing the Zero inflated Poisson continuous mixture distributions which have applications in various fields. The distributions can be formed by either direct integration or integration of moments of the underlying distributions. The two methods are presented in sections one and two. In section four, we present the mixed distributions. We further proved the identities that resulted when the resultant mixed distributions were equated.

Keywords: ZIP, continuous mixed distributions, moments and explicit

1 Introduction

In applications involving count data, it is common to encounter the frequency of observed zeros significantly higher than predicted by the model based on the standard parametric family of discrete distributions. In such situations, zero-inflated Poisson and zero-inflated negative binomial distribution have been widely used in modeling the data, yet other models may be more appropriate in handling the data with excess zeros, as shown by [2] The consequences of this is misspecifying the statistical model leading to erroneous conclusions and bringing uncertainty into research and practice. Therefore the problem is to identify by constructing other alternatives, to the models already present in the literature that may be more appropriate for modeling data with excess zeros. In their paper, [1], presented a method on how to construct the mixed poisson distributions explicitly. The paper gives an alternative method where the distributions can be constructed using the r^{th} moment expectation of the underlying distribution. The identities that result from the two methods are also proved. A mixed distribution is constructed when two probability distributions are mixed. Consider a probability distribution whose parameter is varying and also has a distribution. Then the integral or summation of these two distributions forms a mixed probability distribution. In [3], constructed a Negative Binomial distribution by mixing a Poisson distribution with its parameter. In this work, the ZIP (Zero- Inflated Poisson) distribution was mixed with various continuous distributions to construct the ZIP mixture distributions.

2 Construction of mixed ZIP distribution using moments of the mixing distribution

2.1 General case

Suppose $g(\lambda)$ is the mixing distribution, then ZIP mixed distribution, which can be written in terms of the r^{th} moment of the mixing distribution as

$$= \begin{cases} \rho + (1-\rho)\sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \int_0^{\infty} \lambda^j g(\lambda)d\lambda, & \text{k=0;} \\ (1-\rho)\sum_{j=0}^{\infty} \frac{(-1^j}{j!k!} \int_0^{\infty} \lambda^{j+k} g(\lambda)d\lambda, & \text{k=1}, 2, \ldots. \end{cases} \tag{1}$$

Let$j = x$, then $f(x)$ can be written as

$$Pr(Y = x) = \begin{cases} \rho + (1-\rho)\sum_{x=0}^{\infty} \frac{(-1)^x}{x!} E(\Lambda^x), & \text{k=0;} \\ (1-\rho)\sum_{x=0}^{\infty} \frac{(-1)^x}{x!k!} E(\Lambda^{x+k}), & \text{k=1}, 2, \ldots. \end{cases} \tag{2}$$

where $E(\Lambda^r)$ is the r^{th} moment of the mixing distribution.

3 Construction of the ZIP distributions by direct integration

A random variable Y follows a zero-inflated Mixed Poisson distribution with mixing distribution having probability density function g if its probability function is given by;

$$Prob(Y = k) \quad = \quad \begin{cases} \int_0^{\infty} \left[\rho + (1-\rho)e^{-\lambda}\right] g(\lambda)d\lambda, & \text{k=0;} \\ \int_0^{\infty} \left[(1-\rho)\frac{e^{-\lambda}\lambda^k}{k!}\right] g(\lambda)d\lambda, & \text{k=1}, 2, \ldots. \end{cases} \tag{3}$$

If we equate equation 4 and equation 5, we get identities. This is so because the resultant distributions should be identical.

4 Special cases

4.1 Exponential distribution

For an exponential distribution, the r^{th} moment is

$$\begin{aligned} E(\Lambda^r) &= \int_0^\infty \lambda^r \mu\, e^{-\mu\,\lambda} d\lambda \\ &= \mu \int_0^\infty \lambda^r e^{-\mu\,\lambda} d\lambda \end{aligned}$$

Let

$$y = \mu\lambda, \Longrightarrow \lambda = \frac{\mu}{\lambda}, \text{ and} d\lambda = \frac{dy}{\mu}$$

$$\begin{aligned} E(\Lambda^r) &= \mu \int_0^\infty \left(\frac{y}{\mu}\right)^r e^{-y} \frac{dy}{\mu} \\ &= \frac{r!}{\mu^r} \end{aligned}$$

Thus from equation 2, we have

$$Pr(Y = x) = \begin{cases} \rho + (1-\rho)\sum_{x=0}^{\infty} \frac{(-1)^x}{x!}\frac{x!}{\mu^x}, & \text{k=0;} \\ (1-\rho)\sum_{x=0}^{\infty} \frac{(-1)^x}{x!k!}\frac{(x+k)!}{\mu^{x+k}}, & \text{k=1, 2, \dots.} \end{cases} \tag{4}$$

From equation 3 we construct the mixed ZIP distribution as follows

$$Pr(Y = k) = \begin{cases} \rho + (1-\rho)\frac{\mu}{(1+\mu)}, & \text{k=0;} \\ (1-\rho)\left(\frac{\mu}{1+\mu}\right)\left(\frac{1}{1+\mu}\right)^k, & \text{k=1, 2, \dots.} \end{cases} \tag{5}$$

From equation 2 and equation 3, we have the following identities

1. $$\rho + (1-\rho)\sum_{x=0}^{\infty} \frac{(-1)^x}{x!}\frac{x!}{\mu^x} = \rho + (1-\rho)\left(\frac{\mu}{1+\mu}\right)$$

2. $$(1-\rho)\sum_{x=0}^{\infty} \frac{(-1)^x}{x!k!}\frac{(x+k)!}{\mu^{x+k}} = (1-\rho)\left(\frac{\mu}{1+\mu}\right)\left(\frac{1}{1+\mu}\right)^k$$

<u>Proofs</u>

From the first identity,

$$\begin{aligned}\sum_{x=0}^{\infty}\frac{(-1)^x}{x!}\frac{x!}{\mu^x} &= \sum_{x=0}^{\infty}\begin{pmatrix} -1 \\ x \end{pmatrix}\left(\frac{1}{\mu}\right)^x \\ &= \left(1+\frac{1}{\mu}\right)^{-1} \\ &= \frac{\mu}{1+\mu}\end{aligned}$$

The second identity

$$\begin{aligned}\frac{\Gamma(k+1)}{k!\mu^k}\sum_{x=0}^{\infty}(-1)^x\frac{(x+k)!}{x!\Gamma(k+1)\mu^x} &= \frac{\Gamma(k+1)}{k!\mu^k}\sum_{x=0}^{\infty}(-1)^x\begin{pmatrix} (x+k+1-1) \\ x \end{pmatrix}\frac{1}{\mu^x} \\ &= \frac{\Gamma(k+1)}{k!\mu^k}\sum_{x=0}^{\infty}\begin{pmatrix} -(k+1) \\ x \end{pmatrix}\left(\frac{1}{\mu}\right)^{k+1} \\ &= \frac{\Gamma(k+1)}{k!\mu^k}\left(1+\frac{1}{\mu}\right)^{-(k+1)} \\ &= \frac{k!}{k!\mu^k}\left(\frac{\mu}{1+\mu}\right)^{k+1} \\ &= \left(\frac{\mu}{1+\mu}\right)\left(\frac{1}{1+\mu}\right)^k\end{aligned}$$

4.2 Gamma distribution

From equation 5,

$$Pr(Y=k)=\begin{cases} \rho+(1-\rho)(\frac{\beta}{1+\beta})^{\alpha}, & \text{k=0;} \\ (1-\rho)\begin{pmatrix} \alpha+k-1 \\ k \end{pmatrix}\left(\frac{\beta}{1+\beta}\right)^{\alpha}\left(\frac{1}{1+\beta}\right)^{k}, & \text{k=}1,2,\ldots. \end{cases} \tag{6}$$

The mixed distribution is a ZINB distribution with parameters α, ρ and $\frac{\beta}{1+\beta}$. From equation 4, The mixed distribution becomes

$$Pr(Y=k) = \begin{cases} \rho+(1-\rho)\sum_{x=0}^{\infty}\frac{(-1)^x}{x!}\frac{\beta^{\alpha}}{\Gamma\alpha}\frac{\Gamma(\alpha+x)}{\beta^{\alpha+x}}, & \text{k=0;} \\ (1-\rho)\sum_{x=0}^{\infty}\frac{(-1)^x}{x!k!}\frac{\beta^{\alpha}}{\Gamma\alpha}\frac{\Gamma(\alpha+x+k)}{\beta^{\alpha+x+k}}, & \text{k=}1,2,\ldots. \end{cases} \tag{7}$$

Identities

(a)

$$\rho+(1-\rho)\sum_{x=0}^{\infty}\frac{(-1)^x}{x!}\frac{\beta^{\alpha}}{\Gamma\alpha}\frac{\Gamma(\alpha+x)}{\beta^{\alpha+x}}=\rho+(1-\rho)\left(\frac{\beta}{1+\beta}\right)^{\alpha}$$

(b)

$$(1-\rho)\sum_{x=0}^{\infty}\frac{(-1)^x}{x!k!}\frac{\beta^\alpha}{\Gamma\alpha}\frac{\Gamma(\alpha+x+k)}{\beta^{\alpha+x+k}} = (1-\rho)\left(\begin{array}{c}\alpha+k-1\\ k\end{array}\right)\frac{\beta^\alpha}{(1+\beta)^{\alpha+k}}$$

Proof;

$$\begin{aligned}
\sum_{x=0}^{\infty}\frac{(-1)^x}{x!}\frac{1}{\Gamma\alpha}\frac{\Gamma(\alpha+x)}{\beta^x} &= \sum_{x=0}^{\infty}(-1)^x\left(\begin{array}{c}\alpha+x-1\\ x\end{array}\right)\frac{1}{\beta^x}\\
&= \sum_{x=0}^{\infty}\left(\begin{array}{c}-\alpha\\ x\end{array}\right)\left(\frac{1}{\beta}\right)^\alpha\\
&= \left(1+\frac{1}{\beta}\right)^{-\alpha}\\
&= \left(\frac{\beta}{1+\beta}\right)^\alpha
\end{aligned}$$

For the second identity

$$\begin{aligned}
\sum_{x=0}^{\infty}(-1)^x\frac{\Gamma(\alpha+x+k)}{x!k!\Gamma\alpha\beta^{x+k}} &= \frac{\Gamma(\alpha+k)}{\beta^k k!\Gamma\alpha}\sum_{x=0}^{\infty}(-1)^x\frac{\Gamma(\alpha+x+k)}{x!\Gamma(\alpha+k)}\frac{1}{\beta^{x+k}}\\
&= \frac{\Gamma(\alpha+k)}{k!\Gamma\alpha}\frac{1}{\beta^k}\sum_{x=0}^{\infty}(-1)^x\left(\begin{array}{c}\alpha+x+k-1\\ x\end{array}\right)\left(\frac{1}{\beta}\right)^x\\
&= \frac{\Gamma(\alpha+k)}{k!\Gamma\alpha}\frac{1}{\beta^k}\sum_{x=0}^{\infty}\left(\begin{array}{c}-(\alpha+k)\\ x\end{array}\right)\left(\frac{1}{\beta}\right)^{\alpha+k}\\
&= \frac{\Gamma(\alpha+k)}{k!\Gamma\alpha}\frac{1}{\beta^k}\left(1+\frac{1}{\beta}\right)^{-(\alpha+k)}\\
&= \left(\begin{array}{c}\alpha+k-1\\ k\end{array}\right)\frac{\beta^\alpha\beta^k}{(1+\beta)^{\alpha+k}}\frac{1}{\beta^k}\\
&= \left(\begin{array}{c}\alpha+k-1\\ k\end{array}\right)\left(\frac{\beta}{1+\beta}\right)^\alpha\left(\frac{1}{1+\beta}\right)^k
\end{aligned}$$

4.3 Generalized Lindley distribution

The mixed distribution is

$$Pr(Y=k) = \begin{cases} \rho+(1-\rho)\frac{\theta^{\alpha+1}}{(\theta+1)^\alpha}\left[\frac{1}{(1+\theta)^\alpha}+\frac{1}{(1+\theta)^{\alpha+1}}\right] & \text{k=0;}\\ \frac{(1-\rho)}{k!}\frac{\theta^{\alpha+1}}{(\theta+1)^{\alpha+k+1}}\frac{\Gamma(\alpha+k)}{\Gamma(\alpha+1)}\left[\alpha+\frac{\alpha+k}{1+\theta}\right] & \text{k=}1,2,\ldots. \end{cases} \quad (8)$$

which is a Zero-Inflated Generalized Poisson Lindley distribution with two parameters. Using the method of moments,

$$Pr(Y=k)=\begin{cases}\rho+(1-\rho)\sum_{x=0}^{\infty}\frac{(-1)^x}{x!}\frac{\theta^{1+\alpha}}{(1+\theta)\Gamma(\alpha+1)}\left[\frac{\alpha\Gamma(\alpha+x)}{\theta^{\alpha+x}}+\frac{\Gamma(\alpha+x+1)}{\theta^{\alpha+x+1}}\right], & \text{k=0;}\\ (1-\rho)\sum_{x=0}^{\infty}\frac{(-1)^x}{x!k!}\frac{\theta^{1+\alpha}}{(1+\theta)\Gamma(\alpha+1)}\left[\frac{\alpha\Gamma(\alpha+x+k)}{\theta^{\alpha+x+k}}+\frac{\Gamma(\alpha+x+k+1)}{\theta^{\alpha+x+k+1}}\right], & \text{k=1,}2,\dots.\end{cases} \quad (9)$$

Identities

(a)

$$\rho+(1-\rho)\sum_{x=0}^{\infty}\frac{(-1)^x}{x!}\frac{\theta^{1+\alpha}}{(1+\theta)\Gamma(\alpha+1)}\left[\frac{\alpha\Gamma\alpha+x}{\theta^{\alpha+x}}+\frac{\Gamma\alpha+x+1}{\theta^{\alpha+x+1}}\right]$$

$$=\rho+(1-\rho)\frac{\theta^{\alpha+1}}{(1+\theta)}\left[\frac{1}{(1+\theta)^{\alpha}}+\frac{1}{(1+\theta)^{\alpha+1}}\right] \quad (10)$$

(b)

$$(1-\rho)\sum_{x=0}^{\infty}\frac{(-1)^x}{x!k!}\frac{\theta^{1+\alpha}}{(1+\theta)\Gamma(\alpha+1)}\left[\frac{\alpha\Gamma(\alpha+x+k)}{\theta^{\alpha+x+k}}+\frac{\Gamma(\alpha+x+k+1)}{\theta^{\alpha+x+k+1}}\right]$$

$$=\frac{(1-\rho)}{k!}\frac{\theta^{\alpha+1}}{(1+\theta)\Gamma(\alpha+1)}\left[\frac{\alpha\Gamma(\alpha+k)}{(1+\theta)^{\alpha+k}}+\frac{\Gamma(\alpha+k+1)}{(1+\theta)^{\alpha+k+1}}\right] \quad (11)$$

<u>Proof</u>;

$$\frac{\alpha}{\theta^{\alpha}}\sum_{x=0}^{\infty}(-1)^x\frac{\Gamma(\alpha+x)}{x!\Gamma(\alpha+1)}\frac{1}{\theta^x}+\frac{1}{\theta^{\alpha+1}}\sum_{x=0}^{\infty}(-1)^x\frac{\Gamma(\alpha+x)}{x!\Gamma(\alpha+1)}\frac{1}{\theta^x}$$

$$=\frac{\alpha}{\theta^{\alpha}\alpha}\sum_{x=0}^{\infty}(-1)^x\frac{\Gamma(\alpha+x)}{x!\Gamma\alpha}\frac{1}{\theta^x}+\frac{1}{\theta^{\alpha+1}}\sum_{x=0}^{\infty}(-1)^x\frac{\Gamma(\alpha+x)}{x!\Gamma(\alpha+1)}\frac{1}{\theta^x} \quad (12)$$

$$=\frac{1}{\theta^{\alpha}}\sum_{x=0}^{\infty}(-1)^x\binom{x+\alpha-1}{x}\left(\frac{1}{\theta}\right)^x+\frac{1}{\theta^{\alpha+1}}\sum_{x=0}^{\infty}(-1)^x\binom{x+\alpha+1-1}{x}\left(\frac{1}{\theta}\right)^x \quad (13)$$

$$=\frac{1}{\theta^{\alpha}}\sum_{x=0}^{\infty}\binom{-\alpha}{x}\left(\frac{1}{\theta}\right)^{\alpha}+\frac{1}{\theta^{\alpha+1}}\sum_{x=0}^{\infty}\binom{-(\alpha+1)}{x}\left(\frac{1}{\theta}\right)^{\alpha+1} \quad (14)$$

$$=\frac{1}{\theta^{\alpha}}\left(1+\frac{1}{\theta}\right)^{-\alpha}+\frac{1}{\theta^{\alpha+1}}(1+\frac{1}{\theta})^{-(\alpha+1)} \quad (15)$$

$$=\frac{1}{\theta^{\alpha}}\left(\frac{\theta}{1+\theta}\right)^{\alpha}+\frac{1}{\theta^{\alpha+1}}\left(\frac{\theta}{1+\theta}\right)^{\alpha+1} \quad (16)$$

$$=\frac{1}{(1+\theta)^{\alpha}}+\frac{1}{(1+\theta)^{\alpha+1}} \quad (17)$$

Second item

$$\frac{\alpha}{\theta^{k+\alpha}}\sum_{x=0}^{\infty}(-1)^x\frac{\Gamma(x+\alpha+k)}{x!\theta^x}+\frac{1}{\theta^{\alpha+k+1}}\sum_{x=0}^{\infty}(-1)^x\frac{\Gamma(x+k+\alpha+1)}{x!\theta^x} \qquad (19)(18)$$

$$= \frac{\alpha\Gamma(k+\alpha)}{\theta^{k+\alpha}}\sum_{x=0}^{\infty}(-1)^x\frac{\Gamma(x+\alpha+k)}{x!\Gamma(k+\alpha)}\frac{1}{\theta^x}+\frac{\Gamma(k+\alpha+1)}{\theta^{\alpha+k+1}}\sum_{x=0}^{\infty}(-1)^x\frac{\Gamma(x+k+\alpha+1)}{x!\Gamma(k+\alpha+1)}\frac{1}{\theta^x} \qquad (20)$$

$$= \frac{\alpha\Gamma(\alpha+k)}{\theta^{k+\alpha}}\sum_{x=0}^{\infty}(-1)^x\begin{pmatrix} x+k+\alpha-1 \\ x \end{pmatrix}\left(\frac{1}{\theta}\right)^x$$

$$+\frac{\Gamma(k+\alpha+1)}{\theta^{\alpha+k+1}}\sum_{x=0}^{\infty}(-1)^x\begin{pmatrix} x+k+\alpha+1-1 \\ x \end{pmatrix}\left(\frac{1}{\theta}\right)^x \qquad (21)$$

$$= \frac{\alpha\Gamma(\alpha+k)}{\theta^{k+\alpha}}\sum_{x=0}^{\infty}\begin{pmatrix} -(k+\alpha) \\ x \end{pmatrix}\left(\frac{1}{\theta}\right)^{k+\alpha}+\frac{\Gamma(k+\alpha+1)}{\theta^{\alpha+k+1}}\sum_{x=0}^{\infty}\begin{pmatrix} -(k+\alpha+1) \\ x \end{pmatrix}\left(\frac{1}{\theta}\right)^{k+\alpha+1} \qquad (22)$$

$$= \frac{\alpha\Gamma(\alpha+k)}{\theta^{k+\alpha}}\left(1+\frac{1}{\theta}\right)^{-(k+\alpha)}+\frac{\Gamma(k+\alpha+1)}{\theta^{\alpha+k+1}}\left(1+\frac{1}{\theta}\right)^{-(k+\alpha+1)} \qquad (23)$$

$$= \frac{\alpha\Gamma(\alpha+k)}{\theta^{k+\alpha}}\left(\frac{\theta}{1+\theta}\right)^{k+\alpha}+\frac{\Gamma(k+\alpha+1)}{\theta^{\alpha+k+1}}\left(\frac{\theta}{1+\theta}\right)^{k+\alpha+1} \qquad (24)$$

$$= \frac{\alpha\Gamma(\alpha+k)}{(1+\theta)^{k+\alpha}}+\frac{\Gamma(k+\alpha+1)}{(1+\theta)^{\alpha+k+1}} \qquad (25)$$

4.4 Lindley distribution

The mixed distribution by direct integration gives

$$Pr(Y=k)=\begin{cases} \rho+(1-\rho)\frac{\theta^2}{\theta+1}\left[\frac{1}{(1+\theta)^2}+\frac{1}{1+\theta}\right], & \text{k=0;} \\ (1-\rho)\theta^2\left[\frac{2+k+\theta}{(1+\theta)^{k+3}}\right], & \text{k=}1,2,\ldots. \end{cases} \qquad (26)$$

By the method of moments, we have

$$Pr(Y=k)=\begin{cases} \rho+(1-\rho)\sum_{x=0}^{\infty}\frac{(-1)^x}{x!}\frac{\theta^2}{1+\theta}\left[\frac{\Gamma(x+2)}{\theta^{x+2}}+\frac{\Gamma(x+1)}{\theta^{x+1}}\right], & \text{k=0;} \\ (1-\rho)\sum_{x=0}^{\infty}\frac{(-1^x}{x!k!}\frac{\theta^2}{1+\theta}\left[\frac{\Gamma(x+k+2)}{\theta^{x+k+2}}+\frac{\Gamma(x+k+1)}{\theta^{x+k+1}}\right], & \text{k=}1,2,\ldots. \end{cases} \qquad (27)$$

This implies that we have two identities, i.e

1.

$$\rho+(1-\rho)\sum_{x=0}^{\infty}\frac{(-1)^x}{x!}\frac{\theta^2}{1+\theta}\left[\frac{\Gamma(x+2)}{\theta^{x+2}}+\frac{\Gamma(x+1)}{\theta^{x+1}}\right]$$

$$=\rho+(1-\rho)\frac{\theta^2}{1+\theta}\left(\frac{1}{(1+\theta)^2}+\frac{1}{1+\theta}\right) \qquad (28)$$

2.

$$(1-\rho)\sum_{x=0}^{\infty}\frac{(-1)^x}{x!k!}\frac{\theta^2}{1+\theta}\left[\frac{\Gamma(x+k+2)}{\theta^{x+k+2}}+\frac{\Gamma(x+k+1)}{\theta^{x+k+1}}\right]=(1-\rho)\frac{\theta^2}{1+\theta}\left[\frac{2+k+\theta}{(1+\theta)^{k+2}}\right]$$

Proofs;

$$\begin{aligned}\sum_{x=0}^{\infty}\frac{(-1)^x}{x!}\left[\frac{\Gamma(x+2)}{\theta^{x+2}}+\frac{\Gamma(x+1)}{\theta^{x+1}}\right] &= \frac{\Gamma 2}{\theta^2}\sum_{x=0}^{\infty}(-1)^x\frac{\Gamma(x+2)}{x!\Gamma 2\theta^x}+\frac{1}{\theta}\sum_{x=0}^{\infty}(-1)^x\left(\frac{1}{\theta}\right)^x \\ &= \frac{1\Gamma 1}{\theta^2}\left(1+\frac{1}{\theta}\right)^{-2}+\frac{1}{\theta}\left(1+\frac{1}{\theta}\right)^{-1} \\ &= \frac{1}{(1+\theta)^2}+\frac{1}{1+\theta}\end{aligned}$$

For part (ii)

$$\frac{\Gamma(k+2)}{\theta^{k+2}}\sum_{x=0}^{\infty}(-1)^x\frac{\Gamma(x+k+2)}{\Gamma(k+2)x!k!}\frac{1}{\theta^x}+\frac{\Gamma(k+1)}{\theta^{k+1}}\sum_{x=0}^{\infty}(-1)^x\frac{\Gamma(x+k+1)}{\Gamma(k+1)x!}\frac{1}{\theta^x} \tag{29}$$

$$\begin{aligned} &= \frac{\Gamma(k+2)}{k!\theta^{k+2}}\sum_{x=0}^{\infty}(-1)^x\begin{pmatrix} x+k+2-1 \\ x\end{pmatrix}\left(\frac{1}{\theta}\right)^x+\frac{\Gamma(k+1)}{k!\theta^{k+1}}\sum_{x=0}^{\infty}(-1)^x\begin{pmatrix} x+k+1-1 \\ x\end{pmatrix}\left(\frac{1}{\theta}\right)^x \\ &= \frac{\Gamma(k+2)}{k!\theta^{k+2}}\sum_{x=0}^{\infty}\begin{pmatrix} -(k+2) \\ x\end{pmatrix}\left(\frac{1}{\theta}\right)^{k+2}+\frac{\Gamma(k+1)}{k!\theta^{k+1}}\sum_{x=0}^{\infty}\begin{pmatrix} -(k+1) \\ x\end{pmatrix}\left(\frac{1}{\theta}\right)^{k+1} \\ &= \frac{\Gamma(k+2)}{k!\theta^{k+2}}\left(1+\frac{1}{\theta}\right)^{-(k+2)}+\frac{\Gamma(k+1)}{k!\theta^{k+1}}\left(1+\frac{1}{\theta}\right)^{-(k+1)} \\ &= \frac{(k+1)k!}{k!\theta^{k+2}}\left(\frac{\theta}{1+\theta}\right)^{k+2}+\frac{\Gamma(k+1)}{k!\theta^{k+1}}\left(\frac{\theta}{1+\theta}\right)^{k+1} \\ &= \frac{2+k+\theta}{(1+\theta)^{k+2}}\end{aligned}$$

5 Discussion

The continuous mixture distributions can be formed by direct integration. This applies to most distributions since there are no restrictions imposed during integration. In [4], showed that their is a relationship between the moment of an underlying distribution and the distribution function of the mixed distribution. This paper has presented such a relationship by using Exponential, Gamma with two parameters, Lindley and the Generalized Lindley distributions as the underlying distributions being mixed with the ZIP distribution. Whether direct integration is used or the method of expectation of moments, the resulting distributions must be identical but not necessarily the same. We went ahead and proved identities that resulted by equating mixed distributions formed by direct integration and those formed through r^{th} moment expectation of the mixing distributions.

Conclusion

This work concentrated on purely construction. Since we did not exhaust all the available continuous distributions, therefore more work can be done by considering the distributions which we did not use, by using the methods of construction already used and also other methods can be studied or researched on. Mixed Poisson distributions exhibit several interesting properties as given by [1]. These properties include; Identifiability and Shape properties, Infinite divisibility, Posterior Moments, etc. The study of these properties can form a good basis of further research on the mixed ZIP distributions constructed in this work. In this study, we restricted ourselves to continuous mixing distribution. Discrete or countable mixtures where we have discrete prior distributions could be of interest to a researcher , thus, research can be carried out on this.

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