Finite Solvable Groups with 4-Regular Prime Graphs

Donnie Munyao Kasyoki (munyao@aims.ac.za)
African Institute for Mathematical Sciences (AIMS)

Supervised by: Dr. Hung Phi Tong-Viet
University of KwaZulu Natal, South Africa

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Abstract

In this research project, we seek to address the problem of deciding which finite simple graph is a prime graph of a finite group. In particular, we only focus on showing which 4-regular graphs can be prime graphs of some finite solvable group.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Donnie Munyao Kasyoki, 23 May 2013

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1. Introduction

The study of character theory has proved to be an important tool in the study of finite groups. It is widely applied in the classification of simple groups. We can study the structure of finite groups by studying their character degrees. The study of the character degrees of a finite group by attaching graphs to them is considered a nice way to study them.

There are two different graphs that can be attached to the character degree set of a group G. The character degree set of a group G is denoted by c.d.(G). The first graph attached to c.d.(G) is called the character degree graph denoted by $\Gamma(G)$. The vertex set is the set c.d.$(G) \setminus \{1\}$. There is an edge between any two vertices a and b if they are not co-prime. That is gcd$(a, b) \neq 1$. The other graph attached to c.d.(G) and which is the center of our discussion is called the prime graph denoted by $\triangle(G)$ which is the graph defined in 4.1.14.

In most papers, the character degree graph is normally used to refer to the prime graph. Many scholars have studied the character degree graphs and prime graphs in the past 30 or so years. As a result, many results have been obtained in this field. For example, in Huppert (1991), Huppert listed the prime graphs with at most 4 vertices that can arise as $\triangle(G)$ of some solvable groups G. In Lewis (2008), Lewis classified the prime graphs of solvable groups with 5 vertices.

In H.P.Tong-Viet (2013b), Hung P. Tong Viet studied the 3-regular graphs which might occur as prime graphs of some group G. In the same paper, he also conjectured that the only 4-regular graphs that can arise are the complete graph of order 5 and the 4-regular graph of order 6. This forms the main agenda of our discussion. We will review the already obtained results, that is, n-regular graphs that can be prime graphs of some solvable groups for $n \leq 3$.

2. Review of Group Theory

In this essay, G is a finite group unless otherwise stated.

In order to study Character theory of finite groups we need to review some elements of group theory that will be widely used in this essay. Most of the definitions and results in this chapter can be found in most group theory texts. Let's begin by defining a group.

2.1.1 Definition. Let G be a non-empty set. The pair $(G, *)$ is called a group if the following properties hold:

1. $g * h \in G$ for all $g,\ h \in G$
2. $(g * h) * k = g * (h * k)$ for all $g,\ h,\ k \in G$
3. There exist $e \in G$ such that $g * e = e * g$ for all $g \in G$. The element e is called the identity of the group.
4. For each $g \in G$, there exist $g^{-1} \in G$ such that $g * g^{-1} = g^{-1} * g = e$. The element g^{-1} is called the inverse of g in G.

We simply write G for a group instead of the pair $(G, *)$.

If the number of elements in the group G is finite then we say that G is a finite group. Let's list some examples of groups.

2.1.2 Example.

The set of all invertible $n \times n$ matrices over a field F together with matrix multiplication is a group. This group is called the General Linear Group denoted by $GL(n, F)$.

The set of all $n \times n$ matrices whose determinant is 1 together with matrix multiplication is a group. This group is called Special Linear Group denoted by $SL(n, F)$.

The set of all integers is a group under addition. The group is denoted by $\mathbb{Z}$.

A subset H of a group G is called a subgroup if its a group on its own right.

Henceforth, we will omit $*$ and denote $g * h$ by gh.

2.1.3 Definition. Let N be a subgroup of a group G. N is called a normal subgroup if

$$gxg^{-1} \in N, \quad \text{for all } x \in N \text{ and } g \in G.$$

We write $N \lhd G$ to mean N is a normal subgroup of G.

2.1.4 Definition. Let G be a group. G is called an abelian group if

$$hg = gh \quad \text{for all } h,\ g \in G$$

Let H be a subgroup of a group G. The sets Hx and xH for all $x \in G$ are subsets of G called the right coset and the left coset of H in G, respectively. The index $|G : H|$ is the number of left cosets (or right cosets) of H in G.

2.1.5 Definition. Let G be a group and $N \lhd G$. Let

$$G/N = \{Nx | x \in G\}$$

be the set of right cosets of N in G. Define the operation $*$ as follows:

$$Nx * Ny = Nxy \quad \text{for all } x,\, y \in G.$$

The pair $(G/N, *)$ is called the quotient group of G by N.

The order of the quotient group G/N is given by

$$|G/N| = |G : N| = \frac{|G|}{|N|}$$

2.1.6 Definition. Let G be a group. The conjugacy class of an element $g \in G$ is the set of elements conjugate to it. That is

$$K_g = \{xgx^{-1} \,|\, x \in G\}.$$

2.1.7 Lemma. A group G is an abelian group if and only if each element $g \in G$ is its own conjugacy class. In particular, the number of conjugacy classes equals the size of the group if and only if the group is abelian.

Proof. Let G be an abelian group. It follows that $gh = hg$, for all $g,\, h \in G$. By post-multiplying both sides by h^{-1}, we obtain that $ghh^{-1} = hgh^{-1}$. This simply implies that $g = hgh^{-1}$. To prove the reverse we simply reverse the steps in the proof. □

Let G be any group. For any two elements of G, say h and g, an element $[h, g] = h^{-1}g^{-1}hg \in G$ is called the commutator of h and g. The subgroup G' generated by all the commutators of the elements of G is called the commutator subgroup or the derived subgroup of G. That is;

$$G' = \langle [h, g] | h, g \in G \rangle$$

We define $G = G^{(0)}$, $G' = G^{(1)}$ and $G^{(i+1)} = (G^{(i)})'$.

2.1.8 Definition. Let G be a group. If $G^{(d)} = \{1\}$ for some integer $d \geq 0$, then G is said to be solvable. The least such d is called the derived length of G denoted by d.l.(G).

Let's define the Frobenius group. Let H be a nontrivial subgroup of a group G. H is called a Frobenius complement of G if the following property holds:

$$H \cap H^g = 1 \qquad \forall\, g \in G \setminus H$$

where H^g is the conjugate of H under g. That is, $H^g = \{h^g | h \in H\}$ and $h^g = g^{-1}hg$.

2.1.9 Definition. A group G is called a Frobenius group if it contains a Frobenius complement H as a nontrivial subgroup.

The Frobenius kernel of G with respect to H, is $N \lhd G$ defined by

$$N = \left(G - \bigcup_{g \in G} H^g \right) \cup \{1\}$$

2.1.10 Definition. Let G and H be groups. A group homomorphism from G to H is a map $\varphi : G \to H$ which satisfies:

$$\varphi(g)\varphi(h) = \varphi(gh) \quad \text{for all } g,\, h \in G.$$

A group G is called a p-group if the order of every element of G is a power of a prime p. A subgroup P of G is a p-subgroup if it is a p-group.

The following theorem is a basic result found in most group theory text books.

2.1.11 Theorem. *Let G be a group. Then G is a p-group if and only if $|G| = p^n$ for some positive integer n.*

2.1.12 Definition. Let G be a group and P be the maximal p-subgroup of G. Then P is called a Sylow p-subgroup of the group G.

2.1.13 Definition. Let G be a group. Let $N \lhd G$ such that $|N|$ is co-prime to a prime p. If $|G : N|$ is a power of p, then N is called a normal p-complement of G.

2.1.14 Definition. Let G be a finite abelian group. Then G is called elementary abelian if every non-identity element has order p.

2.1.15 Definition. A p-group G is extraspecial if $G' = Z(G) = \Phi(G)$ is cyclic, where $\Phi(G)$ is the Frattini subgroup of G, that is, the intersection of all maximal subgroups of G. $Z(G)$ denotes the center of the group defined as

$$Z(G) = \{g \in G \,|\, gh = hg \quad \text{for all } h \in G\}.$$

2.1.16 Definition. Let G be a group. Then the exponent of G is the smallest positive integer $\exp(G)$ such that $x^{\exp(G)} = 1$ for all $x \in G$.

2.1.17 Definition. Let G be a group. An automorphism of G is a map $\sigma : G \to G$ such that σ satisfies the following conditions:

(a) σ is a bijection and

(b) σ is a homomorphism. That is $\sigma(xy) = \sigma(x)\sigma(y)$ for all $x, y \in G$.

2.1.18 Proposition. Let G be a group and $\text{Aut}(G) = \{\sigma \,|\, \sigma : G \to G \text{ is an isomorphism}\}$. Then $\text{Aut}(G)$ is a group. $\text{Aut}(G)$ is called the automorphism group of G.

Let n be an integer. We will denote the set of all integers that divide n by $\pi(n)$.

3. Character Theory

3.1 Review of Representation Theory

Before considering the prime graphs of finite groups, we first study characters and character degrees. Let's start by reviewing the representation theory of finite groups. Representations can be studied via the use of modules or vector spaces. We won't go far behind to study modules, algebras and vector spaces. We will assume that the theory is well understood and thus we will go ahead and use it.

3.1.1 Definition. Let G be a group and $\mathbb{F}$ be a field. Let V be a finite dimensional vector space over $\mathbb{F}$. Then a representation of G over V is a group homomorphism $\pi : G \to GL(V)$.

A representation $\pi : G \to GL(V) \cong GL(n, \mathbb{C})$ of a group G defined by

$$\pi(g) = I, \ \forall g \in G$$

where I is the identity in $GL(V)$, is called the trivial representation.

The degree of the representation is the dimension of the vector space. That is:

$$\text{deg}(\pi) = \text{dim}V$$

.

A representation π induces a group action of G on V by linear transformations. That is,

$$\pi(g_1 g_2)v = (\pi(g_1)\pi(g_2))\, v = \pi(g_1)\,(\pi(g_2)v)$$

Let W be a subspace of a vector space V and G be a group. Let π be a representation of G on V. W is said to be π-invariant if $\pi(g)(W) \subseteq W$, for all $g \in G$.

3.1.2 Definition. Let $\pi : G \to GL(V)$ be a representation of G over V. A subspace $W \subseteq V$ is called a G-subspace of V if it is π- invariant.

We say that a representation $\pi : G \to GL(V)$ is irreducible if V has no proper G-subspace.

3.2 Characters and Character Degrees

3.2.1 Definition. Let π be a representation of a group G. Then the character χ of G afforded by π is defined by $\chi(g) = \text{tr}\,(\pi(g))$.

The character afforded by the trivial representation of a group G is called **trivial** or principle character, denoted by 1_G.

If ρ and σ are two representations of a group G affording characters ϑ and θ respectively, then the tensor product of the two representations, $\rho \otimes \sigma$, afford the character χ given by

$$\chi(g) = \vartheta(g)\theta(g)$$

In this essay we will only consider the case in which $\mathbb{F} = \mathbb{C}$, the field of complex numbers. We will use the term character to mean complex character. If χ is a character of G afforded by a representation π, then the character degree of χ denoted by $\chi(1)$ is $\text{deg}(\pi)$.

3.2.2 The lifting process. Let G be a group and $N \lhd G$. Let ρ_0 be a representation of G/N. The following properties hold:

$$\rho_0(Ng)\rho_0(Nh) = \rho_0(Ngh) \text{ and}$$
$$\rho_0(N) = I \qquad \text{for } g, h \in G$$

Let ϑ_0 be the character afforded by ρ_0. We can use ρ_0 to define a representation ρ of G by,

$$\rho(g) = \rho_0(Ng) \quad \forall\, g \in G.$$

The character ϑ afforded by ρ is given by

$$\vartheta(g) = \vartheta_0(Ng)$$

This process can also be used to define representations as well as characters of the quotient group G/N given that of G. The process is called the lifting process.

3.2.3 Properties of character degrees. Now let's study some properties of character degrees of groups.

We will use a special property of the trace function of matrices to obtain a special property of characters of a group. If A and B are two matrices,

$$\text{tr}(AB) = \text{tr}(BA) \;\Rightarrow\; \text{tr}(ABA^{-1}) = \text{tr}(B). \tag{3.2.1}$$

Using equation 3.2.1, given that π is a representation of G affording character χ, then

$$\chi(ghg^{-1}) = \text{tr}\big(\pi(ghg^{-1})\big) = \text{tr}(\pi(g)\pi(h)\pi(g)^{-1}) = \text{tr}(\pi(h)) = \chi(h) \text{ for all } g, h \in G. \tag{3.2.2}$$

We notice that χ is conjugation invariant. Functions satisfying that condition are called class functions.

A character of a group G afforded by π is irreducible if π is. The set of all irreducible characters of a group G is denoted by $\text{Irr}(G)$

The following two results are considered important properties of irreducible characters.

3.2.4 Theorem. *[see Isaacs (2006) Corollary 2.7] Let G be a group. Then the number of irreducible characters of G is equal to the number of its conjugacy classes and*

$$\sum_{\chi \in \mathit{Irr}(G)} \chi(1)^2 = |G|.$$

3.2.5 Theorem. *[see Isaacs (2006) Corollary 3.11] Let G be a group and $\chi \in \mathit{Irr}(G)$. Then $\chi(1)$ divides the order of G.*

Let $\pi : G \to GL(V)$ and $\rho : G \to GL(W)$ be two representations of a finite group G over V and W respectively. We say that the two are equivalent if there exist an isomorphism $x : V \to W$ such that for all $g \in G$,

$$\rho(g) = x\pi(g)x^{-1}.$$

3.2.6 Lemma. Two representations, π and ρ are equivalent if and only if they afford the same character.

Notice that if χ is a character of G afforded by a representation π and H is a subgroup of G, then the restriction χ_H of χ is a character of H afforded by the restriction π_H of π, where,

$$\chi_H = \sum_{\varphi \in \mathrm{Irr}(H)} z_\varphi \varphi$$

for some nonnegative integers z_φ.

3.2.7 Definition. Let $N \lhd G$ be a subgroup and ϑ be a class function of N, Then the function ϑ^g defined by, $\vartheta^g(s) = \vartheta(gsg^{-1}),\ s \in N$ and $g \in G$, is called a conjugate to ϑ in G.

Let G be a group and $\mathrm{Irr}(G)$ be the set of all irreducible characters of G. The set of the character degrees of the irreducible characters of G is denoted by c.d.(G). That is,

$$\text{c.d.}(G) = \{\chi(1) | \chi \in \mathrm{Irr}(G)\}.$$

Let G and H be two groups and $G \times H$ be their direct product. If $\varphi \in \mathrm{Irr}(G)$ and $\vartheta \in \mathrm{Irr}(H)$, then their product $\varphi\vartheta \in \mathrm{Irr}(G \times H)$

In particular, if $G \times H$ is the direct product of two groups G and H, then

$$\text{c.d.}(G \times H) = \{\varphi(1)\vartheta(1) \,|\, \varphi \in \mathrm{Irr}(G) \text{ and } \vartheta \in \mathrm{Irr}(H)\}$$

We also notice that the set $\rho(G \times H) = \rho(G) \cup \rho(H)$.

Here are some results considered important in the study of character theory of finite groups. One of the most celebrated results in this field of study is the theorem below due to Ito.

3.2.8 Theorem (Ito)**.** *[see (Isaacs, 2006) Corollary 12.34] Let G be a solvable group. Then G has a normal abelian Sylow p-subgroup if and only if p does not divide any element of c.d.(G).*

Another result on the structure of a group obtained by studying the character degrees of the characters of the group is due to Thompson.

3.2.9 Theorem. *[See (Thompson, 1970)] If G is a group and p is a prime such that $p|\chi(1) \neq 1$ for all character $\chi \in$ Irr(G), then G has a normal p-complement.*

The following two result are also very useful in this essay.

3.2.10 Theorem (Gallagher)**.** *[see (Isaacs, 2006) Corollary 6.17] Let G be a group with $N \lhd G$. If $\vartheta \in$ Irr(N) is extendible to $\vartheta_0 \in$ Irr(G), then the character $\vartheta_0\lambda$ for $\lambda \in$ Irr(G/N) are all of the irreducible constituents of ϑ^G. In particular, $\vartheta(1)\lambda(1) \in$ c.d.(G) for all $\lambda \in$ Irr(G/N).*

3.2.11 Lemma. [See (Wu and Zhang, 2007)] Let G be a nonabelian solvable group. Then either all the nonlinear irreducible characters of G have the same degree, or, there exists a non-trivial abelian normal subgroup N of G such that G/N is nonabelian.

The following lemma is useful in proofs of some results considered important in the study of character degree graphs and prime graphs of groups.

3.2.12 Lemma. [see (Isaacs, 2006) Lemma 12.3] Let G be a solvable group and N be the unique minimal normal subgroup of G. Then all nonlinear irreducible characters of G have equal degree r and one of the following situations holds:

(i) G is a p-group, or

(ii) G is a Frobenius group with an abelian Frobenius complement of order r. Also N is the Frobenius kernel and is an elementary abelian p-group.

The following theorem will be used together with Lemma 3.2.12.

3.2.13 Theorem. *[see (Isaacs, 2006) Theorem 12.4] Let $K \lhd G$ be such that G/K is the Frobenius group with Frobenius kernel N/K, an elementary abelian p-group. Let $\chi \in$ Irr(N). Then one of the following holds.*

(i) $|G : N|\chi(1) \in$ *c.d.*(G), *or*

(ii) $|N : K|$ *divides* $\chi(1)^2$.

The following result will be used to prove the succeeding result.

3.2.14 Theorem. *[See (Isaacs, 2006) Corollary 11.29] Let $N \lhd G$and $\chi \in$ Irr(G). Let $\varphi \in$ Irr(N) be an irreducible constituent of χ_N. Then $\chi(1)/\varphi(1)$ divides $|G : N|$.*

The following result is an application of Theorem 3.2.13 and Theorem 3.2.14.

3.2.15 Lemma. [See (Lewis, 1998) Lemma 6.2] Let K be a normal subgroup of G such that G/K is a Frobenius group with kernel N/K an elementary abelian p-group for some prime p. Suppose that $a \in$ c.d.(G) is relatively prime to $n = |G : N|$. Then one of the following must hold:

(a) $na \in$ c.d.(G)

(b) $p|a$

Proof. For the proof see Lewis (1998). □

To prove the existence of a solvable group G that satisfy Proposition 5.4.1, we will require the following two results.

3.2.16 Theorem. *[see Isaacs (2006) Corollary 2.6] A group G is abelian if and only if every irreducible character is linear. In particular, G is abelian if and only if $\rho(G) = \emptyset$.*

Proof. The proof is more or less the one in Isaacs (2006). By Lemma 2.1.7, a group G is abelian if and only if the number of conjugacy classes is equal to the order of G. By Theorem 3.2.4, $\sum_{i=1}^{k} \chi_i(1)^2 = |G|$ where k is the number of conjugacy classes. But $\chi_i(1) \geq 1$ for all i. Thus $k = |G|$ if and only if $\chi_i(1) = 1$ for all i. □

3.2.17 Proposition. For each prime p, there exists a solvable group G such that $\rho(G) = \{p\}$

Proof. We know that every p-group is solvable. Clearly, there exists a nonabelian solvable group G such that $|G| = p^m$ for some $m \geq 3$. By Theorem 3.2.16, $\rho(G) \neq \emptyset$. By Theorem 3.2.5, we obtain $\rho(G) = \{p\}$. The proof is complete. □

It is also important to note the following result.

3.2.18 Theorem. *[see (Dolfi, Pacifici, and Sanus, 2008) Theorem 2.1] Let G be a group, and assume that every element in c.d.$(G) \setminus \{1\}$ is a prime number. Then G is solvable.*

4. Prime Graphs of Finite Groups

4.1 Review of Graph Theory

Let's study some graph theory that will used in this essay.

4.1.1 Definition. A graph Γ is a pair (V, E), where V is the set of vertices and E is the set of unordered pairs of the elements of V.

We say that a pair $\{x, y\} \in E$ if and only if there is an edge between x and y in Γ. For a pair $\{x, y\} \in E$, we will simply write xy. For $x, y \in V$, we say that x and y are adjacent if and only if $xy \in E$. We can also say that x is a neighbour of y.

4.1.2 Definition. Let $\Gamma = (V, E)$ be a graph. The neighbourhood of a vertex $v \in V$ denoted by $N(v)$, is the set of vertices of Γ to which v is adjacent. That is,

$$N(v) = \{x \in V | xv \in E\}.$$

The number of neighbours of a vertex in a graph is called the degree of the vertex. Each of the vertices in a graph could be having different degrees. In the case where all the vertices have the same degree, we say that the graph is regular and the degree of the graph denoted by d is the degree of the vertices.

4.1.3 Definition. A graph Γ is called k-regular if it is regular of degree k.

A graph with n vertices is said to be of order n.

4.1.4 Definition. Let Γ be a graph of order n. If Γ is such that all the vertices are adjacent, then Γ is called a complete graph denoted by K_n.

4.1.5 Definition. Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be any two graphs. We say that Γ_1 and Γ_2 are isomorphic if and only if the following two conditions holds:

(a) We can relabel the vertex sets $V_1 = \{u_1, \ldots, u_t\}$ with $V_2 = \{v_1, \ldots, v_s\}$ for some positive integers $s = t \geq 1$ such that (b) holds.

(b) For any $i, j \in \{1, 2, \ldots, t\}$, $u_i \in N(u_j)$ in Γ_1 if and only if $v_i \in N(v_j)$ in Γ_2. We say that u_i corresponds to v_i.

The sequence $v_1, v_2, \ldots, v_k$ in a graph $\Gamma = (V, E)$, for $v_1, v_2, \ldots, v_k \in V$, is a path if $v_i v_{i+1} \in E$ for $i = 1, 2, \ldots, k - 1$.

A graph is said to be connected if for every pair of the vertices we can find a path between them.

Using the path we can find distance between two vertices of a graph. The distance between two vertices u and v denoted by $\partial(u, v)$, is the length of the shortest path joining them.

4.1.6 Definition. Let $\Gamma = (V, E)$ be a connected graph. The diameter of Γ denoted by diam(Γ), is the maximum distance over all pairs of the vertices.

$$d = \max_{x,y \in V} \partial(x, y)$$

4.1.7 Definition. Let $\Gamma = (V, E)$ be a graph and S be the sequence $v_0, v_1, v_2, \ldots, v_k$ for $\{v_0, v_1, v_2, \ldots, v_k\} \subset V$. The sequence S is called a cycle denoted by C_k if its edge set contains v_0v_k and v_iv_{i+1}, $i = 0, \ldots, k-1$ only.

Let $\Gamma = (V, E)$ be a graph. Let $\Gamma = (V', E')$ be a graph such that $V' \subseteq V$ and $E' \subseteq E$. Then Γ' is called a subgraph of Γ. A subgraph may be obtained by deleting vertices, edges or both from the original graph.

4.1.8 Definition. Let Γ be a graph. A subgraph $\mathcal{G}$ is called a proper subgraph of Γ if it is not equal to Γ.

4.1.9 Definition. A subgraph obtained by deleting vertices and the edges attached to the deleted vertices is called an induced subgraph.

4.1.10 Definition. Let K_r be a complete graph of order r. A graph is said to be K_r-free if it does not have K_r as an induced subgraph.

4.1.11 Definition. Let $\Gamma = (V, E)$ be a graph and $I \subset V$ such that if $x, y \in I$, then $x \notin N(y)$. The subset I of V is called an independent set.

The independent number of a graph Γ denoted by Ind(Γ) is the maximal size of independent sets in a graph Γ.

A graph Γ is said to be k-colorable if its vertices can be colored using k colors such that if $x \in N(y)$, then x and y are colored using different colors.

4.1.12 Definition. The chromatic number of a graph Γ usually denoted by $\chi(\Gamma)$ is the minimum number k such that Γ is k-colorable.

Let's consider the following result due to Brooks,

4.1.13 Theorem. *[(Brooks, 1941) Theorem] Let $\Gamma = (V, E)$ be a graph such that $|V| = n$ and maximal degree $d \geq 3$. Suppose that Γ is K_{d+1}-free, then $n \leq$ Ind$(\Gamma)d$.*

Proof. For the proof see Brooks (1941). □

Now we need to define a special kind of graph associated with the character degree set of a finite group. Given a finite group G, we can obtain a set of prime numbers $\rho(G)$ which divide some character degree in the character degree set c.d.(G) of the group G.

4.1.14 Definition. Let G be a group and $\rho(G)$ be the set of primes numbers which divide some character degree in the character degree set c.d.(G) of the group G. The graph $\triangle(G)$ whose vertex set is $\rho(G)$ and the edge set contains unordered pairs $\{a, b\}$ such that the product ab divides some character degree in c.d.(G). The graph $\triangle(G)$ is called the prime graph of the group G.

The following property is considered an important result in graph theory and can also be useful in the study of prime graphs of odd degree.

4.1.15 Lemma. [See (Diestel, 2000) Proposition 1.2.1] Let Γ be a graph. Then Γ has an even number of vertices of odd degree.

As defined in many group theory texts, a permutation is a bijective function from a set to itself. That is, if X is a nonempty set, then a permutation is a bijective map $\sigma : X \to X$.

4.1.16 Definition. An automorphism of a graph Γ is a permutation σ on the vertex set $V(\Gamma)$, such that $uv \in E(\Gamma)$ if and only if $\sigma(u)\sigma(v) \in E(\Gamma)$.

The set of all automorphisms of a graph forms a group called an automorphism group.

4.1.17 Definition. Let Γ be a graph and A be the automorphism group. Γ is said to be vertex transitive if for any $u, v \in V(\Gamma)$ there exists $\sigma \in A$ such that $\sigma(u) = v$.

4.2 Prime Graphs

It is easy to obtain the prime graph of a finite group of small order. Using the program GAP, we can obtain the character degree set of such groups. The problem is to determine which finite graph could be a prime graph of a group. This is one of the new subjects of study attracting a lot of academics interested in the study of structure of finite groups by studying c.d.(G) and $\triangle(G)$. Famous results have been obtained concerning the structure of finite groups by simply studying the set of the character degrees of the group.

Let's consider some examples of prime graphs. We will use GAP to obtain the set c.d.(G) as well as the set $\rho(G)$.

4.2.1 Examples of Prime Graphs.

4.2.2 Example. Let $G = S_7$, the symmetric group with 7! elements. Using GAP, we can obtain the c.d.(G) set.

```
gap> G:=SymmetricGroup(7);
Sym( [ 1 .. 7 ] )

gap> cd:=List(CharacterDegrees(G),x->x[1]);
[ 1, 6, 14, 15, 20, 21, 35 ]
```

We obtain that c.d.$(G) = \{1, 6, 14, 15, 20, 21, 35\}$ and thus the vertex set of $\Delta(G)$ is the set $\rho(G) = \{2, 3, 5, 7\}$ of $\triangle(G)$. We can also calculate the edge set using the command below.

```
gap> List(cd,x->Set(Factors(x)));
[ [ 1 ], [ 2, 3 ], [ 2, 7 ], [ 3, 5 ], [ 2, 5 ], [ 3, 7 ], [ 5, 7 ] ]
```

This implies that the edge set E of $\triangle(G)$ is the set $\{\{2,3\},\{2,7\},\{3,5\},\{2,5\},\{3,7\},\{5,7\}\}$

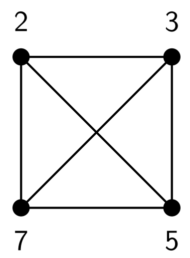


Figure 4.1: Prime graph of $G = S_7$

4.2.3 Example. Let $G = A_8$, the alternating group with $\frac{8!}{2}$ elements. Again using GAP we obtain the set c.d.(G).

```
gap> G:=AlternatingGroup(8);
Alt( [ 1 .. 8 ] )

gap> cd:=List(CharacterDegrees(G),x->x[1]);
[ 1, 7, 14, 20, 21, 28, 35, 45, 56, 64, 70 ]

gap> List(cd,x->Set(Factors(x)));
[ [ 1 ], [ 7 ], [ 2, 7 ], [ 2, 5 ], [ 3, 7 ], [ 2, 7 ], [ 5, 7 ], [ 3, 5 ],
  [ 2, 7 ], [ 2 ], [ 2, 5, 7 ] ]
```

From the above result we obtain that c.d.$(G) = \{1, 7, 14, 20, 21, 28, 35, 45, 56, 64, 70\}$, $V = \rho(G) = \{2, 3, 5, 7\}$ and the edge set $E = \{\{2, 5\}, \{2, 7\}, \{3, 5\}, \{3, 7\}, \{5, 7\}\}$. Thus we obtain $\triangle(G)$ as in the graph in figure 4.2 below.

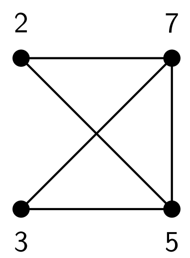


Figure 4.2: Prime graph of $G = A_8$

Let's consider our last example of a prime graph.

4.2.4 Example. Let $G = PSL_3(4)$, the Projective Special Linear group over F_4 the finite field with 4 elements. $|G| = 20160$. Here is how it is constructed in GAP.

```
gap> G:=PSL(3,4);
Group([ (3,4,5)(7,9,8)(10,14,18)(11,17,20)(12,15,21)(13,16,19),
  (1,2,6,7,11,3,10)(4,14,8,15,16,20,13)(5,18,9,19,21,17,12) ])
```

The character degree set c.d.$(G) = \{1, 20, 35, 45, 63, 64\}$ of G as given below

```
gap> cd:=List(CharacterDegrees(G),x->x[1]);
[ 1, 20, 35, 45, 63, 64 ]

gap> List(cd,x->Set(Factors(x)));
[ [ 1 ], [ 2, 5 ], [ 5, 7 ], [ 3, 5 ], [ 3, 7 ], [ 2 ] ]
```

The vertex set of $\triangle(G)$ $\rho(G) = \{2, 3, 5, 7\}$ with the edge set $E = \{\{2, 5\}, \{5, 7\}, \{3, 5\}, \{3, 7\}\}$. Thus $\triangle(G)$ is as shown in Figure 4.3 below.

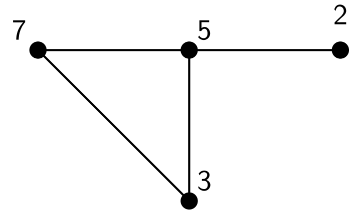


Figure 4.3: Prime graph of G

4.3 Important Results

There have been a few breakthroughs in this field in the last 30 years or so. We will only consider the most important results from the study of prime graphs to determine the structure of finite groups.

The following result is considered an important result in the study to determine which finite graphs are prime graphs of some solvable groups. The result is due to Pálfy.

4.3.1 Theorem (Pálfy's condition)**.** *[(Pálfy, 1998) Theorem]*

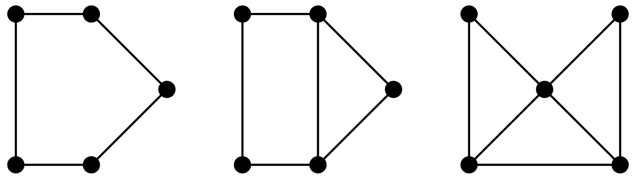
Let G be a solvable group and $\rho(G)$ be the set of primes that divide a character degree $\chi(1)$ for some $\chi \in$ Irr(G). Let π be a set of primes contained in $\rho(G)$. If $|\pi| \geq 3$, then there exists an irreducible character whose degree is divisible by two primes from π.

This condition asserts that if we take any three vertices of $\triangle(G)$ where G is solvable, then at least two of these primes are adjacent. A more general result for any finite group G was later proved.

4.3.2 Theorem (Moretó -Tiep's condition)**.** *[See (Moretó and Tiep, 2008) Main Theorem] Let G be a group and π be a set of primes contained in $\rho(G)$. If $|\pi| \geq 4$, then there exists an irreducible character whose degree is divisible by at least two primes from π.*

Let's consider some classification of prime graphs as in Lewis (2004). Lewis classified prime graphs of solvable groups with $|\rho(G)| = 5$.

4.3.3 Theorem. *[See Lewis (2004) Main Theorem] The graphs with 5 vertices that arise as $\triangle(G)$ for some solvable group G are precisely the graphs with diameter at most 2 that satisfy Theorem 4.3.1 except graphs (1), (2) and possibly (3) of Figure 4.4*



(1) (2) (3)

Figure 4.4: 5-vertex graphs that do not occur

Proof. For the proof see Lewis (2004). In this paper, Lewis only proved the cases for the graph (1) and (2). For graph (3), we are not able to construct a solvable group with graph (3) as a prime graph. However, we have not been in a position to show that no such group exists. □

4.3.4 Lemma. [See (Lewis, 2004) Lemma 3.2] Let $\mathscr{G}$ be a graph of order n. Suppose that no proper subgraph of $\mathscr{G}$ of order n or $n-1$ occurs as $\triangle(G)$ for some solvable group G. Then $\mathscr{G}$ is not a prime graph $\triangle(G)$ for some solvable group G.

Proof. See Lewis (2004) for the proof of this Lemma. □

We claim that the following result holds.

4.3.5 Proposition. Let Γ_1 be a prime graph of some solvable group and Γ be the join of Γ_1 and a one vertex graph. Then Γ is also a prime graph of some solvable group.

Proof. We need to show the existence of a solvable group whose prime graph is as described. Let G and H be two solvable groups whose prime graphs are Γ_1 and one vertex graph respectively. By Lemma 3.2.17, we can obtain a group H such that $\rho(H) = \{p\}$ and $p \notin \rho(G)$. We know that the direct product of solvable groups is also solvable. Thus $G \times H$ is also solvable. Define Γ as the prime graph of $G \times H$. By definition, c.d.$(G \times H) = \{\varphi(1)\vartheta(1) \mid \varphi \in \mathrm{Irr}(G) \text{ and } \vartheta \in \mathrm{Irr}(H)\}$. Also, observe that $\rho(G \times H) = \rho(G) \cup \rho(H)$. Let $\rho(G) = \{q_1, \ldots, q_s\}$ and $\rho(H) = \{p\}$. It is not difficult to see that the edges in Γ_1 also exist in Γ. Now, we need to show that for each q_i for $i = 1, \ldots, s$ and p, there is an edge between them. Let $q_i | \varphi(1)$ for some $\varphi \in \mathrm{Irr}(G)$ and $p | \vartheta(1)$ for $\vartheta \in \mathrm{Irr}(H)$. By definition, $\varphi(1)\vartheta(1) \in$ c.d.$(G \times H)$. This implies that for each i, there is an edge between q_i and p. □

The following result due to M. Lewis and Q. Meng played a role in proving Lemma 4.3.7.

4.3.6 Lemma. [See (Lewis and Meng, 2012) Lemma 2.1] Let G be a solvable group. If $\triangle(G)$ has at least 4 vertices, then either $\triangle(G)$ contains a triangle or $\triangle(G)$ is a square.

One of the most useful result in this essay is the following result.

4.3.7 Lemma. [See (H.P.Tong-Viet, 2013a) Lemma 2.2] If G is a solvable and $\triangle(G)$ has no triangles, then $|\rho(G)| \leq 4$.

Proof. The proof is more or less the proof in H.P.Tong-Viet (2013a). Let G be a solvable group. We only need to show that if $|\rho(G)| > 4$ then $\triangle(G)$ must contain a triangle. Let $|\rho(G)| \geq 4$. By Lemma 4.3.6, $\triangle(G)$ contains a triangle or $\triangle(G)$ is a square. If $\triangle(G)$ contains a triangle, then we are done. If $\triangle(G)$ is a square, then $|\rho(G)| = 4$. Thus if $|\rho(G)| > 4$, then $\triangle(G)$ must contain a triangle. □

The following is a more general result for any finite group due to Hung P. Tong-Viet.

4.3.8 Lemma. [see (H.P.Tong-Viet, 2013a) Theorem A] Let G be a group, $\triangle(G)$ be the prime graph of G and $\rho(G)$ the set of all prime numbers that divide $\vartheta(1)$ for some $\vartheta \in \mathrm{Irr}(G)$. If $\triangle(G)$ has no triangle, then $|\rho(G)| \leq 5$.

5. Regular Prime Graphs of Finite Groups

In this chapter, we will study the n-regular graphs, for $n \leq 4$, which occur as prime graphs of some solvable groups. We will just review the cases when $n \leq 3$. Our main focus is when $n = 4$.

5.1 Background Results

By Theorem 4.1.13, Theorem 4.3.2 and Theorem 4.3.1, the following result is obtained.

5.1.1 Corollary. [see (H.P.Tong-Viet, 2013b) Lemma 2.3] Let G be a group and $\triangle(G)$ be its prime graph. Then the independent number Ind($\Delta(G)$) is at most 3 in general and at most 2 if G is solvable.

The following result follows from Corollary 5.1.1. We will rely on the result below to come up with an upper bound in the number of vertices of a 4-regular graph to be a prime graph of a solvable group.

5.1.2 Corollary. [see (H.P.Tong-Viet, 2013b) Corollary 2.4] Let G be a group and $\triangle(G)$ be the prime graph of G. Suppose that the maximal degree of $\triangle(G)$ is $d \geq 3$ and $\triangle(G)$ is K_{d+1}-free, then $|\rho(G)| \leq 2d$ when G is solvable and $|\rho(G)| \leq 3d$ in general.

In particular, if $\triangle(G)$ is a connected k-regular graph for some integer $k \geq 3$ but it is not K_{d+1}, then $|\rho(G)| \leq 2k$ when G is solvable and $|\rho(G)| \leq 3k$ in general.

The following two results will be used to narrow down our study of 4-regular graphs which can be prime graphs of solvable groups.

5.1.3 Lemma. Let G be a solvable group with a prime graph $\triangle(G)$. If $\triangle(G)$ is disconnected, then $\triangle(G)$ contains 2 connected components which are both complete graphs.

Proof. By Theorem 4.3.1, if G is solvable, then $\triangle(G)$ cannot have more than two connected components. Let $\triangle(G)$ contain two connected components A and B. It suffices to show that if one of the two is not complete then $\triangle(G)$ is not a prime graph of a solvable group. Suppose A has the vertex set $V(A) = \{u_1, \ldots, u_k\}$ where $k \geq 2$. If A is not complete, then $\exists\, u_r, u_s \in V(A)$ for some $1 \leq s, r \leq k$ such that $u_r \notin N(u_s)$. Suppose B has the vertex set $V(B) = \{v_1, \ldots, v_t\}$ where $t \geq 1$. Now we notice that there is no edge between any two of the vertices $u_r, u_s, v_i \in V(\triangle(G))$ for all $i = 1, \ldots, t$. Thus by Theorem 4.3.1, $\triangle(G)$ cannot be a prime graph of a solvable group. We are done. □

We will require the following result to prove the succeeding result.

5.1.4 Theorem. *[see (Lewis, 2008) Theorem 4.3] Let G be a solvable group and $\triangle(G)$ be the associated prime graph with two connected components. Let the size of the vertex set of the two components be n and N with $n \leq N$. Then $N \geq 2^n - 1$.*

5.1.5 Lemma. [(H.P.Tong-Viet, 2013b) Lemma 2.5] Let G be a group and let $k \geq 0$ be an integer. If $\triangle(G)$ is a disconnected k-regular graph, then $k = 0$.

Proof. For the complete proof see H.P.Tong-Viet (2013b). We will only concentrate on the case when G is solvable. If $\triangle(G)$ is connected, then we are done. Suppose that $\triangle(G)$ is disconnected, by Lemma 5.1.3, $\triangle(G)$ contains two connected components which are both complete graphs. Let the two components be of order n and N such that $n \leq N$. By Theorem 5.1.4, $N \geq 2^n - 1$. But we know that complete k-regular graphs are unique for some positive integer k. Notice that each component must be of order $k+1$. We obtain that $n = N = k+1$, which implies that $k+1 \geq 2^{k+1} - 1$. By inspection, we notice that $k = 0$.

□

5.2 1-Regular and 2-Regular Graphs

It is easy to see which 1-regular and 2-regular graphs are prime graphs of some solvable groups. By Theorem 4.3.1 and Lemma 5.1.5, it is clear that no 1-regular and 2-regular graphs with more than 2 and 4 vertices respectively can be prime graphs of some solvable groups. The cases for graphs with 4 vertices and less that can be prime graphs of solvable groups were analysed by Huppert in his paper Huppert (1991). According to the paper and our analysis, the only 1-regular and 2-regular graphs that can occur as prime graphs are the graphs shown in Figure 5.1.

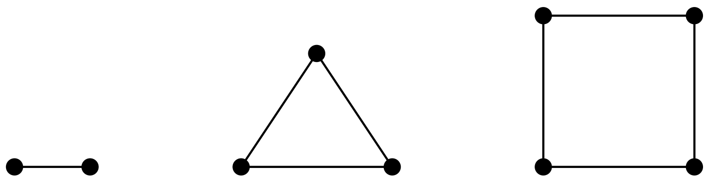


Figure 5.1: 1-regular and 2-regular graphs that occur as prime graphs of some solvable groups

5.3 3-Regular Graphs

As in definition 4.1.3, a 3-regular graph is a graph $\mathscr{G}$ in which every vertex has degree 3. In a recent paper by Hung P. Tong-Viet, it was shown that the following result holds.

5.3.1 Proposition. [see (H.P.Tong-Viet, 2013b) Proposition 2.7] Let G be a group and $\triangle(G)$ be the prime graph of G. Suppose that $\triangle(G)$ is a 3-regular graph with $|\rho(G)| \geq 6$. then $\triangle(G)$ is isomorphic to one of the following graphs.

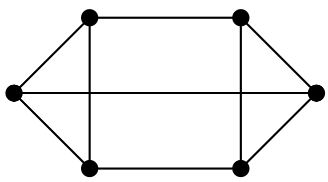


Figure 5.2: Cubic graph of order 6 with 2 triangles

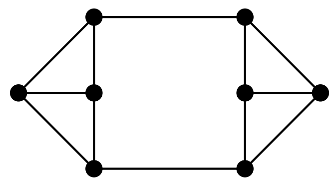


Figure 5.3: Cubic graph of order 8 with 4 triangles

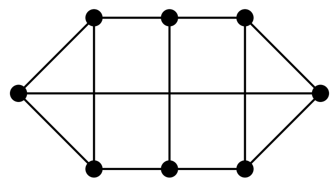


Figure 5.4: Cubic graph of order 8 with 2 triangles

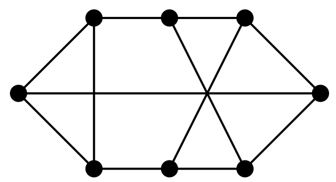


Figure 5.5: Cubic graph of order 8 with 1 triangles

Proof. See H.P.Tong-Viet (2013b) for the proof. □

In H.P.Tong-Viet (2013b), it is also shown that the following result holds.

5.3.2 Lemma. [see (H.P.Tong-Viet, 2013b) Lemma 3.1] Let G be a group and the prime graph $\triangle(G)$ be isomorphic to the graph in Figure 5.2. Then G is nonsolvable.

Proof. For the proof see H.P.Tong-Viet (2013b). □

From the above results Tong-Viet obtained the following result.

5.3.3 Lemma. [see (H.P.Tong-Viet, 2013b) Theorem 3.2] Let G be a solvable group. If the prime graph $\triangle(G)$ is cubic, then $\triangle(G)$ is isomorphic to K_4.

Proof. G is solvable and $\triangle(G)$ cubic, thus by Lemma 5.1.5, $\triangle(G)$ is connected. By Corollary 5.1.2, $|\rho(G)| \leq 6$. Thus $|\rho(G)| = 4, 5$ or 6. By Lemma 4.1.15, it follows that $|\rho(G)| \neq 5$. If $|\rho(G)| = 4$ then we are done. We need to show that $|\rho(G)| \neq 6$. Suppose $|\rho(G)| = 6$ then by Proposition 5.3.1, $\triangle(G)$ is isomorphic to the graph in Figure 5.2. But by Lemma 5.3.2, G is nonsolvable which is a contradiction. □

5.4 4-Regular Prime Graphs of Finite Solvable Groups

Let us consider the least number of vertices required to form a 4-regular graph. We cannot obtain a 4-regular graph with less than 5 vertices. Our first case is when $|\rho(G)| = 5$. To obtain a 4-regular graph with 5 vertices we require all the vertices to be adjacent to each of the other vertices. This produces a complete graph of 5 vertices denoted by K_5, see Figure 5.6.

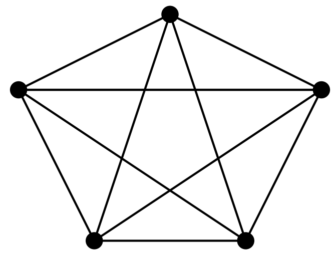


Figure 5.6: 4-regular graph of order 5 (K_5)

We claim that the following result holds.

5.4.1 Proposition. For each complete graph K_m where m is a positive integer, there exists a solvable group G whose prime graph $\triangle(G)$ is isomorphic to K_m.

Proof. By Proposition 3.2.17 there exists a solvable group G such that $\rho(G) = \{p\}$ for any prime p. Now let $G = G_1 \times \cdots \times G_k$, $k \geq 1$, the direct product of finite solvable groups G_i's with $\rho(G_i) = \{p_i\}$ for distinct primes p_i's. It follows that $\rho(G) = \{p_1, \ldots, p_k\}$. Clearly, $\triangle(G)$ has k vertices. We need to show that $\triangle(G)$ is a complete graph. We notice that c.d.$(G_i) = \{p_i, p_i^2, \ldots, p_i^j\}$ for some integer $j \geq 1$ not necessarily the same for each i. To obtain c.d.(G) we multiply every element of c.d.(G_r) with all the elements in c.d.(G_t) for $t \neq r$ and $1 \leq r, t \leq k$. Thus for each pair $(p_n, p_s) \in \rho(G)$ we can find $c_l \in$ c.d.(G) such that the product $p_n p_s | c_l$ for $1 \leq n, s \leq k$ and $1 \leq l \leq |$c.d.$(G)|$. In this case, notice that $m = k - 1$. □

From Proposition 5.4.1 it follows that there is a solvable group G such that $\triangle(G)$ is the graph in Figure 5.6.

For the case of $|\rho(G)| = 6$. Let's begin by ruling out some of the graphs that cannot be possible prime graphs. Mostly we will make use of Lemma 4.3.1 and Lemma 4.3.8. We assume that we have a triangle with vertices p_1, p_2 and p_3.

Let the remaining vertices be q_1, q_2 and q_3. Let all the p_i's be adjacent to one of the q_i's. Then clearly there is no way to complete the graph to form a 4-regular graph of order 6. Now, let two of the p_i's, say p_1 and p_2 be adjacent to one of the q_i's, say q_1. It follows that we must have p_3 and one of the other p_i's, say p_2 being adjacent to one of the vertices in $\{q_2, q_3\}$, say q_3. There is only one way to complete this graph. We let p_1 and p_3 be adjacent to q_2 and then have the q_i's form a triangle. We obtain the graph in Figure 5.7

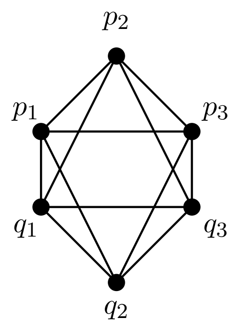


Figure 5.7: 4-regular graph of order 6

5.4.2 Proposition. There exists a solvable group G whose prime graph $\triangle(G)$ is isomorphic to the graph in Figure 5.7.

Proof. It suffices to show the existence of groups G_i for $i = 1, 2, 3$ such that c.d.$(G_1) = \{1, p_1, q_3\}$, c.d.$(G_2) = \{1, p_2, q_2\}$ and c.d.$(G_3) = \{1, p_3, q_1\}$. for some distinct primes $\{p_j\}_{j=1}^3$ and $\{q_k\}_{k=1}^3$.

Actually it can be shown that such groups exists. For instance, for the pairs $\{p_1, q_3\}$. If they satisfy the condition $p_1|q_3 - 1$. Let P be an extraspecial group of order q_3^3 and exponent q_3. One can show that P has an automorphism σ of order p_1 that centralizes the center of P. If you take G_1 to be the semi direct product of the group generated by σ acting on P, then c.d.$(G_1) = \{1, p_1, q_3\}$. For each such pair there exists a solvable group with such character degree set. Then by Theorem 3.2.18, we deduce that they must be solvable. The group $G = G_1 \times G_2 \times G_3$ is solvable and its prime graph $\triangle(G)$ is isomorphic to the graph in Figure 5.7. The proof is complete. □

Let us consider when the cases when $|\rho(G)| \geq 7$.

5.4.3 Proposition. There is no solvable group G whose prime graph $\Delta(G)$ is a 4-regular graph with $|\rho(G)| \geq 7$.

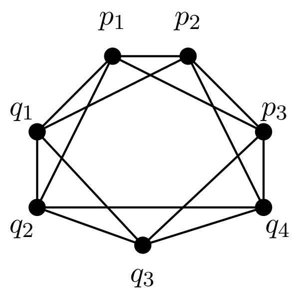


Figure 5.8: 4-regular graph of order 7 with 7 triangles

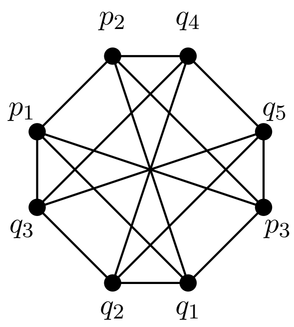


Figure 5.9: 4-regular graph of order 8 with 2 K_4

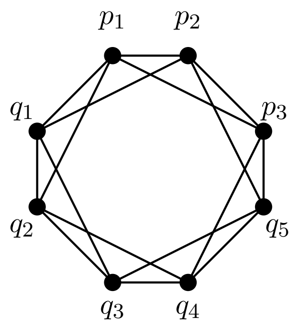


Figure 5.10: 4-regular graph of order 8 with 8 triangles

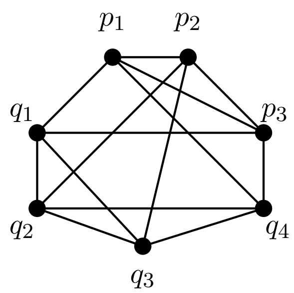


Figure 5.11: 4-regular graph of order 7 with 6 triangles

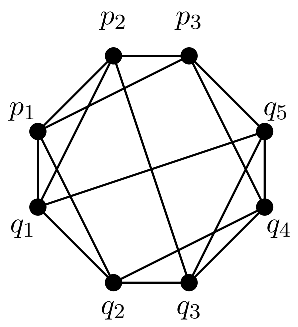


Figure 5.12: 4-regular graph of order 8 with 6 triangles

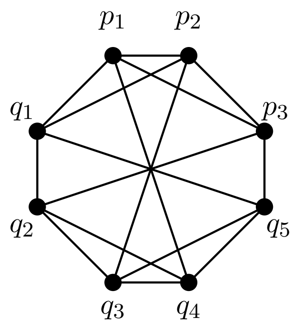


Figure 5.13: 4-regular graph of order 8 with 4 triangles

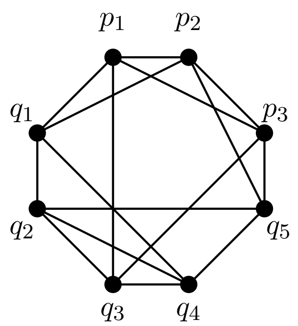


Figure 5.14: 4-regular graph of order 8 with 7 triangles

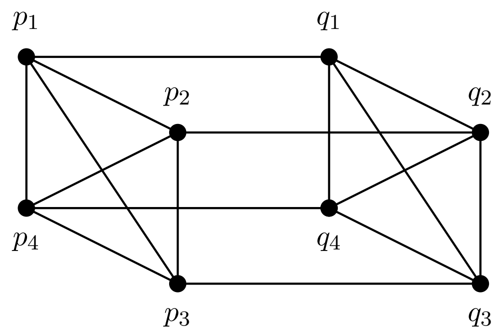


Figure 5.15: Graph Γ, isomorphic to graph in Fig 5.9

Proof. Let G be a solvable group and $\triangle(G)$ be the associated prime graph. If $\triangle(G)$ is 4-regular, by Lemma 5.1.5, the graph should be connected. By Corollary 5.1.2, the cardinality, $|V| = |\rho(G)| \leq 8$. Thus $|\rho(G)| = 5, 6, 7$ or 8. If $|\rho(G)| \geq 7$, then $|\rho(G)| = 7$ or 8. By Lemma 4.3.8, the prime graph must contain a triangle. Let $\triangle(G) = \mathscr{G}$ be the graph. We suppose that $\mathscr{G}$ contains a triangle $\{p_1, p_2, p_3\}$.

Let's consider the case when $\mathscr{G}$ is a 4-regular graph of order 7. Let the remaining vertices be $\{q_1, q_2, q_3, q_4\}$. There are two cases.

Case 1. Suppose that some q_i, say q_1 is adjacent to all the p_i's. Now deg$(p_i) = 3$, $\forall i$. We now consider 3 possibilities.

1. Let $q_2 \in N(p_i)$, $\forall i$, then we have that deg$(p_i) = 4$, $\forall i$. Clearly there is no way to complete the graph to be a 4-regular graph with 7 vertices.
2. Let q_2 be adjacent to 2 vertices in the set p_1, p_2, p_3 say p_1 and p_2. Notice that p_3 is adjacent to either q_3 or q_4. Without loss of generality, let p_3 be adjacent to q_3 and thus deg$(p_i) = 4$, $\forall i$. Again there is no way to complete $\mathscr{G}$ to obtain a 4-regular graph of degree 7.
3. Let each p_i be adjacent to different q_j for $j = 2, 3, 4$, that is $q_2 \in N(p_1)$, $q_3 \in N(p_2)$ and $q_4 \in N(p_3)$. This implies that deg$(p_i) = 4$ $\forall i$. Therefore again we cannot obtain a 4-regular graph of degree 7.

Case 2. Suppose that there is no j such that q_j is adjacent to all the $p_i's$. Then it follows that there are two of the q_j's such that each of them is adjacent to two of the p_i's. There are two possibilities:

1. Let two q_i's, say q_1 and q_4 be adjacent to two vertices in the set $\{p_1, p_2, p_3\}$, say p_1 and p_3. Then it follows that q_2 and q_3 are adjacent to p_3. Notice that deg$(p_i) = 4$, $\forall i$. To complete the graph, we need to have both q_2 and q_3 being adjacent to both q_1 and q_4, which gives the graph in Figure 5.11.
2. Let one of the q_i's, $i = 2, 3, 4, 5$, say q_1 be adjacent to two of the vertices in $\{p_1, p_2, p_3\}$, say p_1 and p_2. Also let p_3 and one of $\{p_1, p_2\}$, say p_2 be adjacent to q_4. There are two possibilities:
   (a) We have that p_1 and p_3 are both adjacent to q_3. This implies that deg$(p_i) = 4$, $\forall i$. Now there is no way to complete $\mathscr{G}$ such that $\mathscr{G}$ is a 4-regular graph.
   (b) Suppose instead we have p_1 and p_3 adjacent to q_2 and q_3 respectively. To complete $\mathscr{G}$, we need to have q_3 being adjacent to q_1, q_2 and q_4 and both q_1 and q_4 adjacent to q_2. Thus we obtain the graph in Figure 5.8

Let's now consider the case when $\mathscr{G}$ is a 4-regular graph of order 8. Assume the remaining vertices are $\{q_i\}_{i=1}^5$

In this case, notice that there must be an edge between one of the q_i's and two of the p_i's. There are 3 cases:

Case 1. Let all the p_i's be adjacent to one of the q_i's, say q_1. There are three possibilities:

1. Let all the p_i's be adjacent to q_2. We obtain that deg$(p_i) = 4, \ \forall i$ and deg$(q_1) =$ deg$(q_2) = 3$. The other three vertices have degree 0. If we let q_3 be adjacent to all the other four q_j's, we obtain that deg$(q_i = 4$ for $i = 1, 2, 3$. Notice that we cannot complete $\mathscr{G}$ to obtain a 4-regular graph of degree 8.

2. Let one of the q_i's, $i = 2, 3, 4, 5$, say q_2 be adjacent to two of the vertices in $\{p_1, p_2, p_3\}$, say p_1 and p_2 and the remaining p_i, which is p_3, be adjacent to one q_i's for $i = 3, 4, 5$, say q_3. We obtain that deg$(p_i) = 4, \ \forall i$, deg$(q_1) = 3$ and deg$(q_2) = 2$. Let q_4 be adjacent to all the q_j's so that deg$(q_1) =$ deg$(q_4) = 4$. We notice that we cannot complete $\mathscr{G}$ to make it 4-regular.

3. Let each of the p_i's be adjacent to different q_j's for $j = 3, 4, 5$, say p_1, p_2 and p_3 be adjacent to q_3, q_4 and q_5 respectively. We already have that the degrees of all the p_i's are 4 and that of q_1 is 3. Let q_2 be adjacent to all the q_j's. The only possibility to complete the graph $\mathscr{G}$ is to have q_3 adjacent to both q_4 and q_5 and q_4 be adjacent to q_5. Hence we obtain the graph in Figure 5.9.

Case 2. Let two of the vertices in the set $\{p_1, p_2, p_3\}$, say p_1 and p_2 be adjacent to one vertex in the set $\{q_i\}_{i=1}^5$, say q_1 and one of the p_i's for $i = 1, 2$, say p_2 together with p_3 be adjacent to one of the q_i's for $i = 2, 3, 4, 5$, say q_5. Then there are 2 possibilities,

1. Let both p_1 and p_3 be adjacent to one of the q_i's for $i = 2, 3, 4$, say q_3. There is only one possibility of completing the graph. Let the remaining of the q_i's, that is q_2 and q_4 be adjacent to each other and all the other q_i's. Hence we obtain the graph in Figure 5.14.

2. Let each of p_1 and p_3 be adjacent to one of the q_i's for $i = 2, 3, 4$, say q_2. We obtain that the deg$(p_i) = 4$ for all p_i's. There is only one possibility to complete $\mathscr{G}$ to obtain a 4-regular graph of degree 4. Let q_3 be adjacent to all the other q_i's to obtain deg$(q_3) = 4$. We complete the graph by letting $q_4 \in N(q_2)$ and q_1 and q_5 be adjacent to q_2 and q_4 respectively. We thus obtain the graph in Figure 5.10.

Case 3. Let two of the p_i's, say p_1 and p_2 be adjacent to one of the q_i's, say q_1. Also let p_3 and one of $\{p_1, p_2\}$, say p_2 be adjacent to q_5 and q_3 respectively. Then there are 2 possibilities,

1. Let p_3 be adjacent to one of $\{q_2, q_4\}$, say q_4 and $p_1 \in N(q_2)$. Let q_1 be connected to q_2 and q_5 such that we obtain the degree of all the p_i's and q_1 is 4. Then there is only one way to complete $\mathscr{G}$. If we let q_3 be joined to q_2, q_4 and q_5 and q_4 be adjacent to q_5 and q_2 obtaining the graph in Figure 5.12, or

2. Let each of p_1 and p_3 be adjacent to one of the vertices in the set $\{q_2, q_4\}$, that is they are not both adjacent to one of them. Then, we have one way to complete the graph. Let q_1 be joined to q_2 and q_5. If we connect q_2 to q_3 and q_4 and let q_3, q_4 and q_5 be all adjacent to each other, we obtain the graph in Figure 5.13

Claim 1

Graphs in Figures 5.8 and 5.10 cannot be prime graphs of some solvable groups.

Proof. Assume that G is solvable and $\triangle(G)$ is isomorphic to either the graph in Figure 5.8 or 5.10. Let K be a maximal normal subgroup of G such that G/K is nonabelian. By Lemma 3.2.12, either of the following two cases can occur:

(a) G/K is a nonabelian p-group for some prime p. In this case, we show that p is adjacent to every other vertex in $\rho(G)$, which is impossible. Indeed, let $q \in \rho(G) \setminus \{p\}$. Then there exists $\varphi \in \mathrm{Irr}(G)$ such that $q|\varphi(1)$. Since G/K is nonabelian, it has a nontrivial character $\theta \in \mathrm{Irr}(G/K)$. Now if $p|\varphi(1)$, then $pq|\varphi(1)$ hence p and q are adjacent and we are done. So assume that $p \nmid \varphi(1)$. Then $\gcd(|G:K|,\ \varphi(1)) = 1$ and thus $\varphi_K \in \mathrm{Irr}(K)$. By Theorem 3.2.10, $\varphi\theta \in \mathrm{Irr}(G)$ and thus $pq|\varphi(1)\theta(1) \in \mathrm{c.d.}(G)$, so p and q are adjacent. Therefore, p is adjacent to every vertex in $\rho(G) \setminus \{p\}$ as wanted.

(b) G/K is a Frobenius group with Frobenius kernel N/K. We know that N/K is an elementary abelian p-group for some prime p and $|G:N| = n \in \mathrm{c.d.}(G)$ with $\gcd(p,n) = 1$. Furthermore, if $\theta \in \mathrm{Irr}(N)$, then either $n\theta(1) \in \mathrm{c.d.}(G)$ or $p|\theta(1)$.

Observe that the graphs in Figure 5.8 and 5.10 have no complete square so that $|\pi(\chi(1))| \leq 3$ for every $\chi \in \mathrm{Irr}(G)$. In particular, $|\pi(n)| \leq 3$.

We observe that for each possibility of p (whether $p \in \rho(G)$ or not) and each possibility of $\pi(n)$, we can find a pair of vertices u and v such that $u,\ v$ are adjacent in $\triangle(G)$, $u \notin \pi(n), \pi(n) \nsubseteq \{u,\ v\}$, and finally $\{p, u, v\}$ and $\{u,\ v,\ r\}$ are not triangles in $\triangle(G)$, where $r \in \pi(n) \setminus \{u,\ v\}$.

Notice that we only need to show that the above property holds in the graph in Figure 5.8. It also suffices to show that it is true when $|\pi(n)| = 3$. Consider, $p = q_1$, $r = p_3$ and $\pi(n) = \{p_1, p_2, p_3\}$. If we take two vertices not in $\pi(n)$, say $\{q_3, q_4\}$ such that there is an edge between them. Clearly, q_1, q_3 and q_4 do not form a triangle. Since the graphs considered in this case are both vertex transitive, with proper choice of $\pi(n)$, the property is true for each p and r chosen.

Let $\chi \in \mathrm{Irr}(G)$ such that $uv \,|\, \chi(1)$ and let $\theta \in \mathrm{Irr}(N)$ be an irreducible constituent of χ_N. Clearly $u \,|\, \theta(1)$ since $\gcd(u,\ |G:N|) = 1$. Moreover, $p \nmid \theta(1)$, otherwise $p \,|\, \chi(1)$ which implies that $\{p,\ u,\ v\}$ is a triangle. Therefore, $f := n\theta(1) \in \mathrm{c.d.}(G)$. Now if $v \in \pi(n)$, then $ruv \,|\, f$ and if $v \notin \pi(n)$, then $v \,|\, \theta(1)$ so that $ruv \,|\, f$. In both cases, $\{r,\ u,\ v\}$ is a triangle in $\triangle(G)$. This contradiction completes the proof.

□

Claim 2.

There is no solvable group G such that the prime graph $\triangle(G)$ is isomorphic to the graph in Figure 5.9.

Proof. Notice that graph Γ is isomorphic to the graph in Figure 5.9. To show that the graph in Figure 5.9 cannot be a prime graph of some solvable group, it suffices to show that graph Γ does not occur as a prime graph of some solvable group.

We claim that if G is a group with $\Delta(G)$ isomorphic to Γ, then $G' = G''$. Let's prove by contradiction. Suppose $G' \neq G''$, then G/G'' is a nonabelian solvable group. Let K be a maximal normal subgroup of G such that G/K is a minimal nonabelian solvable group.

By Lemma 3.2.12, either of the following cases hold:

(a) G/K is a nonabelian p-group for some prime p. Also, G/K has an irreducible character $\vartheta \in \mathrm{Irr}(G/K)$ with $\vartheta(1) = p^a$ for some positive integer a.

We can show that in this case, p is connected to any other prime in $\rho(G) \setminus \{p\}$.We know that for every $q \in \rho(G)$, there exists $\varphi \in \mathrm{Irr}(G)$ such that $q|\varphi(1)$.

Now, if $p \,|\, \varphi(1)$, then it follows that $pq \in E(\Gamma)$ and we are done.

Now, suppose $p \nmid \varphi(1)$, then $\varphi_K \in \mathrm{Irr}(K)$. By Theorem 3.2.10, $\varphi\vartheta \in \mathrm{Irr}(G)$ and thus $pq \,|\, \varphi(1)\vartheta(1)$. Hence, $pq \in E(\Gamma)$, we are done.

Notice that in graph Γ there is no vertex connected to all the other 7 vertices and hence we obtain a contradiction.

(b) G/K is a Frobenius group with Frobenius kernel N/K, an elementary abelian p-group for some prime p. Let $|G : N| = n$. Notice that $n \in \mathrm{c.d.}(G)$ and $\gcd(n, p) = 1$. By Theorem 3.2.13, we have that, for every $\vartheta \in \mathrm{Irr}(N)$, either $n\vartheta(1) \in \mathrm{c.d.}(G)$ or $p \,|\, \vartheta(1)$. Clearly, Γ does not contain a subgraph isomorphic to K_5 and thus, for every $\chi \in \mathrm{Irr}(G)$, $|\pi(\chi(1))| \leq 4$. Notice that Γ contains only 2 complete subgraphs K_4 with vertices $\{p_i\}_{i=1}^4$ and $\{q_j\}_{j=1}^4$ together with the remaining edges p_kq_k for $k = 1, 2, 3, 4$.

Now, since $n \in \mathrm{c.d.}(G)$, $|\pi(n)| \leq 4$ and thus $|\pi(n) \cup \{p\}| \leq 5$. This implies that there is some index $1 \leq \ell \leq 4$ such that $p \notin \{p_\ell, q_\ell\}$ and $\pi(n) \not\subseteq \{p_\ell, q_\ell\}$. Hence, there exists $r \in \pi(n) \setminus \{p_\ell, q_\ell\}$. Now, let $\chi \in \mathrm{Irr}(G)$ such that $p_\ell q_\ell \,|\, \chi(1)$. Considering graph Γ, we notice that $\pi(\chi(1)) = \{p_\ell, q_\ell\}$. Let $\theta \in \mathrm{Irr}(N)$ be an irreducible constituent of χ_N. Since $\theta(1) \,|\, \chi(1)$, we deduce that $p \nmid \theta(1)$. Hence, $n\theta(1) \in \mathrm{c.d.}(G)$. But in Γ, the triangle $\{r, p_\ell, q_\ell\}$ does not occur. Hence, we obtain a contradiction.

Thus, we conclude that $G' = G''$. If G is solvable, then it follows that $G' = G'' = 1$. This implies that G is abelian. This is not possible since $\rho(G) = 8$. Hence, Γ cannot be a prime graph of some solvable group.

□

Claim 3.

There is no solvable group G such that the prime graph $\triangle(G)$ is isomorphic to the graphs in Figures 5.11-5.14.

Proof. By Theorem 4.3.1, for any three vertices in $\triangle(G)$ of a solvable group G, then at least two of these vertices must be adjacent.

Now consider the set $\pi = \{q_1, p_2, q_4\}$ in Figure 5.11. There is no edge between any two of the three vertices contained in π. Similarly, for the graph in Figure 5.12, take $\pi = \{q_1, p_3, q_3\}$. For the graph in Figure 5.13 we take $\pi = \{q_2, p_2, q_5\}$ the same condition applies. Finally for the graph in Figure 5.14, we take $\pi = \{q_3, q_1, q_5\}$ and we are done.

□

Following all the claims, we are now through with the proof.

□

5.4.4 Corollary. Let G be a solvable group with prime graph $\Delta(G)$. If $\triangle(G)$ is 4-regular, then $|\rho(G)| \leq 6$.

Proof. The proof follows directly from Proposition 5.4.1, Proposition 5.4.2 and Proposition 5.4.3.

□

6. Conclusion

It has always been a challenge in the study of prime graphs of finite groups to determine which finite graphs can be prime graphs of solvable groups. In this essay, our main aim was to single out which 4-regular graphs can be prime graphs of solvable groups.

By showing the existence of a solvable group G such that $\rho(G) = \{p\}$ for a prime p, we were able to show that for every complete graph K_m of m vertices, there exists a solvable group G such that $\triangle(G)$ is isomorphic to K_m. We have also showed that there is a solvable group whose prime graph is the graph in Figure 5.7.

We have also shown that given any graph Γ which is a prime graph of a solvable group, then the join of Γ and a one vertex graph is a prime graph of some solvable group.

So far we don't know of any solvable group G such that $\triangle(G)$ is isomorphic to graph (3) of Figure 4.4. It has not been proved that no such group exists either.

We were also able to show that as suspected the only 4-regular graphs that can be prime graphs of some solvable groups are graphs of order less than 6. In particular, there are only two such graphs. The complete graph with 5 vertices and the graph of order 6.

As stated in H.P.Tong-Viet (2013b), it is conjectured that the only k-regular graphs that can be prime graphs of solvable groups are the complete graph of order $k+1$ and the graph of order $k+2$ when k is even. For $k \geq 5$ and odd the complete graph of order $k+1$ is the only graph that can occur.

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