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A Functional Calculus for $(\alpha,\ \alpha+1)-type\ \mathbb{R}$ Operators

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Abstract

A closed densely defined operator H, on a Banach space $\mathcal{X}$, whose spectrum is contained in $\mathbb{R}$ and satisfies

$$\left\|(z-H)^{-1}\right\| \leq c\frac{\langle z\rangle^{\alpha}}{|\Im z|^{\beta}} \qquad \forall\ z\ \notin \mathbb{R} \quad \text{with } \langle z\rangle^{\alpha} := \sqrt{|z|^2+1} \qquad (1)$$

for some α , $\beta \geq 0$; $c > 0$, is said to be of $(\alpha,\ \beta) - type\ \mathbb{R}$ (notation introduced in [10]). For $(\alpha,\ \alpha+1) - type\ \mathbb{R}$ operators we constructed an $\mathfrak{A}$-functional calculus in a more general Banach space setting (where $\mathfrak{A}$ is the algebra of smooth functions on $\mathbb{R}$ that decay like $(\sqrt{1+x^2})^{\beta}$ as $|x| \to \infty$, for some $\beta < 0$. This algebra is fully characterized in [9]). We then show that our functional calculus coincides with C_0-functional calculus for an unbounded operator acting on a Hilbert space.

Mathematics Subject Classification: 47A60

Keywords: functional calculus, spectral operator, extension

1 The definition

Let $\mathfrak{A}$ be the algebra of smooth functions on $\mathbb{R}$ that decay like $(\sqrt{1+x^2})^\beta$ as $|x| \to \infty$, for some $\beta < 0$. This algebra is fully characterized in [9]. Next let H be $(\alpha,\ \alpha+1) - type$ $\mathbb{R}$ operator (introduced in [10]). The motivation for our definition of $f(H)$ comes from two ideas. Firstly, a version of Hörmander's concept of almost analytic extensions [6, 7], as contained in the following definition.

Definition 1.1 Given $f \in \mathfrak{A}$ and $n \geq 0$, an **almost analytic extension** of f to $\mathbb{C}$ is

$$\begin{aligned} \tilde{f}_{\varphi,n}(x,y) &:= \sum_{r=0}^{n} \frac{f^{(r)}(x)(iy)^r}{r!}\varphi(x,y) \qquad (2) \\ &:= \left\{ f(x) + \sum_{r=1}^{n} \frac{f^{(r)}(x)(iy)^r}{r!} \right\} \varphi(x,y) \end{aligned}$$

where

$$\varphi(x,y) = \tau\left(\frac{y}{\langle x \rangle}\right) \quad \text{with } \langle x \rangle := \sqrt{x^2+1} \qquad (3)$$

and τ is non-negative $C_c^\infty(\mathbb{R})$ function such that $\tau(s) = \begin{cases} 1, & |s| < 1 \\ 0, & |s| > 2. \end{cases}$

The second idea in our definition of $f(H)$ comes from the
Helffer and Sjöstrand [5] integral formula,

$$f(H) := -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial f}{\partial \bar{z}}(z-H)^{-1}dxdy \qquad (4)$$

for a suitable function and operator H.

Lemma 1.2 *Let $f \in \mathfrak{A}$, then $\left|\frac{\partial}{\partial \bar{z}}\tilde{f}_{\varphi,n}(x,y)\right| = O(|y|^n)$ as $|y| \to 0$ for a fixed x. Moreover we can find $c' \in \mathbb{R}$ such that*

$$\left|\frac{\partial}{\partial \bar{z}}\tilde{f}_{\varphi,n}(x,y)\right| \leq c'\, |y|^n \ \text{ as } z \to x \in \mathbb{R}.$$

PROOF.

$$\begin{aligned}\frac{\partial}{\partial x}(\tilde{f}_{\varphi,n}(z)) &= \sum_{r=0}^{n}\frac{f^{(r+1)}(x)(iy)^r}{r!}\varphi(x,y)+\sum_{r=0}^{n}\frac{f^{(r)}(x)(iy)^r}{r!}\varphi_x(x,y)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial y}(\tilde{f}_{\varphi,n}(z)) &= \sum_{r=1}^{n}\frac{f^{(r)}(x)i(iy)^{r-1}}{(r-1)!}\varphi(x,y)+\sum_{r=0}^{n}\frac{f^{(r)}(x)(iy)^r}{r!}\varphi_y(x,y)\\
\text{thus } \frac{\partial}{\partial \bar{z}}(\tilde{f}_{\varphi,n}(z)) &= \frac{1}{2}\left(\frac{\partial \tilde{f}_{\varphi,n}}{\partial x}+i\frac{\partial \tilde{f}_{\varphi,n}}{\partial y}\right)(z)\\
&= \frac{1}{2}\left(\sum_{r=0}^{n}\frac{f^{(r)}(x)(iy)^r}{r!}\right)(\varphi_x+i\varphi_y)(z)+\frac{1}{2}f^{(n+1)}(x)\frac{(iy)^n}{n!}\varphi(z) \quad (5)\end{aligned}$$

Now,

$$\begin{aligned}\operatorname{supp}(\varphi_x+i\varphi_y) &\subseteq \left\{(x,y)\ :\ 1\le\frac{|y|}{\langle x\rangle}\le 2\right\}\\
&= \{(x,y)\ :\ \langle x\rangle\le |y|\le 2\langle x\rangle\} \qquad (6)\\
&\subset \{(x,y)\ :\ 1\le |y|\le 2\langle x\rangle\}.\end{aligned}$$

Therefore $\varphi_x+i\varphi_y$ vanishes on the strip $\Omega := \{(x,y)\ :\ -1\le y\le 1\}$. So

$$\begin{aligned}\left|\frac{\partial}{\partial \bar{z}}(\tilde{f}_{\varphi,n}(x,y))\right| &= \frac{1}{2}\left|f^{(n+1)}(x)\right|\frac{|y|^n}{n!} \quad \text{for } (x,y)\in\Omega\\
&= M_x\,|y|^n\end{aligned}$$

With $M_x=\frac{|f^{(n+1)}(x)|}{2n!}$. Thus, $\left|\frac{\partial}{\partial \bar{z}}\tilde{f}_{\varphi,n}(x,y)\right| = O(|y|^n)$ *as* $|y| \to 0$ for a fixed x.

Moreover, since $f\in\mathfrak{A}$ we can find some $\beta<0$ and $c'>0$ such that

$$\begin{aligned}M_x &= \frac{|f^{(n+1)}(x)|}{2n!}\\
&\le c'\langle x\rangle^{\beta-n-1} \qquad \text{for all } (x,y)\in\Omega\\
&\le c'\\
&< \infty\end{aligned}$$

since $\langle x\rangle^{\beta-n-1}\le 1$ for all $x\in\mathbb{R}$. Therefore $\left|\frac{\partial}{\partial\bar{z}}(\tilde{f}_{\varphi,n}(x,y))\right|\le c'|y|^n$ as $z\to x\in\mathbb{R}$. □

Lemma 1.3 *If* $\varphi(x,y):=\tau\left(\frac{y}{\langle x\rangle}\right)$ *with* τ*, a non-negative* $C_c^\infty(\mathbb{R})$ *function such that* $\tau(s)=\begin{cases}1, & |s|<1\\ 0, & |s|>2,\end{cases}$
then $|(\varphi_x+i\varphi_y)(x,y)|\le\frac{K}{\langle x\rangle}$ *for some* $K>0$.

Proof.

$$\begin{aligned} |\varphi_x(x,y)| &= \left|\frac{\partial}{\partial x}\tau\left(\frac{y}{\langle x\rangle}\right)\right| \\ &= \left|\tau'\left(\frac{y}{\langle x\rangle}\right)\cdot\frac{\partial}{\partial x}(y\langle x\rangle^{-1})\right| \\ &\leq \left|-y\tau'\left(\frac{y}{\langle x\rangle}\right)\langle x\rangle^{-2}\right|. \end{aligned}$$

Also

$$\begin{aligned} \varphi_y(x,y) &= \frac{\partial}{\partial y}\tau\left(\frac{y}{\langle x\rangle}\right) \\ &= \tau'\left(\frac{y}{\langle x\rangle}\right)\cdot\frac{\partial}{\partial y}(y\langle x\rangle^{-1}) \\ &= \tau'\left(\frac{y}{\langle x\rangle}\right)\langle x\rangle^{-1}. \end{aligned}$$

Therefore, since τ' is bounded on $\mathbb{R}$, we can set $K := 3\sup_{s\in\mathbb{R}}|\tau'(s)|$ to obtain,

$$\begin{aligned} |(\varphi_x + i\varphi_y)(x,y)| &\leq \frac{K}{3}\left[\frac{|y|}{\langle x\rangle^2} + \frac{1}{\langle x\rangle}\right] \\ &\leq \frac{K}{3}\left[\frac{2\langle x\rangle}{\langle x\rangle^2} + \frac{1}{\langle x\rangle}\right] \quad \text{(using (6))} \\ &\leq \frac{K}{\langle x\rangle}. \end{aligned}$$

□

Theorem 1.4 *Let $n > \alpha \geq 0$, $f \in \mathfrak{A}$ and H be of $(\alpha,\ \alpha+1) - type\ \mathbb{R}$. Then the integral*

$$f(H) := -\frac{1}{\pi}\int_{\mathbb{C}}\frac{\partial\tilde{f}}{\partial\bar{z}}(z-H)^{-1}dxdy \tag{7}$$

is norm convergent and defines an operator in $\mathfrak{B}(\mathcal{X})$ with

$$\|f(H)\| \leq c_\alpha\,\|f\|_{n+1} \quad \text{for some } c_\alpha > 0. \tag{8}$$

Proof. Suppose $\|(z-H)^{-1}\| \leq c\frac{\langle z\rangle^\alpha}{|\Im z|^{\alpha+1}}$ for all $z \notin \mathbb{R}$ (hypothesis). We will use the notation $(x,y) = x + iy := z$.

We observe that by (5), $\frac{\partial\tilde{f}}{\partial\bar{z}}$ is continuous and hence the integrand is norm continuous for $z \notin \mathbb{R}$.

Further,

(A)
$$\begin{aligned} \operatorname{supp}(\varphi) &\subseteq \{(x,y)\ :\ \tau\left(\frac{y}{\langle x\rangle}\right)>0\} \\ &\subseteq \left\{(x,y)\ :\ \frac{|y|}{\langle x\rangle}\le 2\right\} \\ &= \{(x,y)\ :\ 0\le |y|\le 2\langle x\rangle\} \\ &=: V \end{aligned}$$

(B)
$$\begin{aligned} \operatorname{supp}(\varphi_x+i\varphi_y) &\subseteq \left\{(x,y)\ :\ 1\le\frac{|y|}{\langle x\rangle}\le 2\right\} \\ &= \{(x,y)\ :\ \langle x\rangle\le |y|\le 2\langle x\rangle\} \\ &=: U. \end{aligned}$$

(C) For $z\in[\operatorname{supp}(\varphi)\cup\operatorname{supp}(\varphi_x+\varphi_y)]\setminus\mathbb{R}$,

$$\begin{aligned} \left\|(z-H)^{-1}\right\| &\le c\frac{\langle z\rangle^{\alpha}}{|\Im z|^{\alpha+1}} \\ &\le c\frac{\left(1+|x|^2+4\langle x\rangle^2\right)^{\alpha/2}}{|y|^{\alpha+1}} \\ &\le c5^{\alpha/2}\frac{\langle x\rangle^{\alpha}}{|y|^{\alpha+1}}. \end{aligned}$$

(D) $|(\varphi_x+i\varphi_y)(x,y)|\le\frac{K}{\langle x\rangle}$ for some $K>0$, Lemma 1.3.
Also, since φ is bounded, let $M:=\sup_{z\in\mathbb{C}}\{|\varphi(z)|\}$.
Therefore, using the expansion (5) and the estimates above, we have

$$\begin{aligned} \|f(H)\| &\le \frac{c5^{\frac{\alpha}{2}}}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\sum_{r=0}^{n}\left|f^{(r)}(x)\right||y|^r\frac{K}{\langle x\rangle}\chi_U(z)+M\left|f^{(n+1)}(x)\right||y|^n\chi_V(z)\right)\frac{\langle x\rangle^{\alpha}}{|y|^{\alpha+1}}dxdy \\ &= \frac{c5^{\frac{\alpha}{2}}}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\sum_{r=0}^{n}|f^{r}(x)||y|^{r-\alpha-1}K\langle x\rangle^{\alpha-1}\chi_U(z)+\right. \\ &\qquad \left.+M\left|f^{(n+1)}(x)\right||y|^{n-\alpha-1}\langle x\rangle^{\alpha}\chi_V(z)\right)dxdy. \end{aligned}$$

Integrating with respect to y yields the bound

$$\begin{aligned} \|f(H)\| &\le \frac{c'5^{\frac{\alpha}{2}}}{\pi}\int_{-\infty}^{\infty}\left(\sum_{r=0}^{n}|f^{r}(x)|\left[|y|^{r-\alpha}\right]_{\langle x\rangle}^{2\langle x\rangle}\cdot\langle x\rangle^{\alpha-1}+\left|f^{(n+1)}(x)\right|\left[|y|^{n-\alpha}\right]_{0}^{2\langle x\rangle}\cdot\langle x\rangle^{\alpha}\right)dx \\ &= c_\alpha\int_{-\infty}^{\infty}\left(\sum_{r=0}^{n}\left|f^{(r)}(x)\right|\langle x\rangle^{r-1}+\left|f^{(n+1)}\right|\langle x\rangle^{n}\right)dx \\ &= c_\alpha\|f\|_{n+1} \qquad \text{with } c_\alpha:=\frac{c5^{\alpha/2}2^{n-\alpha}}{\pi}\cdot\max\{K,M\}. \end{aligned}$$

□

Similar integrals to that in (4) play a central role in the theory of uniform algebras, Gamelin [4].

It may seem from the computation above that our definition of $f(H)$ depends implicitly on the cut-off function φ and n. However we will prove shortly that $f(H)$ is independent of both φ and n, provided $n > \alpha$.

Lemma 1.5 *If $F \in C_c^\infty(C)$ and $F(z) = O(|y|^\beta)$ as $y \to 0$ for some $\beta > \alpha + 1$, then*

$$-\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial F}{\partial \bar{z}}(z - H)^{-1} dxdy = 0. \tag{9}$$

Proof. Let F have support in $\{z = (x, y) : |x| < N \text{ and } |y| < N\}$ and define Ω_δ for small $\delta > 0$ to be the region $\{z = (x, y) : |x| < N \text{ and } \delta < |y| < N\}$ (see figure 1).

Figure 1: Close up on the support of F by compact regions.

$$
\begin{aligned}
A &:= -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial F}{\partial \bar{z}}(z-H)^{-1}dxdy \\
&= -\lim_{\delta\to 0}\frac{1}{\pi}\int_{\Omega_\delta} \frac{\partial F}{\partial \bar{z}}(z-H)^{-1}dxdy \\
&= \lim_{\delta\to 0}\frac{i}{2\pi}\int_{\partial\Omega_\delta} F(z)(z-H)^{-1}dz \quad \text{(Green's Theorem)} \\
&= \lim_{\delta\to 0}\frac{i}{2\pi}\sum_{r=1}^{8}\int_{L_r} F(z)(z-H)^{-1}dz \\
&= \lim_{\delta\to 0}\frac{i}{2\pi}\left(\int_{L_1} F(z)(z-H)^{-1}dz + \int_{L_5} F(z)(z-H)^{-1}dz\right) \\
&\quad \text{since } [\operatorname{supp}(F)] \cap \left[\left(\cup_{r=2}^{4} L_r\right) \cup \left(\cup_{r=6}^{8} L_r\right)\right] = \emptyset.
\end{aligned}
$$

Now for $(x,y)\in L_1\cup L_5\subset\mathbb{C}\setminus\mathbb{R}$,

$$
\left\|(z-H)^{-1}\right\| \le c\frac{\langle z\rangle^{\alpha}}{|\Im z|^{\alpha+1}} = c\frac{(1+|x|^2+\delta^2)^{\alpha/2}}{\delta^{\alpha+1}} \le c\frac{2^{\alpha/2}\langle N\rangle^{\alpha}}{\delta^{\alpha+1}}.
$$

Therefore

$$
\|A\| \le c2^{\alpha/2}\langle N\rangle^{\alpha}\lim_{\delta\to 0}\int_{-N}^{N}\{|F(x+i\delta)|+|F(x-i\delta)|\}\delta^{-\alpha-1}dx = 0,
$$

since by hypothesis the integrand is $O(\delta^{\beta-\alpha-1})$. □

Theorem 1.6 *The operator* $f(H)$ *is independent of* n *and the cut-off function* φ *defined in* (3), *provided* $n>\alpha$.

Proof. $C_c^\infty(\mathbb{R})$ is dense in $\mathfrak{A}$ with respect to each norm $\|\cdot\|_{n+1}$ [9, Lemma 1.5]. This result together with (8) imply that it is sufficient to prove this for $f\in C_c^\infty$.

If $f\in C_c^\infty(\mathbb{R})$ while φ_1 and φ_2 are cut-off functions define in terms of say τ_1 and τ_2, let

$$
\begin{aligned}
\Omega_1 &:= \left\{(x,y)\ :\ \frac{|y|}{\langle x\rangle}<\epsilon_1\right\} \quad \text{for some } \epsilon_1>0 \\
&= \{(x,y)\ :\ -\epsilon_1\langle x\rangle < y < \epsilon_1\langle x\rangle\} \\
&\subseteq \{z\ :\ \varphi_1(z)=1\}.
\end{aligned}
$$

Similarly let

$$
\begin{aligned}
\Omega_2 &:= \{(x,y)\ :\ -\epsilon_2\langle x\rangle < y < \epsilon_2\langle x\rangle\} \quad \text{for some } \epsilon_2>0 \\
&\subseteq \{z\ :\ \varphi_2(z)=1\}.
\end{aligned}
$$

$$
\begin{aligned}
\text{Now set}\quad \Omega &:= \Omega_1\cap\Omega_2 \\
&= \{(x,y)\ :\ -\epsilon\langle x\rangle < y < \epsilon\langle x\rangle\} \quad \text{with } \epsilon:=\min\{\epsilon_1,\epsilon_2\} \\
&\ne \emptyset.
\end{aligned}
$$

Then for $z \in \Omega$,

$$\begin{aligned}\tilde{f}_{\varphi_1,n}(z) - \tilde{f}_{\varphi_2,n}(z) &= \sum_{r=0}^{n} \frac{f^{(r)}(x)(iy)^r}{r!}\left[\varphi_1(z) - \varphi_2(z)\right] \\ &= 0 \quad \text{since } \varphi_1(z) = \varphi_2(z) = 1 \ \text{ for all } \ z \in \Omega.\end{aligned}$$

This exceeds the hypothesis of lemma 1.5, so invoking lemma 1.5, we have $\tilde{f}_{\varphi_1,n}(H) = \tilde{f}_{\varphi_2,n}(H)$. That is $\tilde{f}_{\varphi,n}(H)$ is independent of φ.

On the other hand, if $m > n > \alpha$ then

$$\begin{aligned}\tilde{f}_{\varphi_1,m}(z) - \tilde{f}_{\varphi_1,n}(z) &= \left(\sum_{r=0}^{m} \frac{f^{(r)}(x)(iy)^r}{r!} - \sum_{r=0}^{n} \frac{f^{(r)}(x)(iy)^r}{r!}\right)\varphi_1(z) \\ &= \sum_{r=n+1}^{m} \frac{f^{(r)}(x)(iy)^r}{r!}\varphi_1(z) \\ &=: y^{n+1}K(z) \qquad (\text{some bounded } \ K : \mathbb{C} \to \mathbb{C})\end{aligned}$$

and since $n+1 > \alpha+1$ we invoke Lemma 1.5 to conclude that $\tilde{f}_{\varphi_1,m}(H) = \tilde{f}_{\varphi_1,n}(H)$. That is $\tilde{f}_{\varphi,n}(H)$ is independent of n. □

Henceforth we will assume that the condition of theorem 1.6 holds and write $\tilde{f}$ instead of $\tilde{f}_{\varphi,n}$ unless a specific cut-off function or n is needed for some purpose which will be stated.

2 The homomorphism $\mathfrak{A} \ni f \mapsto f(H) \in \mathfrak{B}(\mathcal{X})$

In this paper, **support of** f will be understood to be the set

$$\text{supp}\,(f) := \overline{\{x \in \mathbb{R} \ : \ f(x) \neq 0\}}.$$

Thus $\text{supp}\left(\tilde{f}\right)$ is a closed set.

Theorem 2.1 *Let H be an operator of $(\alpha,\ \alpha+1)-$type $\mathbb{R}$ for some $\alpha \geq 0$. If $f \in C_c^\infty(\mathbb{R})$ has support disjoint from $\sigma(H)$, then $f(H) = 0$.*

Proof. By regularity of $\mathbb{C}$, we can find an open set G with $\text{supp}\left(\tilde{f}\right) \subset G$ and $G \cap \sigma(H) = \emptyset$. Since by hypothesis, $\text{supp}\left(\tilde{f}\right)$ is compact, there exists a finite set of smooth curves, $\{\Upsilon_r\}_{r=1}^m$ 'enclosing' $\text{supp}\left(\tilde{f}\right)$ in G. Thus if we put

$\Gamma := \cup_{r=1}^{m}\Upsilon_r$, and $D := ins\Gamma$ then

$$\begin{aligned} f(H) &= -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial \tilde{f}(z)}{\partial \bar{z}}(z-H)^{-1}dz \\ &= -\frac{1}{\pi}\int_{D} \frac{\partial \tilde{f}(z)}{\partial \bar{z}}(z-H)^{-1}dz \\ &= \frac{i}{2\pi}\int_{\Gamma} \tilde{f}(z)(z-H)^{-1}dz \quad \text{(Green's Theorem)} \\ &= \frac{i}{2\pi}\sum_{r=1}^{m}\int_{\Upsilon_r} \tilde{f}(z)(z-H)^{-1}dz \\ &= 0 \quad \text{since } \tilde{f}(z)=0 \ \text{ for all } \ z \in \Upsilon_r \ r=1,2,\ldots,m \end{aligned}$$

□

Corollary 2.2 *Let H be an operator of $(\alpha,\ \alpha+1)-$type $\mathbb{R}$ for some $\alpha \geq 0$. If $f \in \mathfrak{A}$ has support disjoint from $\sigma(H)$ then $f(H)=0$.*

Proof. Follows from theorem 2.1, inequality (8) and density of $C_c^\infty(\mathbb{R})$ in $\mathfrak{A}$ [9, lemma 1.5]. □

Theorem 2.3 *If $f, g \in \mathfrak{A}$ and H is of $(\alpha,\ \alpha+1)-$type $\mathbb{R}$ then*

$$(fg)(H) = f(H)g(H).$$

Proof. We first assume that f and g lie in $C_c^\infty(\mathbb{R})$. Let $K := \operatorname{supp}\left(\tilde{f}\right)$ and $L := \operatorname{supp}(\tilde{g})$ so that K and L are compact subsets of $\mathbb{C}$ and write

$$\begin{aligned} f(H) &= -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}}(z-H)^{-1}dxdy, \quad z =: x+iy \\ \text{and} \quad g(H) &= -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{w}}(w-H)^{-1}dudv, \quad w =: u+iv \end{aligned}$$

$$\text{Then} \quad f(H)g(H) = \frac{1}{\pi^2}\int\int_{K\times L} \frac{\partial \tilde{f}}{\partial \bar{z}}\frac{\partial \tilde{g}}{\partial \bar{w}}(z-H)^{-1}(w-H)^{-1}dxdydudv.^{1}$$

Using resolvent equation,

$$(z-H)^{-1}(w-H)^{-1} = (z-w)^{-1}(w-H)^{-1} - (z-w)^{-1}(z-H)^{-1},$$

we may expand $f(H)g(H)$ as

[1] $K \times L := \{(k,l)\ :\ k \in K, l \in L\}$

$$\begin{aligned} f(H)g(H) &= \frac{1}{\pi^2}\int\int_{K\times L}\frac{\partial\tilde{f}}{\partial\bar{z}}\frac{\partial\tilde{g}}{\partial\bar{w}}\left[(z-w)^{-1}(w-H)^{-1}-(z-w)^{-1}(z-H)^{-1}\right]dxdydudv \\ &= \frac{-1}{\pi}\int_{K\times L}\left(\frac{\partial\tilde{g}}{\partial\bar{w}}(w-H)^{-1}\frac{-1}{\pi}\int_K\frac{\partial\tilde{f}}{\partial\bar{z}}(z-w)^{-1}dxdy\right)dudv \\ &\quad -\frac{-1}{\pi}\int_{K\times L}\left(\frac{\partial\tilde{f}}{\partial\bar{z}}(z-H)^{-1}\frac{-1}{\pi}\int_L\frac{\partial\tilde{g}}{\partial\bar{w}}(z-w)^{-1}dudv\right)dxdy \end{aligned}$$

But

$$\begin{aligned} -\frac{1}{\pi}\int_K\frac{\partial\tilde{f}}{\partial\bar{z}}(z-w)^{-1}dxdy &= -\frac{1}{\pi}\int_{\mathbb{C}}\frac{\partial\tilde{f}}{\partial\bar{z}}(z-w)^{-1}dxdy \\ &= \tilde{f}(w) \quad \text{(Cauchy-Green Theorem).} \end{aligned}$$

Also

$$\begin{aligned} \frac{1}{\pi}\int_L\frac{\partial\tilde{g}}{\partial\bar{w}}(z-w)^{-1}dudv &= -\frac{1}{\pi}\int_{\mathbb{C}}\frac{\partial\tilde{g}}{\partial\bar{w}}(w-z)^{-1}dudv \\ &= \tilde{g}(z) \quad \text{(Cauchy-Green Theorem).} \end{aligned}$$

These lead to the identity

$$\begin{aligned} f(H)g(H) &= -\frac{1}{\pi}\int_{K\times L}\frac{\partial\tilde{g}(w)}{\partial\bar{w}}(w-H)^{-1}\tilde{f}(w)dudv+\frac{-1}{\pi}\int_{K\times L}\frac{\partial\tilde{f}(z)}{\partial\bar{z}}\tilde{g}(z)(z-H)^{-1}dxdy \\ &= -\frac{1}{\pi}\int_{K\times L}\frac{\partial\tilde{g}(z)}{\partial\bar{z}}(z-H)^{-1}\tilde{f}(z)dxdy+\frac{-1}{\pi}\int_{K\times L}\frac{\partial\tilde{f}(z)}{\partial\bar{z}}\tilde{g}(z)(z-H)^{-1}dxdy \\ &= -\frac{1}{\pi}\int_{K\times L}\left\{\tilde{f}(z)\frac{\partial\tilde{g}(z)}{\partial\bar{z}}+\frac{\partial\tilde{f}(z)}{\partial\bar{z}}\tilde{g}(z)\right\}(z-H)^{-1}dxdy \\ &= -\frac{1}{\pi}\int_{K\times L}\frac{\partial(\tilde{f}\tilde{g})(z)}{\partial\bar{z}}(z-H)^{-1}dxdy. \end{aligned}$$

In order to prove that

$$fg(H)=-\frac{1}{\pi}\int_{K\times L}\frac{\partial(\tilde{f}\tilde{g})(z)}{\partial\bar{z}}(z-H)^{-1}dxdy,$$

we must prove that

$$-\frac{1}{\pi}\int_{\mathbb{C}}\frac{\partial k(z)}{\partial\bar{z}}(z-H)^{-1}dxdy=0,$$

where $k(z):=\tilde{f}(z)\tilde{g}(z)-\widetilde{(fg)}(z)$. Since k is of compact support and by Theorem 1.6 and Lemma1.2 may be assumed to satisfy the hypothesis of Lemma 1.5, this follows by invoking Lemma 1.5.

Finally, suppose that $f, g \in \mathfrak{A}$ and let $\phi \in C_c^\infty$ such that

$$\phi(s) = \begin{cases} 1, & |s| < 1 \\ 0, & |s| > 2 \end{cases}$$

Set $\phi_m(s) := \phi(s/m)$ and $f_m := \phi_m f,\ g_m := \phi_m g$, and $h_m := \phi_m^2 fg$.
Then $f_m \to f,\ g_m \to g$ and $h_m \to fg$ in the norm $\|.\|_p$ for some $p > \alpha$. See proof of [9, Lemma 1.5].
From above we have

$$h_m(H) = f_m(H) g_m(H) \quad \text{for all} \ \ m.$$

We finally use the density of $C_c^\infty(\mathbb{R})$ in $\mathfrak{A}$ and (8) to complete the proof. □

Lemma 2.4 *Let* $g \in \mathfrak{A}$ *with* $g = 0$ *on* $[0, \infty)$. *If* H *is of* $(\alpha,\ \alpha+1)-type\ \mathbb{R}$ *and* $\sigma(H) \subseteq [0, \infty)$ *then* $g(H) = 0$.

Proof. For $\epsilon \in (0, \infty)$, let $H_\epsilon := \epsilon + H$. Then H_ϵ is of $(\alpha,\ \alpha+1)-type\ \mathbb{R}$ (since $(\alpha,\ \beta)-type\ \mathbb{R}$ operators are stable under perturbations by reals, [10, Theorem 3.6]). But $\sigma(H) \subset [0, \infty)$ implies that $\sigma(H_\epsilon) \subset [\epsilon, \infty)$, and since $\mathrm{supp}\,(g) \subseteq (\infty, 0]$ we must have $g(H_\epsilon) = 0$ by Theorem 2.1.

Now

$$\begin{aligned} 0 \quad = \quad g(H_\epsilon) &= -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial}{\partial \bar{w}} \tilde{g}(w)(w - (\epsilon + H))^{-1} dudv \\ &= -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial}{\partial \bar{z}} \tilde{g}(z + \epsilon)(z - H)^{-1} dxdy \\ &= -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\partial}{\partial \bar{z}} \tilde{g}_\epsilon(z)(z - H)^{-1} dxdy \\ &= g_\epsilon(H) \end{aligned}$$

where $z := w - \epsilon$ and $g_\epsilon := \tau_\epsilon g \in \mathfrak{A}$ (since $\mathfrak{A}$ is invariant under translations [9, Lemma 1.2]).

So by (8)

$$\|g_\epsilon(H) - g(H)\| = \|g(H)\| \le c_\alpha \|g_\epsilon - g\|_{n+1}, \quad \text{for some } n > \alpha,\ c_\alpha > 0 \text{ for all } \epsilon > 0.$$

Suppose $g_\epsilon \in \mathfrak{S}^{\beta_\epsilon}$ for some $\beta_\epsilon < 0$ and $\epsilon \ge 0$, where we set $g_0 := g$. Then $|g_\epsilon(x)| \le c_{r,\epsilon} \langle x \rangle^{\beta_\epsilon - r}$ for each $x \in \mathbb{R}$.

Let $\beta := \sup\{\beta_\epsilon \ : \ \epsilon \in (0, \infty)\} < 0$ and $c := \max\limits_{0<r\le n} \sup\limits_{\epsilon\in(0,\infty)} \{c_{r,\epsilon}\} > 0$.

β and c exist and are finite, see the proof of [9, Lemma 1.2]. Thus

$$\left|g_\epsilon^{(r)}(x)\right| \langle x \rangle^{r-1} \le c \langle x \rangle^{\beta - r} \langle x \rangle^{r-1} = c \langle x \rangle^{\beta - 1}$$

and the function $h(x) = c \langle x \rangle^{\beta-1}$ is integrable and $\left|\frac{d^r}{dx^r}(g_\epsilon(x) - g(x))\right| \langle x \rangle^{r-1} \leq h(x)$ for each ϵ. Therefore by dominated convergence theorem we have

$$\int_0^\infty \left|g_\epsilon^{(r)}(x) - g^{(r)}(x)\right| \langle x \rangle^{r-1} dx \to 0 \text{ as } \epsilon \to 0,$$

that is $\|g_\epsilon - g\|_{n+1} \to 0$ as $\epsilon \to 0$. Thus $\|g(H)\| = 0$. □

Let $E : C^\infty([0, \infty)) \to C^\infty(\mathbb{R})$ be the Seeley's extension operator [11]. For $f \in C^\infty([0, \infty))$ such that $Ef \in \mathfrak{A}$ we define $f(H)$ to be $Ef(H)$ where $Ef(H)$ is given by (7) with appropriate condition on $\|(z - H)^{-1}\|$.

Theorem 2.5 *If $f : [0, \infty) \to \mathbb{C}$ is such that*

$$\left|\frac{d^r}{dx^r} f(x)\right| \leq c_r \langle x \rangle^{\beta - r} \tag{10}$$

for some $\beta < 0$, for all $r \geq 0$ and for all $x \geq 0$; and H is of $(\alpha,\ \alpha+1) -$ type $\mathbb{R}$ with $\sigma(H) \subseteq [0, \infty)$, then $f(H)$ is uniquely determined and

$$\|f(H)\| \leq k \|f\|_{n+1}^{+}, \quad k > 0, \quad \text{whenever } n > \alpha.$$

Proof. Observe that $Ef \in \mathfrak{A}$, where E is Seeley's extension operator (Lemma 3.3[9]). So that
$f(H) \equiv Ef(H)$ is defined and $f(H) \in \mathfrak{B}(\mathcal{X})$.

Moreover,

$$\begin{aligned} \|f(H)\| &= \|(Ef)(H)\| \\ &\leq c_{n+1} \|Ef\|_{n+1} \quad [(8)] \\ &\leq c' c_{n+1} \|f\|_{n+1}^{+} \quad ([9, \text{Theorem } 3.4]) \\ &=: K \|f\|_{n+1}^{+}. \end{aligned}$$

Finally if $g \in \mathfrak{A}$ is another extension of f, set

$$h := g - Ef \in \mathfrak{A}$$

which implies $h = 0$ on $[0, \infty)$ and thus by Lemma 2.4

$$h(H) = 0.$$

□

Corollary 2.6 *Let $f, g \in C^\infty([0, \infty))$ satisfy (10) with H of $(\alpha,\ \alpha+1) -$ type $\mathbb{R}$ and $\sigma(H) \subseteq [0, \infty)$. Then*

$$(fg)(H) = f(H)g(H).$$

Proof.

$$(Ef)(t) := \begin{cases} \sum_{k=0}^{\infty} a_k\phi(b_k t) f(b_k t), & t<0 \\ f(t), & t \geq 0 \end{cases}$$

and

$$(Eg)(t) := \begin{cases} \sum_{k=0}^{\infty} a_k\phi(b_k t) g(b_k t), & t<0 \\ g(t), & t \geq 0 \end{cases}$$

Thus $(Eg)(t)(Ef)(t) := \begin{cases} \left(\sum_{k=0}^{\infty} a_k\phi(b_k t) g(b_k t)\right)\left(\sum_{k=0}^{\infty} a_k\phi(b_k t) f(b_k t)\right), & t<0 \\ g(t)f(t), & t \geq 0. \end{cases}$ Clearly gf satisfies (10) and

$$E(gf)(t) := \begin{cases} \sum_{k=0}^{\infty} a_k\phi(b_k t)(gf)(b_k t), & t<0 \\ g(t)f(t), & t \geq 0. \end{cases}$$

Thus, $(Eg)(t)\cdot(Ef)(t) - E(gf)(t) = 0, \quad t \geq 0.$
Therefore by Lemma 2.4 $(Eg)(H)\cdot(Ef)(H) = E(gf)(H)$.
i.e. $g(H)\cdot f(H) = gf(H)$. □

Remark 2.7 Theorem 2.3 and Corollary 2.6 show that the map

$$\begin{aligned} \kappa : \mathfrak{A} &\to \mathfrak{B}(\mathcal{X}) \\ f &\mapsto f(H) \end{aligned}$$

is an algebra homomorphism. We prove one more result to verify that κ is a functional calculus.

Theorem 2.8 *Let* H *be an operator of* $(\alpha,\ \alpha+1)-$*type* $\mathbb{R}$ *for some* $\alpha \geq 0$. *If* $w \notin \mathbb{R}$ *and* $r_w(x) := (w-x)^{-1}$ *for all* $x \in \mathbb{R}$ *then* $r_w \in \mathfrak{A}$ *and*

$$r_w(H) = (w-H)^{-1}.$$

Proof. Clearly $r_w \in \mathfrak{A}$, and without loss of generality, suppose that $\Im w > 0$. For large $m>0$ define $\Omega_m := \{(x,y) \ : \ |x| < m \text{ and } \frac{\langle x\rangle}{m} < |y| < 2m\}$.

The boundary of Ω_m consists of two closed curves, both traversed in the anti-clockwise direction, see figure 2.

With τ as in definition 1.1, put

$$\varphi(z) := \tau\left(\frac{\lambda\,|y|}{\langle x\rangle}\right)$$

where $\lambda > 0$ is chosen

1. large enough to ensure that $w \notin \operatorname{supp}(\varphi)$.

Figure 2: Close up on $\mathbb{C}$, over which $r_w(z)$ is integrated.

2. so that for each $m \geq 1$, $|y| < \frac{2\langle m\rangle}{\lambda} \leq 2m$, for all $(x, y) \in \Omega_m$. Thus, $\langle 1/m\rangle \leq \lambda$. Since $\langle 1/m\rangle < 2$ for all $m \geq 1$ we may assume that $\lambda \geq 2$.

An application of Green's Theorem yields

$$\begin{aligned} r_w(H) &= -\lim_{m \to \infty} \frac{1}{\pi} \int_{\Omega_m} \frac{\partial \tilde{r}_w}{\partial \bar{z}} (z-H)^{-1} dx dy \\ &= \lim_{m \to \infty} \frac{i}{2\pi} \int_{\partial\Omega_m} \tilde{r}_w(z)(z-H)^{-1} dz. \end{aligned}$$

We next show that

$$\lim_{m \to \infty} \left\| \int_{\partial\Omega_m} \{r_w(z) - \tilde{r}_w(z)\}(z-H)^{-1} dz \right\| = 0.$$

$\partial\Omega_m$ consists of four vertical straight lines, two horizontal straight lines and two curves. The integral is estimated separately on each of these.

1. Vertical lines: Suppose γ_1 is the vertical line in the first quadrant. Using Taylor's approximation theorem to expand $r_w(z)$ at $(m, 0)$, we obtain, for all $z \in \gamma_1$,

$$r_w(z) = \sum_{s=0}^{n} \frac{r_w^{(s)}(m)(iy)^s}{s!} + R(z; m)$$

with $R(z;m) := \frac{r_w^{(n+1)}(d)(iy)^{n+1}}{(n+1)!}$, $\quad d = m + \epsilon iy$ for some $\epsilon \in (0,1)$.

Therefore, for any $z \in \gamma_1$ we have

$$r_w(z) = \sum_{s=0}^{n} \frac{r_w^{(s)}(m)(iy)^s}{s!} + \frac{r_w^{(n+1)}(d)(iy)^{n+1}}{(n+1)!}$$

which implies

$$\begin{aligned}
&|r_w(z) - \tilde{r}_w(z)| \\
&\quad \leq \ |(1-\varphi(z))r_w(z)| + \varphi(z)\left| r_w(z) - \frac{\tilde{r}_w(z)}{\varphi(z)} \right| \\
&\quad = \ c_1\chi(z)\langle z\rangle^{-1} + \varphi(z)\left| \sum_{s=0}^{n} \frac{r_w^{(s)}(m)(iy)^s}{s!} + \frac{r_w^{(n+1)}(d)(iy)^{n+1}}{(n+1)!} - \sum_{s=0}^{n} r_w^{(s)}(m)\frac{(iy)^s}{s!} \right| \\
&\quad \leq \ c_1\chi(z)\langle z\rangle^{-1} + c_w\frac{|y|^{n+1}}{\langle d\rangle^{n+2}}
\end{aligned}$$

where $\chi(z) := \begin{cases} 1 & if\ \ \langle x\rangle < \lambda|y| \\ 0 & \text{otherwise.} \end{cases}$

But $z = m + iy,\ d = m + i\epsilon y$ implies $\langle z\rangle \geq \langle m\rangle$ and $\langle m\rangle \leq \langle d\rangle$. So,

$$|r_w(z) - \tilde{r}_w(z)| \leq c_1\chi(z)\langle m\rangle^{-1} + c_w\frac{|y|^{n+1}}{\langle m\rangle^{n+2}}$$

Also, for $z := m + iy \in \gamma_1$, $\quad \frac{\langle m\rangle}{m} \leq |y| \leq 2m$ and hence

$$\begin{aligned}
\langle z\rangle^2 &= 1 + |m|^2 + |y|^2 \\
&\leq 1 + m^2 + 4m^2 \\
&\leq 5\langle m\rangle^2.
\end{aligned}$$

Therefore

$$\left\|(z-H)^{-1}\right\| \leq \frac{c5^{\alpha/2}\langle m\rangle^{\alpha}}{|y|^{\alpha+1}} \qquad \text{for some } c > 0. \tag{11}$$

Hence

$$
\begin{aligned}
&\left\|\int_{\gamma_1} \{r_w(z) - \tilde{r}_w(z)\}(z-H)^{-1} dz\right\| \\
\leq\ & cc_1 5^{\alpha/2} \int_{\frac{\langle m\rangle}{\lambda}}^{2m} \langle m\rangle^{-1} \frac{\langle m\rangle^{\alpha}}{y^{\alpha+1}} dy + cc_w 5^{\alpha/2} \int_{\frac{\langle m\rangle}{m}}^{2m} |y|^{n-\alpha} \langle m\rangle^{\alpha-n-2} dy \\
\leq\ & cc_1 5^{\alpha/2} \langle m\rangle^{\alpha-1} \int_{\frac{\langle m\rangle}{\lambda}}^{2m} \frac{\lambda^{\alpha+1}}{\langle m\rangle^{\alpha+1}} dy + cc_w 5^{\alpha/2} \langle m\rangle^{\alpha-n-2} \int_{\frac{\langle m\rangle}{m}}^{2m} |2m|^{n-\alpha} dy \\
\leq\ & cc_1 5^{\alpha/2} \langle m\rangle^{\alpha-1} \frac{\lambda^{\alpha+1}}{\langle m\rangle^{\alpha+1}} \left|2m - \frac{\langle m\rangle}{\lambda}\right| + cc_w 5^{\alpha/2} \langle m\rangle^{\alpha-n-2} |2m|^{n-\alpha} \left|2m - \frac{\langle m\rangle}{m}\right| \\
=\ & (m^{-1}) \left\{ c_1' \langle 1/m\rangle^{-2} \left|2 - \frac{\langle 1/m\rangle}{\lambda}\right| + c_w' \langle 1/m\rangle^{\alpha-n-2} \left|2 - \frac{\langle 1/m\rangle}{m}\right| \right\} \\
=\ & O(m^{-1}) \quad \text{as } m \to \infty, \quad \text{provided } n > \alpha.
\end{aligned}
$$

The estimate is valid for all vertical lines.

2. The curves: Let γ_2 be the curve in the upper half plane, i.e $\gamma_2 := \{(x,y) \ : \ y = \frac{\langle x\rangle}{m}\}$.

   Since $\frac{1}{m}\langle x\rangle < \frac{1}{\lambda}\langle x\rangle$ for all $m > \lambda$, $\varphi(z) = 1$ for all $z \in \gamma_2$ and $m > \lambda$. Therefore using Taylor's approximation at $(x,0)$ with $d := x + \epsilon i y$ for some $\epsilon \in (0,1)$ we have,

$$
\begin{aligned}
|r_w(z) - \tilde{r}_w(z)| &\leq |(1-\varphi(z)) r_w(z)| + \varphi(z) \left| r_w(z) - \frac{\tilde{r}_w(z)}{\varphi(z)} \right| \\
&= \varphi(z) \left| r_w(z) - \frac{\tilde{r}_w(z)}{\varphi(z)} \right| \\
&\leq c_w \frac{r_w^{(n+1)}(d) |y|^{n+1}}{(n+2)!} \\
&\leq c_w \frac{|y|^{n+1}}{\langle d\rangle^{n+2}}, \quad z \in \gamma_2, \quad m > \lambda
\end{aligned}
$$

where $\langle d\rangle \geq \langle x\rangle$. Also, for $z \in \gamma_2$,

$$
\begin{aligned}
\langle z\rangle^2 &= 1 + |x|^2 + |y|^2 \\
&= \langle x\rangle^2 + \frac{\langle x\rangle^2}{m^2} \\
&= \frac{\langle m\rangle^2}{m^2} \langle x\rangle^2 .
\end{aligned}
$$

Hence $\|(z-H)^{-1}\| \leq \frac{c\langle m\rangle^{\alpha}\langle x\rangle^{\alpha}}{m^{\alpha}|y|^{\alpha+1}}$ for some $c>0$ and

$$\begin{aligned}
\left\|\int_{\gamma_2}\{r_w(z)-\tilde{r}_w(z)\}(z-H)^{-1}dz\right\| &\leq c_\omega c\int_{\gamma_2}\frac{|y|^{n+1}}{\langle x\rangle^{n+2}}\frac{\langle m\rangle^{\alpha}\langle x\rangle^{\alpha}}{m^{\alpha}|y|^{\alpha+1}}dz \\
&= c_1\int_{\gamma_2}\frac{\langle m\rangle^{\alpha}}{m^{\alpha}}\langle x\rangle^{\alpha-n-2}|y|^{n-\alpha}\,dz \\
&= c_1\int_{\gamma_2}\frac{\langle m\rangle^{\alpha}}{m^{\alpha}}\langle x\rangle^{\alpha-n-2}\frac{\langle x\rangle^{n-\alpha}}{m^{n-\alpha}}dz \\
&= c_1\frac{\langle m\rangle^{\alpha}}{m^{n}}\int_{\gamma_2}\langle x\rangle^{-2}\,dz \\
&= O(m^{\alpha-n}) \quad \text{as } m\ \rightarrow\ \infty
\end{aligned}$$

provided $n>\alpha$. The estimate here is also valid for the other curve.

3. <u>The horizontal lines</u> Let γ_3 be the horizontal line in the upper half plane, i.e $\gamma_3 := \{(x,y)\ :\ y=2m\}$. Now supp $(\varphi)\subset\left\{(x,y)\ :\ \frac{\lambda|y|}{\langle x\rangle}\leq 2\right\}$ Thus $\varphi(z)=0$ for all $z\in\overline{\Omega_m}$ with $|y|>\frac{2\langle m\rangle}{\lambda}$.

   So, for $z\in\gamma_3$, $\varphi(z)=0$ if $2m>\frac{2\langle m\rangle}{\lambda}$, that is $\lambda>\langle 1/m\rangle$.
   Therefore if we choose m large enough so that $\lambda>\langle 1/m\rangle$, then for $z\in\gamma_3$,

$$\begin{aligned}
|r_w(z)-\tilde{r}_w(z)| &= |r_w(z)| \\
&\leq \frac{c_1}{\langle z\rangle} \\
&= \frac{c_1}{\sqrt{1+|x|^2+|2m|^2}} \\
&\leq \frac{c_1}{\langle 2m\rangle}.
\end{aligned}$$

   Also, for $z\in\gamma_3$,

$$\begin{aligned}
\langle z\rangle^2 &= 1+|x|^2+|y|^2 \\
&\leq 1+m^2+4m^2 \\
&\leq 5\langle m\rangle^2.
\end{aligned}$$

Hence $\|(z-H)^{-1}\| \quad \leq \quad \frac{c5^{\alpha/2}\langle m\rangle^{\alpha}}{|2m|^{\alpha+1}}$ for some $c>0$ and

$$\begin{aligned}\left\|\int_{\gamma_3}\{r_w(z)-\tilde{r}_w(z)\}(z-H)^{-1}dz\right\| &\leq c_1c5^{\alpha/2}\int_{\gamma_3}\frac{1}{\langle 2m\rangle}\frac{\langle m\rangle^{\alpha}}{(2m)^{\alpha+1}}dz\\ &\leq c_\omega m^{-2}\left\langle 1/2m\right\rangle^{-1}\left\langle 1/m\right\rangle^{\alpha}\int_{-m}^{m}dx\\ &= 2c_\omega m^{-2}\left\langle 1/2m\right\rangle^{-1}\left\langle 1/m\right\rangle^{\alpha}m\\ &= O(m^{-1}) \quad \text{as } m \ \to \ \infty\end{aligned}$$

provided $n>\alpha$. The estimate here is also valid for the other horizontal line.

Combining all the cases we obtain

$$r_w(H)=\frac{i}{2\pi}\lim_{m\to\infty}\int_{\partial\Omega_m}r_w(z)(z-H)^{-1}dz.$$

The integrand is holomorphic on and inside the part of $\partial\Omega_m$ in the lower half plane, so the contribution of that integral is zero by Cauchy's theorem. The integrand is meromorphic in the upper half plane with a single pole at $z=w$. Therefore

$$\begin{aligned}r_w(H) &= -\text{Res}_{z=w}\{r_w(z)(z-H)^{-1}\}\\ &= (w-H)^{-1}\end{aligned}$$

where $\text{Res}_{z=w}f(z)$ denotes the residue of f at the pole w. □

Definition 2.9 By a **$\mathfrak{A}$-functional calculus** for an operator H of $(\alpha,\ \alpha+1)-type\ \mathbb{R}$ we will mean a continuous linear map κ from $\mathfrak{A}$ into $\mathfrak{B}(\mathcal{X})$ such that

1. $\kappa(fg)=\kappa(f)\kappa(g)$, for all $f,g,\in\mathfrak{A}$.
2. If $w\notin\mathbb{R}$ then $r_w\in\mathfrak{A}$ and $\kappa(r_w)=(w-H)^{-1}$ (r_w is defined in Theorem 2.8).

Note that in this definition $\kappa(f)\equiv f(H)$.

Lemma 2.10 *Let $f\in C_0(\mathbb{R})$, H a closed operator with $\sigma(H)\subset\mathbb{R}$ and $\lambda\in\mathbb{R}\setminus\{0\}$ such that $f^{-1}(\lambda)\neq\emptyset$ and $f^{-1}(\lambda)\cap\sigma(H)=\emptyset$. Then there exists a smooth function $\phi\in C^\infty(\mathbb{R})$ and a neighbourhood G of $\sigma(H)$ such that*

$$\phi(t)=\begin{cases}0 & if\ t\in f^{-1}(\lambda)\\ 1 & if\ t\in G.\end{cases}$$

PROOF. Let $x_0 \in \mathbb{R}$ be such that $f(x_0) = \lambda$ and $d := \mathrm{dist}(x_0, \sigma(H)) > 0$. Choose $\epsilon_0 \in \mathbb{R}\ :\quad 0 < \epsilon_0 < d$. Let G_0 be an open set such that

$$[x_0 - \epsilon_0, x_0 + \epsilon_0] \subset G_0 \subset [x_0 - d, x_0 + d].$$

We can choose a smooth function ψ_0 such that

$$\psi_0(t) = \begin{cases} 1 & if\ t \in [x_0 - \epsilon_0, x_0 + \epsilon] \\ 0 & if\ t \in \mathbb{R} \setminus G_0. \end{cases}$$

Next, set $\phi_0 := 1 - \psi_0$, then clearly ϕ_0 is smooth and

$$\phi_0(t) = \begin{cases} 0 & if\ t = x_0 \\ 1 & if\ t \in \mathbb{R} \setminus G_0. \end{cases}$$

$G_0 := (a, b)$

Now set $O_{x_0} := (x_0 - \epsilon_0, x_0 + \epsilon_0)$. Similarly choose open sets O_x for each $x \in f^{-1}(\lambda)$. Since $\lambda \neq 0$ and $f(x) \rightarrow 0$ as $|x| \rightarrow \infty$, $f^{-1}(\lambda)$ is a compact set and $\{O_x\ :\ x \in f^{-1}(\lambda)\}$ is an open cover for $f^{-1}(\lambda)$, hence we can find a finite sub-cover $\{O_i\ :\ i = 1, \ldots, m\} \subset \{O_x\ :\ x \in f^{-1}(\lambda)\}$. Corresponding to each O_i, let ϕ_i be the smooth function constructed above. So $\phi_i = 1$ on O_i, and $\phi_i = 1$ on $\mathbb{R} \setminus G_i$. Finally, put $\phi := \prod_{i=1}^{m} \phi_i$, $G := (\mathbb{R} \cup_{i=1}^{m} G_i)^o \supset \sigma(H)$, whence

1. ϕ is smooth on $\mathbb{R}$.
2. $\phi \equiv 0$ *on* $f^{-1}(\lambda)$.
3. $\phi \equiv 1$ *on* G.

□

Theorem 2.11 (Spectral Mapping Theorem) *Let* $f \in \mathfrak{A}$ *and* H *is of* $(\alpha,\ \alpha+1) -$ *type* $\mathbb{R}$ *, then*

$$f(\sigma(H)) = \sigma(f(H)).$$

PROOF. Let $\lambda \in \sigma(H) \subset \mathbb{R}$ and suppose if possible

$$f(\lambda) \notin \sigma(f(H)). \tag{12}$$

Then $[f(\lambda) - f(H)]^{-1} \in \mathfrak{B}(\mathcal{X})$.

If $\acute{f}_\lambda(x) := \begin{cases} \frac{f(\lambda)-f(x)}{\lambda - x}, & x \neq \lambda \\ f'(\lambda), & x = \lambda, \end{cases}$

then $\acute{f}_\lambda \in \mathfrak{A}$ [9, Theorem 2.6], and
$(\lambda - H)\acute{f}_\lambda(H)(i - H)^{-1} = (f(\lambda) - f(H))(i - H)^{-1}$.

Thus
$(\lambda - H)\acute{f}_\lambda(H)(i-H)^{-1}(i-H)(f(\lambda)-f(H))^{-1} =$
$= (f(\lambda)-f(H))(i-H)^{-1}(i-H)(f(\lambda)-f(H))^{-1}$
$\iff (\lambda - H)\acute{f}_\lambda(H)(f(\lambda)-f(H))^{-1} = I.$

Therefore $(\lambda - H)^{-1} = \acute{f}_\lambda(H)(f(\lambda)-f(H))^{-1} \in \mathfrak{B}(\mathcal{X})$!![2]

This contradicts the choice of λ. Hence (12) is not possible. Thus $f(\lambda) \in \sigma(f(H))$ implies $f(\sigma(H)) \subseteq \sigma(f(H))$.

Conversely, if $\lambda \notin f(\sigma(H))$ then $h(x) := \frac{1}{\lambda - f(x)}$ is finite for all $x \in \sigma(H)$. Moreover at each $x \in \sigma(H)$ (and $x \in G$ where G is the neighbourhood of $\sigma(H)$ constructed in lemma 2.10)

$$\begin{aligned}
h'(x) &= [\lambda - f(x)]^{-2} f'(x) \\
&= [h(x)]^2 f'(x) \\
h^{(2)}(x) &= f^{(2)}(x)[h(x)]^2 + 2f'(x)h(x)[h(x)]^2 f' \\
&= f^{(2)}[h(x)]^2 + 2[f'(x)]^2[h(x)]^3 \\
h^{(3)}(x) &= f^{(3)}(x)[h(x)]^2 + f^{(2)}(x)2f'(x)h(x)[h(x)]^2 f'(x) + \\
&\quad +2\{2f'(x)f^{(2)}(x)[h(x)]^3 + [f'(x)]^2 + 3[h(x)]^2[h(x)]^2 f'(x)\} \\
&= f^{(3)}(x)[h(x)]^2 + 6f'(x)f^{(2)}(x)[h(x)]^3 + 6[f'(x)]^3[h(x)]^4 \\
&\vdots \\
h^{(m)}(x) &= \sum_{k=2}^{m+1} [h(x)]^k \sum_{s=1}^{r(k)} \prod_{i=1}^{m} [f^{(i)}(x)]^{p(s,i)} l_s
\end{aligned}$$

where $l_s \in \mathbb{Z}, 1 \le r(k) < m, 0 \le p(s,i) \le m$ and $\sum_{i=1}^m ip(s,i) = m$. Therefore since $f \in \mathfrak{A}$, we can find some $\beta < 0$ such that

$$\begin{aligned}
\left|h^{(m)}(x)\right| &\le \sum_{k=2}^{m+1} |h(x)|^k \sum_{s=1}^{r(k)} \prod_{i=1}^{m} \left|f^{(i)}(x)\right|^{p(s,i)} |l_s| \\
&\le \sum_{k=2}^{m+1} |h(x)|^k \sum_{s=1}^{r(k)} \prod_{i=1}^{m} [c_i \langle x\rangle^{\beta - i}]^{p(s,i)} |l_s| \\
&\le \langle x\rangle^{\beta - m} \sum_{k=2}^{m+1} |h(x)|^k b_k \\
&\le c\langle x\rangle^{\beta - m}, \; c > 0, \; \beta < 0
\end{aligned} \tag{13}$$

(Here we have used the fact that $\sum_{i=1}^m p(s,i) \ge 1$ and $|h|_G < \infty$.)

If ϕ is the smooth function such that

$$\phi(t) = \begin{cases} 0 & if\ t \in f^{-1}(\lambda) \\ 1 & if\ t \in G \end{cases}$$

[2]We denote contradiction by !!

also constructed in lemma 2.10, set

$$g(x) := (i-x)^{-1}\phi(x)h(x).$$

Then using (13) and since $(x+c)(w-x)^{-1}q,\ (q+c)(w-x)^{-1} \in \mathfrak{A}$ for $q \in \mathfrak{A}$ and $c, w \in \mathbb{C}$ with $\Im w \neq 0$ [9, Lemma 2.5] we conclude that $g \in \mathfrak{A}$ and

$$(\lambda - f(H))g(H)(i-H) = I.$$

That is, $\lambda - f(H)$ has an inverse. Therefore, $\lambda \notin \sigma(f(H))$. Hence $\sigma(f(H)) \subseteq f(\sigma(H))$. □

3 Extending the functional calculus to $C_0(\mathbb{R})$

Let $C_0(\mathbb{R})$ denote the algebra of all continuous functions $f : \mathbb{R} \to \mathbb{C}$ such that $f(x) \to 0$ as $|x| \to \infty$ with the supremum norm $\|.\|_\infty$. Then $\mathfrak{A}$ is a dense sub-algebra of $C_0(\mathbb{R})$ [10, Corollary 2.4].

In this section we extend the $\mathfrak{A}$-functional calculus to $C_0(\mathbb{R})$. For more general extensions, see DeLaubenfels [1]. First, we have the following preliminaries.

Lemma 3.1 *If $f \in \mathfrak{A}$ and H is self-adjoint on Hilbert space $\mathcal{H}$, then $\|f(H)\| \leq \|f\|_\infty$.*

Proof. First, observe that H is of $(0,\ 1) - type\ \mathbb{R}$. Also from (7),

$$\bar{f}(H) = f(H)^*$$

in this case. Now choose $d \in \mathbb{R}$ such that $d > \|f\|_\infty$ and set

$$g(t) := d - \sqrt{\left(d^2 - |f(t)|^2\right)}$$

then clearly $0 \leq g \in \mathfrak{A}$, and

$$(d - g(t))^2 = d^2 - |f(t)|^2\,, \text{ for each } t \in \mathbb{R}.$$

so

$$f\bar{f} - 2dg + g^2 = 0 \in \mathfrak{A}.$$

Thus

$$f(H)^*\, f(H) - dg(H) - dg(H)^* + g(H)^*\, g(H) = 0$$

implies $$f(H)^*\, f(H) + \{d - g(H)\}^*\, \{d - g(H)\} = d^2.$$

If $\psi \in \mathcal{H}$, then

$$\|f(H)\psi\|^2 + \|\{d - g(H)\}\psi\|^2 = d^2 \|\psi\|$$

and therefore
$$\|f(H)\psi\| \leq d \|\psi\| .$$

□

We are now in a position to describe the $\mathfrak{A}$-functional calculus for a self-adjoint operator in a standard fashion.

Corollary 3.2 *If $f \in \mathfrak{A}$ and H is self-adjoint on Hilbert space $\mathcal{H}$, then the functional calculus*

$$\kappa : \mathfrak{A} \ni f \mapsto f(H) \in \mathfrak{B}(\mathcal{H})$$

can be extended to a unique map

$$\tilde{\kappa} : C_0(\mathbb{R}) \ni f \mapsto f(H) \in \mathfrak{B}(\mathcal{H})$$

such that:

1. *$\tilde{\kappa}$ is an algebra homomorphism.*
2. *$\bar{f}(H) = f(H).^*$*
3. *$\|f(H)\| \leq \|f\|_\infty$.*
4. *if $w \in \mathbb{C} \setminus \mathbb{R}$ and $r_w := (w - s)^{-1}$ then $r_w(H) = (w - H)^{-1}$.*

Proof. The existence follows from Theorem 2.3, Corollary 2.6, Theorem 2.8 and Lemma 3.1. So we need only to establish the uniqueness.

Suppose η is another extension of κ to $C_0(\mathbb{R})$ and let $\mathfrak{X} \subseteq C_0((R)$ be the set of f for which $\tilde{\kappa}(f) = \eta(f)$. Then $\mathfrak{X}$ is norm closed sub-algebra of $C_0(\mathbb{R})$ which contains r_w for all $w \notin \mathbb{R}$. Thus whenever $x, y \in \mathbb{R}$,

$$x \neq y \iff r_w(x) \neq r_w(y) \quad \text{for some w } \notin \mathbb{R}$$

Therefore, by Stone - Weierstrass Theorem, $\mathfrak{X} = C_0(\mathbb{R})$. □

Remark 3.3 H is of $(0,\ 1)-type\,\mathbb{R}$ with the constant $c = 1$ if and only if iH is a generator of a one-parameter group of isometries on $\mathcal{X}$[10, Theorem 3.1]. This together with Corollary 3.2 provide a proof to a version of the spectral theorem for a self-adjoint operator on a Hilbert space, which asserts:

> iH generates a uniformly bounded strongly continuous group if and only if H has a $C_0(\mathbb{R})$ functional calculus.

The most natural infinite dimensional analogue of a diagonalizable matrix is a **scalar operator** (short for spectral operator of scalar type in the sense of Dunford [3, Chapter XVIII]). For an operator H with real spectrum, this means that there exits a projection-valued measure F such that

$$Hx = \int_{\mathbb{R}} t dF(t)x$$

with maximal domain.

The class of scalar operators includes (but is not limited to) self-adjoint operators on a Hilbert space. However on a general Banach space, it is hard to find a scalar operator. If H is an operator with $\sigma(H) \subset \mathbb{R}$ and acting on a reflexive Banach space $\mathcal{X}$, then H is scalar if and only if iH generates a uniformly bounded strongly continuous group [8, page 155]. So, via the spectral theorem, a self-adjoint operator H on a Hilbert space $\mathcal{H}$ is scalar if and only if H has a $C_0(\mathbb{R})$ functional calculus. In fact this is true in general. That is;

> an operator acting on a reflexive Banach space is scalar if and only if it has a $C_0(\mathbb{R})$ functional calculus [2, Theorem 6.10].

In the light of the forgoing, it is therefore reasonable to have the following conjecture:

Conjecture 3.4 *A densely define closed linear operator H, acting on a reflexive Banach space $\mathcal{X}$, is scalar if it is of* $(0,\ 1)-type\ \mathbb{R}$ *and* $\|f(H)\| \leq \|f\|_\infty$ *for each* $f \in \mathfrak{A}$.

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