Shell structure of the SU (N) generator spectrum: interpretation as spin angular momentum operators

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Shell structure of the $SU(N)$ generator spectrum: interpretation as spin angular momentum operators

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Abstract

We have established that $SU(N)$ symmetry group generators occur in a spectrum with a quantum structure composed of $N-1$ configuration shells each containing a definite number of symmetric and antisymmetric pairs of generators specified by quantum numbers $l = 1, ..., N-1$; $m = 0, 1, ..., l$. Interpreting the generators as spin angular momentum operators brings the generator spectrum to a form precisely similar to the spectrum of orbital angular momentum states composed of orbital configuration shells containing definite numbers of orbital states specified by orbital and magnetic quantum numbers $l = 0, 1, ..., n-1$; $m = 0, \pm 1, ..., \pm l$ in the n^{th}-energy level of an atom, thus revealing that the quantum state space of an $SU(N)$ symmetry group corresponds directly to the quantum state space of the n^{th}-energy level of an atom. Within each configuration shell containing specified generators in the $SU(N)$ generator spectrum, we have determined the associated quadratic and Fubini-Veneziano spin angular momentum operators to general order, which we have finally used to obtain the corresponding universal $SU(N)$ quadratic and Fubini-Veneziano spin operators. Basic algebraic relations of the resulting Cartan-Weyl generators have been determined explicitly for general $SU(N)$ symmetry groups. Considering applications to gauge field theories, we easily establish that $SU(N)$ gauge fields have quantum structure corresponding directly to the $SU(N)$ generator spectrum. We have provided elaborate explanations of the important implications of the expanded algebraic properties and quantum structure of the $SU(N)$ generator spectrum to the existing $SU(N)$ gauge field theories.

1 Introduction

In two recent articles [1 , 2], we presented an exact mathematical method for determining $SU(N)$ symmetry group generators and established that the generators occur in a spectrum composed of $N-1$ state transition subspaces each containing a definite number of specified generators, similar to the electronic state configuration shells in the energy level spectrum of an atom. In particular, in [2] where we developed the $SU(N)$ generator spectrum property, we identified the $N-1$ state transition subspaces as focal state transition spaces (FSTS).

The $SU(N)$ symmetry group is defined in an N-dimensional (integer $N = 2, 3, 4, ...$) state space spanned by N mutually orthonormal state vectors $|n\rangle$, $n = 1, 2, 3, ..., N$, which we call group basis vectors, defined as column matrices, i.e., $N \times 1$ matrices, with entries 0 in all rows except entry 1 in the n-th row according to

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix} \;;\quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix} \;;\quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ . \\ . \\ . \\ 0 \end{pmatrix} \;;\quad ..... \;;\quad |N-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ . \\ 1 \\ 0 \end{pmatrix} \;;\quad |N\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ . \\ 1 \end{pmatrix} \qquad (1a)$$

satisfying orthonormalization relation

$$\langle n|m\rangle = \delta_{nm} \qquad (1b)$$

The $SU(N)$ symmetry group generators are determined as tensor products of pair-wise coupled group basis vectors defined in hermitian symmetric and antisymmetric forms within the respective focal state transition spaces. We established in [2] that a focal state transition space characterized by a group basis vector denoted by FSTS-$|m\rangle$, $m = 2, 3, ..., N$, contains $2m - 1$ traceless non-diagonal and diagonal symmetric and antisymmetric generators determined explicitly as $N \times N$ matrices as presented in the generator spectra of the $SU(2)$, $SU(3)$, $SU(4)$, $SU(5)$ symmetry groups as examples.

In the present article, we follow the orbital shell interpretation suggested in [2] to provide a complete shell structure of the *general* $SU(N)$ generator spectrum, which includes the $N-1$ non-traceless diagonal symmetric generators defined within each of the $N-1$ focal state transition spaces FSTS-$|m\rangle$ of the spectrum and then proceed by taking a weighted sum of these $N - 1$ non-traceless diagonal symmetric generators to determine the $N \times N$ identity matrix I_N of the $SU(N)$ symmetry group, thus allowing us to reduce the general $SU(N)$ generator spectrum to the *standard* $SU(N)$ generator spectrum composed of the identity matrix and the standard $N^2 - 1$ traceless non-diagonal and diagonal symmetric and antisymmetric generators. The emerging picture suggests a reinterpretation of the focal state transition spaces FSTS-$|m\rangle$ as *generator configuration shells* containing definite numbers of specified generators. This is now a comprehensive quantum structure in which each of the $N-1$ configuration shells in an $SU(N)$ generator spectrum is specified by a quantum number l taking $N - 1$ values $l = 1, ..., N - 1$ and contains a definite number of generators determined in symmetric and antisymmetric pairs specified by a quantum number m taking $l + 1$ values $m = 0, 1, ..., l$. The emerging quantum structure of an $SU(N)$ generator spectrum is similar to the quantum structure of an (orbital) angular momentum state spectrum composed of orbital state configuration shells specified by orbital angular momentum quantum number $l = 0, ..., n - 1$ containing definite numbers of orbital angular momentum states specified by magnetic quantum number $m = 0, \pm 1, \pm 2, ..., \pm l$ in the n^{th}-energy level of an atom, leading to an important physical picture that an $SU(N)$ generator spectrum corresponds precisely to the orbital angular momentum state spectrum in an atomic energy level, as we describe in detail in section 4 below.

For clarity, we have chosen to develop the quantum structure and algebraic properties of an $SU(N)$ generator spectrum in three interrelated stages. The purpose here is to open up our minds to the general composition of an $SU(N)$ generator spectrum, which, in addition to the familiar $N^2 - 1$ traceless generators denoted in the Gell-Mann basis by $\lambda_1, \lambda_2, ..., \lambda_{N^2-1}$, also contains $N - 1$ non-traceless diagonal symmetric generators arising from the basic algebraic definition of $SU(N)$ generators. In this respect, we also note that, except for the presentation given in [3], the common understanding of $SU(N)$ generators as presented in the wider physics literature generally ignores the $N \times N$ identity matrix of the group. These issues become clear in the present work.

In section 2, we present a general method for enumerating and determining $SU(N)$ symmetry group generators in the Gell-Mann basis specified by quantum numbers $l = 1, ..., N-1$, $m = 0, 1, ..., l$. The underlying algebraic property is that $SU(N)$ symmetry group generators are determined as pairs of hermitian symmetric and antisymmetric pair-wise tensor products of the group state basis vectors $|1\rangle$, $|2\rangle$, ... , $|N\rangle$. The resulting generator spectrum is then composed of the standard N^2-1 traceless symmetric and antisymmetric generators, plus the $N-1$ non-traceless diagonal symmetric generators, all distributed in $N-1$ configuration shells $l = 1, ..., N-1$ each containing $2(l+1)$ symmetric and antisymmetric generators, which constitute a *general* $SU(N)$ generator spectrum. Hence, according to the basic algebraic definition of generators stated above, the general $SU(N)$ generator spectrum contains a total of $N^2 - 1 + N - 1 = (N-1)(N+2)$ basic generators.

In section 3, we reduce the general generator spectrum to the familiar *standard* $SU(N)$ generator spectrum by taking an appropriately weighted sum of the $N-1$ non-traceless diagonal symmetric generators to form an $N \times N$ identity matrix, which we identify as the identity generator. Each of the $N-1$ shells $l = 1, ..., N-1$ in the standard $SU(N)$ generator spectrum now contains $2l+1$ traceless symmetric and antisymmetric generators, but the spectrum is now extended to include an $l = 0$ shell (0^{th}-shell) containing the single ($2 \times 0 + 1 = 1$) identity generator. Hence, the standard $SU(N)$ generator spectrum has N configuration shells $l = 0, 1, ..., N-1$, each containing $2l+1$ standard generators including the identity generator, giving a total of $(N^2-1)+1 = N^2$ generators. The property that each shell specified by quantum number $l = 0, 1, ..., N-1$ contains $2l+1$ brings the quantum structure of the standard $SU(N)$ generator spectrum into direct correspondence with the quantum structure of orbital angular momentum state spectrum in an atomic n^{th}-energy level composed of configuration shells each containing $2l+1$ ($l = 0, 1, ..., n-1$) orbital angular momentum states.

Section 4 contains the main results of the present work. We begin by refining the correspondence of the $SU(N)$ generator spectrum to the spectrum of orbital angular momentum states in the n^{th}-energy level of an atom noted in section 3 in a precise form, leading to a reinterpretation of $SU(N)$ symmetry group generators as spin angular momentum operators. The $SU(N)$ generators in the Gell-Mann basis are now enumerated and determined as symmetric and antisymmetric pairs of spin operators specified by quantum numbers l , m in each of the $N-1$ configuration shells $l = 1, ..., N-1$, in one-to-one correspondence with orbital angular momentum states similarly specified by a corresponding pair of quantum numbers l , m defined in similar manner. The $2l+1$ traceless generators in the l^{th}-shell are interpreted as components of a $(2l+1)$-component l^{th}-shell spin angular momentum vector, which we square to determine a quadratic spin operator. The generators raised to even and odd powers take simple forms which we use to introduce l^{th}-shell quadratic and Fubini-Veneziano spin operators of general order. Universal $SU(N)$ quadratic Casimir and Fubini-Veneziano spin operators of general order are easily determined. Finally, we identify an extended Cartan-Weyl basis, which we use to generate basic algebraic relations for general $SU(N)$ symmetry groups.

In section 5, we make some critical observations on the physical implications of the expanded algebraic space and quantum structure of the $SU(N)$ generator spectrum on the existing and new models of $SU(N)$ gauge field theories of elementary particle interactions. This section provides an important algebraic foundation for reviewing the physical content of existing $SU(N)$ gauge field theories such as $SU(2) \times U(1)$, $SU(3)_c$, $SU(3)_c \times SU(2)_L \times U(1)_Y$, $SU(5)_{GUT}$, among others.

2 Configuration shell structure in the general $SU(N)$ generator spectrum

As in [1 , 2], we adopt the standard Gell-Mann notation λ for $SU(N)$ symmetry group generators, but now we reinterpret the $N-1$ focal state transition spaces (FSTS) in the generator spectrum introduced in [2] as *configuration shells* containing definite numbers of generators specified by quantum numbers $l = 1, ..., N-1$, $m = 0, 1, ..., l$, which now provides a well defined quantum picture. In this quantum picture, a general $SU(N)$ generator spectrum is composed of $N-1$ configuration shells specified by a *shell quantum number* l taking $N-1$ integer values $l = 1, ..., N-1$. A configuration shell specified by a quantum number l, referred to as the l^{th}-shell, contains a definite number $2(l+1)$ of specified generators occurring as hermitian non-diagonal or diagonal *symmetric-antisymmetric* pairs. Each generator in the l^{th}-shell is specified by the shell number l and a *symmetric-antisymmetric generator pair* quantum number m taking $l+1$ values $m = 0, 1, ..., l$. Specifically, in the Gell-Mann notation λ, the l^{th}-shell specified by $l = 1, ..., N-1$, contains $2l$ hermitian *non-diagonal* generators occurring as l symmetric-antisymmetric pairs $(\lambda_{l^2+2m}\ ,\ \lambda_{l^2+2m+1})$ specified by the shell quantum number l and the symmetric-antisymmetric pair quantum number $m = 0, 1, ..., l-1$, plus 2 hermitian *diagonal* generators occurring as 1 symmetric-antisymmetric pair $(\lambda_{l^2+2l}\ ,\ \overline{I}_{l^2+2l})$ specified by the shell quantum number l and the symmetric-antisymmetric pair quantum number $m = l$. The quantum number specification of the symmetric-antisymmetry generator pairs in each of the $N-1$ shells $l = 1, ..., N-1$ enumerates all the standard traceless non-diagonal and diagonal $SU(N)$ generators in the Gell-Mann basis in the expected serial order $\lambda_1, \lambda_2, ..., \lambda_{N^2-1}$ as we demonstrate below in worked examples.

There are $2l$ traceless non-diagonal symmetric and antisymmetric generators, 1 traceless diagonal antisymmetric generator and 1 non-traceless diagonal symmetric generator, making the specified total of $2(l+1)$ symmetric and antisymmetric generators in the l^{th}-shell. The l traceless non-diagonal symmetric and antisymmetric generator pairs $(\lambda_{l^2+2m}\ ,\ \lambda_{l^2+2m+1})$ specified by the l pair quantum numbers $m = 0, 1, ..., l-1$ are determined as non-diagonal tensor products of the pair-wise coupled state basis vectors $|m+1\rangle$ and $|l+1\rangle$ obtained in hermitian form

$$l = 1, ..., N-1 \quad ; \quad m = 0, 1, ..., l-1$$

$$\lambda_{l^2+2m} = |m+1\rangle\langle l+1| + |l+1\rangle\langle m+1| \quad ; \quad \lambda_{l^2+2m+1} = -i(\ |m+1\rangle\langle l+1| - |l+1\rangle\langle m+1|\ ) \quad (2ai)$$

The traceless diagonal antisymmetric generator and its non-traceless diagonal symmetric partner occur at $m = l$ as the last antisymmetric and symmetric pair $(\lambda_{l^2+2l}\ ,\ \overline{I}_{l^2+2l})$ determined as normalized superpositions of diagonal tensor products of the pair-wise coupled state basis vectors $|m+1\rangle$ and $|l+1\rangle$ obtained in hermitian form

$$l = 1, ..., N-1 \quad ; \quad m = l$$

$$\lambda_{l^2+2l} = \frac{1}{\sqrt{\frac{1}{2}l(l+1)}} \sum_{m=0}^{l-1} (\ |m+1\rangle\langle m+1| - |l+1\rangle\langle l+1|\ )$$

$$\overline{I}_{l^2+2l} = \frac{1}{\sqrt{\frac{1}{2}l(l+1)}} \sum_{m=0}^{l-1} (\ |m+1\rangle\langle m+1| + |l+1\rangle\langle l+1|\ ) \quad (2aii)$$

We note that in the algebraic method of enumerating and determining generators as symmetric-antisymmetric pairs in equations $(2ai)$-$(2aii)$, non-diagonal symmetric generators are enumerated as

λ_{l^2+2m}, while their partner antisymmetric generators are enumerated as λ_{l^2+2m+1} for $m = 0, ..., l-1$ as specified in equation $(2ai)$, but for the diagonal symmetric-antisymmetric pair specified by $m = l$, we have reorganized the enumeration for the traceless diagonal antisymmetric generator as λ_{l^2+2l} as given in equation $(2aii)$ to agree with the standard Gell-Mann notation, while for the partner non-traceless diagonal symmetric generator, we have introduced an appropriate notation $\overline{I}_{l^2+2l}$, which will prove convenient in the determination and algebraic interpretation of the $N \times N$ identity matrix I_N as an identity generator of a standard $SU(N)$ symmetry group in the next section.

Using equations $(2ai)$-$(2aii)$, we run through all the $l+1$ values of the symmetric-antisymmetric generator pair quantum number $m = 0, 1, ..., l$ to enumerate and determine in explicit forms all the $2(l+1)$ symmetric and antisymmetric generators $\lambda_l^2, \lambda_{l^2+1}, \lambda_{l^2+2}, ..., \lambda_{l^2+2l}, \overline{I}_{l^2+2l}$ in the l^{th}-shell of a general $SU(N)$ generator spectrum as presented below, where we have used abbreviation PRST-$|l+1\rangle$ for the *principal state basis vector* which characterizes the l^{th}-shell :

Generators in the l^{th}-shell of a general $SU(N)$ spectrum : $N-1$ shells, $l = 1, ..., N-1$

$$
l^{th} - \text{shell} \ : \ PRST - |l+1\rangle \quad
\begin{cases}
m = 0 \ : \quad \lambda_{l^2} = |1\rangle\langle l+1| + |l+1\rangle\langle 1| \\
\qquad\qquad \lambda_{l^2+1} = -i(\ |1\rangle\langle l+1| - |l+1\rangle\langle 1|\ ) \\
\\
m = 1 \ : \quad \lambda_{l^2+2} = |2\rangle\langle l+1| + |l+1\rangle\langle 2| \\
\qquad\qquad \lambda_{l^2+3} = -i(\ |2\rangle\langle l+1| - |l+1\rangle\langle 2|\ ) \\
.... \\
.... \\
.... \\
\\
m = l-1 \ : \quad \lambda_{l^2+2(l-1)} = |l\rangle\langle l+1| + |l+1\rangle\langle l| \\
\qquad\qquad \lambda_{l^2+2l-1} = -i(\ |l\rangle\langle l+1| - |l+1\rangle\langle l|\ ) \\
\\
m = l \ : \ \lambda_{l^2+2l} = \frac{1}{\sqrt{\frac{1}{2}l(l+1)}} \sum_{m=0}^{l-1} |m+1\rangle\langle m+1| - |l+1\rangle\langle l+1| \\
\qquad\qquad \overline{I}_{l^2+2l} = \frac{1}{\sqrt{\frac{1}{2}l(l+1)}} \sum_{m=0}^{l-1} |m+1\rangle\langle m+1| + |l+1\rangle\langle l+1|
\end{cases}
\tag{2b}
$$

In the Gell-Mann basis, the generators expressed in tensor product forms are evaluated in explicit $N \times N$ matrix forms using the definitions of the $SU(N)$ symmetry group basis vectors $|1\rangle$, $|2\rangle$, ... , $|N\rangle$ given in equation $(1a)$, which we present in the general $SU(2)$, $SU(3)$, $SU(4)$, $SU(5)$ generator spectra as examples in the next subsection.

2.1 Shell structure of the general $SU(2)$, $SU(3)$, $SU(4)$, $SU(5)$ generator spectra

Let us now give examples to enumerate and determine generators in the configuration shells of the general generator spectra of the $SU(2)$, $SU(3)$, $SU(4)$ and $SU(5)$ symmetry groups which have been generally used in formulating gauge theories of particle interactions in quantum field theory. The general generator spectrum of each symmetry group $SU(2)$, $SU(3)$, $SU(4)$, $SU(5)$ is composed of

$N-1$ shells, specified by shell numbers $l=1,...,N-1$. All the $2(l+1)$ symmetric and antisymmetric generators in each shell are enumerated and determined explicitly by setting the shell quantum numbers $l=1,...,N-1$ as appropriate in the tensor product forms given in equation $(2b)$ and using the definitions of group basis vectors $|1\rangle$, $|2\rangle$, ... , $|N\rangle$ given in equation $(1a)$ to evaluate the tensor products in $N\times N$ matrix forms. Since each of the $N-1$ shells contains $2(l+1)$ generators, the total number of symmetric and antisymmetric generators in a general $SU(N)$ generator spectrum is obtained as $\sum_{l=1}^{N-1}2(l+1)=(N-1)(N+2)$, composed of the familiar $\sum_{l=1}^{N-1}(2l+1)=N^2-1$ traceless non-diagonal and diagonal symmetric-antisymmetric generators λ_1 , λ_2 , ... , λ_{N^2-1} and the $N-1$ non-traceless diagonal symmetric generators $\overline{I}_3$, ... , $\overline{I}_{N^2-1}$ introduced in the present work as we determine below for the $SU(2)$, $SU(3)$, $SU(4)$ and $SU(5)$ symmetry groups. For each symmetry group, we have presented the calculations in full detail in the equally useful operator forms, skipping only the straightforward evaluations of the tensor products giving the matrix forms.

2.1.1 General $SU(2)$ generator spectrum

$$N=2\ :\qquad \text{no. of shells}=1\ :\quad \text{shell numbers } l=1$$
$$\text{group basis vectors}\ :\quad |1\rangle=\begin{pmatrix}1\\0\end{pmatrix}\quad ;\quad |2\rangle=\begin{pmatrix}0\\1\end{pmatrix} \tag{2ci}$$

The general $SU(2)$ generator spectrum is composed of $2-1=1$ configuration shell specified by $l=1$. This single 1^{st}-shell contains $1+1=2$ pairs of symmetric and antisymmetric generators specified by pair numbers $m=0,1$. The $m=0$ pair are enumerated and determined by setting $l=1$ in equation $(2b)$ as $\lambda_1=|1\rangle\langle 2|+|2\rangle\langle 1|$, $\lambda_2=-i(\ |1\rangle\langle 2|-|2\rangle\langle 1|\ )$, while the $m=1$ pair are enumerated and determined by setting $l=1$ in the last pair in equation $(2b)$ as $\lambda_3=\frac{1}{\sqrt{\frac{1}{2}1(1+1)}}(\ \sum_{m=0}^{1-1}|m+1\rangle\langle m+1|-|2\rangle\langle 2|\ )=|1\rangle\langle 1|-|2\rangle\langle 2|$, $\overline{I}_3=\frac{1}{\sqrt{\frac{1}{2}1(1+1)}}(\ \sum_{m=0}^{1-1}|m+1\rangle\langle m+1|+|2\rangle\langle 2|\ )=|1\rangle\langle 1|+|2\rangle\langle 2|$. We use the definitions of the $SU(2)$ group basis vectors given in equation $(2ci)$ above to evaluate the tensor products explicitly as 2×2 matrices. The $(2-1)(2+2)=4$ symmetric and antisymmetric generators $\lambda_1,\lambda_2,\lambda_3,\overline{I}_3$ contained in the single 1^{st}-shell of the general $SU(2)$ generator spectrum are presented below, where the principal group basis vector characterizing the 1^{st}-shell is abbreviated as PRST-$|1+1\rangle$=PRST-$|2\rangle$.

General $SU(2)$ generator spectrum : single shell, $l=1$

$$1^{st}-\text{shell}\ :\ PRST-|2\rangle\quad \begin{cases} m=0\ :\quad \lambda_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}\quad ;\quad \lambda_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix}\\ m=1\ :\quad \lambda_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\quad ;\quad \overline{I}_3=\begin{pmatrix}1&0\\0&1\end{pmatrix}\end{cases} \tag{2cii}$$

2.1.2 General $SU(3)$ generator spectrum

$$N=3\ :\qquad \text{no. of shells}=2\ :\quad \text{shell numbers}\ l=1\ ,\ 2$$

$$\text{group basis vectors}\ :\quad |1\rangle=\begin{pmatrix}1\\0\\0\end{pmatrix}\quad ;\quad |2\rangle=\begin{pmatrix}0\\1\\0\end{pmatrix}\quad ;\quad |3\rangle=\begin{pmatrix}0\\0\\1\end{pmatrix}\qquad (2di)$$

The general $SU(3)$ generator spectrum is composed of $3-1=2$ configuration shells specified by $l=1\ ,\ 2$.

The 1^{st}-shell contains $1+1=2$ pairs of symmetric and antisymmetric generators specified by pair numbers $m=0,1$; the $m=0$ pair are enumerated and determined by setting $l=1$ in equation $(2b)$ as $\lambda_1=|1\rangle\langle 2|+|2\rangle\langle 1|$, $\lambda_2=-i(\ |1\rangle\langle 2|-|2\rangle\langle 1|\ )$, while the $m=1$ pair are enumerated and determined by setting $l=1$ in the last pair in equation $(2b)$ as $\lambda_3=\frac{1}{\sqrt{\frac{1}{2}1(1+1)}}(\ \sum_{m=0}^{1-1}|m+1\rangle\langle m+1|-|2\rangle\langle 2|\ )=|1\rangle\langle 1|-|2\rangle\langle 2|$, $\overline{I}_3=\frac{1}{\sqrt{\frac{1}{2}1(1+1)}}(\ \sum_{m=0}^{1-1}|m+1\rangle\langle m+1|+|2\rangle\langle 2|\ )=|1\rangle\langle 1|+|2\rangle\langle 2|$.

The 2^{nd}-shell contains $2+1=3$ pairs of symmetric and antisymmetric generators specified by pair numbers $m=0,1,2$; the $m=0$ pair are enumerated and determined by setting $l=2$ in equation $(2b)$ as $\lambda_4=|1\rangle\langle 3|+|3\rangle\langle 1|$, $\lambda_5=-i(\ |1\rangle\langle 3|-|3\rangle\langle 1|\ )$, the $m=1$ pair are enumerated and determined by setting $l=2$ in the second pair in equation $(2b)$ as $\lambda_6=|2\rangle\langle 3|+|3\rangle\langle 2|$, $\lambda_7=-i(\ |2\rangle\langle 3|-|3\rangle\langle 2|\ )$, while the $m=2$ pair are enumerated and determined by setting $l=2$ in the last pair in equation $(2b)$ as $\lambda_8=\frac{1}{\sqrt{\frac{1}{2}2(2+1)}}(\ \sum_{m=0}^{2-1}|m+1\rangle\langle m+1|-|3\rangle\langle 3|\ )=\frac{1}{\sqrt{3}}(\ (\ |1\rangle\langle 1|-|3\rangle\langle 3|\ )+(\ |2\rangle\langle 2|-|3\rangle\langle 3|\ )\ )$, $\overline{I}_8=\frac{1}{\sqrt{\frac{1}{2}2(2+1)}}(\ \sum_{m=0}^{2-1}|m+1\rangle\langle m+1|+|3\rangle\langle 3|\ )=\frac{1}{\sqrt{3}}(\ (\ |1\rangle\langle 1|+|3\rangle\langle 3|\ )+(\ |2\rangle\langle 2|+|3\rangle\langle 3|\ )\ )$. We use the definitions of the $SU(3)$ group basis vectors given in equation $(2di)$ above to evaluate the tensor products explicitly as 3×3 matrices. The $(3-1)(3+2)=10$ symmetric and antisymmetric generators $\lambda_1,\lambda_2,\lambda_3,\overline{I}_3,...,\lambda_8,\overline{I}_8$ contained in the 2 shells of the general $SU(3)$ generator spectrum are presented below, where the principal group basis vectors characterizing the 1^{st} and 2^{nd} shells are abbreviated as PRST-$|1+1\rangle$=PRST-$|2\rangle$ and PRST-$|2+1\rangle$=PRST-$|3\rangle$, respectively.

General $SU(3)$ generator spectrum : 2 shells, $l=1,2$

$$1^{st}-\text{shell}\ :\ PRST-|2\rangle\quad \begin{cases} m=0\ :\quad \lambda_1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\quad ;\quad \lambda_2=\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}\\ m=1\ :\quad \lambda_3=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}\quad ;\quad \overline{I}_3=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\end{cases}$$

$$2^{nd}-\text{shell}\ :\ PRST-|3\rangle\quad\begin{cases} m=0\ :\quad \lambda_4=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}\ ;\ \lambda_5=\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix}\\ m=1\ :\quad \lambda_6=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}\ ;\ \lambda_7=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix}\\ m=2\ :\lambda_8=\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}\ ;\ \overline{I}_8=\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix}\end{cases}\tag{2dii}$$

2.1.3 General $SU(4)$ generator spectrum

$$N=4\ :\qquad \text{number of shells}=3\ :\quad \text{shell numbers}\ l=1,2,3$$

$$\text{group basis vectors}\ :\ |1\rangle=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\ ;\ |2\rangle=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\ ;\ |3\rangle=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\ ;\ |4\rangle=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\tag{2ei}$$

The general $SU(4)$ generator spectrum is composed of $4-1=3$ configuration shells specified by $l=1,2,3$.

The 1^{st}-shell contains $1+1=2$ pairs of symmetric and antisymmetric generators specified by pair numbers $m=0,1$; the $m=0$ pair are enumerated and determined by setting $l=1$ in equation $(2b)$ as $\lambda_1=|1\rangle\langle 2|+|2\rangle\langle 1|$, $\lambda_2=-i(\ |1\rangle\langle 2|-|2\rangle\langle 1|\ )$, while the $m=1$ pair are enumerated and determined by setting $l=1$ in the last pair in equation $(2b)$ as $\lambda_3=\frac{1}{\sqrt{\frac{1}{2}1(1+1)}}(\ \sum_{m=0}^{1-1}|m+1\rangle\langle m+1|-|2\rangle\langle 2|\ )=|1\rangle\langle 1|-|2\rangle\langle 2|$, $\overline{I}_3=\frac{1}{\sqrt{\frac{1}{2}1(1+1)}}(\ \sum_{m=0}^{1-1}|m+1\rangle\langle m+1|+|2\rangle\langle 2|\ )=|1\rangle\langle 1|+|2\rangle\langle 2|$.

The 2^{nd}-shell contains $2+1=3$ pairs of symmetric and antisymmetric generators specified by pair numbers $m=0,1,2$; the $m=0$ pair are enumerated and determined by setting $l=2$ in equation $(2b)$ as $\lambda_4=|1\rangle\langle 3|+|3\rangle\langle 1|$, $\lambda_5=-i(\ |1\rangle\langle 3|-|3\rangle\langle 1|\ )$, the $m=1$ pair are enumerated and determined by setting $l=2$ in the second pair in equation $(2b)$ as $\lambda_6=|2\rangle\langle 3|+|3\rangle\langle 2|$, $\lambda_7=-i(\ |2\rangle\langle 3|-|3\rangle\langle 2|\ )$, while the $m=2$ pair are enumerated and determined by setting $l=2$ in the last pair in equation $(2b)$ as $\lambda_8=\frac{1}{\sqrt{\frac{1}{2}2(2+1)}}(\ \sum_{m=0}^{2-1}|m+1\rangle\langle m+1|-|3\rangle\langle 3|\ )=\frac{1}{\sqrt{3}}(\ (\ |1\rangle\langle 1|-|3\rangle\langle 3|\ )+(\ |2\rangle\langle 2|-|3\rangle\langle 3|\ )\ )$, $\overline{I}_8=\frac{1}{\sqrt{\frac{1}{2}2(2+1)}}(\ \sum_{m=0}^{2-1}|m+1\rangle\langle m+1|+|3\rangle\langle 3|\ )=\frac{1}{\sqrt{3}}(\ (\ |1\rangle\langle 1|+|3\rangle\langle 3|\ )+(\ |2\rangle\langle 2|+|3\rangle\langle 3|\ )\ )$.

The 3^{rd}-shell contains $3+1=4$ pairs of symmetric and antisymmetric generators specified by pair numbers $m=0,1,2,3$; the $m=0$ pair are enumerated and determined by setting $l=3$ in equation $(2b)$ as $\lambda_9=|1\rangle\langle 4|+|4\rangle\langle 1|$, $\lambda_{10}=-i(\ |1\rangle\langle 4|-|4\rangle\langle 1|\ )$, the $m=1$ pair are enumerated and determined by setting $l=3$ in the second pair in equation $(2b)$ as $\lambda_{11}=|2\rangle\langle 4|+|4\rangle\langle 2|$, $\lambda_{12}=-i(\ |2\rangle\langle 4|-|4\rangle\langle 2|\ )$, the $m=2$ pair are enumerated and determined by setting $l=3$ in the third pair (not indicated) in equation $(2c)$ as $\lambda_{13}=|3\rangle\langle 4|+|4\rangle\langle 3|$, $\lambda_{14}=-i(\ |3\rangle\langle 4|-|4\rangle\langle 3|\ )$, while the $m=3$ pair are enumerated and determined by setting $l=3$ in the last pair in equation $(2b)$ as $\lambda_{15}=\frac{1}{\sqrt{\frac{1}{2}3(3+1)}}(\ \sum_{m=0}^{3-1}|m+$

$1\rangle\langle m+1| - |4\rangle\langle 4|\ ) = \frac{1}{\sqrt{6}}(\ (\ |1\rangle\langle 1| - |4\rangle\langle 4|\ ) + (\ |2\rangle\langle 2| - |4\rangle\langle 4|\ )\ ) + (\ |3\rangle\langle 3| - |4\rangle\langle 4|\ )\ )\ ,\ \overline{I}_{15} = \frac{1}{\sqrt{\frac{1}{2}3(3+1)}}(\ \sum_{m=0}^{3-1} |m+1\rangle\langle m+1| + |4\rangle\langle 4|\ ) = \frac{1}{\sqrt{6}}(\ (\ |1\rangle\langle 1| + |4\rangle\langle 4|\ ) + (\ |2\rangle\langle 2| + |4\rangle\langle 4|\ )\ ) + (\ |3\rangle\langle 3| + |4\rangle\langle 4|\ )\ ).$
We use the definitions of the $SU(4)$ group basis vectors given in equation $(2ei)$ above to evaluate the tensor products explicitly as 4×4 matrices. The $(4-1)(4+2) = 18$ symmetric and antisymmetric generators $\lambda_1, \lambda_2, \lambda_3, \overline{I}_3, ..., \lambda_{15}, \overline{I}_{15}$ contained in the 3 shells of the general $SU(4)$ generator spectrum are presented below, where the principal group basis vectors characterizing the 1^{st} , 2^{nd} and 3^{rd} shells are abbreviated as PRST-$|1+1\rangle$=PRST-$|2\rangle$, PRST-$|2+1\rangle$=PRST-$|3\rangle$ and PRST-$|3+1\rangle$=PRST-$|4\rangle$, respectively.

General $SU(4)$ generator spectrum : 3 shells, $l = 1, 2, 3$

$$1^{st} - \text{shell} \ : \ PRST - |2\rangle \quad \begin{cases} m=0 \ : \quad \lambda_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad ; \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ m=1 \ : \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad ; \quad \overline{I}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{cases}$$

$$2^{nd} - \text{shell} \ : \ PRST - |3\rangle \quad \begin{cases} m=0 \ : \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad ; \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ m=1 \ : \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad ; \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ m=2 \ : \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ; \ \overline{I}_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{cases}$$

$$
3^{rd}-\text{shell} \;:\; PRST-|4\rangle \quad \begin{cases} m=0 \;:\; \lambda_9 = \begin{pmatrix} 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0\\ 1&0&0&0 \end{pmatrix} \;;\; \lambda_{10} = \begin{pmatrix} 0&0&0&-i\\ 0&0&0&0\\ 0&0&0&0\\ i&0&0&0 \end{pmatrix} \\ m=1 \;:\; \lambda_{11} = \begin{pmatrix} 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&1&0&0 \end{pmatrix} \;;\; \lambda_{12} = \begin{pmatrix} 0&0&0&0\\ 0&0&0&-i\\ 0&0&0&0\\ 0&i&0&0 \end{pmatrix} \\ m=2 \;:\; \lambda_{13} = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{pmatrix} \;;\; \lambda_{14} = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&-i\\ 0&0&i&0 \end{pmatrix} \\ m=3 \;:\lambda_{15} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&-3 \end{pmatrix} ;\; \overline{I}_{15} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&3 \end{pmatrix} \end{cases} \tag{2eii}
$$

2.1.4 General $SU(5)$ generator spectrum

$$
N=5 \;:\quad \text{number of shells} = 4 \;:\quad \text{shell numbers } l = 1,2,3,4
$$

$$
\text{group basis vectors} \;:\; |1\rangle = \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} \;;\; |2\rangle = \begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix} \;;\; |3\rangle = \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} \;;\; |4\rangle = \begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix} \;;\; |5\rangle = \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} \tag{2fi}
$$

The general $SU(5)$ generator spectrum is composed of $5-1=4$ configuration shells specified by $l=1,2,3,4$.

The 1^{st}-shell contains $1+1=2$ pairs of symmetric and antisymmetric generators specified by pair numbers $m=0,1$; the $m=0$ pair are enumerated and determined by setting $l=1$ in equation $(2b)$ as $\lambda_1 = |1\rangle\langle 2| + |2\rangle\langle 1|$, $\lambda_2 = -i(\ |1\rangle\langle 2| - |2\rangle\langle 1|\ )$, while the $m=1$ pair are enumerated and determined by setting $l=1$ in the last pair in equation $(2b)$ as $\lambda_3 = \frac{1}{\sqrt{\frac{1}{2}1(1+1)}}(\ \sum_{m=0}^{1-1} |m+1\rangle\langle m+1| - |2\rangle\langle 2|\ ) = |1\rangle\langle 1| - |2\rangle\langle 2|$, $\overline{I}_3 = \frac{1}{\sqrt{\frac{1}{2}1(1+1)}}(\ \sum_{m=0}^{1-1} |m+1\rangle\langle m+1| + |2\rangle\langle 2|\ ) = |1\rangle\langle 1| + |2\rangle\langle 2|$.

The 2^{nd}-shell contains $2+1=3$ pairs of symmetric and antisymmetric generators specified by pair numbers $m=0,1,2$; the $m=0$ pair are enumerated and determined by setting $l=2$ in equation $(2b)$ as $\lambda_4 = |1\rangle\langle 3| + |3\rangle\langle 1|$, $\lambda_5 = -i(\ |1\rangle\langle 3| - |3\rangle\langle 1|\ )$, the $m=1$ pair are enumerated and determined by setting $l=2$ in the second pair in equation $(2b)$ as $\lambda_6 = |2\rangle\langle 3| + |3\rangle\langle 2|$, $\lambda_7 = -i(\ |2\rangle\langle 3| - |3\rangle\langle 2|\ )$, while the $m=2$ pair are enumerated and determined by setting $l=2$ in the last pair in equation

(2b) as $\lambda_8 = \frac{1}{\sqrt{\frac{1}{2}2(2+1)}}(\ \sum_{m=0}^{2-1} |m+1\rangle\langle m+1| - |3\rangle\langle 3| \ ) = \frac{1}{\sqrt{3}}(\ (\ |1\rangle\langle 1| - |3\rangle\langle 3| \ ) + (\ |2|\langle 2| - |3\rangle\langle 3| \ ) \ )$, $\overline{I}_8 = \frac{1}{\sqrt{\frac{1}{2}2(2+1)}}(\ \sum_{m=0}^{2-1} |m+1\rangle\langle m+1| + |3\rangle\langle 3| \ ) = \frac{1}{\sqrt{3}}(\ (\ |1\rangle\langle 1| + |3\rangle\langle 3| \ ) + (\ |2\rangle\langle 2| + |3\rangle\langle 3| \ ) \ )$.

The 3^{rd}-shell contains $3+1=4$ pairs of symmetric and antisymmetric generators specified by pair numbers $m = 0,1,2,3$; the $m=0$ pair are enumerated and determined by setting $l=3$ in equation (2b) as $\lambda_9 = |1\rangle\langle 4| + |4\rangle\langle 1|$, $\lambda_{10} = -i(\ |1\rangle\langle 4| - |4\rangle\langle 1| \ )$, the $m=1$ pair are enumerated and determined by setting $l=3$ in the second pair in equation (2b) as $\lambda_{11} = |2\rangle\langle 4| + |4\rangle\langle 2|$, $\lambda_{12} = -i(\ |2\rangle\langle 4| - |4\rangle\langle 2| \ )$, the $m=2$ pair are enumerated and determined by setting $l=3$ in the third pair (not indicated) in equation (2c) as $\lambda_{13} = |3\rangle\langle 4| + |4\rangle\langle 3|$, $\lambda_{14} = -i(\ |3\rangle\langle 4| - |4\rangle\langle 3| \ )$, while the $m=3$ pair are enumerated and determined by setting $l=3$ in the last pair in equation (2b) as $\lambda_{15} = \frac{1}{\sqrt{\frac{1}{2}3(3+1)}}(\ \sum_{m=0}^{3-1} |m+1\rangle\langle m+1| - |4\rangle\langle 4| \ ) = \frac{1}{\sqrt{6}}(\ (\ |1\rangle\langle 1| - |4\rangle\langle 4| \ ) + (\ |2\rangle\langle 2| - |4\rangle\langle 4| \ ) \ ) + (\ |3\rangle\langle 3| - |4\rangle\langle 4| \ ) \ )$, $\overline{I}_{15} = \frac{1}{\sqrt{\frac{1}{2}3(3+1)}}(\ \sum_{m=0}^{3-1} |m+1\rangle\langle m+1| + |4\rangle\langle 4| \ ) = \frac{1}{\sqrt{6}}(\ (\ |1\rangle\langle 1| + |4\rangle\langle 4| \ ) + (\ |2\rangle\langle 2| + |4\rangle\langle 4| \ ) \ ) + (\ |3\rangle\langle 3| + |4\rangle\langle 4| \ ) \ )$.

The 4^{th}-shell contains $4+1=5$ pairs of symmetric and antisymmetric generators specified by pair numbers $m = 0,1,2,3,4$; the $m=0$ pair are enumerated and determined by setting $l=4$ in equation (2b) as $\lambda_{16} = |1\rangle\langle 5| + |5\rangle\langle 1|$, $\lambda_{17} = -i(\ |1\rangle\langle 5| - |5\rangle\langle 1| \ )$, the $m=1$ pair are enumerated and determined by setting $l=4$ in the second pair in equation (2b) as $\lambda_{18} = |2\rangle\langle 5| + |5\rangle\langle 2|$, $\lambda_{19} = -i(\ |2\rangle\langle 5| - |5\rangle\langle 2| \ )$, the $m=2$ pair are enumerated and determined by setting $l=4$ in the third (not indicated) pair in equation (2b) as $\lambda_{20} = |3\rangle\langle 5| + |5\rangle\langle 3|$, $\lambda_{21} = -i(\ |3\rangle\langle 5| - |5\rangle\langle 3| \ )$, the $m=3$ pair are enumerated and determined by setting $l=4$ in the fourth pair (not indicated) in equation (2b) as $\lambda_{22} = |4\rangle\langle 5| + |5\rangle\langle 4|$, $\lambda_{23} = -i(\ |4\rangle\langle 5| - |5\rangle\langle 4| \ )$, while the $m=4$ pair are enumerated and determined by setting $l=4$ in the last pair in equation (2b) as $\lambda_{24} = \frac{1}{\sqrt{\frac{1}{2}4(4+1)}}(\ \sum_{m=0}^{4-1} |m+1\rangle\langle m+1| - |5\rangle\langle 5| \ ) = \frac{1}{\sqrt{10}}(\ (\ |1\rangle\langle 1| - |5\rangle\langle 5| \ ) + (\ |2\rangle\langle 2| - |5\rangle\langle 5| \ ) \ ) + (\ |3\rangle\langle 3| - |5\rangle\langle 5| \ ) \ ) + (\ |4\rangle\langle 4| - |5\rangle\langle 5| \ ) \ )$, $\overline{I}_{24} = \frac{1}{\sqrt{\frac{1}{2}4(4+1)}}(\ \sum_{m=0}^{4-1} |m+1\rangle\langle m+1| + |5\rangle\langle 5| \ ) = \frac{1}{\sqrt{10}}(\ (\ |1\rangle\langle 1| + |5\rangle\langle 5| \ ) + (\ |2\rangle\langle 2| + |5\rangle\langle 5| \ ) \ ) + (\ |3\rangle\langle 3| + |5\rangle\langle 5| \ ) \ ) + (\ |4\rangle\langle 4| + |5\rangle\langle 5| \ ) \ )$.
We use the definitions of the $SU(5)$ group basis vectors given in equation (2fi) above to evaluate the tensor products explicitly as 5×5 matrices. The $(5-1)(5+2) = 28$ symmetric and antisymmetric generators $\lambda_1, \lambda_2, \lambda_3, \overline{I}_3, ..., \lambda_{24}, \overline{I}_{24}$ contained in the 4 shells of the general $SU(5)$ generator spectrum are presented below, where the principal group basis vectors characterizing the 1^{st} , 2^{nd} , 3^{rd} and 4^{th} shells are abbreviated as PRST-$|1+1\rangle$=PRST-$|2\rangle$, PRST-$|2+1\rangle$=PRST-$|3\rangle$, PRST-$|3+1\rangle$=PRST-$|4\rangle$ and PRST-$|4+1\rangle$=PRST-$|5\rangle$, respectively.

General $SU(5)$ generator spectrum : 4 shells, $l = 1,2,3,4$

$$1^{st}-\text{shell} \ : \ PRST - |2\rangle \quad \begin{cases} m=0 \ : \quad \lambda_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad ; \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ m=1 \ : \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad ; \quad \overline{I}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{cases}$$

$$2^{nd}-\text{shell} \;:\; PRST-|3\rangle \quad \begin{cases} m=0 \;:\quad \lambda_4 = \begin{pmatrix} 0&0&1&0&0\\ 0&0&0&0&0\\ 1&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix} \quad ; \quad \lambda_5 = \begin{pmatrix} 0&0&-i&0&0\\ 0&0&0&0&0\\ i&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix} \\ m=1 \;:\quad \lambda_6 = \begin{pmatrix} 0&0&0&0&0\\ 0&0&1&0&0\\ 0&1&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix} \quad ; \quad \lambda_7 = \begin{pmatrix} 0&0&0&0&0\\ 0&0&-i&0&0\\ 0&i&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix} \\ m=2 \;:\; \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&-2&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix} ; \; \overline{I}_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&2&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix} \end{cases}$$

$$3^{rd}-\text{shell} \;:\; PRST-|4\rangle \quad \begin{cases} m=0 \;:\quad \lambda_9 = \begin{pmatrix} 0&0&0&1&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 1&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix} \quad ; \quad \lambda_{10} = \begin{pmatrix} 0&0&0&-i&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ i&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix} \\ m=1 \;:\quad \lambda_{11} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&0 \end{pmatrix} \quad ; \quad \lambda_{12} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&-i&0\\ 0&0&0&0&0\\ 0&i&0&0&0\\ 0&0&0&0&0 \end{pmatrix} \\ m=2 \;:\quad \lambda_{13} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&1&0\\ 0&0&1&0&0\\ 0&0&0&0&0 \end{pmatrix} \quad ; \quad \lambda_{14} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&-i&0\\ 0&0&i&0&0\\ 0&0&0&0&0 \end{pmatrix} \\ m=3 \;:\; \lambda_{15} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&-3&0\\ 0&0&0&0&0 \end{pmatrix} ; \; \overline{I}_{15} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&3&0\\ 0&0&0&0&0 \end{pmatrix} \end{cases}$$

$$
4^{th}-\text{shell}\ :\ PRST-|5\rangle\quad
\begin{cases}
m=0\ :\quad \lambda_{16}=\begin{pmatrix}0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\end{pmatrix}\quad ;\ \lambda_{17}=\begin{pmatrix}0&0&0&0&-i\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\i&0&0&0&0\end{pmatrix}\\
m=1\ :\quad \lambda_{18}=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&1&0&0&0\end{pmatrix}\quad ;\ \lambda_{19}=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&-i\\0&0&0&0&0\\0&0&0&0&0\\0&i&0&0&0\end{pmatrix}\\
m=2\ :\quad \lambda_{20}=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&1&0&0\end{pmatrix}\quad ;\ \lambda_{21}=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&-i\\0&0&0&0&0\\0&0&i&0&0\end{pmatrix}\\
m=3\ :\quad \lambda_{22}=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\end{pmatrix}\quad ;\ \lambda_{23}=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&-i\\0&0&0&i&0\end{pmatrix}\\
m=4\ :\lambda_{24}=\frac{1}{\sqrt{10}}\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&-4\end{pmatrix}\ ;\ \overline{I}_{24}=\frac{1}{\sqrt{10}}\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&4\end{pmatrix}
\end{cases}
\tag{2fii}
$$

The $SU(2)$, $SU(3)$, $SU(4)$, $SU(5)$ generator spectra which we have enumerated and determined explicitly in the Gell-Mann basis in equations $(2cii)$, $(2dii)$, $(2eii)$, $(2fii)$ display the property that a general $SU(N)$ generator spectrum is composed of the standard N^2-1 hermitian symmetric and antisymmetric traceless generators $\lambda_1, \lambda_2, ..., \lambda_{N^2-1}$ plus $N-1$ hermitian non-traceless symmetric generators $\overline{I}_{l^2+2l}$, $l = 1, ..., N-1$. According to the basic algebraic method for enumerating and determining generators in non-diagonal and diagonal symmetric and antisymmetric pairs given in equations $(2ai)$, $(2aii)$, the generators $\overline{I}_{l^2+2l}$ are the symmetric partners of the standard $N-1$ traceless diagonal antisymmetric generators λ_{l^2+2l} , $l = 1, ..., N-1$ as displayed in the $SU(2)$, $SU(3)$, $SU(4)$, $SU(5)$ spectra in equations $(2cii)$, $(2dii)$, $(2eii)$, $(2fii)$. It is then clear that the $N-1$ hermitian non-traceless symmetric generators $\overline{I}_{l^2+2l}$, $l = 1, ..., N-1$ are part of the general generator spectrum of an $SU(N)$ symmetry group, which cannot just be ignored without proper mathematical or physical justification.

In the next section, we demonstrate how the non-traceless symmetric generators $\overline{I}_l$ can be summed up into the identity matrix to reduce the general $SU(N)$ generator spectrum to the familiar standard $SU(N)$ generator spectrum, while in section 4 where we reinterpret the $SU(N)$ generators as spin angular momentum operators, we restore the generators $\overline{I}_{l^2+2l}$ back into the $SU(N)$ generator spectrum, noting that they satisfy Cartan subalgebra and therefore identify them as Cartan generators in an extended Cartan-Weyl basis.

3 Configuration shell structure in the standard $SU(N)$ generator spectrum

The enumeration of the general $SU(N)$ generator spectrum which we have developed here differs significantly from standard descriptions, which consider only the N^2-1 *traceless* symmetric and antisymmetric $N \times N$ matrices λ_1 , λ_2 , λ_3 , ... , λ_{N^2-2} , λ_{N^2-1}, composed of $N(N-1)$ non-diagonal and $N-1$ diagonal matrices, as the standard generators of an $SU(N)$ symmetry group. The algebraic property we have applied here for enumerating and determining $SU(N)$ symmetry group generators as non-diagonal and diagonal hermitian symmetric-antisymmetric pairs according to the formula given in equations $(2ai)$-$(2aii)$ identifies and includes the $N-1$ *non-traceless* diagonal symmetric generators $\overline{I}_{l^2+2l}$ as the symmetric partners of the traceless diagonal antisymmetric generators λ_{l^2+2l} in each shell $l = 1, ..., N-1$ of a general $SU(N)$ generator spectrum.

To reduce the unfamiliar general $SU(N)$ generator spectrum determined in general form in equations $(2a)$-$(2b)$ to the standard $SU(N)$ generator spectrum, we apply an important algebraic property that the $N-1$ non-traceless diagonal symmetric generators $\overline{I}_{l^2+2l}$ can be combined as a weighted sum to form the $N \times N$ identity matrix I_N of the $SU(N)$ symmetry group according to

$$\frac{1}{N-1}\sum_{l=1}^{N-1}\sqrt{\frac{1}{2}l(l+1)}\;\overline{I}_{l^2+2l} = I_N \quad ; \quad N \geq 2 \quad ; \quad I_N = N \times N \text{ identity matrix} \qquad (3a)$$

We may then drop the $N-1$ non-traceless diagonal symmetric generators $\overline{I}_{l^2+2l}$ from the respective $N-1$ shells as specified in equation $(2c)$ and effectively represent them by the resultant group identity matrix I_N, which we identify as an *identity generator* λ_0 determined as

$$\lambda_0 = I_N \qquad \Rightarrow \qquad \lambda_0 = \frac{1}{N-1}\sum_{l=1}^{N-1}\sqrt{\frac{1}{2}l(l+1)}\;\overline{I}_{l^2+2l} \quad ; \quad N \geq 2 \qquad (3b)$$

Considering that the $N-1$ non-traceless diagonal symmetric generators $\overline{I}_{l^2+2l}$, $l = 1, ..., N-1$ are absorbed in the definition of the identity matrix I_N and therefore replacing them with the identity generator $\lambda_0 = I_N$ reduces the general $SU(N)$ generator spectrum in equation $(2b)$ to a standard $SU(N)$ generator spectrum composed of the familiar N^2-1 traceless generators in the Gell-Mann basis λ_1 , λ_2 , λ_3 , ... , λ_{N^2-2} , λ_{N^2-1} and the identity generator λ_0.

Due to its unique algebraic property that it is non-traceless and commutes with all the other N^2-1 traceless generators, the identity generator λ_0 is placed in a separate shell, which we classify as the 0^{th}-shell, for consistency in the enumeration of the definite numbers of specified generators in each of the $N-1+1 = N$ configuration shells of the resulting standard $SU(N)$ generator spectrum.

Dropping the non-traceless diagonal symmetric generator $\overline{I}_{l^2+2l}$ from the l^{th}-shell of the general $SU(N)$ generator spectrum in equation $(2b)$ and introducing the 0^{th}-shell containing the identity generator $\lambda_0 = I_N$, we obtain the expected standard $SU(N)$ generator spectrum presented here in the general form:

Generators in the l^{th}-shell of a standard $SU(N)$ spectrum : N shells, $l = 1, ..., N-1$

$$0^{th}-\text{shell} \ : \ PRST-|1\rangle \qquad m = 0 \ : \qquad \lambda_0 = I_N$$

$$l^{th}-\text{shell}, \ l \neq 0 \ : \ PRST-|l+1\rangle \quad \begin{cases} m = 0 \ : & \lambda_{l^2} = |1\rangle\langle l+1| + |l+1\rangle\langle 1| \\ & \lambda_{l^2+1} = -i(\ |1\rangle\langle l+1| - |l+1\rangle\langle 1|\ ) \\ \\ m = 1 \ : & \lambda_{l^2+2} = |2\rangle\langle l+1| + |l+1\rangle\langle 2| \\ & \lambda_{l^2+3} = -i(\ |2\rangle\langle l+1| - |l+1\rangle\langle 2|\ ) \\ \cdots\cdots \\ \cdots\cdots \\ \cdots\cdots \\ \\ m = l-1 \ : & \lambda_{l^2+2(l-1)} = |l\rangle\langle l+1| + |l+1\rangle\langle l| \\ & \lambda_{l^2+2l-1} = -i(\ |l\rangle\langle l+1| - |l+1\rangle\langle l|\ ) \\ \\ m = l : & \lambda_{l^2+2l} = \frac{1}{\sqrt{\frac{1}{2}l(l+1)}} \sum_{m=0}^{l-1} |m+1\rangle\langle m+1| - |l+1\rangle\langle l+1| \end{cases}$$

$$(3c)$$

The quantum structure of a standard $SU(N)$ generator spectrum is composed of N shells specified by a quantum number l taking N integer values $l = 0, 1, ..., N-1$. In each of the N shells specified by a quantum number l, the generators are distributed as symmetric-antisymmetric pairs specified by a symmetric-antisymmetric generator pair quantum number m taking $l+1$ integer values $m = 0, 1, ..., l$. Since the symmetric partner of the antisymmetric diagonal generator λ_{l^2+2l} , $l \neq 0$, is contained in the identity generator λ_0 in the 0^{th}-shell, each of the N shells specified by a quantum number $l = 0, 1, ..., N-1$ contains $2l+1$ generators as demonstrated in the standard $SU(N)$ generator spectrum in equation $(3c)$ above. The total number of generators in the standard $SU(N)$ generator spectrum is therefore obtained as $\sum_{l=0}^{N-1}(2l+1) = N^2$, consisting of the single (1) identity generator λ_0 and the familiar N^2-1 traceless non-diagonal and diagonal symmetric and antisymmetric generators λ_1 , λ_2 , ... , λ_{N^2-1}. We note that the enumeration of the N^2-1 traceless non-diagonal and diagonal symmetric and antisymmetric generators λ_1 , λ_2 , ... , λ_{N^2-1}, together with the $N \times N$ identity matrix I_N, as a complete set of $SU(N)$ generators has also been presented in general form in [3], but without the quantum structure which we have developed here and earlier in [2].

Using the explicit forms of the $N-1$ non-traceless generators $\bar{I}_{l^2+2l}$, $l = 1, ..., N-1$ already determined in the general $SU(2)$, $SU(3)$, $SU(4)$, $SU(5)$ generator spectra in equations $(2cii)$, $(2dii)$, $(2eii)$, $(2fii)$ to determine the respective identity generators λ_0 according to the definition in equation $(3b)$, we take account of the explicit forms of the N^2-1 traceless generators λ_1 , λ_2 , ... , λ_{N^2-1} already determined in equations $(2cii)$, $(2dii)$, $(2eii)$, $(2fii)$, to present the shell structure of the standard $SU(2)$, $SU(3)$, $SU(4)$, $SU(5)$ generator spectra as examples below.

3.1 Shell structure of the standard $SU(2)$ generator spectrum

$$N = 2 \ : \qquad \text{no. of shells} = 2 \ : \quad \text{shell numbers } l = 0 \ , \ 1$$

The group basis vectors $|1\rangle$, $|2\rangle$ are defined in equation $(2ci)$.

Using equation ($3b$) with the non-traceless generator $\overline{I}_3$ already determined in equation ($2cii$), we determine the $SU(2)$ identity generator λ_0 in the form

$$\lambda_0 = \overline{I}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3di}$$

which taken together with the $2^2 - 1 = 3$ generators $\lambda_1, \lambda_2, \lambda_3$ already determined in equation ($2cii$) constitute the standard $SU(2)$ generator spectrum presented below for ease of reference.

Standard $SU(2)$ generator spectrum : 2 shells, $l = 0, 1$

$$\begin{array}{llll} 0^{th} - \text{shell} \ : & PRST - |1\rangle \ : & m = 0 & \lambda_0 \\ 1^{st} - \text{shell} \ : & PRST - |2\rangle & \begin{cases} m = 0 \ : & \lambda_1 \quad ; \quad \lambda_2 \\ m = 1 \ : & \lambda_3 \end{cases} & \end{array} \tag{3dii}$$

3.2 Shell structure of the standard $SU(3)$ generator spectrum

$$N = 3 \ : \qquad \text{no. of shells} = 3 \ : \quad \text{shell numbers } l = 0, 1, 2$$

The group basis vectors $|1\rangle$, $|2\rangle$, $|3\rangle$ are defined in equation ($2di$).

Using equation ($3b$) with the non-traceless generators $\overline{I}_3$, $\overline{I}_8$ already determined in equation ($2dii$), we determine the $SU(3)$ identity operator λ_0 in the form

$$\lambda_0 = \frac{1}{2}(\ \overline{I}_3 + \sqrt{3} \ \overline{I}_8 \ ) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3ei}$$

which taken together with the $3^2 - 1 = 8$ generators $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_7, \lambda_8$ already determined in equation ($2dii$) constitute the standard $SU(3)$ generator spectrum presented below for ease of reference.

Standard $SU(3)$ generator spectrum : 3 shells, $l = 0, 1, 2$

$$\begin{array}{llll} 0^{th} - \text{shell} \ : & PRST - |1\rangle \ : & m = 0 & \lambda_0 \\ 1^{st} - \text{shell} \ : & PRST - |2\rangle & \begin{cases} m = 0 \ : & \lambda_1 \quad ; \quad \lambda_2 \\ m = 1 \ : & \lambda_3 \end{cases} & \\ 2^{nd} - \text{shell} \ : & PRST - |3\rangle & \begin{cases} m = 0 \ : & \lambda_4 \quad ; \quad \lambda_5 \\ m = 1 \ : & \lambda_6 \quad ; \quad \lambda_7 \\ m = 2 \ : & \lambda_8 \end{cases} & \end{array} \tag{3eii}$$

3.3 Shell structure of the standard $SU(4)$ generator spectrum

$$N = 4 \ : \qquad \text{number of shells} = 4 \ : \quad \text{shell numbers } l = 0, 1, 2, 3$$

The group basis vectors $|1\rangle$, $|2\rangle$, $|3\rangle$, $|4\rangle$ are defined in equation $(2ei)$.

Using equation $(3b)$ with the non-traceless generators $\overline{I}_3$, $\overline{I}_8$, $\overline{I}_{15}$ already determined in equation $(2eii)$, we determine the $SU(4)$ identity operator λ_0 in the form

$$\lambda_0 = \frac{1}{3}(\ \overline{I}_3 + \sqrt{3} \ \overline{I}_8 + \sqrt{6} \ \overline{I}_{15} \ ) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3fi}$$

which taken together with the $4^2 - 1 = 15$ generators $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, ..., \lambda_{13}, \lambda_{14}, \lambda_{15}$ already determined in equation $(2eii)$ constitute the standard $SU(4)$ generator spectrum presented below for ease of reference.

The standard $SU(4)$ generator spectrum : 4 shells, $l = 0, 1, 2, 3$

$$0^{th} - \text{shell} \ : \qquad PRST - |1\rangle \ : \qquad m = 0 \qquad \lambda_0$$

$$1^{st} - \text{shell} \ : \ PRST - |2\rangle \quad \begin{cases} m = 0 \ : & \lambda_1 \quad ; \quad \lambda_2 \\ m = 1 \ : & \lambda_3 \end{cases}$$

$$2^{nd} - \text{shell} \ : \ PRST - |3\rangle \quad \begin{cases} m = 0 \ : & \lambda_4 \quad ; \quad \lambda_5 \\ m = 1 \ : & \lambda_6 \quad ; \quad \lambda_7 \\ m = 2 \ : & \lambda_8 \end{cases}$$

$$3^{rd} - \text{shell} \ : \quad PRST - |4\rangle \quad \begin{cases} m = 0 \ : & \lambda_9 \quad ; \quad \lambda_{10} \\ m = 1 \ : & \lambda_{11} \quad ; \quad \lambda_{12} \\ m = 2 \ : & \lambda_{13} \quad ; \quad \lambda_{14} \\ m = 3 \ : & \lambda_{15} \end{cases} \tag{3fii}$$

3.4 Shell structure of the standard $SU(5)$ generator spectrum

$$N = 5 \ : \qquad \text{number of shells} = 5 \ : \quad \text{shell numbers } l = 0, 1, 2, 3, 4$$

The group basis vectors $|1\rangle$, $|2\rangle$, $|3\rangle$, $|4\rangle$, $|5\rangle$ are defined in equation $(2fi)$.

Using equation $(3b)$ with the non-traceless generators $\overline{I}_3$, $\overline{I}_8$, $\overline{I}_{15}$, $\overline{I}_{24}$ already determined in equation $(2fii)$, we determine the $SU(5)$ identity generator λ_0 in the form

$$\lambda_0 = \frac{1}{4}(\ \overline{I}_3 + \sqrt{3} \ \overline{I}_8 + \sqrt{6} \ \overline{I}_{15} + \sqrt{10} \ \overline{I}_{24} \ ) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3gi}$$

which taken together with the $5^2-1=24$ generators $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, ..., \lambda_{22}, \lambda_{23}, \lambda_{24}$ already determined in equation $(2fii)$ constitute the standard $SU(5)$ generator spectrum presented below for ease of reference.

The standard $SU(5)$ *generator spectrum* : 5 shells, $l=0,1,2,3,4$

$$
\begin{array}{lll}
0^{th}-\text{shell} \ : \ PRST-|1\rangle \ : & m=0 & \lambda_0 \\
1^{st}-\text{shell} \ : \ PRST-|2\rangle & \begin{cases} m=0 \ : & \lambda_1 \quad ; \quad \lambda_2 \\ m=1 \ : & \lambda_3 \end{cases} & \\
2^{nd}-\text{shell} \ : \ PRST-|3\rangle & \begin{cases} m=0 \ : & \lambda_4 \quad ; \quad \lambda_5 \\ m=1 \ : & \lambda_6 \quad ; \quad \lambda_7 \\ m=2 \ : & \lambda_8 \end{cases} & \\
3^{rd}-\text{shell} \ : \ PRST-|4\rangle & \begin{cases} m=0 \ : & \lambda_9 \quad ; \quad \lambda_{10} \\ m=1 \ : & \lambda_{11} \quad ; \quad \lambda_{12} \\ m=2 \ : & \lambda_{13} \quad ; \quad \lambda_{14} \\ m=3 \ : & \lambda_{15} \end{cases} & \\
4^{th}-\text{shell} \ : \ PRST-|5\rangle & \begin{cases} m=0 \ : & \lambda_{16} \quad ; \quad \lambda_{17} \\ m=1 \ : & \lambda_{18} \quad ; \quad \lambda_{19} \\ m=2 \ : & \lambda_{20} \quad ; \quad \lambda_{21} \\ m=3 \ : & \lambda_{22} \quad ; \quad \lambda_{23} \\ m=4 \ : & \lambda_{24} \end{cases} &
\end{array}
\tag{3gii}
$$

These $SU(2)$, $SU(3)$, $SU(4)$, $SU(5)$ examples clearly display a complete shell structure of a standard $SU(N)$ generator spectrum. Each shell specified by a quantum number $l=0,1,...,N-1$ contains a definite number $2l+1$ generators specified by the symmetric-antisymmetric generator pair quantum number $m=0,1,...,l$. This quantum structure of a standard $SU(N)$ generator spectrum is similar to the orbital angular momentum state spectrum in the n^{th}-energy level in an atom where each orbital state configuration shell specified by orbital angular momentum quantum number $l=0,1,...,n-1$ contains a definite number $2l+1$ of orbital angular momentum states specified by magnetic quantum number $m=0,\pm1,\pm2,...,\pm l$.

An important mismatch which emerges between the quantum structure of the $SU(N)$ generator spectrum and the quantum structure of the orbital angular momentum state spectrum in the n^{th}-energy level of an atom is that, in the Gell-Mann basis where the generators are enumerated serially as $\lambda_1, \lambda_2, ..., \lambda_{N^2-1}$ displayed in the spectra in equations $(3dii)$, $(3eii)$, $(3fii)$, $(3gii)$, the specification by the quantum numbers l , m is lost once the quantum numbers are used in the enumeration formulae in equations $(2ai)$, $(2aii)$ to generate the serial numbering of the Gell-Mann matrices. To maintain explicit specification of the generators by the quantum numbers l , m, we take advantage of the formulae for enumerating and determining the generators as symmetric and antisymmetric pairs in equations $(2ai)$, $(2aii)$ to reinterpret the generators in the Gell-Mann basis as *spin angular momentum operators*, composed of hermitian conjugate spin state raising and lowering operators specified by quantum numbers l , m. This reinterpretation of the generators in the Gell-Mann basis as spin angular momentum basis explicitly specified by the quantum numbers l , m ensures that the $SU(N)$ generator spectrum corresponds precisely to the orbital angular momentum state spectrum also specified by a corresponding pair of quantum numbers l , m taking similar values in the n^{th}-energy level of an atom as we now establish in the next section.

4 Shell structure of the $SU(N)$ generator spectrum in spin angular momentum interpretation

The algebraic property that generators in a standard $SU(N)$ generator spectrum are specified by a configuration shell quantum number $l = 0, 1, ..., N-1$ and a symmetric-antisymmetric generator pair quantum number $m = 0, 1, ..., l$ similar to the specification of orbital angular momentum states by an orbital shell quantum number $l = 0, 1, ..., n-1$ and a magnetic quantum number $m = -l, -(l-1), ..., 0, 1, ..., l-1, l$ in the n^{th}-energy level in an atom provides motivation to seek a precise correspondence between an $SU(N)$ symmetry group generator spectrum and an atomic n^{th}-energy level orbital angular momentum state spectrum.

An atomic energy level specified by a principal quantum number n, normally referred to as the n^{th}-energy level, is composed of n orbital state configuration shells specified by an orbital angular momentum quantum number l taking n values $l = 0, 1, ..., n-1$. Each orbital shell specified by an orbital quantum number l contains $2l+1$ orbital angular momentum states, each described by a spherical harmonic function Y_l^m specified by orbital shell quantum number l and a magnetic quantum number m taking $2l+1$ integer values $m = -l, -(l-1), ..., 0, 1, ..., (l-1), l$.

Noting that the specifications of the configuration shell quantum numbers l in both $SU(N)$ generator and atomic n^{th}-energy level orbital state spectra are precisely consistent, we harmonize the specifications of the $SU(N)$ symmetric-antisymmetric generator pair and the atomic magnetic quantum numbers, m, by considering that in the atomic orbital state spectrum, the single $l=0$, $m=0$ state Y_0^0 takes a symmetrically neutral unit value obtained as

$$l = 0 \quad ; \quad m = 0 \;\; : \qquad Y_0^0 = 1 \tag{4a}$$

while the remaining $2l$ orbital states Y_l^m , $l \neq 0$, specified by $m = \mp 1, ..., \mp l$ can be reinterpreted as l conjugate pairs (Y_l^{-m} , Y_l^m), now specified by l values of the magnetic quantum number $m = 1, ..., l$ according to the standard relation [4]

$$Y_l^{-m} = (-1)^m Y_l^{m*} \qquad ; \qquad m = 1, ..., l \tag{4b}$$

Taking the single symmetrically neutral orbital state Y_0^0 in equation $(4a)$ and the l conjugate pairs $Y_l^{\pm m}$ related according to equation $(4b)$ together, we now redefine the atomic magnetic quantum number m as a conjugate orbital state pair quantum number taking $l+1$ values $m = 0, 1, ..., l$ including the unit state Y_0^0, which is now precisely consistent with the specification of the $SU(N)$ symmetric-antisymmetric generator pair quantum number m also taking $l+1$ values $m = 0, 1, ..., l$ including the identity generator λ_0.

To achieve complete harmony in the comparison of the shell structures of the spectra of $SU(N)$ generators and atomic n^{th}-energy level orbital angular momentum states, we reorganize the notation for the atomic orbital angular momentum states Y_0^0 , $Y_l^{\pm m}$ in the equivalent form Y_{00} , $Y_{lm}^{\pm}$ according to the redefinitions

$$Y_{00} = Y_0^0 \quad ; \quad Y_{lm}^{+} = Y_l^m \quad ; \quad Y_{lm}^{-} = Y_l^{-m} \qquad \Rightarrow \qquad Y_{lm}^{-} = (-1)^m Y_{lm}^{+*} \quad ; \quad m = 1, ..., l \tag{4c}$$

We now redefine the $SU(N)$ generators and introduce an appropriate notation specified by the quantum numbers l , m corresponding directly to the specification and notation of the atomic orbital angular momentum states Y_{00} , $Y^{\pm}_{lm}$. Such a redefinition of $SU(N)$ generators is easily achieved in the spin angular momentum basis, where we follow the formulae for enumerating and determining $SU(N)$ generators in symmetric-antisymmetric pairs in equations $(2ai)$, $(2aii)$ to introduce hermitian conjugate spin angular momentum state raising and lowering operators $S^{\pm}_{lm}$ defined by

$$l = 1, ..., N-1 \quad ; \quad m = 0, 1, ..., l-1$$

$$S^{+}_{lm} = |m+1\rangle\langle l+1| \qquad ; \qquad S^{-}_{lm} = |l+1\rangle\langle m+1| \qquad ; \qquad S^{-}_{lm} = (S^{+}_{lm})^{\dagger} \qquad (5ai)$$

Using the $SU(N)$ symmetry group basis state vector orthonormality relation given in equation $(1b)$, noting

$$m = 0, ..., l-1 \quad \Rightarrow \quad l+1 > m+1 \quad ; \quad \langle m+1|l+1\rangle = 0 \quad ; \quad \langle l+1|m+1\rangle = 0 \qquad (5aii)$$

we obtain the algebraic relations

$$S^{+\,2}_{lm} = 0 \quad ; \quad S^{-\,2}_{lm} = 0 \qquad ; \qquad S^{+}_{lm}S^{-}_{lm} = |m+1\rangle\langle m+1| \quad ; \quad S^{-}_{lm}S^{+}_{lm} = |l+1\rangle\langle l+1| \qquad (5aiii)$$

The $SU(N)$ generators in the Gell-Mann basis $\lambda_0, \lambda_1, \lambda_2, ..., \lambda_{N^2-1}$ are now interpreted as hermitian spin angular momentum operators specified by quantum numbers l , m. In particular, the non-diagonal symmetric-antisymmetric generator pair $(\lambda_{l^2+2m}$, $\lambda_{l^2+2m+1})$ enumerated and determined according to the formula in equation $(2ai)$ is now interpreted as the non-diagonal symmetric-antisymmetric hermitian spin operator pair $(\sigma^{x}_{lm}$, $\sigma^{y}_{lm})$ determined according to equation $(2ai)$ in the form

$$l = 1, ..., N-1 \quad ; \quad m = 0, 1, ..., l-1 \ : \qquad \lambda_{l^2+2m} = \sigma^{x}_{lm} \quad ; \quad \lambda_{l^2+2m} = \sigma^{y}_{lm}$$

$$\sigma^{x}_{lm} = |m+1\rangle\langle l+1| + |l+1\rangle\langle m+1| \quad ; \quad \sigma^{y}_{lm} = -i(\ |m+1\rangle\langle l+1| - |l+1\rangle\langle m+1|\ ) \qquad (5bi)$$

while the diagonal symmetric-antisymmetric generator pair $(\lambda_{l^2+2l}$, $\overline{I}_{l^2+2l})$ enumerated and determined according to the formula in equation $(2aii)$ is now interpreted as the diagonal symmetric-antisymmetric generator hermitian spin operator pair $(\sigma^{z}_{l}$, $\sigma^{0}_{l})$ determined according to equation $(2aii)$ in the form

$$l = 1, ..., N-1 \quad ; \quad m = l \ : \qquad \lambda_{l^2+2l} = \sigma^{z}_{l} \quad ; \quad \overline{I}_{l^2+2l} = \sigma^{0}_{l}$$

$$\sigma^{z}_{l} = \frac{1}{\sqrt{\frac{1}{2}l(l+1)}} \sum_{m=0}^{l-1} \sigma^{z}_{lm} \qquad ; \qquad \sigma^{z}_{lm} = |m+1\rangle\langle m+1| - |l+1\rangle\langle l+1|$$

$$\sigma^{0}_{l} = \frac{1}{\sqrt{\frac{1}{2}l(l+1)}} \sum_{m=0}^{l-1} I_{lm} \qquad ; \qquad I_{lm} = |m+1\rangle\langle m+1| + |l+1\rangle\langle l+1| \qquad (5bii)$$

It follows from the definitions in equations $(5ai)$, $(5aii)$, $(5bi)$, $(5bii)$ that the non-diagonal generators σ^{x}_{lm} , σ^{y}_{lm} and the diagonal generators are expressed in terms of the spin state raising and lowering operators $S^{\pm}_{lm}$ in the form

$$l = 1, ..., N-1 \quad ; \quad m = 0, 1, ..., l-1$$

$$\sigma^{x}_{lm} = S^{+}_{lm} + S^{-}_{lm} \ ; \quad \sigma^{y}_{lm} = -i(\ S^{+}_{lm} - S^{-}_{lm}\ ) \ \Rightarrow \ S^{\pm}_{lm} = S^{x}_{lm} \pm i\, S^{y}_{lm} \ ; \quad S^{x}_{lm} = \frac{1}{2}\sigma^{x}_{lm} \ ; \quad S^{y}_{lm} = \frac{1}{2}\sigma^{y}_{lm} \quad (5biii)$$

$$\sigma_l^z = \frac{1}{\sqrt{\frac{1}{2}l(l+1)}} \sum_{m=0}^{l-1} \sigma_{lm}^z \qquad ; \qquad \sigma_{lm}^z = S_{lm}^+ S_{lm}^- - S_{lm}^- S_{lm}^+$$

$$\sigma_l^0 = \frac{1}{\sqrt{\frac{1}{2}l(l+1)}} \sum_{m=0}^{l-1} I_{lm} \qquad ; \qquad I_{lm} = S_{lm}^+ S_{lm}^- + S_{lm}^- S_{lm}^+ \tag{5biv}$$

For completeness, we introduce S_l^z , S_l^0 defined by

$$l = 1, ..., N-1 \quad ; \quad m = l \ : \qquad S_l^z = \frac{1}{2}\sigma_l^z \quad ; \quad S_l^0 = \frac{1}{2}\sigma_l^0 \tag{5bv}$$

We interpret the $2l$ traceless non-diagonal generators σ_{lm}^x , σ_{lm}^y (enumerated by $m = 0, 1, ..., l-1$) and the single traceless diagonal generator σ_l^z (enumerated by $m = l$) as components of a $(2l+1)$-component l^{th}-shell spin angular momentum vector $\vec{\sigma}_l$ defined by

$$\vec{\sigma}_l = (\ \sigma_{l0}^x \ , \ \sigma_{l0}^y \ , \ \sigma_{l1}^x \ , \ \sigma_{l1}^y \ , \ ... \ , \ \sigma_{l\ l-1}^x \ , \ \sigma_{l\ l-1}^y \ , \ \sigma_l^z \ ) \quad ; \quad l = 1, ..., N-1 \tag{5ci}$$

which leads to the introduction of an l^{th}-shell quadratic spin angular momentum operator σ_l^2 obtained as

$$\sigma_l^2 = \vec{\sigma}_l \cdot \vec{\sigma}_l = \sum_{m=0}^{l-1} ((\sigma_{lm}^x)^2 + (\sigma_{lm}^y)^2) \ + \ (\sigma_l^z)^2 \tag{5cii}$$

Using σ_{lm}^x , σ_{lm}^y from equation $(5biii)$ gives

$$(\sigma_{lm}^x)^2 + (\sigma_{lm}^y)^2 = 2(\ S_{lm}^+ S_{lm}^- + S_{lm}^- S_{lm}^+ \ ) \tag{5ciii}$$

which we substitute into equation $(5cii)$ and introduce $\sigma_l^z = 2S_l^z$ to obtain the form

$$\sigma_l^2 = 4 \left(\sum_{m=0}^{l-1} \frac{1}{2}(\ S_{lm}^+ S_{lm}^- + S_{lm}^- S_{lm}^+ \ ) \ + \ (S_l^z)^2 \right) \tag{5civ}$$

We can express σ_l^2 in the form

$$\sigma_l^2 = 4S_l^2 \qquad ; \qquad S_l^2 = \sum_{m=0}^{l-1} \frac{1}{2}(\ S_{lm}^+ S_{lm}^- + S_{lm}^- S_{lm}^+ \ ) \ + \ (S_l^z)^2 \tag{5cv}$$

after using equations $(5biii)$, $(5biv)$ to redefine the l^{th}-shell spin angular momentum vector $\vec{\sigma}_l$ as $\mathbf{S}_l$ according to

$$\mathbf{S}_l = \frac{1}{2}\vec{\sigma}_l \quad \Rightarrow \quad \mathbf{S}_l = (\ S_{l0}^x \ , \ S_{l0}^y \ , \ S_{l1}^x \ , \ S_{l1}^y \ , \ ... \ , \ S_{l\ l-1}^x \ , \ S_{l\ l-1}^y \ , \ S_l^z \ ) \quad ; \quad S_l^2 = \mathbf{S}_l \cdot \mathbf{S}_l = \frac{1}{4}\sigma_l^2 \tag{5cvi}$$

To introduce some higher order spin operators, we use the algebraic relations obtained in equations $(5aii)$, $(5aiii)$, $(5biii)$, $(5biv)$ to obtain the following algebraic relations

$$I_{lm} = S_{lm}^+ S_{lm}^- + S_{lm}^- S_{lm}^+ \qquad ; \qquad [S_{lm}^\pm \ , \ I_{lm}] = 0 \qquad ; \qquad I_{lm}^{\ k} = I_{lm} \quad ; \quad k = 1, 2, 3, ... \tag{6a}$$

$$(\sigma_{lm}^x)^2 = I_{lm} \ ; \quad [\ \sigma_{lm}^x \ , \ I_{lm} \ ] = 0 \quad \Rightarrow \quad (\sigma_{lm}^x)^{2k} = I_{lm} \ ; \quad (\sigma_{lm}^x)^{2k+1} = \sigma_{lm}^x$$

$$(\sigma^y_{lm})^2 = I_{lm}; \quad [\ \sigma^y_{lm}\ ,\ I_{lm}\ ] = 0 \quad \Rightarrow \quad (\sigma^y_{lm})^{2k} = I_{lm}; \quad (\sigma^y_{lm})^{2k+1} = \sigma^y_{lm}\ ; \quad k = 1,2,3,... \qquad (6b)$$

We can now use these general algebraic properties of the $SU(N)$ generators in the spin angular momentum basis to introduce generalizations of the l^{th}-shell quadratic spin angular momentum σ_l^2 to higher order spin operators. Noting that the l^{th}-shell quadratic spin angular momentum operator σ_l^2 as defined in equation $(5cii)$ is an even-power spin operator, we introduce generalizations to l^{th}-shell *even-power spin operator* $Q_{l:2n}$ and *odd-power spin operator* $\mathcal{F}_{l:2n+1}$ of n^{th}-order, $n = 0,1,2,3,...$ defined by

$$Q_{l:2n} = \sum_{m=0}^{l-1}((\sigma^x_{lm})^{2n} + (\sigma^y_{lm})^{2n}) \ + \ (\sigma^z_l)^{2n} \quad ; \quad n = 0,1,2,3,... \qquad (7ai)$$

$$\mathcal{F}_{l:2n+1} = \sum_{m=0}^{l-1}((\sigma^x_{lm})^{2n+1} + (\sigma^y_{lm})^{2n+1}) \ + \ (\sigma^z_l)^{2n+1} \quad ; \quad n = 0,1,2,3,... \qquad (7aii)$$

Using equation $(6b)$ and substituting equations $(5biii)$, $(5biv)$, $(5bv)$, $(5ciii)$ as appropriate, we obtain

$$(\sigma^x_{lm})^{2n} + (\sigma^y_{lm})^{2n} = 2I_{lm} = 2(S^+_{lm}S^-_{lm} + S^-_{lm}S^+_{lm}) \qquad (7bi)$$

$$(\sigma^x_{lm})^{2n+1} + (\sigma^y_{lm})^{2n+1} = \sigma^x_{lm} + \sigma^y_{lm} = \alpha S^+_{lm} + \alpha^* S^-_{lm} \quad ; \quad \alpha = 1 - i = \sqrt{2}\ e^{-i\frac{\pi}{4}} \qquad (7bii)$$

$$(\sigma^z_l)^{2n} = 2^{2n}(S^z_l)^{2n} \quad ; \quad (\sigma^z_l)^{2n+1} = 2^{2n+1}(S^z_l)^{2n+1} \qquad (7biii)$$

which we substitute into equations $(7ai)$ and $(7aii)$ as appropriate to express the even-power and odd-power spin operators in the form

$$Q_{l:2n} = 2^{2n}\left(\sum_{m=0}^{l-1} \frac{1}{2^{2n-1}}(S^+_{lm}S^-_{lm} + S^-_{lm}S^+_{lm}) \ + \ (S^z_l)^{2n} \right) \quad ; \quad n = 0,1,2,3,... \qquad (7ci)$$

$$\mathcal{F}_{l:2n+1} = 2^{2n+1}\left(\sum_{m=0}^{l-1} \frac{1}{2^{2n+1}}(\alpha S^+_{lm} + \alpha^* S^-_{lm}) \ + \ (S^z_l)^{2n+1} \right) \quad ; \quad n = 0,1,2,3,... \qquad (7cii)$$

Setting $n = 1$ in equation $(7ci)$, we obtain the l^{th}-shell even-power spin operator of 1^{st}-order, $Q_{l:2}$, taking the form

$$n = 1\ : \qquad Q_{l:2} = 4\left(\sum_{m=0}^{l-1} \frac{1}{2}(S^+_{lm}S^-_{lm} + S^-_{lm}S^+_{lm}) \ + \ (S^z_l)^2 \right) \quad \Rightarrow \quad Q_{l:2} = \sigma_l^2 \qquad (7d)$$

which we identify as the l^{th}-shell quadratic spin angular momentum operator σ_l^2 obtained in equation $(5civ)$. We therefore identify the general l^{th}-shell even-power spin operator $Q_{l:2n}$ in equation $(7ci)$ as the l^{th}-shell quadratic spin angular momentum operator of n^{th}-order.

Similarly, we set $n = 0$ in equation $(7cii)$ to obtain the l^{th}-shell odd-power spin operator of 0^{th}-order, $\mathcal{F}_{l:1}$, taking the form

$$n = 0\ : \qquad \mathcal{F}_{l:1} = \sum_{m=0}^{l-1}(\alpha S^+_{lm} + \alpha^* S^-_{lm}) \ + \ \sigma^z_l \quad ; \quad \alpha = 1 - i = \sqrt{2}\ e^{-i\frac{\pi}{4}} \qquad (7e)$$

after reintroducing $2S^z_l = \sigma^z_l$. We observe that the l^{th}-shell odd-power spin operator of 0^{th}-order, $\mathcal{F}_{l:1}$, obtained here in equation $(7e)$ takes a form precisely similar to the form of the Fubini-Veneziano momentum operator [5]. We therefore identify this l^{th}-shell odd-power spin operator of 0^{th}-order, $\mathcal{F}_{l:1}$ as a *Fubini-Veneziano spin angular momentum operator*, which does not seem to have been determined elsewhere in the earlier literature on Lie, Kac-Moody or Virasoro algebras that we are aware of. In general, we now identify the l^{th}-shell odd-power spin operator $\mathcal{F}_{l:2n+1}$ in equation $(7cii)$ as the l^{th}-shell Fubini-Veneziano spin angular momentum operator of n^{th}-order.

For ease of evaluation of the general even-power and odd-power second parts, $(S_l^z)^{2n}$, $(S_l^z)^{2n+1}$, of the l^{th}-shell quadratic and Fubini-Veneziano spin angular momentum operators $Q_{l:2n}$, $\mathcal{F}_{l:2n+1}$ of n^{th}-order, $n = 0, 1, 2, 3, ...$, in equations $(7ci)$, $(7cii)$, we express the l^{th}-shell diagonal generator S_l^z and similarly the diagonal generator S_l^0 and the quadratic spin angular momentum operator σ_l^2 in terms of diagonal projection operators I_l , I_{l+1} defined within the l^{th}-shell as

$$I_l = \sum_{m=0}^{l-1} |m+1\rangle\langle m+1| \; ; \quad I_{l+1} = \sum_{m=0}^{l} |m+1\rangle\langle m+1| \; = I_l + |l+1\rangle\langle l+1| ; \quad |l+1\rangle\langle l+1| = I_{l+1} - I_l \quad (8a)$$

Using the definitions of σ_{lm}^z , I_{lm} given in equation $(5bii)$, we express

$$\sum_{m=0}^{l-1} \sigma_{lm}^z = I_l - l\,|l+1\rangle\langle l+1| = (1+l)\; I_l - l\; I_{l+1} \quad ; \quad \sum_{m=0}^{l-1} I_{lm} = I_l + l\,|l+1\rangle\langle l+1| = (1-l)\; I_l + l\; I_{l+1} \quad (8b)$$

which we substitute into equation $(5bii)$ and reorganize to obtain

$$S_l^z = \frac{1}{2}\sigma_l^z \; : \qquad S_l^z = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{l+1}{l}}\; I_l \; - \; \sqrt{\frac{l}{l+1}}\; I_{l+1} \right)$$

$$S_l^0 = \frac{1}{2}\sigma_l^0 \; : \qquad S_l^0 = \frac{1}{\sqrt{2}}\left(\frac{1-l}{\sqrt{l(l+1)}}\; I_l \; + \; \sqrt{\frac{l}{l+1}}\; I_{l+1} \right) \qquad (8c)$$

Substituting $S_{lm}^+ S_{lm}^- + S_{lm}^- S_{lm}^+ = I_{lm}$ into equation $(5civ)$ and using equation $(8b)$, we express

$$\sigma_l^2 = 2\,(\; (1-l)\; I_l + l\; I_{l+1} \;) \; + \; 4(S_l^z)^2 \qquad (8d)$$

Using S_l^z from equation $(8c)$ and noting

$$I_l^2 = I_l \quad ; \quad I_{l+1}^2 = I_{l+1} \quad ; \quad I_l I_{l+1} = I_{l+1} I_l = I_l \quad \Rightarrow \quad [\; I_l \; , \; I_{l+1} \;] = 0 \qquad (8e)$$

we obtain

$$(S_l^z)^2 = \frac{1}{2}\left(\frac{1-l}{l}\; I_l + \frac{l}{l+1}\; I_{l+1} \right) \qquad (8f)$$

which we substitute into equation $(8d)$ and reorganize as appropriate to express the quadratic spin angular momentum operator σ_l^2 in the form

$$\sigma_l^2 = 2\left(\frac{(l+1)^2 - 1}{l+1}\; I_{l+1} \; - \; \frac{l^2-1}{l}\; I_l \right) \qquad (8g)$$

Finally, we note that the property that the diagonal projection operators I_l , I_{l+1} commute according to equation $(8e)$ allows application of binomial expansions of S_l^z , S_l^0 , σ_l^2 in equations $(8c)$, $(8g)$ raised to any power in repeated multiplication, which are easily reorganized to obtain the general forms

$$k = 0, 1, 2, 3, ...$$

$$(S_l^z)^k = \left(\frac{1}{\sqrt{2}}\right)^k \left(\left(\left(\sqrt{\frac{l+1}{l}} - \sqrt{\frac{l}{l+1}} \right)^k - \left(\sqrt{\frac{l}{l+1}} \right)^k \right) I_l \ + \ \left(\sqrt{\frac{l}{l+1}} \right)^k \ I_{l+1} \right) \qquad (8hi)$$

$$(S_l^0)^k = \left(\frac{1}{\sqrt{2}}\right)^k \left(\left(\left(\frac{1-l}{\sqrt{l(l+1)}} + \sqrt{\frac{l}{l+1}} \right)^k - \left(\sqrt{\frac{l}{l+1}} \right)^k \right) I_l \ + \ \left(\sqrt{\frac{l}{l+1}} \right)^k \ I_{l+1} \right) \qquad (8hii)$$

$$(\sigma_l^2)^k = 2^k \left(\left(\left(\frac{1-l^2}{l} + \frac{(l+1)^2-1}{l+1} \right)^k - \left(\frac{(l+1)^2-1}{l+1} \right)^k \right) I_l \ + \ \left(\frac{(l+1)^2-1}{l+1} \right)^k \ I_{l+1} \right) \qquad (8hii)$$

We observe that $Q_{l:2n}$, $\mathcal{F}_{l:2n+1}$ in equations $(7ci)$, $(7cii)$ can be expressed in terms of the diagonal projection operators I_l , I_{l+1} by setting k to even and odd integer values $k = 2n$, $k = 2n+1$, $n = 0, 1, 2, 3, ...$, respectively. In particular, $Q_{l:2n}$ in equation $(7ci)$ can be expressed entirely in terms of I_l , I_{l+1} by substituting $S_{lm}^+ S_{lm}^- + S_{lm}^- S_{lm}^+ = I_{lm}$ and using $\sum_{m=0}^{l-1} I_{lm} = (1-l)\ I_l + l\ I_{l+1}$ from equation $(8b)$.

In the set of equations $(5ai)$-$(5biv)$, $(5civ)$, $(7ci)$-$(7cii)$, we have used the quantum numbers $l = 1, ..., N-1$, $m = 0, 1, ..., l$ to enumerate and determine the $2l+1$ traceless symmetric and antisymmetric generators σ_{lm}^x , σ_{lm}^y , σ_l^z, the single non-traceless symmetric generator σ_l^0 and the corresponding quadratic and Fubini-Veneziano spin angular momentum operators of n^{th}-order, $Q_{l:2n}$, $\mathcal{F}_{l:2n+1}$, $n = 0, 1, 2, 3, ...$, contained in the l^{th}-shell, $l = 1, ..., N-1$, of an $SU(N)$ generator spectrum in the spin angular momentum basis.

Using equations $(5ai)$, $(5bii)$-$(5biii)$ and $(7ci)$-$(7cii)$, noting that $\sigma_l^2 = Q_{l:2}$, $\mathcal{F}_{l:1}$ are contained in the respective n^{th}-order forms $Q_{l:2n}$, $\mathcal{F}_{l:2n+1}$, we run through all the $l+1$ values of the symmetric and antisymmetric generator pair quantum number $m = 0, 1, ..., l$ to enumerate and determine the explicit forms of all the $2(l+1)$ symmetric and antisymmetric generators in the spin angular momentum basis $(\sigma_{l0}^x\ ,\ \sigma_{l0}^y)$, $(\sigma_{l1}^x\ ,\ \sigma_{l1}^y)$, ... , $(\sigma_{l\ l-1}^x\ ,\ \sigma_{l\ l-1}^y)$, $(\sigma_l^z\ ,\ \sigma_l^0)$, together with the corresponding quadratic and Fubini-Veneziano spin angular momentum operators of n^{th}-order, $Q_{l:2n}$, $\mathcal{F}_{l:2n+1}$, $n = 0, 1, 2, 3, ...$, in the l^{th}-shell of a general $SU(N)$ generator spectrum as presented below, where we have used abbreviation PRST-$|l+1\rangle$ for the *principal state basis vector* of the l^{th}-shell :

Generators in the l^{th}-shell of a general $SU(N)$ spectrum in spin angular momentum basis: $N-1$ shells, $l = 1, ..., N-1$

$$N \geq 2 \ : \quad l = 1, ..., N-1 \quad ; \quad m = 0, 1, ..., l$$

$$l^{th} - \text{shell} \ : \ PRST - |l+1\rangle \ \begin{cases} m = 0 \ : \quad \sigma^x_{l0} = S^+_{l0} + S^-_{l0} \quad ; \quad \sigma^y_{l0} = -i(\ S^+_{l0} - S^-_{l0}\ ) \\ m = 1 \ : \quad \sigma^x_{l1} = S^+_{l1} + S^-_{l1} \quad ; \quad \sigma^y_{l1} = -i(\ S^+_{l1} - S^-_{l1}\ ) \\ \ldots\ldots\ldots \\ \ldots\ldots\ldots \\ \ldots\ldots\ldots \\ m = l-1 \ : \ \sigma^x_{l\ l-1} = S^+_{l\ l-1} + S^-_{l\ l-1} \ ; \ \sigma^y_{l\ l-1} = -i(S^+_{l\ l-1} - S^-_{l\ l-1}) \\ m = l \ : \quad \sigma^z_l = \frac{1}{\sqrt{\frac{1}{2}l(l+1)}} \sum_{m=0}^{l-1} \sigma^z_{lm} \quad ; \quad \sigma^0_l = \frac{1}{\sqrt{\frac{1}{2}l(l+1)}} \sum_{m=0}^{l-1} I_{lm} \\ Q_{l:2n} = 2^{2n} \left(\ \sum_{m=0}^{l-1} \frac{1}{2^{2n-1}} (S^+_{lm} S^-_{lm} + S^-_{lm} S^+_{lm}) \ + \ (S^z_l)^{2n} \ \right) \\ \mathcal{F}_{l:2n+1} = 2^{2n+1} \left(\ \sum_{m=0}^{l-1} \frac{1}{2^{2n+1}} (\alpha S^+_{lm} + \alpha^* S^-_{lm}) \ + \ (S^z_l)^{2n+1} \ \right) \end{cases} \tag{9a}$$

As examples, we present the general $SU(2)$, $SU(3)$, $SU(4)$, $SU(5)$ generator spectra in the spin angular momentum basis, including the respective l^{th}-shell quadratic and Fubini-Veneziano spin angular momentum operators of n^{th}-order, $Q_{l:2n}$, $\mathcal{F}_{l:2n+1}$, $n = 0, 1, 2, 3, ...$, below.

$SU(2)$

General $SU(2)$ generator spectrum : single shell, $l = 1$

$$1^{st} - \text{shell} \ : \ PRST - |2\rangle \quad \begin{cases} m = 0 \ : \quad \sigma^x_{10} = S^+_{10} + S^-_{10} \quad ; \quad \sigma^y_{10} = -i(S^+_{10} - S^-_{10}) \\ m = 1 \ : \quad \sigma^z_1 \quad ; \quad \sigma^0_1 \quad ; \quad Q_{1:2n} \quad ; \quad \mathcal{F}_{1:2n+1} \end{cases} \tag{9b}$$

$SU(3)$

General $SU(3)$ generator spectrum : 2 shells, $l = 1, 2$

$$1^{st} - \text{shell} \ : \ PRST - |2\rangle \quad \begin{cases} m = 0 \ : \quad \sigma^x_{10} = S^+_{10} + S^-_{10} \quad ; \quad \sigma^y_{10} = -i(S^+_{10} - S^-_{10}) \\ m = 1 \ : \quad \sigma^z_1 \quad ; \quad \sigma^0_1 \quad ; \quad Q_{1:2n} \quad ; \quad \mathcal{F}_{1:2n+1} \end{cases}$$

$$2^{nd} - \text{shell} \ : \ PRST - |3\rangle \quad \begin{cases} m = 0 \ : \quad \sigma^x_{20} = S^+_{20} + S^-_{20} \quad ; \quad \sigma^y_{20} = -i(S^+_{20} - S^-_{20}) \\ m = 1 \ : \quad \sigma^x_{21} = S^+_{21} + S^-_{21} \quad ; \quad \sigma^y_{21} = -i(S^+_{21} - S^-_{21}) \\ m = 2 \ : \quad \sigma^z_2 \quad ; \quad \sigma^0_2 \quad ; \quad Q_{2:2n} \quad ; \quad \mathcal{F}_{2:2n+1} \end{cases} \tag{9c}$$

$$SU(4)$$

General $SU(4)$ *generator spectrum* : 3 shells, $l = 1, 2, 3$

$$1^{st} - \text{shell} \; : \; PRST - |2\rangle \quad \begin{cases} m = 0 \; : \quad \sigma_{10}^{x} = S_{10}^{+} + S_{10}^{-} \quad ; \quad \sigma_{10}^{y} = -(S_{10}^{+} - S_{10}^{-}) \\ m = 1 \; : \quad \sigma_{1}^{z} \quad ; \quad \sigma_{1}^{0} \quad ; \quad Q_{1:2n} \quad ; \quad \mathcal{F}_{1:2n+1} \end{cases}$$

$$2^{nd} - \text{shell} \; : \; PRST - |3\rangle \quad \begin{cases} m = 0 \; : \quad \sigma_{20}^{x} = S_{20}^{+} + S_{20}^{-} \quad ; \quad \sigma_{20}^{y} = -i(S_{20}^{+} - S_{20}^{-}) \\ m = 1 \; : \quad \sigma_{21}^{x} = S_{21}^{+} + S_{21}^{-} \quad ; \quad \sigma_{21}^{y} = -i(S_{21}^{+} - S_{21}^{-}) \\ m = 2 \; : \quad \sigma_{2}^{z} \quad ; \quad \sigma_{2}^{0} \quad ; \quad Q_{2:2n} \quad ; \quad \mathcal{F}_{2:2n+1} \end{cases}$$

$$3^{rd} - \text{shell} \; : \; PRST - |4\rangle \quad \begin{cases} m = 0 \; : \quad \sigma_{30}^{x} = S_{30}^{+} + S_{30}^{-} \quad ; \quad \sigma_{30}^{y} = -i(S_{30}^{+} - S_{30}^{-}) \\ m = 1 \; : \quad \sigma_{31}^{x} = S_{31}^{+} + S_{31}^{-} \quad ; \quad \sigma_{31}^{y} = -i(S_{31}^{+} - S_{31}^{-}) \\ m = 2 \; : \quad \sigma_{32}^{x} = S_{32}^{+} + S_{32}^{-} \quad ; \quad \sigma_{32}^{y} = -i(S_{32}^{+} - S_{32}^{-}) \\ m = 3 \; : \quad \sigma_{3}^{z} \quad ; \quad \sigma_{3}^{0} \quad ; \quad Q_{3:2n} \quad ; \quad \mathcal{F}_{3:2n+1} \end{cases} \qquad (9d)$$

$$SU(5)$$

General $SU(5)$ *generator spectrum* : 4 shells, $l = 1, 2, 3, 4$

$$1^{st} - \text{shell} \; : \; PRST - |2\rangle \quad \begin{cases} m = 0 \; : \quad \sigma_{10}^{x} = S_{10}^{+} + S_{10}^{-} \quad ; \quad \sigma_{10}^{y} = -i(S_{10}^{+} - S_{10}^{-}) \\ m = 1 \; : \quad \sigma_{1}^{z} \quad ; \quad \sigma_{1}^{0} \quad ; \quad Q_{1:2n} \quad ; \quad \mathcal{F}_{1:2n+1} \end{cases}$$

$$2^{nd} - \text{shell} \; : \; PRST - |3\rangle \quad \begin{cases} m = 0 \; : \quad \sigma_{20}^{x} = S_{20}^{+} + S_{20}^{-} \quad ; \quad \sigma_{20}^{y} = -i(S_{20}^{+} - S_{20}^{-}) \\ m = 1 \; : \quad \sigma_{21}^{x} = S_{21}^{+} + S_{21}^{-} \quad ; \quad \sigma_{21}^{y} = -i(S_{21}^{+} - S_{21}^{-}) \\ m = 2 \; : \quad \sigma_{2}^{z} \quad ; \quad \sigma_{2}^{0} \quad ; \quad Q_{2:2n} \quad ; \quad \mathcal{F}_{2:2n+1} \end{cases}$$

$$3^{rd} - \text{shell} \; : \; PRST - |4\rangle \quad \begin{cases} m = 0 \; : \quad \sigma_{30}^{x} = S_{30}^{+} + S_{30}^{-} \quad ; \quad \sigma_{30}^{y} = -i(S_{30}^{+} - S_{30}^{-}) \\ m = 1 \; : \quad \sigma_{31}^{x} = S_{31}^{+} + S_{31}^{-} \quad ; \quad \sigma_{31}^{y} = -i(S_{31}^{+} - S_{31}^{-}) \\ m = 2 \; : \quad \sigma_{32}^{x} = S_{32}^{+} + S_{32}^{-} \quad ; \quad \sigma_{32}^{y} = -i(S_{32}^{+} - S_{32}^{-}) \\ m = 3 \; : \quad \sigma_{3}^{z} \quad ; \quad \sigma_{3}^{0} \quad ; \quad Q_{3:2n} \quad ; \quad \mathcal{F}_{3:2n+1} \end{cases}$$

$$4^{th} - \text{shell} \ : \ PRST - |5\rangle \quad \begin{cases} m=0 \ : \quad \sigma^x_{40} = S^+_{40} + S^-_{40} \quad ; \quad \sigma^y_{40} = -i(S^+_{40} - S^-_{40}) \\ m=1 \ : \quad \sigma^x_{41} = S^+_{41} + S^-_{41} \quad ; \quad \sigma^y_{41} = -i(S^+_{41} - S^-_{41}) \\ m=2 \ : \quad \sigma^x_{42} = S^+_{42} + S^-_{42} \quad ; \quad \sigma^y_{42} = -i(S^+_{42} - S^-_{42}) \\ m=3 \ : \quad \sigma^x_{43} = S^+_{43} + S^-_{43} \quad ; \quad \sigma^y_{43} = -i(S^+_{43} - S^-_{43}) \\ m=4 \ : \quad \sigma^z_4 \quad ; \quad \sigma^0_4 \quad ; \quad Q_{4:2n} \quad ; \quad \mathcal{F}_{4:2n+1} \end{cases} \tag{9e}$$

The generators are determined in explicit forms using the definitions of the spin state raising and lowering operators $S^{\pm}_{lm}$ in equation $(5ai)$ for $l = 1, ..., N-1$, $m = 0, 1, ..., l$ in each case.

4.1 The $SU(N)$ identity, Casimir and Fubini-Veneziano spin operators

Having enumerated and determined $SU(N)$ generators, including the general quadratic and Fubini-Veneziano spin angular momentum operators of n^{th}-order in the spin angular momentum basis defined within the $N-1$ configuration shells, we now complete the specification of the algebraic structure by determining the universal symmetry group operators, namely, the $SU(N)$ identity operator, Casimir and Fubini-Veneziano spin angular momentum operators.

4.1.1 The $SU(N)$ identity operator

We identify the l^{th}-shell non-traceless symmetric diagonal generator σ^0_l defined in equation $(5bii)$ as the l^{th}-shell part of the $SU(N)$ identity operator I_N, which is the $N \times N$ identity matrix, determined as a weighted sum of the $N-1$ parts from the $N-1$ shells specified by $l = 1, ..., N-1$ using the algebraic relation following from equation $(5bii)$ in the form

$$\sqrt{\frac{1}{2}l(l+1)} \ \sigma^0_l = \sum_{m=0}^{l-1} I_{lm} \qquad \Rightarrow \qquad \sum_{l=1}^{N-1} \sqrt{\frac{1}{2}l(l+1)} \ \sigma^0_l = \sum_{l=1}^{N-1} \sum_{m=0}^{l-1} I_{lm} \tag{10a}$$

which on using the relation

$$\sum_{l=1}^{N-1} \sum_{m=0}^{l-1} I_{lm} = (N-1) I_N \tag{10b}$$

provides the $SU(N)$ identity operator I_N defined in the spin angular momentum basis in the form

$$I_N = \frac{1}{N-1} \sum_{l=1}^{N-1} \sqrt{\frac{1}{2}l(l+1)} \ \sigma^0_l \tag{10c}$$

noting that the $SU(N)$ identity operator I_N, generally evaluated as an $N \times N$ matrix, takes the basic operator form

$$I_N = \sum_{k=1}^{N} |k\rangle\langle k| \tag{10d}$$

4.1.2 The $SU(N)$ quadratic Casimir spin angular momentum operators

We identify the l^{th}-shell quadratic spin angular momentum operator σ_l^2 determined in final form in equation $(5civ)$ as the l^{th}-shell part of the $SU(N)$ quadratic Casimir spin angular momentum operator Q_N determined as a sum of the l^{th}-shell parts σ_l^2 from the $N-1$ shells specified by $l = 1, ..., N-1$ in the form

$$Q_N = \sum_{l=1}^{N-1} \sigma_l^2 \tag{11a}$$

which on substituting $S_{lm}^+ S_{lm}^- + S_{lm}^- S_{lm}^+ = I_{lm}$ from equation $(5biv)$ into equation $(5civ)$ takes the form

$$Q_N = 4 \left(\sum_{l=1}^{N-1} \sum_{m=0}^{l-1} \frac{1}{2} I_{lm} \; + \; \sum_{l=1}^{N-1} (S_l^z)^2 \right) \tag{11b}$$

Using the definition in equation $(5bii)$ to determine the explicit forms of $S_l^z = \frac{1}{2}\sigma_l^z$ in the $SU(N)$, $N = 2, 3, 4, 5$, examples, we obtain a general relation

$$\sum_{l=1}^{N-1} (S_l^z)^2 = \frac{N-1}{2N} I_N \tag{11c}$$

Substituting the relations from equations $(10b)$ and $(11c)$ into equation $(11b)$ provides the $SU(N)$ quadratic Casimir spin angular momentum operator Q_N in the form

$$Q_N = 2(N-1)\left(1 + \frac{1}{N}\right) I_N \qquad \Rightarrow \qquad Q_N = 4 \binom{N}{2} \, \frac{1}{N} \left(1 + \frac{1}{N}\right) I_N \tag{11d}$$

where we have obtained the final form by expressing

$$N - 1 = N(N-1)\frac{1}{N} = 2\frac{N!}{2!(N-2)!}\frac{1}{N} \qquad ; \qquad \frac{N!}{2!(N-2)!} = \binom{N}{2} \tag{11e}$$

We may also reorganize equation $(11d)$ to express the $SU(N)$ quadratic Casimir operator in the form

$$Q_N = 2 \, \frac{N^2 - 1}{N} I_N \tag{11f}$$

where we now identify $N^2 - 1$ as the total number of the traceless $SU(N)$ generators σ_{lm}^x , σ_{lm}^y , σ_l^z , $l = 1, ..., N-1$, $m = 0, 1, ..., l$.

We observe that, in contrast to its l^{th}-shell parts σ_l^2, $l = 1, ..., N-1$, determined in final form in equation $(5civ)$, the $SU(N)$ quadratic Casimir spin angular momentum operator Q_N determined in final form in equation $(11d)$ or $(11f)$ is a multiple of the identity operator I_N, which commutes with all the $SU(N)$ generators. In subsection 4.2 below, we establish that the l^{th}-shell quadratic spin angular momentum operators σ_l^2, $l = 1, ..., N-1$, are Cartan generators satisfying Cartan subalgebra and generating eigenvalue equations on the spin state raising and lowering operators $S_{lm}^{\pm}$.

Identifying the general l^{th}-shell quadratic spin angular momentum operator of n^{th}-order $Q_{l:2n}$ determined in final form in equation $(7ci)$ as the l^{th}-shell part of the general n^{th}-order $SU(N)$ quadratic Casimir spin angular momentum operator $Q_{N:2n}$, we obtain a generalization of the $SU(N)$ quadratic Casimir spin angular momentum operator Q_N to the n^{th}-order $SU(N)$ quadratic Casimir spin angular momentum operator determined as a sum of the l^{th}-shell parts $Q_{l:2n}$ from the $N-1$ shells specified by $l = 1, ..., N-1$ in the form

$$Q_{N:2n} = \sum_{l=1}^{N-1} Q_{l:2n} \quad ; \quad n = 1, 2, 3, ... \tag{11g}$$

which on substituting $Q_{l:2n}$ from equation $(7ci)$, introducing $S_{lm}^{+}S_{lm}^{-} + S_{lm}^{-}S_{lm}^{+} = I_{lm}$ and using the relation obtained in equation $(10b)$ takes the final form (reintroducing $(\sigma_l^z)^{2n} = 2^{2n}(S_l^z)^{2n}$)

$$Q_{N:2n} = 2(N-1)I_N \ + \ 2^{2n} \sum_{l=1}^{N-1} (S_l^z)^{2n} \quad ; \quad n = 1, 2, 3, ... \tag{11h}$$

where $(S_l^z)^{2n}$ under summation in the second component can be evaluated explicitly in terms of projection operators I_l , I_{l+1} by setting $k = 2n$ in equation $(8hi)$.

Notice that, setting $n = 1$ in equation $(11h)$ and using the relation obtained in equation $(11c)$ provides the $SU(N)$ quadratic Casimir spin angular momentum operator Q_N obtained in equation $(11d)$, which we may now interpret as an $SU(N)$ general quadratic Casimir spin angular momentum operator of 1^{st}-order. The property that for $n \geq 2$, the summation in the second term in equation $(11h)$ is not a multiple of the identity operator I_N means that for $n \geq 2$, the $SU(N)$ n^{th}-order quadratic Casimir spin angular momentum operator $Q_{N:2n}$, $n \geq 2$, is not a multiple of the $SU(N)$ identity operator I_N. Only the 1^{st}-order $(n = 1)$ $SU(N)$ quadratic Casimir spin angular momentum operator $Q_N = Q_{N:2}$ is a multiple of the identity operator I_N as determined in equation $(11d)$.

4.1.3 The $SU(N)$ Fubini-Veneziano spin operator

We identify the general l^{th}-shell Fubini-Veneziano spin angular momentum operator of n^{th}-order, $\mathcal{F}_{l:2n+1}$, obtained in final form in equation $(7cii)$ as the l^{th}-shell part of an $SU(N)$ n^{th}-order Fubini-Veneziano spin angular momentum operator $\mathcal{F}_{N:2n+1}$, determined as the sum of the l^{th}-shell parts $\mathcal{F}_{l:2n+1}$ from the $N-1$ shells specified by $l = 1, ..., N-1$ in the form

$$\mathcal{F}_{N:2n+1} = \sum_{l=1}^{N-1} \mathcal{F}_{l:2n+1} \quad ; \quad n = 1, 2, 3, ... \tag{12a}$$

which on substituting $\mathcal{F}_{l:2n+1}$ from equation $(7cii)$ takes the form

$$\mathcal{F}_{N:2n+1} = \sum_{l=1}^{N-1} \sum_{m=0}^{l-1} (\alpha S_{lm}^{+} + \alpha^{*} S_{lm}^{-}) \ + \ 2^{2n+1} \sum_{l=1}^{N-1} (S_l^z)^{2n+1} \quad ; \quad n = 1, 2, 3, ... \tag{12b}$$

where $(S_l^z)^{2n+1}$ under summation in the second component can be evaluated explicitly in terms of projection operators I_l , I_{l+1} by setting $k = 2n+1$ in equation $(8hi)$.

Setting $n = 0$ in equations $(12a)$, $(12b)$ provides the $SU(N)$ Fubini-Veneziano spin angular momentum operator $\mathcal{F}_{N:1}$ in the form

$$n = 0 \ : \qquad \mathcal{F}_{N:1} = \sum_{l=1}^{N-1} \mathcal{F}_{l:1} \qquad \Rightarrow \qquad \mathcal{F}_{N:1} = \sum_{l=1}^{N-1} \sum_{m=0}^{l-1} (\alpha S_{lm}^{+} + \alpha^{*} S_{lm}^{-}) \ + \ \sum_{l=1}^{N-1} \sigma_l^z \tag{12c}$$

where we have reintroduced $\sigma_l^z = 2S_l^z$.

4.2 Algebraic properties of the $SU(N)$ generator spectrum : Cartan-Weyl basis

Let us now determine the basic algebraic properties of a general $SU(N)$ generator spectrum in the spin angular momentum basis. In this respect, we have used the explicit forms of the generators determined in the $SU(2)$, $SU(3)$, $SU(4)$, $SU(5)$ examples in equations $(9b)$-$(9e)$ to establish that the traceless antisymmetric diagonal generator S_l^z, its non-traceless symmetric partner S_l^0 and the quadratic spin angular momentum operator σ_l^2 satisfy Cartan subalgebra within and across the $N-1$ configuration shells specified by $l = 1, ..., N-1$. We therefore identify the generators S_l^z , S_l^0 , σ_l^2, $l = 1, ..., N-1$, determined according to the definitions in equations $(5bii)$, $(5bv)$, $(5civ)$ as Cartan generators. The set of the Cartan generators S_l^z , S_l^0 , σ_l^2, taken together with the spin state raising and lowering operators (generally called step operators) S_{lm}^+ , S_{lm}^-, constitute an *extended* Cartan-Weyl basis S_l^z , S_{lm}^+ , S_{lm}^- , S_l^0 , σ_l^2, $l = 1, ..., N-1$, noting that in the standard literature on Lie , Kac-Moody and Virasoro algebras [5 , 6 , 7], the Cartan-Weyl basis has been identified only as the three generators $S_l^z \equiv H_l$, $S_{lm}^{\pm} \equiv E_{\pm\alpha}$ in corresponding notations. In the present work, we have extended the Cartan-Weyl basis in an $SU(N)$ generator spectrum by introducing the non-traceless symmetric diagonal generator S_l^0 and the quadratic spin angular momentum operator σ_l^2 which satisfy the Cartan subalgebra and generate eigenvalue equations on the spin state raising and lowering operators $S_{lm}^{\pm}$, as expected in the Cartan-Weyl basis.

Using the explicit forms of the Cartan-Weyl basis S_l^z , S_{lm}^+ , S_{lm}^- , S_l^0 , σ_l^2 determined in the $SU(2)$, $SU(3)$, $SU(4)$, $SU(5)$ examples using the definitions in equations $(5bii)$, $(5bv)$, $(5civ)$ or equations $(8c)$, $(8g)$ as convenient, we obtain explicit algebraic relations, which we generalize to apply to any general $SU(N)$ generator spectrum for $N \geq 2$. We consider the algebraic relations between generators and spin operators within a given configuration shell specified by a quantum number l and algebraic relations across configuration shells specified by quantum numbers l , l' , $l \neq l'$.

The Cartan generators S_l^z , S_l^0 , σ_l^2 satisfy the Cartan subalgebra within and across the $N-1$ shells obtained as

$$l \ , \ l' \ = 1, ..., N-1$$

$$[\ S_l^z \ , \ S_{l'}^z \ ] = 0 \quad ; \quad [\ S_l^0 \ , \ S_{l'}^0 \ ] = 0 \quad ; \quad [\ \sigma_l^2 \ , \ \sigma_{l'}^2 \ ] = 0$$
$$[\ S_l^z \ , \ S_{l'}^0 \ ] = 0 \quad ; \quad [\ S_l^z \ , \ \sigma_{l'}^2 \ ] = 0 \quad ; \quad [\ S_l^0 \ , \ \sigma_{l'}^2 \ ] = 0 \qquad (13a)$$

The commutation brackets of the Cartan generators S_l^z , S_l^0 , σ_l^2 and the spin state raising and lowering operators $S_{lm}^{\pm}$ within or across configuration shells constitute *eigenvalue equations* for the spin state raising and lowering operators. The sum of the commutation brackets $[\ S_{lm}^+, S_{lm}^-\ ]$ taken over $m = 0, ..., l-1$ provides the Cartan generator S_l^z in the l^{th}-shell. The commutation brackets of the spin state raising and lowering operators $S_{lm}^{\pm}$ within a higher configuration shell provide *all* spin state raising and lowering operators in the lower shells, while the commutation brackets of the spin state raising and lowering operators $S_{lm}^{\pm}$ and $S_{l'm'}^{\pm}$ across configuration shells l , l' , $l \neq l'$ produce spin state raising and lowering operators in the various configuration shells within the generator spectrum, leading to a closed $SU(N)$ generator algebra, which we now present in general form specified by configuration shell and conjugate spin state raising-lowering operator pair quantum numbers l , m below.

Within a configuration shell specified by a quantum number $l = 1, ..., N-1$, the three Cartan generators S_l^z , S_l^0 , σ_l^2 and the $2l$ spin state raising and lowering operators $S_{lm}^{\pm}$ are correlated by algebraic relations (noting $[\ S^+_{lm}, S^-_{lm}\ ] = \sigma^z_{lm}$)

$$[\ S^z_l\ ,\ S^\pm_{lm}\ ] = \pm\frac{1+l}{\sqrt{2l(l+1)}}\ S^\pm_{lm}\ ;\quad [\ S^0_l\ ,\ S^\pm_{lm}\ ] = \pm\frac{1-l}{\sqrt{2l(l+1)}}\ S^\pm_{lm}\ ;\quad [\ \sigma^2_l\ ,\ S^\pm_{lm}\ ] = \pm 2\ \frac{1-l^2}{l}\ S^\pm_{lm}$$

$$\sum_{m=0}^{l-1}[\ S^+_{lm}\ ,\ S^-_{lm}\ ] = \sqrt{2l(l+1)}\ S^z_l \qquad (13bi)$$

$$[\ S^\pm_{lm'}\ ,\ S^\mp_{lm}\ ] = \pm\ |m+1\rangle\langle m'+1| \qquad ; \qquad [\ S^\pm_{lm'}\ ,\ S^\pm_{lm}\ ] = 0$$

$$l = 1,...,N-1 \quad ; \quad m' < m\ :\quad m' = 0,...,l-2 \quad ; \quad m = 1,...,l-1 \qquad (13bii)$$

Across the configuration shells, Cartan generators S^z_l , S^0_l , σ^2_l in a *higher* shell specified by $l = 2,...,N-1$ commute with all step operators $E_{l'm'}$ in the *lower* shells specified by $l' = 1,...,l-1$ according to the algebraic relations

$$l > l'\ :\qquad l = 2,...,N-1 \quad ; \quad l' = 1,...,l-1\ ; \quad m' = 0,...,l'-1$$

$$[\ S^z_l\ ,\ S^\pm_{l'm'}\ ] = 0 \qquad ; \qquad [\ S^0_l\ ,\ S^\pm_{l'm'}\ ] = 0 \qquad ; \qquad [\ \sigma^2_l\ ,\ S^\pm_{l'm'}\ ] = 0 \qquad (13c)$$

while Cartan generators S^z_l , S^0_l , σ^2_l in a lower shell specified by $l = 1,...,l'-1$ has mixed algebraic relations with Cartan-Weyl basis spin state raising and lowering operators $S_{l'm'}$ in the upper shells specified by $l' = 2,...,N-1$; $m' = 0,...,l'-1$ according to

$$l < l'\ :\qquad l = 1,...,l'-1 \quad ; \quad l' = 2,...,N-1 \quad ; \quad m' = 0,...,l'-1$$

$$[\ S^z_l\ ,\ S^\pm_{l'm'}\ ] = \begin{cases} \pm\ \frac{1}{\sqrt{2l(l+1)}}\ S^\pm_{l'm'} & ;\quad m' < l \\ \mp\ \frac{l}{\sqrt{2l(l+1)}}\ S^\pm_{l'm'} & ;\quad m' = l \\ 0\quad ;\quad m' > l \end{cases} \qquad (13di)$$

$$[\ S^0_l\ ,\ S^\pm_{l'm'}\ ] = \begin{cases} \pm\ \frac{1}{\sqrt{2l(l+1)}}\ S^\pm_{l'm'} & ;\quad m' < l \\ \mp\ \frac{l}{\sqrt{2l(l+1)}}\ S^\pm_{l'm'} & ;\quad m' = l \\ 0\quad ;\quad m' > l \end{cases} \qquad (13dii)$$

$$[\ \sigma^2_l\ ,\ S^\pm_{l'm'}\ ] = \begin{cases} \pm 2\ \frac{l(l+1)+1}{l(l+1)}\ S^\pm_{l'm'} & ;\quad m' < l \\ \pm 2\ \frac{(l+1)^2-1}{l+1}\ S^\pm_{l'm'} & ;\quad m' = l \\ 0\quad ;\quad m' > l \end{cases} \qquad (13diii)$$

Finally, the commutation bracket relations across the shells between spin state raising and lowering operators $S^{\pm}_{lm}$ in lower shells specified by $l = 1, ..., l' - 1$; $m = 0, ..., l - 1$ and spin state raising and lowering operators $S_{l'm'}$ in the higher shells specified by $l' = 2, ..., N - 1$; $m' = 0, 1, ..., l' - 1$ *vanish*, except the commutation brackets for raising and lowering operators specified by $m' = m, l$ taking the general forms

$$l < l' \ : \qquad l = 1, ..., l' - 1 \quad ; \quad l' = 2, ..., N - 1 \ ; \ m = 0, ..., N - 2$$

$$m' = m \ , \ l \ : \qquad [\ S^{\pm}_{lm} \ , \ S^{\mp}_{l'm}\ ] = \mp S^{\mp}_{l'l} \qquad ; \qquad [\ S^{\pm}_{lm} \ , \ S^{\pm}_{l'l}\ ] = \pm S^{\pm}_{l'm} \tag{13ei}$$

$$m' \neq m \ , \ l \ : \qquad [\ S^{\pm}_{lm} \ , \ S^{\mp}_{l'm'}\ ] = 0 \tag{13eii}$$

We have thus determined a complete set of the basic algebraic relations in an $SU(N)$ generator spectrum in the extended Cartan-Weyl basis S^z_l , S^+_{lm} , S^-_{lm} , S^0_l , σ^2_l, composed of the Cartan subalgebra of S^z_l , S^0_l , σ^2_l in equation (13*a*), the eigenvalue equations generated by the Cartan generators S^z_l , S^0_l , σ^2_l on the spin state raising and lowering operators $S^{\pm}_{lm}$ in equations (13*b*)-(13*d*) and the spin state raising and lowering operator algebraic relations in equation (13*e*). The $SU(N)$ generator spectrum satisfying the set of basic algebraic relations in equations (13*a*)-(13*e*) constitutes a closed $SU(N)$ Lie algebra. We observe that the extended Cartan-Weyl basis should be expanded to include the l^{th}-shell Fubini-Veneziano spin angular momentum operator $\mathcal{F}_{l:1}$, which generates some more general algebraic relations. Further generalizations of the basic algebraic relations involving the general n^{th}-order quadratic and Fubini-Veneziano spin angular momentum operators $Q_{l:2n}$, $\mathcal{F}_{l:2n+1}$, $n = 0, 1, 2, 3, ...$ as defined in equations (7*ci*) , (7*cii*) and the universal $SU(N)$ quadratic Casimir and Fubini-Veneziano spin angular momentum operators $Q_{N:2n}$, $\mathcal{F}_{N:2n+1}$ in equations (11*h*) , (12*b*) are easily determined using the basic algebraic relations obtained here in equations (13*a*)-(13*e*).

The closed algebraic relations in equations (13*a*)-(13*e*) provide the minimal framework for determining the spectrum of roots , weights and particle states of an $SU(N)$ generator spectrum, as demonstrated in elaborate calculations in the $SU(2)$, $SU(3)$ and $SU(4)$ Lie algebras generally used in models of gauge field theories of elementary particle interactions [5-9]. The quantum structure of the $SU(N)$ generator spectrum, where each of the $N - 1$ configuration shells in the generator spectrum contains a definite number of generators specified by the shell quantum number $l = 1, ..., N - 1$ and the symmetric-antisymmetric generator pair (or conjugate spin state raising-lowering operator) quantum numbers $m = 0, ..., l$, means that the roots, weights and particle states are *not randomly distributed*, but must be defined and distributed within the corresponding $N-1$ configuration shells of the $SU(N)$ generator spectrum. In particular, the property that the l^{th}-shell quadratic spin angular momentum operator σ^2_l, identified as the l^{th}-shell component of the $SU(N)$ quadratic Casimir spin angular momentum operator Q_N, commutes with both the traceless antisymmetric diagonal generator S^z_l and the non-traceless symmetric diagonal generator S^0_l can be used to determine a spectrum of spin angular momentum state eigenvectors and eigenvalues within and across configuration shells of an $SU(N)$ generator spectrum using standard angular momentum algebra in the basis S^z_l , S^+_{lm} , S^-_{lm} , σ^2_l or the basis S^0_l , S^+_{lm} , S^-_{lm} , σ^2_l, noting that the relation $(S^0_l)^2 = (S^z_l)^2$ means that the quadratic spin angular momentum operator σ^2_l as defined in equations (5*cii*) , (5*civ*) takes the same algebraic form on replacing $(S^z_l)^2$ with $(S^0_l)^2$.

It is obvious from the expanded algebraic space spanned by the extended Cartan-Weyl basis S^z_l , S^+_{lm} , S^-_{lm} , S^0_l , σ^2_l , $\mathcal{F}_{l:1}$ (or the more general extended Cartan-Weyl basis S^z_l , S^+_{lm} , S^-_{lm} , S^0_l , $Q_{l:2n}$

, $\mathcal{F}_{l:2n+1}$, $n = 0, 1, 2, 3, ...$) determined in the present work that the full quantum state space of an $SU(N \geq 2)$ symmetry group is much larger than the quantum state spaces of the existing gauge field theories based on $SU(2)$, $SU(3)$, $SU(4)$, $SU(5)$, $SU(6)$ symmetry groups [5 , 8 , 10-20] where only the basic Cartan-Weyl basis $S^z_l \equiv H_l$, $S^{\pm}_{lm} \equiv E_{\pm\alpha}$ (considering corresponding notation in [5 , 8 , 10-20], noting $N = 2-6$, $l = 1, ..., N-1$, $m = 0, 1, ..., l$ in the present work) have been used to determine the roots , weights and particle states. Hence, for practical applications of $SU(N)$ symmetry groups in formulating accurate models of gauge field theories of particle interactions, it is necessary to determine general algebraic properties and the complete quantum structure of the respective $SU(N)$ generator spectrum using the entire set of the extended Cartan-Weyl basis S^z_l , S^+_{lm} , S^-_{lm} , S^0_l , σ^2_l ($Q_{l:2n}$) , $\mathcal{F}_{l:1}$ ($\mathcal{F}_{l:1;2n+1}$) , together with the associated $SU(N)$ quadratic Casimir and Fubini-Veneziano spin angular momentum operators $Q_{N:2n}$, $\mathcal{F}_{N:2n+1}$, $n = 0, 1, 2, 3, ...$, to determine the entire spectrum of spin angular momentum state eigenvectors and eigenvalues, or, alternatively, to determine the entire spectrum of roots, weights and associated particle states, within the full quantum state space of the $SU(N)$ symmetry group. We highlight important implications of the expanded algebraic space and quantum structure of an $SU(N)$ generator spectrum to models of gauge field theories based on $SU(N)$ symmetry groups in section 4.4 below.

4.3 Spectrum of $SU(2)$-subspaces in an $SU(N)$ state space

We conclude this section by developing a clearer understanding that general $SU(N)$ generators are defined within a spectrum of $\frac{1}{2}N(N-1)$ 2-dimensional $SU(2)$-subspaces. The underlying algebraic property is that an N-dimensional $SU(N)$ space spanned by the N orthonormal state basis vectors $|1\rangle$, ... , $|N\rangle$ is composed of a spectrum of $\frac{1}{2}N(N-1)$ 2-dimensional subspaces spanned by coupled pairs of the N state basis vectors. Generators of the $SU(N)$ symmetry group are determined as symmetric and antisymmetric pairs of off-diagonal and diagonal state projection or spin operators within each of the $\frac{1}{2}N(N-1)$ 2-dimensional subspaces. Since the generators defined within a 2-dimensional subspace spanned by a coupled pair of orthonormal state basis vectors satisfy a closed $SU(2)$ Lie algebra, we refer to the 2-dimensional subspace as an $SU(2)$-subspace of the N-dimensional $SU(N)$ space.

The distribution of $SU(N)$ generators in a spectrum composed of $N-1$ configuration shells each containing a definite number of generators specified by shell quantum number $l = 1, ..., N-1$ and symmetric-antisymmetric generator pair quantum number $m = 0, 1, ..., l$ means that the $\frac{1}{@}N(N-1)$ $SU(2)$-subspaces within which the generators are defined are also distributed in a spectrum of 2-dimensional subspaces specified by the two quantum numbers l , m. It follows from the definitions of generators in the spin angular momentum basis in equations $(5bi)$, $(5bii)$, that in the l^{th}-shell of an $SU(N)$ space, an $SU(2)$-subspace is spanned by a coupled pairs of orthonormal state basis vectors $|m+1\rangle$ and $|l+1\rangle$ specified by $l = 1, ..., N-1$, $m = 0, 1, ..., l-1$, where $|l+1\rangle$ is the principal state basis vector characterizing the l^{th}-shell as defined earlier.

Generally, in the l^{th}-shell of an $SU(N)$ generator spectrum, we specify a 2-dimensional $SU(2)$-subspace spanned by a coupled pair of orthonormal state basis vectors $|m+1\rangle$ and $|l+1\rangle$ by $SU(2)_{lm} = (\ |m+1\rangle\ ,\ |l+1\rangle\ )$, $l = 1, ..., N-1$, $m = 0, 1, ..., l-1$. This l^{th}-shell $SU(2)$-subspace contains a set of generators in the spin angular momentum basis σ^x_{lm} , σ^y_{lm} , σ^z_{lm} , I_{lm} defined in equations $(5bi)$, $(5bii)$ satisfying closed $SU(2)$ Lie algebra according to

$$SU(2)_{lm} = (\ |m+1\rangle\ ,\ |l+1\rangle\ ) \equiv (\ \sigma^x_{lm}\ ,\ \sigma^y_{lm}\ ,\ \sigma^z_{lm}\ ,\ I_{lm}\ ) \quad ; \quad l = 1, ..., N-1\ ,\ m = 0, 1, ..., l-1$$

$$[\ \sigma^x_{lm}\ ,\ \sigma^y_{lm}\ ] = 2i\sigma^z_{lm} \quad ; \quad [\ \sigma^y_{lm}\ ,\ \sigma^z_{lm}\ ] = 2i\sigma^x_{lm} \quad ; \quad [\ \sigma^z_{lm}\ ,\ \sigma^x_{lm}\ ] = 2i\sigma^y_{lm}$$
$$I_{lm}\sigma^j_{lm} = \sigma^j_{lm}I_{lm} = \sigma^j_{lm} \quad ; \quad [\ I_{lm}\ ,\ \sigma^j_{lm}\ ] = 0\ ,\ j = x, y, z \qquad (14a)$$

where we identify I_{lm} as the identity generator within the 2-dimensional subspace $SU(2)_{lm} = (\ |m+1\rangle\ ,\ |l+1\rangle\ )$.

The property that $SU(2)_{lm} = (\ |m+1\rangle\ ,\ |l+1\rangle\ )$ is specified by l values of the quantum number $m = 0, 1, ..., l-1$ means that the l^{th}-shell contains l $SU(2)$-subspaces $SU(2)_{lm} = (\ |m+1\rangle\ ,\ |l+1\rangle\ )$, each specified by a set of generators $\sigma^x_{lm}\ ,\ \sigma^y_{lm}\ ,\ \sigma^z_{lm}\ ,\ I_{lm}$ satisfying closed $SU(2)$ Lie algebra. We are thus led to an equivalent representation of an $SU(N)$ generator spectrum as a spectrum of 2-dimensional $SU(2)$-subspaces $SU(2)_{lm} = (\ |m+1\rangle\ ,\ |l+1\rangle\ )$ specified by generators $\sigma^x_{lm}\ ,\ \sigma^y_{lm}\ ,\ \sigma^z_{lm}\ ,\ I_{lm}$ satisfying closed $SU(2)$ Lie algebra as presented in equation $(14b)$ below.

$SU(2)$-subspaces in the l^{th}-shell of an $SU(N)$ space : $N \geq 2$

$$SU(2)_{lm} = (\ |m+1\rangle\ ,\ |l+1\rangle\ ) \equiv (\ \sigma^x_{lm}\ ,\ \sigma^y_{lm}\ ,\ \sigma^z_{lm}\ ,\ I_{lm}\ ) \quad ; \quad l = 1, ..., N-1\ ,\ m = 0, 1, ..., l-1$$

$$l^{th} - \text{shell} : \begin{cases} m = 0\ :\ SU(2)_{l0} = (|1\rangle\ ,\ |l+1\rangle) \equiv (\sigma^x_{l0}, \sigma^y_{l0}, \sigma^z_{l0}, I_{l0}) \\ m = 1\ :\ SU(2)_{l1} = (|2\rangle\ ,\ |l+1\rangle) \equiv (\sigma^x_{l1}, \sigma^y_{l1}, \sigma^z_{l1}, I_{l1}) \\ .... \\ .... \\ .... \\ m = l-1\ :\ SU(2)_{l\ l-1} = (|l\rangle\ ,\ |l+1\rangle) \equiv (\sigma^x_{l\ l-1}, \sigma^y_{l\ l-1}, \sigma^z_{l\ l-1}, I_{l\ l-1}) \\ ---------------------------------- \\ m = l\ :\qquad \sigma^z_l = \frac{1}{\sqrt{\frac{1}{2}l(l+1)}} \sum_{m=0}^{l-1} \sigma^z_{lm} \qquad ; \qquad \sigma^0_l = \frac{1}{\sqrt{\frac{1}{2}l(l+1)}} \sum_{m=0}^{l-1} I_{lm} \end{cases} \tag{14b}$$

where we note that the basic generators $\sigma^x_{lm}\ ,\ \sigma^y_{lm}\ ,\ \sigma^z_{lm}\ ,\ I_{lm}$ within the $SU(2)$-subspaces are enumerated up to the value $m = l-1$, while the last value $m = l$ enumerates the effective antisymmetric diagonal generator σ^z_l and symmetric diagonal generator σ^0_l, each determined as the sum of the respective antisymmetric generators $\sigma^z_{l0}\ ,\ \sigma^z_{l1}\ ,\ ...\ ,\ \sigma^z_{l\ l-1}$ and symmetric generators $I_{l0}\ ,\ I_{l1}\ ,\ ...\ ,\ I_{l\ l-1}$ from the l $SU(2)$-subspaces in the l^{th}-shell, which provides the definitions in equation $(5bii)$.

We illustrate the quantum structure of an $SU(N)$ generator spectrum based on the distribution of the $\frac{1}{2}N(N-1)$ $SU(2)$-subspaces using the $SU(5)$ generator spectrum as an example. For the $SU(5)$ symmetry group, setting $N = 5$ in equation $(14b)$ gives $5-1 = 4$ shells $l = 1, 2, 3, 4$ containing the total $\frac{1}{2}5(5-1) = 10$ $SU(2)$-subspaces, where the l^{th}-shell contains l $SU(2)$-subspaces as displayed in equation $(14c)$ below.

Spectrum of $SU(2)$-subspaces in the quantum structure of $SU(5)$ space

$$SU(2)_{lm} = (\ |m{+}1\rangle\ ,\ |l{+}1\rangle\ ) \equiv (\ \sigma^x_{lm}\ ,\ \sigma^y_{lm}\ ,\ \sigma^z_{lm}\ ,\ I_{lm}\ ) \quad ; \quad N = 5\ :\quad l = 1,2,3,4\ ,\ m = 0,1,...,l{-}1$$

$$1^{st} - \text{shell} : \begin{cases} m = 0\ :\ SU(2)_{10} = (|1\rangle\ ,\ |2\rangle) \equiv (\sigma^x_{10}, \sigma^y_{10}, \sigma^z_{10}, I_{10}) \\ ---------------------------- \\ m = 1\ :\qquad \sigma^z_1 = \sigma^z_{10} \qquad ; \qquad \sigma^0_1 = I_{10} \end{cases}$$

$$2^{nd} - \text{shell} : \begin{cases} m = 0\ :\ SU(2)_{20} = (|1\rangle\ ,\ |3\rangle) \equiv (\sigma^x_{20}, \sigma^y_{20}, \sigma^z_{20}, I_{20}\ ) \\ m = 1\ :\ SU(2)_{21} = (|2\rangle\ ,\ |3\rangle) \equiv (\sigma^x_{21}, \sigma^y_{21}, \sigma^z_{21}, I_{21}) \\ ---------------------------- \\ m = 2\ :\qquad \sigma^z_2 = \frac{1}{\sqrt{3}} \sum^1_{m=0} \sigma^z_{2m} \qquad ; \qquad \sigma^0_2 = \frac{1}{\sqrt{3}} \sum^1_{m=0} I_{2m} \end{cases}$$

$$3^{rd} - \text{shell} : \begin{cases} m = 0\ :\ SU(2)_{30} = (|1\rangle\ ,\ |4\rangle) \equiv (\sigma^x_{30}, \sigma^y_{30}, \sigma^z_{30}, I_{30}) \\ m = 1\ :\ SU(2)_{31} = (|2\rangle\ ,\ |4\rangle) \equiv (\ \sigma^x_{31}, \sigma^y_{31}, \sigma^z_{31}, I_{31}) \\ m = 2\ :\ SU(2)_{32} = (|3\rangle\ ,\ |4\rangle) \equiv (\sigma^x_{32}, \sigma^y_{32}, \sigma^z_{32}, I_{32}) \\ ---------------------------- \\ m = 3\ :\qquad \sigma^z_3 = \frac{1}{\sqrt{6}} \sum^2_{m=0} \sigma^z_{3m} \qquad ; \qquad \sigma^0_3 = \frac{1}{\sqrt{6}} \sum^2_{m=0} I_{3m} \end{cases}$$

$$4^{th} - \text{shell} : \begin{cases} m = 0\ :\ SU(2)_{40} = (|1\rangle\ ,\ |5\rangle) \equiv (\sigma^x_{40}, \sigma^y_{40}, \sigma^z_{40}, I_{40}) \\ m = 1\ :\ SU(2)_{41} = (|2\rangle\ ,\ |5\rangle) \equiv (\sigma^x_{41}, \sigma^y_{41}, \sigma^z_{41}, I_{41}) \\ m = 2\ :\ SU(2)_{42} = (|3\rangle\ ,\ |5\rangle) \equiv (\sigma^x_{42}, \sigma^y_{42}, \sigma^z_{42}, I_{42}) \\ m = 3\ :\ SU(2)_{43} = (|4\rangle\ ,\ |5\rangle) \equiv (\sigma^x_{43}, \sigma^y_{43}, \sigma^z_{43}, I_{43}) \\ ---------------------------- \\ m = 4\ :\qquad \sigma^z_4 = \frac{1}{\sqrt{10}} \sum^3_{m=0} \sigma^z_{4m} \qquad ; \qquad \sigma^0_4 = \frac{1}{\sqrt{10}} \sum^3_{m=0} I_{4m} \end{cases}$$

(14c)

In summary, we have now provided a complete algebraic and quantum structure of a general $SU(N)$ symmetry group illustrated here explicitly in the $SU(5)$ example in equation (14c) above. An $SU(N)$ space is composed of a spectrum of $\frac{1}{2}N(N{-}1)$ $SU(2)$-subspaces distributed in definite numbers among $N - 1$ configuration shells specified by quantum numbers $l = 1, ..., N - 1$. The l^{th}-shell contains l $SU(2)$-subspaces, where an $SU(2)$-subspace specified by $SU(2)_{lm} = (\ |m + 1\rangle\ ,\ |l + 1\rangle\ )$, is a 2-dimensional state space spanned by a coupled pair of orthonormal state basis vectors $|m + 1\rangle$ and

$|l+1\rangle$, $l = 1, ..., N-1$, $m = 0, ..., l-1$. General $SU(N)$ symmetry group generators are enumerated and determined in sets σ^x_{lm} , σ^y_{lm} , σ^z_{lm} , I_{lm} satisfying closed $SU(2)$ Lie algebra defined within the $SU(2)$-subspaces $SU(2)_{lm} = (\ |m+1\rangle\ ,\ |l+1\rangle\ )$, $l = 1, ..., N-1$, $m = 0, ..., l-1$, such that the full set of the $SU(N)$ generators are distributed in a spectrum composed of $N-1$ configuration shells based on the distribution of the respective $SU(2)$-subspaces among the $N-1$ configuration shells in the $SU(N)$ space. The l^{th}-shell effective antisymmetric and symmetric diagonal generators σ^z_l , σ^0_l at $m = l$ are determined as sums of the respective antisymmetric and symmetric generators σ^z_{lm} , I_{lm} defined within the SU(2-subspaces according to the formulae in equations $(5bii)$, $(14b)$.

The algebraic and quantum structure which we have provided here accounts for the enumeration and determination of the full $SU(N)$ generator spectrum presented with explicit examples in sections 2 and 3 above. The property that $SU(N)$ generators are defined within $SU(2)$-subspaces specified by quantum numbers l , m can be very useful in developing and understanding the dynamical structure of $SU(N)$ gauge field theories of particle interactions, which we comment on in the next section.

5 Some remarks on implications of the expanded $SU(N)$ algebraic space and quantum structure to models of gauge field theories

We begin by observing that theoretical models of elementary particle interactions driven by electromagnetic, weak nuclear and strong nuclear forces, acting separately or as unified forces, in quantum field theory have generally been formulated as gauge field theories based on the algebraic properties of $U(1)$ and $SU(N)$ symmetry groups. For $SU(N)$ gauge field theories such as the unified electroweak interaction, strong nuclear interaction and grand unified interaction models, the driving gauge field forces are characterized by the respective vector bosons (quanta of the gauge field), which are identified directly with the generators of the Lie algebra of the chosen $SU(N)$ symmetry group. In particular, an $SU(N)$ gauge field is specified by N^2-1 vector boson components $\mathcal{A}^\mu_j$, $j = 1, 2, ..., N^2-1$, corresponding to the N^2-1 traceless generators of the $SU(N)$ symmetry group enumerated in the Gell-Mann basis $\lambda_1, \lambda_2, ..., \lambda_{N^2-1}$, such that the $SU(N)$ gauge field potential four-vector $\mathcal{A}^\mu$ is obtained as $\mathcal{A}^\mu = \sum_{j=1}^{N^2-1} \lambda_j \mathcal{A}^\mu_j$ [6 , 8 , 10-20]. In the spin angular momentum basis where the $SU(N)$ generators are specified by quantum numbers l , m, the corresponding gauge field vector boson components are also specified by the same quantum numbers l , m as we demonstrate below. The general dynamics in an $SU(N)$ gauge field is generated by interaction energy arising from the coupling of particle currents and the gauge field. The model of an $SU(N)$ gauge field theory is completed by including appropriately specified scalar fields, generally identified as the Higgs fields, which also couple to the particle currents and the gauge field [6 , 8 , 10-20].

The definition of an $SU(N)$ gauge field potential $\mathcal{A}^\mu$ in terms of the group generators according to $\mathcal{A}^\mu = \sum_{j=1}^{N^2-1} \lambda_j \mathcal{A}^\mu_j$ means that the dynamical structure of an $SU(N)$ gauge field theory is determined by the algebraic properties and quantum structure of the $SU(N)$ generator spectrum. The formulation of an $SU(N)$ gauge field theory must therefore be based on an accurate enumeration and determination of all the generators, together with the corresponding quadratic and Fubini-Veneziano spin angular momentum operators, in the Gell-Mann or spin angular momentum basis, taking into account the quantum structure of the generator spectrum composed of $N-1$ configuration shells each containing a definite number of generators specified by the shell quantum number $l = 1, ..., N-1$ and symmetric-antisymmetric generator pair quantum number $m = 0, 1, ..., l$. The quantum structure of the $SU(N)$ generator spectrum means that the $SU(N)$ gauge field four-vector $\mathcal{A}^\mu$ is composed of a spectrum of N^2-1 vector bosons $\mathcal{A}_j^\mu$, $j=1,...,N^2-1$, distributed in definite numbers among the $N-1$ configuration shells of the generator spectrum. On the other hand, the algebraic properties of the $SU(N)$ symmetry group, starting with the basic relations in equations ($13a$)-($13e$), generated by the full spectrum of generators expressed in the extended Cartan-Weyl basis, which besides the standard traceless basis S_l^z , S_{lm}^+ , S_{lm}^-, now includes the non-traceless symmetric generator S_l^0, quadratic and Fubini-Veneziano spin angular momentum operators σ_l^2 ($Q_{l:2n}$) , $\mathcal{F}_{l:1}$ ($\mathcal{F}_{l:2n+1}$), are used to determine the complete spectrum of particle states within the entire quantum state space of the $SU(N)$ gauge field theory. We note that the expanded algebraic space specified by the extended Cartan-Weyl basis S_l^z , S_{lm}^+ , S_{lm}^- , S_l^0 , σ_l^2 ($Q_{l:2n}$) , $\mathcal{F}_{l:1}$ ($\mathcal{F}_{l:2n+1}$), considered together with the associated universal $SU(N)$ quadratic Casimir and Fubini-Veneziano spin operators $Q_{N:2n}$, $\mathcal{F}_{N:2n+1}$, lead to an expansion of the quantum state space of an $SU(N)$ gauge field theory compared to the quantum state spaces of existing models of $SU(N)$ gauge field theories where only the basic Cartan-Weyl basis S_l^z , S_{lm}^+ , S_{lm}^- has been used to determine the particle states.

We observe that in the current models of electroweak ($SU(2)\times U(1)$), quantum chromodynamics ($SU(3)_c$), the standard model ($SU(3)_c\times SU(2)_L\times U(1)_{em}$), Pati-Salam leptoquark models ($SU(4)_{L+R}\times SU(4)_{L+R}$ and $SU(4)_L\times SU(4)_R\times SU(4')$) and Georgi-Glashow grand unification ($SU(5)$) gauge field theories [6 , 8 , 10-20], all $SU(2)$, $SU(3)$ and $SU(4)$ generators have been determined accurately in agreement with the corresponding Gell-Mann basis determined and presented here for ease of comparison in equations ($2cii$) , ($2dii$) , ($2eii$) of the respective generator spectra, including the correct forms of the respective basic Cartan generators $2S_l^z=\lambda_{l^2+2l}$ in each shell $l=1,...,N-1$ for $N=2,3,4$, but for the $SU(5)$ symmetry group as used in the model of a grand unified theory in [14-20], the third and fourth (last two) Cartan generators derived through fine-tuning in the forms (in the notation of the present work) $S_3^z=\text{diag}(0,0,0,1,-1)$, $S_4^z=\frac{1}{\sqrt{15}}\text{diag}(2,2,2,-3,-3)$ to achieve the desired consistency in the $SU(5)$ grand unified theory deviate completely from the accurate forms $2S_3^z=\lambda_{15}=\frac{1}{\sqrt{6}}\text{diag}(1,1,1,-3,0)$, $2S_4^z=\lambda_{24}=\frac{1}{\sqrt{10}}\text{diag}(1,1,1,1,-4)$ determined and presented here for ease of comparison in equation ($2fii$) of the $SU(5)$ generator spectrum. This discrepancy in the determination of Cartan generators already provides a good reason to review the $SU(5)$ grand unified theory, since, according to the evaluations in [14-20], the specification of the electric charge operator and the prediction of the weak mixing or Weinberg angle θ_W depend on the two Cartan generators $2S_3^z=\lambda_{15}$, $2S_4^z=\lambda_{24}$, which we have enumerated and determined accurately in the present work.

Major challenges on quantum structure and algebraic properties arise in existing gauge field theories based on the $SU(N)$ symmetry groups, which we can illustrate using the simpler $SU(2)$ and $SU(3)$ cases.

For the $SU(2)$ symmetry group applicable in the $SU(2)\times U(1)$ (electroweak) and $SU(3)_c\times SU(2)_L\times U(1)_Y$ (standard model) gauge field theories, all the $2^2-1=3$ generators in the Gell-Mann basis $\lambda_1,\lambda_2,\lambda_3$, have been determined accurately [6 , 8 , 10 , 11 , 14] in agreement with the corresponding matrix forms determined and presented here for ease of comparison in equation ($2cii$) of the general $SU(2)$ generator spectrum, but the quantum structure is not specified. According to equations ($2ai$) , ($2aii$) , ($5ai$) , ($5bi$) , ($5bii$), the 3 $SU(2)$ generators in the Gell-Mann or spin angular momentum basis are specified by quantum numbers $l=1$, $m=0,1$ and occupy the single ($2-1=1$) configuration shell in the $SU(2)$ generator spectrum, containing the $2\times1+1=3$ generators $\lambda_1,\lambda_2,\lambda_3\equiv\sigma_{10}^x,\sigma_{10}^y,\sigma_1^z$ as presented in the $SU(2)$ generator spectra in equations ($3dii$) , ($9b$), noting that the basic Cartan generator is obtained as $2S_1^z=\sigma_1^z=\lambda_3$. But the expected non-traceless symmetric diagonal generator $2S_1^0=\sigma_1^0$, quadratic spin angular momentum operator σ_1^2 ($Q_{1:2n}$, $\sigma_1^2=Q_{1:2}$) and the Fubini-Veneziano spin angular momentum operator $\mathcal{F}_{1:1}$ ($\mathcal{F}_{1:2n+1}$, $n=0,1,2,3,...$), which we have introduced here in the 1^{st}-shell in equation ($9b$), are missing in the specifications of Cartan-Weyl basis used in the existing $SU(2)\times U(1)$ and $SU(3)_c\times SU(2)_L\times U(1)_Y$ gauge field theories. However, we note that the symmetric generator S_1^0 and the quadratic spin angular momentum operator σ_1^2, together with the corresponding $SU(2)$ quadratic Casimir operator Q_2, coincide with the $SU(2)$ identity operator I_2 according to $S_1^0=\frac{1}{2}\sigma_1^0=\frac{1}{2}I_2$ from equations $(5biv)$, $(5bv)$ and $Q_2=2\ \frac{2^2-1}{2}I_2=3I_2$ from equation $(11f)$. It follows that all the $SU(2)$ algebraic properties and the associated spectrum of roots, weights and particle states in an $SU(2)\times U(1)$ or $SU(3)_c\times SU(2)_L\times U(1)_Y$ gauge field theory have always been determined accurately, except now we have discovered that the quantum state space can be expanded by including the additional spectrum of roots, weights and particle states arising from the algebraic relations generated by the $SU(2)$ Fubini-Veneziano spin angular momentum operator $\mathcal{F}_{1:1}=\alpha S_{10}^{+}+S_{10}^{-}+\sigma_1^z$ determined by setting $l=1$, $m=0$ in equation $(7e)$, noting that for $SU(2)$, $\sigma_1^z=\lambda_3$ in the Gell-Mann notation. We observe that for $SU(2)$, the Fubini-Veneziano spin angular momentum operator $\mathcal{F}_{1:2n+1}$ takes the same form in all orders $n=0,1,2,3,...$ according to the relation $(\sigma_1^z)^{2n}=I_2$, giving $(\sigma_1^z)^{2n+1}=\sigma_1^z$, $\mathcal{F}_{1;2n+1}=\mathcal{F}_{1:1}$, which applies to the $SU(2)$ generator spectrum in the spin angular momentum basis in equation $(9b)$.

For the $SU(3)$ symmetry group applicable in the $SU(3)_c$ (quantum chromodynamics) and $SU(3)_c\times SU(2)_L\times U(1)_Y$ (standard model) gauge field theories, all the $3^2-1=8$ generators in the Gell-Mann basis $\lambda_1,\lambda_2,...,\lambda_8$ have been determined accurately [6 , 8 , 10 , 11 , 14] in agreement with the corresponding matrix forms determined and presented here for ease of comparison in equation $(2dii)$ of the general $SU(3)$ generator spectrum, but the quantum structure is not specified. According to equations $(2ai)$, $(2aii)$, $(5ai)$, $(5bi)$, $(5bii)$, the 8 $SU(3)$ generators in the Gell-Mann or spin angular momentum basis are specified by quantum numbers $l=1,2$, $m=0,...,l$ and occupy $3-1=2$ configuration shells in the $SU(3)$ generator spectrum, with $2\times 1+1=3$ generators $\lambda_1,\lambda_2,\lambda_3\equiv\sigma_{10}^x,\sigma_{10}^y,\sigma_1^z$ in the 1^{st}-shell and $2\times 2+1=5$ generators $\lambda_4,\lambda_5,\lambda_6,\lambda_7,\lambda_8\equiv\sigma_{20}^x,\sigma_{20}^y,\sigma_{21}^x,\sigma_{21}^y,\sigma_2^z$ in the 2^{nd}-shell as presented in the $SU(3)$ generator spectra in equations $(3eii)$, $(9c)$, noting that the two basic Cartan generators are obtained as $2S_1^z=\sigma_1^z=\lambda_3$, $2S_2^z=\sigma_2^z=\lambda_8$. But the non-traceless symmetric diagonal generators, quadratic and the Fubini-Veneziano spin angular momentum operators obtained in the present work as S_1^0 , σ_1^2 $(Q_{1:2n})$, $\mathcal{F}_{1;1}$ $(\mathcal{F}_{1:2n+1})$ in the 1^{st}-shell and S_2^0 , σ_2^2 $(Q_{2;2n})$, $\mathcal{F}_{2;1}$ $(\mathcal{F}_{2:2n+1})$ in the 2^{nd}-shell as given in equation $(9c)$ are missing in the specifications of Cartan generators used in the existing $SU(3)_c$ and $SU(3)_c\times SU(2)_L\times U(1)_Y$ gauge field theories. The $SU(3)$ quadratic Casimir operator Q_3 coincides with the identity operator I_3 according to $Q_3=2\ \frac{3^2-1}{3}I_3=\frac{16}{3}I_3$ from equation $(11f)$, which does not affect the algebraic properties of the generator spectrum. We easily arrive at the general understanding that the quantum state space of the existing $SU(3)_c$ and $SU(3)_c\times SU(2)_L\times U(1)_Y$ gauge field theories, which do not include the additional spectrum of roots, weights and particle states arising from the algebraic relations generated by the $SU(3)$ non-traceless symmetric diagonal generators, quadratic and Fubini-Veneziano spin angular momentum operators S_l^0 , σ_l^2 $(Q_{l:2n})$, $\mathcal{F}_{l:1}$ $(\mathcal{F}_{l:2n+1})$ for $l=1,2$, $n=0,1,2,3,...$, is not fully specified. It follows that the quantum state spaces of the existing $SU(3)_c$ and $SU(3)_c\times SU(2)_L\times U(1)_Y$ gauge field theories are seriously limited, noting that the composite $SU(2)$ and $SU(3)$ quantum state spaces are both limited as explained above.

In general, the set of basic algebraic relations in equations $(13a)$-$(13e)$, which are useful in determining the spectrum of roots, weights and particle states, already reveals that an incomplete specification of the Cartan generators, which does not include the non-traceless symmetric diagonal generators, quadratic and the Fubini-Veneziano spin angular momentum operators S_l^0 , σ_l^2 $(Q_{l:2n})$, $\mathcal{F}_{l:1}$ $(\mathcal{F}_{l:2n+1})$ for $l=1,...,N-1$, $n=0,1,2,3,...$ and does not specify the quantum structure of the chosen $SU(N)$ generator spectrum composed of $N-1$ configuration shells each containing definite numbers of specified generators, drastically limits the quantum state space of a gauge field theory based on the

$SU(N)$ symmetry group. According to equations $(13b)$-$(13d)$, the non-traceless symmetric diagonal generator σ_l^0 and the quadratic spin angular momentum operator σ_l^2 generate eigenvalue equations which can be used to determine additional spectra of roots, weights and particle states to expand the quantum state spaces of the existing $SU(3)_c$, $SU(3)_c \times SU(2)_L \times U(1)_Y$, $SU(4)_L \times SU(4)_R \times SU(4')$, $SU(5)$, $SU(6)$ gauge field models.

In particular, the interpretation of the $2l+1$ traceless generators σ_{lm}^x , σ_{lm}^y , S_l^z , $m = 0, 1, ..., l$ in the l^{th}-shell as components of a $(2l+1)$-component spin angular momentum vector $\vec{\sigma}_l$, with quadratic operator $\sigma_l^2 = \vec{\sigma}_l \cdot \vec{\sigma}_l$ commuting with both the traceless antisymmetric diagonal generator S_l^z and the non-traceless symmetric diagonal generator S_l^0, provides an opportunity for applying standard angular momentum algebra using the basis $S_l^z, S_{lm}^+, S_{lm}^-, \mathcal{F}_{l:1}, \sigma_l^2$ and the basis $S_l^0, S_{lm}^+, S_{lm}^-, \mathcal{F}_{l:1}, \sigma_l^2$ to determine the spectrum of spin state eigenvectors and eigenvalues within and across the $N-1$ shells spanning the quantum state space of an $SU(N)$ gauge field theory. Greater detail may be achieved by replacing the quadratic and Fubini-Veneziano spin angular momentum operators σ_l^2 , $\mathcal{F}_{l:1}$ with their respective general n^{th}-order quadratic and Fubini-Veneziano spin angular momentum operators $Q_{l:2n}$, $\mathcal{F}_{l:2n+1}$ and the universal $SU(N)$ quadratic Casimir and Fubini-Veneziano spin angular momentum operators $Q_{N:2n}$, $\mathcal{F}_{N:2n+1}$, $n = 0, 1, 2, 3, ...$ as may be necessary. The construction of the quantum state space spanned by spin state eigenvectors and eigenvalues through the angular momentum algebra can be very useful in characterizing particle transition states and identifying possible selection rules for allowed or forbidden transitions in particle interactions in an $SU(N)$ gauge field model. The selection rules may be more natural mechanisms to account for some of the unexpectedly enhanced or suppressed transitions emerging as discrepancies between theory and experiment in $SU(N)$ gauge field theories, which are generally addressed through speculations by adding hypothetical particle states and scalar fields to expand the quantum state space as desired.

Finally, we consider the implication of the quantum structure of the $SU(N)$ generator spectrum to the nature of the gauge field potential four-vector $\mathcal{A}^\mu$, which corresponds to the $SU(N)$ generator spectrum in the Gell-Mann basis λ_j , $j = 1, 2, ..., N^2 - 1$, according to the standard definition $\mathcal{A}^\mu = \sum_{j=1}^{N^2-1} \lambda_j \mathcal{A}_j^\mu$ [6 , 8 , 10-20], where $\mathcal{A}_j^\mu$ are the component vector bosons of the gauge field potential. As we have pointed out earlier, the quantum structure of the $SU(N)$ generator spectrum in the Gell-Mann or spin angular momentum basis displayed explicitly in the respective examples in equations $(3d)$-$(3g)$ or equations $(9b)$-$(9e)$ means that the $SU(N)$ gauge field potential four-vector $\mathcal{A}^\mu$ is composed of a spectrum of $N^2 - 1$ vector bosons $\mathcal{A}_j^\mu$, $j = 1, ..., N^2 - 1$, distributed in definite numbers among the $N-1$ configuration shells of the generator spectrum. In particular, in the spin angular momentum basis where the l^{th}-shell generators σ_{lm}^x , σ_{lm}^y , σ_l^z , σ_l^0 are specified explicitly by quantum numbers $l = 1, ..., N-1$, $m = 0, 1, ..., l$, the corresponding gauge field vector bosons are also specified by the same quantum numbers l , m as $\mathcal{A}_{lm}^{\mu x}$, $\mathcal{A}_{lm}^{\mu y}$, $\mathcal{A}_l^{\mu z}$, $\mathcal{A}_l^{\mu 0}$, such that the l^{th}-shell gauge field potential four-vector $\mathcal{A}_l^\mu$ is obtained as $\mathcal{A}_l^\mu = \sum_{m=0}^{l-1} \sigma_{lm}^j \mathcal{A}_{lm}^{\mu j} + \sigma_l^z \mathcal{A}_l^{\mu z}$, $j = x, y$, and the total $SU(N)$ gauge field potential four-vector $\mathcal{A}^\mu$ is then obtained as the sum of the $N-1$ l^{th}-shell potentials $\mathcal{A}_l^\mu$ in the form $\mathcal{A}^\mu = \sum_{l=1}^{N-1} \mathcal{A}_l^\mu$. For completeness, we have introduced a hyperphoton vector boson $H_l^{\mu 0}$ corresponding to the non-traceless symmetric generator σ_l^0 in the l^{th}-shell, giving a corresponding l^{th}-shell hyperphoton gauge field potential four-vector $H_l^\mu = \sigma_l^0 H_l^{\mu 0}$.

We present an illustrative picture of the expected quantum structure of an $SU(N)$ gauge field model, using the $SU(5)$ grand unified theory as an example, which also captures all features of the quantum structure of gauge field models based on the lower $SU(2)$, $SU(3)$, $SU(4)$ symmetry groups in equation $(15a)$ below, where the generators are in the spin angular momentum basis according to the general $SU(N)$ generator spectrum in equation $(9a)$ and the $SU(5)$ example in equation $(9e)$. The definitions given in equations $(5ai)$, $(5bi)$, $(5bii)$, $(5civ)$, $(7e)$ are used to enumerate and determine the generators σ^x_{lm} , σ^y_{lm} , σ^z_l , σ^0_l, together with the corresponding quadratic and Fubini-Veneziano spin angular momentum operators σ^2_l , $\mathcal{F}_{l:1}$ in each of the $N-1$ configuration shells specified by $l = 1, ..., N-1$. Noting that the grand unified $SU(5)$ gauge field model is composed of strong, leptoquark and weak interaction sectors, the gauge field vector bosons corresponding to the traceless generators σ^x_{lm} , σ^y_{lm} , σ^z_l in the l^{th}-shell within these sectors are identified as gluons G^μ , leptoquark bosons X^μ and weak bosons $\mathcal{W}^\mu$ based on the quantum structure, while the gauge field vector boson corresponding to the non-traceless generator σ^0_l in the l^{th}-shell is identified as a hyperphoton vector boson $H^{\mu 0}_l$ as clarified below.

Shell structure of $SU(5)$ gauge field model : 5 shells, $l = 1, 2, 3, 4$

$$l = 1 \ : \ 1^{st} - \text{shell} \ : \ PRST - |2\rangle \quad \begin{cases} m = 0 \ : & \sigma^x_{10} \quad ; \quad \sigma^y_{10} \quad \mapsto \quad G^x_{10} \quad ; \quad G^y_{10} \\ m = 1 \ : & \sigma^z_1 \ ; \ \sigma^0_1 \ ; \ (\sigma^2_1 \ ; \ \mathcal{F}_{1:1}) \ \mapsto \ G^z_1 \quad ; \quad H^0_1 \end{cases}$$

$$l = 2 \ : \ 2^{nd} - \text{shell} \ : \ PRST - |3\rangle \quad \begin{cases} m = 0 \ : & \sigma^x_{20} \quad ; \quad \sigma^y_{20} \quad \mapsto \quad G^x_{20} \quad ; \quad G^x_{20} \\ m = 1 \ : & \sigma^x_{21} \quad ; \quad \sigma^y_{21} \quad \mapsto \quad G^x_{21} \quad ; \quad G^y_{21} \\ m = 2 \ : & \sigma^z_2 \ ; \ \sigma^0_2 \ ; \ (\sigma^2_2 \ ; \ \mathcal{F}_{2:1}) \ \mapsto \ G^z_2 \quad ; \quad H^0_2 \end{cases}$$

$$l = 3 \ : \ 3^{rd} - \text{shell} \ : \ PRST - |4\rangle \quad \begin{cases} m = 0 \ : & \sigma^x_{30} \quad ; \quad \sigma^y_{30} \quad \mapsto \quad X^x_{30} \quad ; \quad X^y_{30} \\ m = 1 \ : & \sigma^x_{31} \quad ; \quad \sigma^y_{31} \quad \mapsto \quad X^x_{31} \quad ; \quad X^y_{31} \\ m = 2 \ : & \sigma^x_{32} \quad ; \quad \sigma^y_{32} \quad \mapsto \quad X^x_{32} \quad ; \quad X^y_{32} \\ m = 3 \ : & \sigma^z_3 \ ; \ \sigma^0_3 \ ; \ (\sigma^2_3 \ , \ \mathcal{F}_{3:1}) \ \mapsto \ X^z_3 \quad ; \quad H^0_3 \end{cases}$$

$$l = 4 \ : \ 4^{th} - \text{shell} \ : \ PRST - |5\rangle \quad \begin{cases} m = 0 \ : & \sigma^x_{40} \quad ; \quad \sigma^y_{40} \quad \mapsto \quad \mathcal{W}^x_{40} \quad ; \quad \mathcal{W}^y_{40} \\ m = 1 \ : & \sigma^x_{41} \quad ; \quad \sigma^y_{41} \quad \mapsto \quad \mathcal{W}^x_{41} \quad ; \quad \mathcal{W}^y_{41} \\ m = 2 \ : & \sigma^x_{42} \quad ; \quad \sigma^y_{42} \quad \mapsto \quad \mathcal{W}^x_{42} \quad ; \quad \mathcal{W}^y_{42} \\ m = 3 \ : & \sigma^x_{43} \quad ; \quad \sigma^y_{43} \quad \mapsto \quad \mathcal{W}^x_{43} \quad ; \quad \mathcal{W}^y_{43} \\ m = 4 \ : & \sigma^z_4 \ ; \ \sigma^0_4 \ ; \ (\sigma^2_4 \ , \ \mathcal{F}_{4:1}) \ \mapsto \ \mathcal{W}^z_4 \quad ; \quad H^0_4 \end{cases} \tag{15a}$$

An important physical feature which emerges in the expected quantum structure of the $SU(5)$ gauge field model ($SU(5)$ GUT) in equation ($15a$) is that the 24-component $SU(5)$ gauge field potential four-vector $\mathcal{A}^\mu = (G^\mu\ ,\ X^\mu\ ,\ \mathcal{W}^\mu)$ is composed of $2\times 1+1 = 3$ gluons $G^x_{10}\ ,\ G^y_{10}\ ,\ G^z_1$ corresponding to the 3 traceless generators $\sigma^x_{10}\ ,\ \sigma^y_{10}\ ,\ \sigma^z_1$, respectively, in the 1^{st}-shell, $2\times 2+1 = 5$ gluons $G^x_{20}\ ,\ G^y_{20}\ ,\ G^x_{21}\ ,\ G21^y\ ,\ G^z_2$ corresponding to the 5 traceless generators $\sigma^x_{20}\ ,\ \sigma^y_{20}\ ,\ \sigma^x_{21}\ ,\ \sigma^y_{21}\ ,\ \sigma^z_2$, respectively, in the 2^{nd}-shell, $2\times 3+1 = 7$ leptoquark (or *intermediate interaction*) bosons $X^x_{30}\ ,\ X^y_{30}\ ,\ X^x_{31}\ ,\ X^y_{31}\ ,\ X^x_{32}\ ,\ X^y_{32}\ ,\ X^z_3$ corresponding to the 7 traceless generators $\sigma^x_{30}\ ,\ \sigma^y_{30}\ ,\ \sigma^x_{31}\ ,\ \sigma^y_{31}\ ,\ \sigma^x_{32}\ ,\ \sigma^y_{32}\ ,\ \sigma^z_3$, respectively, in the 3^{rd}-shell and $2\times 4+1 = 9$ weak interaction gauge field bosons $\mathcal{W}^x_{40}\ ,\ \mathcal{W}^y_{40}\ ,\ \mathcal{W}^x_{41}\ ,\ \mathcal{W}^y_{41}\ ,\ \mathcal{W}^x_{42}\ ,\ \mathcal{W}^y_{42}\ ,\ \mathcal{W}^x_{43}\ ,\ \mathcal{W}^y_{43}\ ,\ \mathcal{W}^z_4$ corresponding to the 9 traceless generators $\sigma^x_{40}\ ,\ \sigma^y_{40}\ ,\ \sigma^x_{41}\ ,\ \sigma^y_{41}\ ,\ \sigma^x_{42}\ ,\ \sigma^y_{42}\ ,\ \sigma^x_{43}\ ,\ \sigma^y_{43}\ ,\ \sigma^z_4$ in the 4^{th}-shell, making a total of $5^2-1 = 24$ gauge field vector bosons. The $SU(5)$ gauge field model is thus composed of a strong interaction sector consisting of two sub-sectors driven by 3 gluon vector bosons $\mathcal{A}^\mu_1 \equiv (G^x_{10}, G^y_{10}, G^z_1)$ in the 1^{st}-shell and 5 gluon vector bosons $\mathcal{A}^\mu_2 \equiv (G^x_{20}, G^y_{20}, G^x_{21}, G21^y, G^z_2)$ in the 2^{nd}-shell, an intermediate interaction sector driven by 7 leptoquark vector bosons $\mathcal{A}^\mu_3 \equiv (X^x_{30}, X^y_{30}, X^x_{31}, X^y_{31}, X^x_{32}, X^y_{32}, X^z_3)$ in the 3^{rd}-shell and a weak interaction sector driven by 9 weak vector bosons $\mathcal{A}^\mu_4 \equiv (\mathcal{W}^x_{40}, \mathcal{W}^y_{40}, \mathcal{W}^x_{41}, \mathcal{W}^y_{41}, \mathcal{W}^x_{42}, \mathcal{W}^y_{42}, \mathcal{W}^x_{43}, \mathcal{W}^y_{43}, \mathcal{W}^z_4)$ in the 4^{th}-shell. We note that the weak interaction sector in the 4^{th}-shell is generalized such that, of the 9 weak vector bosons, the first 6, namely, $\mathcal{W}^x_{40}\ ,\ \mathcal{W}^y_{40}\ ,\ \mathcal{W}^x_{41}\ ,\ \mathcal{W}^y_{41}\ ,\ \mathcal{W}^x_{42}\ ,\ \mathcal{W}^y_{42}$, are identified with the (6) Y vector bosons introduced in the original $SU(5)$ grand unified theory [14-19], while the last three, namely, $\mathcal{W}^x_{43}\ ,\ \mathcal{W}^y_{43}\ ,\ \mathcal{W}^z_4$ are identified with the (3) well known $W^\pm\ ,\ W_3$ vector bosons in the $SU(2)$ component of the standard $SU(2)\times U(1)$ electroweak gauge theory [6 , 8 , 10 , 11 , 14].

Considering that the non-traceless symmetric diagonal generator σ^0_l is the algebraic partner of the traceless antisymmetric diagonal generator σ^z_l enumerated and determined in each shell $l = 1,2,3,4$ according to the definitions in equation ($5bii$), we have introduced a corresponding l^{th}-shell hyperphoton gauge field vector boson $H^{\mu 0}_l$, $l = 1,2,3,4$ in each shell as displayed in equation ($15a$). In this respect, we develop an understanding that the electromagnetic field photon A and the associated weak field neutral vector boson Z^0 which we have not specified explicitly in the quantum structure of the $SU(5)$ gauge field model in equation ($15a$) may arise through the hyperphoton vector bosons $H^0_1\ ,\ H^0_2\ ,\ H^0_3\ ,\ H^0_4$ corresponding to the non-traceless symmetric diagonal generators $\sigma^0_1\ ,\ \sigma^0_2\ ,\ \sigma^0_3\ ,\ \sigma^0_4$, respectively.

We get finer detail on the dynamical structure of an $SU(N)$ gauge field if we consider the equivalent, but finer quantum structure where the $SU(N)$ generators are defined within a spectrum of $\frac{1}{2}N(N-1)$ $SU(2)$-subspaces distributed in definite numbers among $N-1$ configuration shells in the $SU(N)$ space as explained in section 4.3 above. In this respect, generators defined within an $SU(2)$-subspace specified by quantum numbers $l\ ,\ m$ determined according to the general spectrum in equation ($14b$) as $SU(2)_{lm} = (\sigma^x_{lm}\ ,\ \sigma^y_{lm}\ ,\ \sigma^z_{lm}\ ,\ I_{lm})$ in the l^{th}-shell are directly associated with a corresponding gauge field vector boson $\mathcal{A}^\mu_{lm} = (\mathcal{A}^x_{lm}\ ,\ \mathcal{A}^y_{lm}\ ,\ \mathcal{A}^z_{lm}\ ,\ h_{lm})$ defined within the same $SU(2)$-subspace in the l^{th}-shell. As an example, we determine the expected finer quantum structure of the grand unified $SU(5)$ gauge field model where the gauge field vector bosons are defined within a spectrum of $\frac{1}{2}5(5-1) = 10$ $SU(2)$-subspaces by setting $N = 5$ giving $l = 1,2,3,4$ in the general spectrum of $SU(2)$-subspaces in equation ($14b$). The expected finer quantum structure of the grand unified $SU(5)$ gauge field model composed of a spectrum of 10 $SU(2)$-subspaces $SU(2)_{lm} = (|m+1\rangle\ ,\ |l+1\rangle)$, $l = 1,2,3,4\ ,\ m = 0,...,l-1$, is displayed in equation ($15b$) below.

Expected spectrum of $SU(2)$-subspaces in the quantum structure of the grand unified $SU(5)$ gauge field model

$$SU(2)_{lm} = (\ |m{+}1\rangle\ ,\ |l{+}1\rangle\ ) \equiv (\ \sigma^x_{lm}\ ,\ \sigma^y_{lm}\ ,\ \sigma^z_{lm}\ ,\ I_{lm}\ ) \quad ; \quad N=5\ : \quad l=1,2,3,4\ ,\ m=0,1,...,l{-}1$$

$$1^{st}-\text{shell}: \begin{cases} m=0\ :\ SU(2)_{10} = (|1\rangle\ ,\ |2\rangle) \equiv (\sigma^x_{10}, \sigma^y_{10}, \sigma^z_{10}, I_{10})\ :\ \mathcal{A}^\mu_{10} = (G^x_{10}, G^y_{10}, G^z_{10}, h_{10}) \\ ----------------------------------- \\ m=1\ :\ \sigma^z_1 = \sigma^z_{10} \quad ; \quad \sigma^0_1 = I_{10} \qquad : \qquad G^z_1 \quad ; \quad H^0_1 \end{cases}$$

$$2^{nd}-\text{shell}: \begin{cases} m=0\ :\ SU(2)_{20} = (|1\rangle\ ,\ |3\rangle) \equiv (\sigma^x_{20}, \sigma^y_{20}, \sigma^z_{20}, I_{20}\ )\ :\ \mathcal{A}^\mu_{20} = (G^x_{20}, G^y_{20}, G^z_{20}, h_{20}) \\ m=1\ :\ SU(2)_{21} = (|2\rangle\ ,\ |3\rangle) \equiv (\sigma^x_{21}, \sigma^y_{21}, \sigma^z_{21}, I_{21})\ :\ \mathcal{A}^\mu_{21} = (G^x_{21}, G^y_{21}, G^z_{21}, h_{21}) \\ ----------------------------------- \\ m=2\ :\ \sigma^z_2 = \frac{1}{\sqrt{3}} \sum_{m=0}^{1} \sigma^z_{2m} \quad ; \quad \sigma^0_2 = \frac{1}{\sqrt{3}} \sum_{m=0}^{1} I_{2m} \qquad : \qquad G^z_2 \quad ; \quad H^0_2 \end{cases}$$

$$3^{rd}-\text{shell}: \begin{cases} m=0\ :\ SU(2)_{30} = (|1\rangle\ ,\ |4\rangle) \equiv (\sigma^x_{30}, \sigma^y_{30}, \sigma^z_{30}, I_{30})\ :\ \mathcal{A}^\mu_{30} = (X^x_{30}, X^y_{30}, X^z_{30}, h_{30}) \\ m=1\ :\ SU(2)_{31} = (|2\rangle\ ,\ |4\rangle) \equiv (\ \sigma^x_{31}, \sigma^y_{31}, \sigma^z_{31}, I_{31})\ :\ \mathcal{A}^\mu_{31} = (X^x_{31}, X^y_{31}, X^z_{31}, h_{31}) \\ m=2\ :\ SU(2)_{32} = (|3\rangle\ ,\ |4\rangle) \equiv (\sigma^x_{32}, \sigma^y_{32}, \sigma^z_{32}, I_{32})\ :\ \mathcal{A}^\mu_{32} = (X^x_{32}, X^y_{32}, X^z_{32}, h_{32}) \\ ----------------------------------- \\ m=3\ :\ \sigma^z_3 = \frac{1}{\sqrt{6}} \sum_{m=0}^{2} \sigma^z_{3m} \quad ; \quad \sigma^0_3 = \frac{1}{\sqrt{6}} \sum_{m=0}^{2} I_{3m} \qquad : \qquad X^z_3 \quad ; \quad H^0_3 \end{cases}$$

$$4^{th}-\text{shell}: \begin{cases} m=0\ :\ SU(2)_{40} = (|1\rangle\ ,\ |5\rangle) \equiv (\sigma^x_{40}, \sigma^y_{40}, \sigma^z_{40}, I_{40})\ :\ \mathcal{A}^\mu_{40} = (\mathcal{W}^x_{40}, \mathcal{W}^y_{40}, \mathcal{W}^z_{40}, h_{40}) \\ m=1\ :\ SU(2)_{41} = (|2\rangle\ ,\ |5\rangle) \equiv (\sigma^x_{41}, \sigma^y_{41}, \sigma^z_{41}, I_{41})\ :\ \mathcal{A}^\mu_{41} = (\mathcal{W}^x_{41}, \mathcal{W}^y_{41}, \mathcal{W}^z_{41}, h_{41}) \\ m=2\ :\ SU(2)_{42} = (|3\rangle\ ,\ |5\rangle) \equiv (\sigma^x_{42}, \sigma^y_{42}, \sigma^z_{42}, I_{42})\ :\ \mathcal{A}^\mu_{42} = (\mathcal{W}^x_{42}, \mathcal{W}^y_{42}, \mathcal{W}^z_{42}, h_{42}) \\ m=3\ :\ SU(2)_{43} = (|4\rangle\ ,\ |5\rangle) \equiv (\sigma^x_{43}, \sigma^y_{43}, \sigma^z_{43}, I_{43})\ :\ \mathcal{A}^\mu_{43} = (\mathcal{W}^x_{43}, \mathcal{W}^y_{43}, \mathcal{W}^z_{43}, h_{43}) \\ ----------------------------------- \\ m=4\ :\ \sigma^z_4 = \frac{1}{\sqrt{10}} \sum_{m=0}^{3} \sigma^z_{4m} \quad ; \quad \sigma^0_4 = \frac{1}{\sqrt{10}} \sum_{m=0}^{3} I_{4m} \qquad : \qquad \mathcal{W}^z_4 \quad ; \quad H^0_4 \end{cases} \tag{15b}$$

The quantum structure demonstrated in the $SU(5)$ gauge field in equation $(15b)$ above characterizes $SU(N)$ gauge field vector bosons as four-vectors $\mathcal{A}^\mu_{lm}$ defined within $\frac{1}{2}N(N-1)$ $SU(2)$-subspaces in the $N-1$ shells of the $SU(N)$ space. Each $SU(2)$-subspace contains only one four-vector boson, meaning that there are $\frac{1}{2}N(N-1)$ four-vector bosons in an $SU(N)$ gauge field. The general dynamics of interacting particles in an $SU(N)$ gauge field model is therefore driven by $\frac{1}{2}N(N-1)$ four-vector bosons. If the gauge field model is based on several $SU(N)$ symmetry groups, then the dynamics is driven by a combination of sets of $\frac{1}{2}N(N-1)$ four-vector bosons from each of the $SU(N)$ gauge fields in the model.

It is clear from the quantum structure in equation $(15b)$ that the grand unified $SU(5)$ gauge field is composed of a total of $\frac{1}{2}5(5-1) = 10$ four-vector bosons, which we have identified as gluons G, leptoquark bosons X and weak bosons $\mathcal{W}$, defined within the respective 10 $SU(2)$-subspaces distributed among the $5-1=4$ configuration shells of the $SU(5)$ generator spectrum. Dynamics in the grand unified $SU(5)$ gauge field model is therefore driven by three gluon four-vector bosons $\mathcal{A}^{\mu}_{10}$, $\mathcal{A}^{\mu}_{20}$, $\mathcal{A}^{\mu}_{21}$ characterizing the strong interaction sector in the 1^{st} and 2^{nd} shells, three leptoquark four-vector bosons $\mathcal{A}^{\mu}_{30}$, $\mathcal{A}^{\mu}_{31}$, $\mathcal{A}^{\mu}_{32}$ characterizing the leptoquark (or intermediate) interaction sector in the 3^{rd} shell and four weak four-vector bosons $\mathcal{A}^{\mu}_{40}$, $\mathcal{A}^{\mu}_{41}$, $\mathcal{A}^{\mu}_{42}$, $\mathcal{A}^{\mu}_{43}$ characterizing the general electroweak interaction sector in the 4^{th} shell.

As we observed earlier, the strong interaction sector is composed of two subsectors of dynamics, one driven by a single gluon four-vector boson $\mathcal{A}^{\mu}_{10}$ within the $SU(2)_{10} = (|1\rangle\ ,\ |2\rangle)$ subspace in the 1^{st}-shell and the other driven by two gluon four-vector bosons $\mathcal{A}^{\mu}_{20}$, $\mathcal{A}^{\mu}_{21}$ within the two $SU(2)_{20} = (|1\rangle\ ,\ |3\rangle)$, $SU(2)_{21} = (|2\rangle\ ,\ |3\rangle)$ subspaces in the 2^{nd}-shell. Whether or not these two interaction mechanisms driven by a single gluon four-vector boson in the 1^{st}-shell or two gluon four-vector bosons in the 2^{nd}-shell represent some internal dynamical features of the strong interaction can only be determined in a formal investigation of the strong interactions in various models of $SU(N \geq 3)$ gauge field theories.

Here again, we make some observations on the weak interaction sector of the grand unified $SU(5)$ gauge field model. In contrast to the standard interpretation in the original model [14-19], the quantum structure in equation $(15b)$ reveals that the electroweak interaction sector in the $SU(5)$ gauge theory is larger, characterized by dynamics driven by four weak four-vector bosons $\mathcal{A}^{\mu}_{40}$, $\mathcal{A}^{\mu}_{41}$, $\mathcal{A}^{\mu}_{42}$, $\mathcal{A}^{\mu}_{43}$ defined within the respective $SU(2)_{40}$, $SU(2)_{41}$, $SU(2)_{42}$, $SU(2)_{43}$ subspaces in the 4^{th}-shell of the $SU(5)$ generator spectrum. We identify the three weak four-vector bosons $\mathcal{A}^{\mu}_{40}$, $\mathcal{A}^{\mu}_{41}$, $\mathcal{A}^{\mu}_{42}$ in the first three $SU(2)$-subspaces with the 6 Y leptoquark vector bosons in the interpretation given in the original $SU(5)$ gauge field model [14-19], while the single weak four-vector boson $\mathcal{A}^{\mu}_{43}$ in the last $SU(2)$-subspaces precisely with the standard interpretation given in [14-19]. The generators $\sigma^{x}_{43} = \lambda_{22}$, $\sigma^{y}_{43} = \lambda_{23}$ determined here in the Gell-Mann matrix form in equation $(2fii)$ and the generator σ^{z}_{43} which we use the $SU(5)$ basis vectors $|4\rangle$, $|5\rangle$ defined in equation $(2fi)$ to determine here in the matrix form

$$\sigma_{43} = |4\rangle\langle 4| - |5\rangle\langle 5| = \mathrm{diag}(0,0,0,1,-1) \tag{15c}$$

agree precisely with the generators identified with the weak vector bosons in the original model in [14-19]. But the challenge now arises that the generator σ^{z}_{43} in equation $(15c)$ is contained in the definition of the effective traceless diagonal generator $\sigma^{z}_{4} = \lambda_{24}$ obtained here in the 4^{th}-shell in equation $(15b)$ according to the summation

$$\sigma^{z}_{4} = \frac{1}{\sqrt{10}} \sum_{m=0}^{3} \sigma^{z}_{4m} = \frac{1}{\sqrt{10}} \mathrm{diag}(1,1,1,1,-4) \qquad \rightarrow \qquad \sigma^{z}_{4} = \lambda_{24} \tag{15d}$$

where the Gell-Mann matrix λ_{24} is determined in the $SU(5)$ generator spectrum in equation $(2fii)$. Using the component generator σ^{z}_{43} defined only in the $SU(2)_{43}$-subspace, in preference to the effective generator σ^{z}_{4} incorporating all the four diagonal generators σ^{z}_{40} , σ^{z}_{41} , σ^{z}_{42} , σ^{z}_{43} defined in the four subspaces $SU(2)_{40}$, $SU(2)_{41}$, $SU(2)_{42}$, $SU(2)_{43}$, to define the weak vector boson automatically poses a problem in the algebraic and dynamical structure of the $SU(5)$ gauge field model. Even more serious is the fact that the original $SU(5)$ gauge field model ignores completely the correct effective traceless diagonal $SU(5)$ generators $\sigma^{z}_{3} = \lambda_{15}$, $\sigma^{z}_{4} = \lambda_{24}$ determined here in equation $(2fii)$, while using "fine-tuned" traceless diagonal generators $\sigma^z_{43} = \text{diag}(0,0,0,1,-1)$ and $\text{diag}(\frac{1}{\sqrt{15}}(2,2,2,-3,-3)$ to define the weak vector boson and hyperphoton fields, respectively. Without giving further detail, it seems very clear that the formulation of the original grand unified $SU(5)$ gauge field model needs a critical review to account for all the correct 24 traceless generators and incorporate the quantum structure of the generator spectrum in the definition of the $SU(5)$ gauge field potential.

We reemphasize that what we have presented here is neither a substantive review nor a reformulation of any existing gauge field theory, but only a picture of the implications of the emerging quantum structure and expanded algebraic properties of an $SU(N)$ generator spectrum, which now include additional eigenvalue equations and associated spectrum of roots, weights and particle states generated through the additional l^{th}-shell $(l = 1, ..., N-1)$ non-traceless symmetric diagonal generators, the quadratic and Fubini-Veneziano spin angular momentum operators, together with the universal $SU(N)$ quadratic Casimir and Fubinin-Veneziano spin angular momentum operators introduced in the present work. Work on such a substantive review or reformulation of $SU(N)$ gauge field theories based on the general algebraic properties and quantum structure of the generator spectrum of $SU(N)$ symmetry groups is in progress to be presented in a forthcoming article.

6 Conclusion

We have provided a mathematical formula for enumerating and determining $SU(N)$ symmetry group generators based on an algebraic property that the generators occur in symmetric and antisymmetric pairs specified by quantum numbers $l = 1, ..., N-1$; $m = 0, 1, ..., l$ according to equations $(2ai)$, $(2aii)$ in the Gell-Mann basis where the tensor products are evaluated in explicit matrix forms and according to equations $(5ai)$, $(5bi)$, $(5bii)$ in the spin angular momentum basis where the generators are identified as standard spin operators. The generators occur in a spectrum composed of $N-1$ configuration shells specified by shell quantum number $l = 1, ..., N-1$, where each shell contains $2l+1$ traceless symmetric and antisymmetric generators plus 1 non-traceless symmetric generator, occurring in symmetric-antisymmetric pairs specified by the symmetric-antisymmetric pair quantum number $m = 0, 1, ..., l$. We have also developed an equivalent finer quantum structure in which the $SU(N)$ generators are defined in sets satisfying closed $SU(2)$ Lie algebra within $SU(2)$-subspaces distributed in definite numbers among $N-1$ configuration shells in a general $SU(N)$ space.

In the spin angular momentum basis, we have interpreted the $2l+1$ traceless symmetric and antisymmetric generators in the l^{th}-shell as components of a $(2l+1)$-component l^{th}-shell spin angular momentum vector, which we have then used, together with the general algebraic properties of the spin operators, to determine the l^{th}-shell quadratic (even-power) and Fubini-Veneziano (odd-power) spin angular momentum operators of general order in each of the $N-1$ shells $l = 1, ..., N-1$. Taking a weighted sum of the non-traceless symmetric generators and similarly the sums of the quadratic and Fubini-Veneziano spin angular momentum operators from each of the $N-1$ shells, we have determined the respective universal $SU(N)$ identity, quadratic Casimir and Fubini-Veneziano spin angular momentum operators.

We have established that, replacing the $N-1$ non-traceless symmetric diagonal generators with the universal $SU(N)$ identity generator as defined above, we reduce the general $SU(N)$ generator spectrum to a standard $SU(N)$ generator spectrum consisting of the single (1) identity generator and the N^2-1 traceless symmetric and antisymmetric generators. It has emerged that, by including a configuration shell specified by $l = 0$ containing the identity generator, the resulting quantum structure of a standard $SU(N)$ generator spectrum composed of N configuration shells each containing $2l+1$ generators now specified by quantum numbers $l = 0, 1, ..., N-1$; $m = 0, 1, ..., l$ is precisely similar to the spectrum of orbital angular momentum states composed of orbital shells $l = 0, 1, ..., n-1$ each containing $2l + 1$ orbital angular momentum states specified by orbital and magnetic quantum numbers $l = 0, 1, ..., n - 1$; $m = 0, \pm 1, ..., \pm l$ in the n^{th}-energy level of an atom, thus revealing an important physical property that the quantum state space of an $SU(N)$ symmetry group corresponds directly to the quantum state space of the n^{th}-energy level of an atom.

Defining the $SU(N)$ generator spectrum in an extended Cartan-Weyl basis, we have determined general basic algebraic relations applicable to general $SU(N)$ symmetry groups. These basic algebraic relations can be used to determine the entire system of roots, weights, Dynkin diagrams and the general spectrum of particle states which span the quantum state space of an $SU(N)$ symmetry group. Alternatively, the l^{th}-shell quadratic and Fubini-Veneziano spin angular momentum operators, taken together with the corresponding spin state raising and lowering operators, constitute two sets of bases for applying standard angular momentum algebra to determine a complete spectrum of spin angular momentum eigenvectors and eigenvalues within and across all the $N - 1$ configuration shells in an $SU(N)$ generator spectrum.

We have provided elaborate explanations on how the expanded algebraic space and quantum structure of an $SU(N)$ generator spectrum, characterized by an extended Cartan-Weyl basis which now includes the non-traceless symmetric diagonal generators and the quadratic and Fubini-Veneziano spin operators introduced in each of the $N - 1$ configuration shells, may affect the dynamical structure of existing or new models of $SU(N)$ gauge field theories. In particular, the algebraic properties of the additional generators in the extended Cartan-Weyl basis automatically expand the quantum state space, while the quantum structure of the generator spectrum, taken to the finer level of generators defined within $SU(2)$-subspaces, leads to an interpretation that the $SU(N)$ gauge field has a quantum spectrum with $N - 1$ configuration shells each containing specified vector bosons of the full gauge field potential. The quantum structure provides a clear dynamical picture of the gauge field. Application of standard angular momentum algebra to determine the entire spectrum of state eigenvectors and eigenvalues within and across the configuration shells in the quantum state space of an $SU(N)$ gauge field provides the possibility of identifying selection rules governing allowed or forbidden state transitions due to particle interactions driven by the gauge fields.

Details of the expanded algebraic properties, quantum structure and further insight into the dynamical properties of existing or new models of gauge field theories of particle interactions based on $SU(N)$ symmetry groups will be presented in later work.

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