# Photospins in the quantum Rabi model

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**Abstract**

This article provides a systematic interpretation of the quantum Rabi model as a model of *photospins* formed in the atom-field Jaynes-Cummings and anti-Jaynes-Cummings interaction mechanisms. A photospin is a quantized photon-carrying two-state quasiparticle mode specified by two qubit state vectors, state eigenvectors, energy eigenvalues and well defined dynamical and symmetry operators. The algebraic properties of a photospin are exactly the same as the algebraic properties of a two-state atomic spin (spin-$\frac{1}{2}$ particle). The time evolving photospin qubit state vectors describe exact Rabi oscillations between the qubit states, while the corresponding time evolving density operator reveals that the geometric configuration of the photospin state space is a circle of unit radius in the $yz$-plane. The internal dynamics of a Jaynes-Cummings interaction generated photospin (*rotating photospin*) is characterized by *red-sideband transitions* specified by frequency detuning $\delta = \omega_0 - \omega$, while the internal dynamics of an anti-Jaynes-Cummings interaction generated photospin (*antirotating photospin*) is characterized by *blue-sideband transitions* specified by frequency detuning $\overline{\delta} = \omega_0 + \omega$. The simple algebraic properties of a photospin allow formulation of exactly solved models of interacting photospins on Jaynes-Cummings and anti-Jaynes-Cummings optical lattices. The physical property that a photospin state transition operator has eigenvalues $\pm 1$ in the eigenstate basis provides models of interacting photospins equivalent to one-dimensional Curie-Weiss or Ising models of interacting spins on a linear crystal lattice. Time evolving state vectors of two interacting photospins have been determined exactly as entangled nonorthogonal state vectors, which have wide applications in quantum information processing, quantum computation, quantum teleportation and communication, quantum state tomography and related quantum technologies.

## 1 Introduction

The work presented in this paper is an elaboration of an earlier work on polariton and antipolariton qubits in the quantum Rabi model [1], where we now focus attention on the definition, algebraic properties and dynamical evolution of emerging physical entities called *photospins*, which we interpret as *quantized photon-carrying two-state quasiparticle excitation modes* arising as polariton qubits in the Jaynes-Cummings interaction mechanism or antipolariton qubits in the anti-Jaynes-Cummings interaction mechanism within the broader quantum Rabi model. The quantum Rabi model describes the dynamics of a quantized electromagnetic field mode interacting with a two-level atom generated by Hamiltonian of the form [2-6]

$$H_R = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \hbar\omega_0 s_z + \hbar g(\hat{a} + \hat{a}^\dagger)(s_+ + s_-) \tag{1a}$$

where $\omega$ , $\hat{a}$ , $\hat{a}^\dagger$ are the quantized field mode angular frequency, annihilation and creation operators, while $\omega_0$ , $s_z$ , $s_+$ , $s_-$ , $\sigma_x = s_- + s_+$ are the atomic spin state transition angular frequency and operators. We identify the two level atom represented here by spin-$\frac{1}{2}$ state operators simply as atomic spin.

We apply normal and antinormal ordering of the basic field mode and atomic spin operators, expressing ($I$ is the $2\times 2$ identity matrix)

$$\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) = \frac{1}{2}(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger) \quad ; \quad s_z = \frac{1}{2}(s_z + s_z) \quad ; \quad s_+s_- = \frac{1}{2}I + s_z \quad ; \quad s_-s_+ = \frac{1}{2}I - s_z \tag{1b}$$

to decompose the Rabi Hamiltonian in equation $(1a)$ into a symmetrized two-component form [1 , 7]

$$H_R = \frac{1}{2}(\ H + \overline{H}\ ) \tag{1c}$$

where the normal order component $H$ is the *rotating component*, generally known as the Jaynes-Cummings Hamiltonian, obtained in this symmetrization as

$$H = \hbar(\ \omega\hat{a}^\dagger\hat{a} + \omega_0 s_z + 2g(\hat{a}s_+ + \hat{a}^\dagger s_-)\ ) \tag{1d}$$

and the anti-normal order component $\overline{H}$ is the *anti-rotating component*, generally known as the anti-Jaynes-Cummings Hamiltonian, obtained as

$$\overline{H} = \hbar(\ \omega\hat{a}\hat{a}^\dagger + \omega_0 s_z + 2g(\hat{a}s_- + \hat{a}^\dagger s_+)\ ) \tag{1e}$$

The factor 2 doubling the coupling constant as $2g$ in the Jaynes-Cummings and anti-Jaynes-Cummings Hamiltonians $H$ , $\overline{H}$ in equations $(1d)$ , $(1e)$ arises from the symmetrization of the Rabi Hamiltonian $H_R$ consistent with normal and antinormal operator ordering in equation $(1b)$. It becomes clear in the calculations below that this doubling of the coupling constant arising through symmetrization of the quantum Rabi Hamiltonian does not adversely affect the dynamical features of the interaction, since the coupling parameter can be redefined as appropriate. We observe that decomposition of the Rabi Hamiltonian into Jaynes-Cummings and anti-Jaynes-Cummings components has been discussed in various contexts in [3-6].

The Jaynes-Cummings and anti-Jaynes-Cummings interaction mechanisms within the quantum Rabi model generate polariton and antipolariton qubits, redefined as photospins, which are quantized two-state quasiparticle excitation modes specified by two qubit state vectors, state transition operators, excitation number operators and Hamiltonians [1]. In particular, we redefine a polariton or an antipolariton qubit as a photospin by *normalizing the state transition operator*. Squaring the normalized state transition operator provides an identity operator. We therefore interpret a photospin as a quantized photon-carrying two-state quasiparticle specified by two qubit state vectors and *normalized* state transition, identity, excitation number, Hamiltonian and symmetry operators defined within a two-dimensional state space spanned by the qubit state vectors.

Specifically, a polariton qubit is formed in a Jaynes-Cummings interaction, while an antipolariton qubit is formed in an anti-Jaynes-Cummings interaction starting from a basic atom-field initial state

$|+n\rangle$ or $|-n\rangle$ in which the atom starts in a spin-up ($|+\rangle$) or spin-down ($|-\rangle$) state and the field mode starts in a number ($|n\rangle$) state. The atomic spin-up (excited) state vector $|+\rangle$ and spin-down (ground) state vector $|-\rangle$ are defined in the standard form

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad ; \qquad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2a}$$

The field mode and atomic spin state vectors satisfy standard algebraic operations

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \qquad ; \qquad \hat{a}^{\dagger}|n\rangle = \sqrt{n+1}|n+1\rangle \qquad ; \qquad \hat{a}^{\dagger}\hat{a}|n\rangle = n|n\rangle$$

$$s_+|+\rangle = 0 \; ; \quad s_+|-\rangle = |+\rangle \; ; \quad s_-|+\rangle = |-\rangle \; ; \quad s_-|-\rangle = 0 \; ; \quad s_z|\pm\rangle = \pm\frac{1}{2}|\pm\rangle \tag{2b}$$

The basic composite atom-field initial $n$-photon spin-up or spin-down state vectors $|+n\rangle$ , $|-n\rangle$ are denoted in this work by $|\psi_{+n}\rangle$ , $|\psi_{-n}\rangle$, respectively, with a convenient consolidated notation $|\psi_{\pm n}\rangle$, according to the definitions

$$|\psi_{+n}\rangle = |+n\rangle \qquad ; \qquad |\psi_{-n}\rangle = |-n\rangle \qquad ; \qquad |\psi_{\pm n}\rangle = |\pm n\rangle \tag{2c}$$

satisfying composite atom-field algebraic relations expressed in convenient form

$$\hat{a}s_+|\psi_{+n}\rangle = 0 \qquad ; \qquad \hat{a}s_+|\psi_{-n}\rangle = \sqrt{n+\frac{1}{2}-\frac{1}{2}}\;|+n-1\rangle$$

$$\hat{a}^{\dagger}s_-|\psi_{+n}\rangle = \sqrt{n+\frac{1}{2}+\frac{1}{2}}\;|-n+1\rangle \qquad ; \qquad \hat{a}^{\dagger}s_-|\psi_{-n}\rangle = 0 \tag{2d}$$

$$\hat{a}s_-|\psi_{+n}\rangle = \sqrt{n+\frac{1}{2}-\frac{1}{2}}\;|-n-1\rangle \quad ; \quad \hat{a}s_-|\psi_{-n}\rangle = 0$$

$$\hat{a}^{\dagger}s_+|\psi_{+n}\rangle = 0 \; ; \quad \hat{a}^{\dagger}s_+|\psi_{-n}\rangle = \sqrt{n+\frac{1}{2}+\frac{1}{2}}\;|+n+1\rangle \tag{2e}$$

$$s_z|\psi_{+n}\rangle = \frac{1}{2}|\psi_{+n}\rangle \qquad ; \qquad s_z|\psi_{-n}\rangle = -\frac{1}{2}|\psi_{-n}\rangle \qquad ; \qquad s_z|\psi_{\pm n}\rangle = \pm\frac{1}{2}|\psi_{\pm n}\rangle \tag{2f}$$

for describing the dynamics in a consolidated form applicable to the cases where the atom is initially in spin-up state $|+\rangle$ or spin-down state $|-\rangle$.

This work is organized in a self-contained format. Each section is complete in itself and can be read independently. We introduce the polariton qubit, redefined as a *rotating photospin* in section 2, where we define all the dynamical operators and determine the qubit state vectors, state eigenvectors, energy eigenvalues, time evolving state vectors and density operator. A model of interacting rotating photospins in a Jaynes-Cummings optical lattice concludes this section. In section 3, we introduce the antipolariton qubit, redefined as an *antirotating photospin*, define all the dynamical operators and determine the qubit state vectors, state eigenvectors, energy eigenvalues, time evolving state vectors and density operator, then conclude with a model of interacting antirotating photospins in an anti-Jaynes-Cummings optical lattice. Section 4 contains the conclusions.

## 2 Polariton qubits

A polariton qubit is formed in an atom-field Jaynes-Cummings interaction. The polariton qubit Hamiltonian is obtained through a redefinition of the generating Jaynes-Cummings Hamiltonian $H$ in equation $(1d)$ by introducing a polariton excitation number operator $\hat{N}$ obtained as the sum of the quantized field mode and atomic spin excitation number operators in normal order form $\hat{N} = \hat{a}^{\dagger}\hat{a} + s_+s_-$, where $\hat{a}$ , $\hat{a}^{\dagger}$ , $s_-$ , $s_+$ are the basic field mode and atomic spin operators.

Adding and subtracting an atomic spin normal order term $\hbar\omega s_+s_-$ in equation $(1d)$ and reorganizing using the algebraic relation $(\frac{1}{2}I \equiv \frac{1}{2})$

$$s_+s_- = \frac{1}{2} + s_z \tag{3a}$$

we introduce the polariton qubit excitation number operator $\hat{N}$ to redefine the Jaynes-Cummings Hamiltonian $H$ as a polariton qubit Hamiltonian in the form

$$H = \hbar\omega\hat{N} + 2\hbar g(\alpha s_z + \hat{a}s_+ + \hat{a}^{\dagger}s_-\ ) - \frac{1}{2}\hbar\omega \quad ; \quad \hat{N} = \hat{a}^{\dagger}\hat{a} + s_+s_- \quad ; \quad \alpha = \frac{\delta}{2g} \quad ; \quad \delta = \omega_0 - \omega \tag{3b}$$

where $\delta$ is a frequency-detuning parameter arising in the Jaynes-Cummings interaction mechanism. Noting that the interaction component of the Hamiltonian $H$ in equation $(3b)$ generates state transitions according to equation $(2d)$, we introduce a *polariton qubit state transition operator* $\hat{A}$ defined by

$$\hat{A} = \alpha s_z + \hat{a}s_+ + \hat{a}^{\dagger}s_- \tag{3c}$$

which on squaring and applying standard atom-field operator algebraic relations provides the polariton qubit excitation number operator $\hat{N}$ , according to

$$\hat{A}^2 = \hat{N} + \frac{1}{4}\alpha^2 \qquad \Rightarrow \qquad \hat{N} = \hat{A}^2 - \frac{1}{4}\alpha^2 \tag{3d}$$

Substituting the state transition operator $\hat{A}$ from equation $(3c)$ and the excitation number operators $\hat{N}$ from equation $(3d)$ into equation $(3b)$ provides the polariton qubit Hamiltonian $H$ in the appropriate form

$$H = \hbar(\ \omega\hat{A}^2 + 2g\hat{A}\ ) - \frac{1}{4}\hbar\omega\alpha^2 - \frac{1}{2}\hbar\omega \tag{3e}$$

The excitation number operator $\hat{N}$ generates a $U(1)$-symmetry operator $U(\theta)$ obtained together with the hermitian conjugate (noting $\hat{N}^{\dagger} = \hat{N}$) as

$$U(\theta) = e^{-i\theta\hat{N}} \qquad ; \qquad U^{\dagger}(\theta) = e^{i\theta\hat{N}} \tag{3f}$$

Setting $\theta = n\pi$, $n = 0, 1, 2, 3, ...$ in equation $(3f)$ provides the polariton qubit $Z_2$-symmetry operator $U_n(\pi)$ and parity-symmetry operator $\hat{\Pi}$ in the form

$$U_n(\pi) = e^{-in\pi\hat{N}} \ = \ (\ \hat{\Pi}\ )^n \quad ; \quad n = 0, 1, 2, 3, ... \qquad ; \qquad \hat{\Pi} = e^{-i\pi\hat{N}} \tag{3g}$$

Since the polariton qubit Hamiltonian $H$ has been obtained in terms of the qubit state transition operator in equation $(3e)$, it is easy to establish that the state transition operator $\hat{A}$ , the excitation number operator $\hat{N}$ in equation $(3d)$ and all the three symmetry operators $U(\theta)$ , $U_n(\pi)$ , $\hat{\Pi}$ in equations $(3f)$ , $(3g)$ commute with the Hamiltonian $H$ and are therefore conserved in the dynamics of the polariton qubit, as proved earlier in [1 , 7].

To determine the state vectors, we consider the polariton qubit to be formed in a Jaynes-Cummings interaction starting with the field mode in a number state $|n\rangle$ and the atom in either spin-up state $|+\rangle$ or spin-down state $|-\rangle$, forming the composite atom-field initial $n$-photon spin-up state $|\psi_{+n}\rangle = |+n\rangle$ or spin-down state $|\psi_{-n}\rangle = |-n\rangle$, collectively denoted by $|\psi_{\pm n}\rangle = |\pm n\rangle$ as defined in equation $(2c)$ for a consolidated description of polariton qubit dynamics starting with the atom in either spin-up (excited) or spin-down (ground) state to avoid the separate repetitive and lengthy presentation in [1].

We determine the polariton qubit state vectors by applying the state transition operator $\hat{A} = \alpha s_z + \hat{a}s_+ + \hat{a}^\dagger s_-$ from equation $(3c)$ to the composite atom-field initial $n$-photon spin-up and spin-down state vectors $|\psi_{+n}\rangle = |+n\rangle$ , $|\psi_{-n}\rangle = |-n\rangle$ and using the algebraic operations in equations $(2d)$ and $(2f)$ to obtain

$$\hat{A}|\psi_{+n}\rangle = \frac{1}{2}\alpha|\psi_{+n}\rangle + \sqrt{n+\frac{1}{2}+\frac{1}{2}}\ |-n+1\rangle \ ; \quad \hat{A}|\psi_{-n}\rangle = -\frac{1}{2}\alpha|\psi_{-n}\rangle + \sqrt{n+\frac{1}{2}-\frac{1}{2}}\ |+n-1\rangle \quad (4a)$$

which we express in a convenient consolidated form for an interaction beginning with the atom initially in spin-up state $|+\rangle$ or spin-down state $|-\rangle$ as

$$\hat{A}|\psi_{\pm n}\rangle = \pm\frac{1}{2}\alpha|\psi_{\pm n}\rangle + \sqrt{n+\frac{1}{2}\pm\frac{1}{2}}\ |\mp n \pm 1\rangle \qquad (4b)$$

Reorganizing the r.h.s of equation $(4b)$, we obtain the polariton qubit state vectors $|\psi_{\pm n}\rangle$ , $|\phi_{\pm n}\rangle$ defined by

$$|\psi_{\pm n}\rangle = |\pm n\rangle \quad ; \quad |\phi_{\pm n}\rangle = \pm c_{\pm n}|\pm n\rangle + s_{\pm n}|\mp n \pm 1\rangle \quad ; \quad A_{\pm n} = \sqrt{(n+\frac{1}{2}\pm\frac{1}{2}) + \frac{1}{4}\alpha^2}$$

$$c_{\pm n} = \frac{\delta}{2R_{\pm n}} \quad ; \quad s_{\pm n} = \frac{2g\sqrt{n+\frac{1}{2}\pm\frac{1}{2}}}{R_{\pm n}} \quad ; \quad R_{\pm n} = 2gA_{\pm n} \qquad (4c)$$

which satisfy qubit state transition algebraic operations

$$\hat{A}|\psi_{\pm n}\rangle = A_{\pm n}|\phi_{\pm n}\rangle \ ; \quad \hat{A}|\phi_{\pm n}\rangle = A_{\pm n}|\psi_{\pm n}\rangle \ \Rightarrow \ \hat{A}^2|\psi_{\pm n}\rangle = A^2_{\pm n}|\psi_{\pm n}\rangle \ ; \quad \hat{A}^2|\phi_{\pm n}\rangle = A^2_{\pm n}|\phi_{\pm n}\rangle \quad (4d)$$

The qubit state vectors $|\psi_{\pm n}\rangle$ , $|\phi_{\pm n}\rangle$ are nonorthonormal (normalized, but nonorthogonal), satisfying nonorthonormality relations

$$\langle\psi_{\pm n}|\psi_{\pm n}\rangle = 1 \quad ; \quad \langle\phi_{\pm n}|\phi_{\pm n}\rangle = 1 \quad ; \quad \langle\psi_{\pm n}|\phi_{\pm n}\rangle = \pm c_{\pm n} \quad ; \quad \langle\phi_{\pm n}|\psi_{\pm n}\rangle = \pm c_{\pm n} \qquad (4e)$$

We obtain the polariton state eigenvectors and energy eigenvalues as simple superpositions of the qubit state vectors $|\psi_{\pm n}\rangle$ , $|\phi_{\pm n}\rangle$ in the form

$$|\Psi^+_{\pm n}\rangle = \frac{1}{\sqrt{2(1\pm c_{\pm n})}}(\ |\psi_{\pm n}\rangle + |\phi_{\pm n}\rangle\ ) \quad ; \quad |\Psi^-_{\pm n}\rangle = \frac{1}{\sqrt{2(1\mp c_{\pm n})}}(\ |\psi_{\pm n}\rangle - |\phi_{\pm n}\rangle\ ) \qquad (4f)$$

which satisfy eigenvalue equations generated by the state transition operator $\hat{A}$ and Hamiltonian $H$ from equations $(3c)$ , $(3e)$ using the qubit state transition algebraic operations in equation $(4d)$ in the form

$$\hat{A}\ |\Psi^\pm_{\pm n}\rangle = \pm A_{\pm n}\ |\Psi^\pm_{\pm n}\rangle \quad ; \quad H\ |\Psi^\pm_{\pm n}\rangle = E^\pm_{\pm n}\ |\Psi^\pm_{\pm n}\rangle \quad ; \quad E^\pm_{\pm n} = \hbar\omega(n\pm\frac{1}{2}) \pm \hbar R_{\pm n} \qquad (4g)$$

which agree exactly with state eigenvectors and energy eigenvalues determined through diagonalization of the Jaynes-Cummings Hamiltonian in standard quantum optics literature [8-15].

The state eigenvectors $|\Psi^{+}_{\pm n}\rangle$ , $|\Psi^{-}_{\pm n}\rangle$ obtained in equation $(4f)$ are orthonormal, satisfying orthonormality relations

$$\langle\Psi^{+}_{\pm n}|\Psi^{+}_{\pm n}\rangle = 1 \quad ; \quad \langle\Psi^{-}_{\pm n}|\Psi^{-}_{\pm n}\rangle = 1 \quad ; \quad \langle\Psi^{+}_{\pm n}|\Psi^{-}_{\pm n}\rangle = 0 \quad ; \quad \langle\Psi^{-}_{\pm n}|\Psi^{+}_{\pm n}\rangle = 0 \qquad (4h)$$

## 2.1 Rotating photospins

We now introduce a *photospin* as a polariton qubit specified by a *normalized* qubit state transition operator and the qubit state vectors. Squaring the normalized state transition operator provides a corresponding *qubit state identity operator*. All the dynamical and symmetry operators of the photospin are defined in terms of its normalized qubit state transition and identity operators. It emerges that the algebraic properties of a photospin within the two-dimensional state space spanned by its two qubit state vectors are similar to the algebraic properties of a two-state atomic spin in the two-dimensional state space spanned by its spin-up and spin-down qubit state vectors. The identification *photospin* originates from this algebraic property. We interpret a photospin as a quantized photon-carrying two-state quasiparticle with algebraic and dynamical properties precisely similar to the algebraic and dynamical properties of a two-state atomic spin. We characterize the photospin arising as a polariton qubit formed in a Jaynes-Cummings interaction where the atomic spin couples to the *rotating positive frequency component* of the field mode from the initial $n$-photon spin-up or spin-down state $|\psi_{\pm n}\rangle = |\pm n\rangle$ as a *rotating photospin*.

The rotating photospin is specified by the qubit state vectors $|\psi_{\pm n}\rangle$ , $|\phi_{\pm n}\rangle$ obtained in equation $(4c)$. The photospin state transition operator $\hat{\mathcal{E}}_{\pm}$ and corresponding state identity operator $\hat{\mathcal{I}}_{\pm}$ are obtained through normalization of the polariton qubit state transition operator $\hat{A}$ in equation $(3c)$ based on the qubit state transition algebraic operations in equation $(4d)$ in the form

$$\hat{\mathcal{E}}_{\pm} = \frac{\hat{A}}{A_{\pm n}} \qquad ; \qquad \hat{\mathcal{I}}_{\pm} = \frac{\hat{A}^2}{A^2_{\pm n}} \qquad \Rightarrow \qquad \hat{\mathcal{E}}^2_{\pm} = \hat{\mathcal{I}}_{\pm} \qquad (5a)$$

which generate state transition algebraic operations on the photospin qubit state vectors $|\psi_{\pm n}\rangle$ , $|\phi_{\pm n}\rangle$ derived easily from the corresponding polariton qubit state transition algebraic operations in equation $(4d)$ in the form

$$\hat{\mathcal{E}}_{\pm}\ |\psi_{\pm n}\rangle = |\phi_{\pm n}\rangle \ ; \quad \hat{\mathcal{E}}_{\pm}\ |\phi_{\pm n}\rangle = |\psi_{\pm n}\rangle \qquad ; \qquad \hat{\mathcal{I}}_{\pm}\ |\psi_{\pm n}\rangle = |\psi_{\pm n}\rangle \ ; \quad \hat{\mathcal{I}}_{\pm}\ |\phi_{\pm n}\rangle = |\phi_{\pm n}\rangle \qquad (5b)$$

General algebraic properties of the rotating photospin qubit state transition operator $\hat{\mathcal{E}}_{\pm}$ are easily determined using equation $(5a)$ in the form $(k = 0, 1, 2, ...)$

$$\hat{\mathcal{E}}^2_{\pm} = \hat{\mathcal{I}}_{\pm} \quad ; \quad \hat{\mathcal{E}}^{2k}_{\pm} = \hat{\mathcal{I}}_{\pm} \quad ; \quad \hat{\mathcal{E}}^{2k+1}_{\pm} = \hat{\mathcal{E}}_{\pm} \quad ; \quad e^{\pm\theta\hat{\mathcal{I}}_{\pm}} = e^{\pm\theta}\hat{\mathcal{I}}_{\pm} \quad ; \quad e^{\pm i\theta\hat{\mathcal{I}}_{\pm}} = e^{\pm i\theta}\hat{\mathcal{I}}_{\pm}$$

$$e^{\pm\theta\hat{\mathcal{E}}_{\pm}} = \cosh\theta\ \hat{\mathcal{I}}_{\pm} \pm \sinh\theta\ \hat{\mathcal{E}}_{\pm} \qquad ; \qquad e^{\pm i\theta\hat{\mathcal{E}}_{\pm}} = \cos\theta\ \hat{\mathcal{I}}_{\pm} \pm i\sin\theta\ \hat{\mathcal{E}}_{\pm} \qquad (5c)$$

where we have applied exponential expansion with separated even and odd power terms, which are expressed as hyperbolic or trigonometric functions as appropriate.

Substituting $\hat{A} = A_{\pm n}\hat{\mathcal{E}}_{\pm}$ , $\hat{A}^2 = A^2_{\pm n}\hat{\mathcal{I}}_{\pm}$ from equation $(5a)$ into equations $(3d)$ , $(3e)$ , $(3f)$ and $(3g)$, using $R_{\pm n} = 2gA_{\pm n}$ and the exponentiation of the identity operator $\hat{\mathcal{I}}_{\pm}$ given in equation $(5c)$ as appropriate, we easily determine the rotating photospin excitation number operator $\hat{N}_\pm$ , Hamiltonian $H_\pm$ , $U(1)$-symmetry operator $U_\pm(\theta)$ and parity-symmetry operator $\hat{\Pi}_\pm$ in the form $(n = 0, 1, 2, ...)$

$$\hat{N}_\pm = \left(n + \frac{1}{2} \pm \frac{1}{2}\right) \hat{\mathcal{I}}_\pm \quad ; \quad H_\pm = \hbar\omega \left(n \pm \frac{1}{2}\right) \hat{\mathcal{I}}_\pm + \hbar R_{\pm n} \hat{\mathcal{E}}_\pm \tag{5d}$$

$$U_\pm(\theta) = e^{\pm i\theta\left(n+\frac{1}{2}\pm\frac{1}{2}\right)} \hat{\mathcal{I}}_\pm \ ; \quad \hat{\Pi}_\pm = e^{\pm i\pi\left(n+\frac{1}{2}\pm\frac{1}{2}\right)} \hat{\mathcal{I}}_\pm = \pm\hat{\mathcal{I}}_\pm \ \Rightarrow \ \hat{\Pi}_+ = \pm\hat{\mathcal{I}}_+ \ ; \quad \hat{\Pi}_- = \pm\hat{\mathcal{I}}_- \tag{5e}$$

where we have specified how the parity-symmetry operator $\hat{\Pi}_\pm$ applies.

It follows from the relation in equation $(3g)$ that the photospin $Z_2$-symmetry operator $U_{n\pm}(\pi)$ just equals the parity-symmetry operator $\hat{\Pi}_\pm$ obtained in equation $(5e)$. In addition, equation $(5e)$ reveals that the parity-symmetry operator obtained as $\hat{\Pi}_\pm = \pm\hat{\mathcal{I}}_\pm$ is proportional to the photospin state identity operator, such that we can introduce a *parity-symmetry number* $p$ taking even or odd integer values $2k$ , $2k+1$ , $k = 0, 1, 2, ...$ to express the photospin parity-symmetry operator obtained in equation $(5e)$ in the form

$$\hat{\Pi}_\pm = (-1)^p \hat{\mathcal{I}}_\pm \ \Rightarrow \ \hat{\mathcal{I}}_\pm = (-1)^p \hat{\Pi}_\pm \ ; \quad p = \begin{cases} 2k \ , \ k = 0, 1, 2, ... \ ; \ (-1)^p = +1 : \ \text{even parity} \\ 2k+1 \ , \ k = 0, 1, 2, ...; \ (-1)^p = -1 : \text{odd parity} \end{cases} \tag{5f}$$

which we can use to introduce the parity-symmetry operator $\hat{\Pi}_\pm$ and number $p$ into the specification of the rotating photospin excitation number operator $\hat{N}_\pm$ and Hamiltonian $H_\pm$ in equation $(5d)$ according to

$$\hat{\mathcal{I}}_\pm = (-1)^p \hat{\Pi}_\pm \quad \Rightarrow \quad \hat{N}_\pm = \left(p + \frac{1}{2} \pm \frac{1}{2}\right) (-1)^p \hat{\Pi}_\pm$$

$$H_\pm = \hbar\omega \left(p \pm \frac{1}{2}\right) (-1)^p \hat{\Pi}_\pm + \hbar R_{\pm n} \hat{\mathcal{E}}_\pm \quad ; \quad p = \begin{cases} 2n \ , \ n = 0, 1, 2, ... : \ \text{even parity states} \\ 2n+1 \ , \ n = 0, 1, 2, ... : \ \text{odd parity states} \end{cases} \tag{5g}$$

The state eigenvectors $|\Psi^\pm_{\pm n}\rangle$ are determined as superpositions of the qubit state vectors $|\psi_{\pm n}\rangle$ , $|\phi_{\pm n}\rangle$ in equation $(4f)$. Applying the photospin state transition operator $\hat{\mathcal{E}}_\pm$ and Hamiltonian $H_\pm$ from equation $(5d)$ on the state eigenvectors $|\Psi^\pm_{\pm n}\rangle$ in equation $(4f)$ and using qubit state transition algebraic operations in equation $(5b)$, we obtain the eigenvalue equations in the form

$$\hat{\mathcal{E}}_+ \ |\Psi^\pm_{+n}\rangle = \pm|\Psi^\pm_{+n}\rangle \quad ; \quad \hat{\mathcal{E}}_- \ |\Psi^\pm_{-n}\rangle = \pm|\Psi^\pm_{-n}\rangle \quad ; \quad \hat{\mathcal{I}}_+ \ |\Psi^\pm_{+n}\rangle = |\Psi^\pm_{+n}\rangle \quad ; \quad \hat{\mathcal{I}}_- \ |\Psi^\pm_{-n}\rangle = |\Psi^\pm_{-n}\rangle$$

$$H_\pm \ |\Psi^\pm_{\pm n}\rangle = E^\pm_{\pm n} \ |\Psi^\pm_{\pm n}\rangle \quad ; \quad E^\pm_{\pm n} = \hbar\omega(n \pm \frac{1}{2}) \pm \hbar R_{\pm n} \tag{5h}$$

which, apart from providing the expected energy eigenvalues $E^\pm_{\pm n}$ of the Hamiltonian $H_\pm$ as obtained earlier in equation $(4g)$, also provide the important algebraic property that the photospin state transition operator $\hat{\mathcal{E}}_\pm$ has eigenvalues $\pm 1$.

The photospin Hamiltonian $H_\pm$ in equation $(5d)$ generates time evolution operator $U_\pm(t)$ obtained in the form (noting $\hat{\mathcal{E}}_\pm \hat{\mathcal{I}}_\pm = \hat{\mathcal{I}}_\pm \hat{\mathcal{E}}_\pm = \hat{\mathcal{E}}_\pm$ )

$$U_\pm(t) = e^{-\frac{i}{\hbar} H_\pm t} \quad \Rightarrow \quad U_\pm(t) = e^{-i\omega t\left(n\pm\frac{1}{2}\right) \ \hat{\mathcal{I}}_\pm} e^{-iR_{\pm n} t \hat{\mathcal{E}}_\pm} \tag{6a}$$

which we apply the algebraic properties in equation $(5c)$ to evaluate in explicit form

$$U_\pm(t) = e^{-i\omega t\left(n\pm\frac{1}{2}\right)} \left(\cos(R_{\pm n} t) \ \hat{\mathcal{I}}_\pm - i \sin(R_{\pm n} t) \ \hat{\mathcal{E}}_\pm\right) \tag{6b}$$

The time evolving rotating photospin state vector $|\Psi_{\pm n}(t)\rangle$ is generated from the initial $n$-photon spin-up or spin-down state vector $|\psi_{\pm n}\rangle$ by applying the time evolution operator $U_\pm(t)$ from equation

(6$b$) and using the qubit state transition algebraic operations in equation (5$b$) to obtain in the explicit form

$$|\Psi_{\pm n}(t)\rangle = U_{\pm}(t)|\psi_{\pm n}\rangle \ ; \quad |\Psi_{\pm n}(t)\rangle = e^{-i\omega(n\pm\frac{1}{2})t}\left(\ \cos(R_{\pm n}t)|\psi_{\pm n}\rangle - i\sin(R_{\pm n}t)|\phi_{\pm n}\rangle\ \right) \tag{6c}$$

Expressing the imaginary number $-i$ in polar form and factoring symmetrically according to

$$-i = e^{-\frac{i}{2}\pi} \ = e^{-\frac{i}{4}\pi}e^{-\frac{i}{4}\pi} \tag{6d}$$

we rewrite the time evolving state vector $|\Psi_{\pm n}(t)\rangle$ in equation (6$c$) in the standard form

$$|\Psi_{\pm n}(t)\rangle = e^{-i(\omega(n\pm\frac{1}{2})t+\frac{1}{4}\pi)}\left(\ \cos(R_{\pm n}t)e^{\frac{i}{4}\pi}\ |\psi_{\pm n}\rangle + \sin(R_{\pm n}t)e^{-\frac{i}{4}\pi}\ |\phi_{\pm n}\rangle\ \right) \tag{6e}$$

where we now identify the coefficient $\cos(R_{\pm n}t)e^{\frac{i}{4}\pi}$ as the probability amplitude to be in the qubit state $|\psi_{\pm n}\rangle$ and the coefficient $\sin(R_{\pm n}t)e^{-\frac{i}{4}\pi}$ as the probability amplitude to be in the qubit state $|\phi_{\pm n}\rangle$. The corresponding probabilities $P^{\psi}_{\pm n}(t)$ , $P^{\phi}_{\pm n}(t)$ to be in the respective qubit states $|\psi_{\pm n}\rangle$ , $|\phi_{\pm n}\rangle$ are obtained as

$$P^{\psi}_{\pm n}(t) = |\cos(R_{\pm n}t)e^{\frac{i}{4}\pi}\ |^2 \ = \cos^2(R_{\pm n}t) \qquad ; \qquad P^{\phi}_{\pm n}(t) = |\sin(R_{\pm n}t)e^{-\frac{i}{4}\pi}\ |^2 \ = \sin^2(R_{\pm n}t) \tag{6f}$$

which satisfy the standard probability relation $P^{\psi}_{\pm n}(t) + P^{\phi}_{\pm n}(t) = 1$.

To complete the specification of the dynamical evolution of the rotating photospin, we apply the time evolution operator $U_{\pm}(t)$ from equation (6$b$) on the entangled qubit state vector $|\phi_{\pm n}\rangle$ and use the qubit state transition algebraic operations from equation (5$b$) to determined the time evolving qubit state vector $|\Phi_{\pm n}(t)\rangle$ in the form

$$|\Phi_{\pm n}(t)\rangle = U_{\pm}(t)|\phi_{\pm n}\rangle \ ; \quad |\Phi_{\pm n}(t)\rangle = e^{-i\omega(n\pm\frac{1}{2})t}\left(\ \cos(R_{\pm n}t)|\phi_{\pm n}\rangle - i\sin(R_{\pm n}t)|\psi_{\pm n}\rangle\ \right) \tag{6g}$$

Applying the photospin state transition operator $\hat{\mathcal{E}}_{\pm}$ on the two time evolving qubit state vectors $|\Psi_{\pm n}(t)$ , $|\Phi_{\pm n}(t)\rangle$ in equations (6$c$) , (6$g$) and using the state transition algebraic operations from equation (5$b$) reveals that the time evolving rotating photospin state vectors satisfy qubit state transition algebraic operations

$$\hat{\mathcal{E}}_{\pm}|\Psi_{\pm n}(t)\rangle = |\Phi_{\pm n}(t)\rangle \qquad ; \qquad \hat{\mathcal{E}}_{\pm}|\Phi_{\pm n}(t)\rangle = |\Psi_{\pm n}(t)\rangle \tag{6h}$$

It follows from the form of the time evolving qubit state vector $|\Psi_{\pm n}(t)\rangle$ in equation (6$e$) that the coupled qubit state vectors $|\psi_{\pm n}\rangle$ , $|\phi_{\pm n}\rangle$ undergo reversible transitions into each other in Rabi oscillations at frequency $R_{\pm n}$. The time evolving qubit state vector $|\Psi_{\pm n}(t)\rangle$ periodically evolves into the qubit state $|\psi_{\pm n}\rangle$ or $|\phi_{\pm n}\rangle$ at respective times $\tau_k = \frac{k}{R_{\pm n}}\pi$ or $\tau_k = \frac{2k+1}{2R_{\pm n}}\pi$ , $k = 0, 1, 2, ....$ The Rabi oscillations between the photospin qubit states determined here agree precisely with the elaborate description of state transitions based on a Bloch sphere description in [15] where, in non-resonant ($\delta \neq 0$) atom-field interaction, effective Rabi state transition oscillations occur only between the bare atom-field state $|\pm n\rangle$ (coinciding with $|\psi_{\pm n}\rangle = |\pm n\rangle$ in the present work) and a superposition of the bare states $|\pm n\rangle$ and $|\mp n \pm 1\rangle$ (coinciding with $|\phi_{\pm n}\rangle = \pm c_{\pm n}|\pm n\rangle + s_{\pm n}|\mp n \pm 1\rangle$ in the present work), with the time evolution similarly following a circular path of unit radius in the $yz$-plane, which we establish below in the density operator representation of the rotating photospin states.

It is important to note that, due to the nonorthogonality of the qubit state vectors $|\psi_{\pm n}\rangle$ , $|\phi_{\pm n}\rangle$ determined by frequency detuning $\delta = \omega_0 - \omega$ according to equations (4$c$) , (4$e$), the periodically time evolving qubit state probabilities $P^{\psi}_{\pm n}(t)$ , $P^{\phi}_{\pm n}(t)$ in equation (6$h$) are generally different from the corresponding *state transition* probabilities $P^{\psi\phi}_{\pm n}(t)$ , $P^{\phi\psi}_{\pm n}(t)$ obtained according to standard definition in the explicit forms (using equation $(4e)$ )

$$P^{\psi\phi}_{\pm n}(t) = |\langle\phi_{\pm n}|\Psi_{\pm n}(t)\rangle|^2 \;= c^2_{\pm n}\cos^2(R_{\pm n}t) + \sin^2(R_{\pm n}t)$$

$$P^{\phi\psi}_{\pm n}(t) = |\langle\psi_{\pm n}|\Psi_{\pm n}(t)\rangle|^2 \;= \cos^2(R_{\pm n}t) + c^2_{\pm n}\sin^2(R_{\pm n}t) \qquad (6i)$$

which we use equation $(6g)$ to express in terms of the qubit state probabilities $P^{\psi}_{\pm n}(t)$ , $P^{\phi}_{\pm n}(t)$ in the form

$$P^{\psi\phi}_{\pm n}(t) = c^2_{\pm n}P^{\psi}_{\pm n}(t) + P^{\phi}_{\pm n}(t) \qquad ; \qquad P^{\phi\psi}_{\pm n}(t) = P^{\psi}_{\pm n}(t) + c^2_{\pm n}P^{\phi}_{\pm n}(t) \qquad (6j)$$

An important property to emphasize here is that, away from resonance $\delta \neq 0$, the state transition probabilities determined in equation $(6i)$ , $(6j)$ do not satisfy the standard probability normalization relation, since their sum is greater than the expected unit value, except at resonance where $\delta = 0$ , $\bar{c}_{\pm n} = 0$, according to

$$P^{\psi\phi}_{\pm n}(t) + P^{\phi\psi}_{\pm n}(t) = 1 + c^2_{\pm n} \qquad \Rightarrow \qquad P^{\psi\phi}_{\pm n}(t) + P^{\phi\psi}_{\pm n}(t) \geq 1 \qquad (6k)$$

meaning that the transition probabilities $P^{\psi\phi}_{\pm n}(t)$ , $P^{\phi\psi}_{\pm n}(t)$ determined according to standard definition as squares of absolute values of state transition probability amplitudes in equation $(6i)$ do not satisfy the definition of state probabilities for the coupled nonorthogonal qubit state vectors $|\psi_{\pm n}\rangle$ , $|\phi_{\pm n}\rangle$, except under the resonance condition $\omega_0 = \omega$ , $\delta = 0$ , $c_{\pm n} = 0$, when they reduce to the corresponding state probabilities $P^{\psi}_{\pm n}(t)$ , $P^{\phi}_{\pm n}(t)$ according to equation $(6j)$.

We gain detailed insight into the internal dynamics of the rotating photospin by noting that the Jaynes-Cummings interaction mechanism generates the photospin through the coupling of the atomic spin to the rotating positive frequency component of the quantized electromagnetic field mode. The transitions between the qubit states in a rotating photospin are therefore driven by positive energy photon emission-absorption processes. To understand this clearly, we substitute the definitions of the qubit state vectors $|\psi_{\pm n}\rangle$ , $|\phi_{\pm n}\rangle$ from equation $(4c)$ into equation $(6e)$, reorganize and introduce a polar form

$$(\cos(R_{\pm n}t) \mp ic_{\pm n}\sin(R_{\pm n}t)\ ) = \sqrt{P^{in}_{\pm n}(t)}e^{\mp i\vartheta_{\pm}(t)} \ ; \ \tan\vartheta_{\pm}(t) = c_{\pm n}\tan(R_{\pm n}t) \ ; \ \sin(R_{\pm n}t) = \sqrt{P^{ea}_{\pm n}(t)}$$

$$P^{in}_{\pm n}(t) = \cos^2(R_{\pm n}t) + c^2_{\pm n}\sin^2(R_{\pm n}t) \qquad ; \qquad P^{ea}_{\pm n}(t) = \sin^2(R_{\pm n}t) \qquad (6l)$$

to express the time evolving qubit state vector in the more transparent bare atom-field state basis $\{|\pm n\rangle\ ,\ |\mp n \pm 1\rangle\}$ in the form

$$|\Psi_{\pm n}(t)\rangle = e^{-i(\omega(n\pm\frac{1}{2})t\pm\frac{1}{2}\varphi_{\pm}(t)\ )}\left(\sqrt{P^{in}_{\pm n}(t)}\ e^{\mp\frac{i}{2}\varphi_{\pm}(t)}\ |\pm n\rangle + \sqrt{P^{ea}_{\pm n}(t)}\ e^{\pm\frac{i}{2}\varphi_{\pm}(t)}\ |\mp n \pm 1\rangle\right)$$

$$\varphi_{\pm n}(t) = \vartheta_{\pm n}(t) \mp \frac{1}{2}\pi \qquad (6m)$$

where $P^{in}_{\pm n}(t)$ is the probability to be in the initial state $|\pm n\rangle$ and $P^{ea}_{\pm n}(t)$ is the probability to be in the photon emission-absorption state $|\mp n \pm 1\rangle$. The form of the time evolving qubit state vector $|\Psi_{\pm n}(t)\rangle$ in the bare atom-field state basis in equation $(6m)$ now reveals that, in a process starting from the $n$-photon spin-up state $|+n\rangle$ where the atom begins in an excited state $|+\rangle$, the excited atom emits a positive energy photon, triggering the rotating positive frequency field mode to absorb a positive energy photon, causing a transition $|+n\rangle \ \rightarrow \ |-n+1\rangle$, while in a process starting from the $n$-photon spin-down state $|-n\rangle$ where the atom begins in a ground state $|-\rangle$, the rotating positive frequency field mode emits a positive energy photon, triggering the atom to absorb a positive energy photon, causing a transition $|-n\rangle \rightarrow |+n-1\rangle$, thus accounting for the dynamical evolution which couples the bare atom-field states $|\pm n\rangle$ and $|\mp n \pm 1\rangle$ as described by the time evolving qubit state vector $|\Psi_{\pm n}(t)\rangle$ in equation $(6m)$. These state transitions driven by emission-absorption of only positive energy photons in a rotating photospin (or a polariton qubit) generated in a Jaynes-Cummings interaction mechanism are identified as *red-sideband transitions* specified by frequency detuning $\delta = \omega_0 - \omega$ [16]. The mathematical property that the bare atom-field state probabilities $P^{in}_{\pm n}(t)$ , $P^{ea}_{\pm n}(t)$ as determined in equation $(6l)$ cannot simultaneously take alternate values 0 and 1 means that, starting from the initial state $|\pm n\rangle$ the state vector $|\Psi_{\pm n}(t)\rangle$ in equation $(6m)$ cannot evolve to the photon emission-absorption state $|\mp n \pm 1\rangle$ and vice-versa. Instead, it has been well established in [15] that exact Rabi oscillations of the time evolving qubit state vector $|\Psi_{\pm n}(t)\rangle$ occur only between the initial state $|\pm n\rangle$ and a superposition state $\beta_{in}|\pm n\rangle + \beta_{ea}|\mp n \pm 1\rangle$, which then agrees precisely with the exact Rabi oscillations between the rotating photospin qubit states $|\psi_{\pm n}\rangle$ , $|\phi_{\pm n}\rangle$ described by the time evolving qubit state vector determined directly in equation $(6c)$ , $(6e)$. This reaffirms the interpretation that the qubit state vectors $|\psi_{\pm n}\rangle = |\pm n\rangle$ , $|\phi_{\pm n}\rangle = \pm c_{\pm n}|\pm n\rangle + s_{\pm n}|\mp n \pm 1\rangle$ as defined in equation $(4c)$ are the natural state vectors of a rotating photospin. Hence, the general dynamical evolution of the rotating photospin is described by the time evolving qubit state vector $|\Psi_{\pm n}(t)\rangle$ as determined in equation $(6c)$ , $(6e)$.

We observe that the specification of the rotating photospin qubit state vectors and transition operator which we have provided in the present work completely solves the problem of determining polariton qubit states in the theoretical and experimental studies in [ 17 , 18 ], where a qubit state transition operator has not been defined. Note that polaritons as defined in [ 17 , 18 ] are the rotating photospins as defined in the present work.

Finally, we note that the time evolving photospin state eigenvectors $|\Psi^{\pm}_{\pm n}(t)\rangle$ are determined by applying the time evolution operator as defined in the form $U_{\pm}(t) = e^{-\frac{i}{\hbar}H_{\pm}t}$ in equation $(6a)$ on the state eigenvectors $|\Psi^{\pm}_{\pm n}\rangle$ in equation $(4f)$ and using the eigenvalue equations $(5h)$ giving the explicit form

$$|\Psi^{\pm}_{\pm n}(t)\rangle = U_{\pm}(t)|\Psi^{\pm}_{\pm n}\rangle \; = \; e^{-\frac{i}{\hbar}E^{\pm}_{\pm n}t}|\Psi^{\pm}_{\pm n}\rangle \tag{6n}$$

which can be used where necessary.

For a comprehensive study of the distribution of states and general statistical properties of the rotating photospin within a geometrical frame specified by the coupled qubit state vectors, we introduce the general time evolving density operator $\hat{\rho}_{\pm n}(t)$ defined by

$$\hat{\rho}_{\pm n}(t) = |\Psi_{\pm n}(t)\rangle\langle\Psi_{\pm n}(t)| \tag{7a}$$

which on substituting the time evolving qubit state vector $|\Psi_{\pm n}(t)\rangle$ from equation $(6c)$ (or $(6e)$) and expanding takes the explicit form

$$\hat{\rho}_{\pm n}(t) = \rho^{11}_{\pm n}(t)|\psi_{\pm n}\rangle\langle\psi_{\pm n}| + \rho^{12}_{\pm n}(t)|\psi_{\pm n}\rangle\langle\phi_{\pm n}| + \rho^{21}_{\pm n}(t)|\phi_{\pm n}\rangle\langle\psi_{\pm n}| + \rho^{22}_{\pm n}(t)|\phi_{\pm n}\rangle\langle\phi_{\pm n}|$$

$$\rho^{11}_{\pm n}(t) = \cos^2(R_{\pm n}t) \quad ; \quad \rho^{12}_{\pm n}(t) = \frac{i}{2}\sin(2R_{\pm n}t) \quad ; \quad \rho^{21}_{\pm n}(t) = -\frac{i}{2}\sin(2R_{\pm n}t)$$

$$\rho^{22}_{\pm n}(t) = \sin^2(R_{\pm n}t) \tag{7b}$$

We interpret the density operator coefficients $\rho^{ij}_{\pm n}(t)$ , $i, j = 1, 2$ as elements of a $2 \times 2$ density matrix $\rho_{\pm n}(t)$, which we express in terms of the standard $2 \times 2$ Pauli spin matrices $I$ , $\sigma_x$ , $\sigma_y$ , $\sigma_z$ in the form

$$\rho_{\pm n}(t) = \begin{pmatrix} \rho^{11}_{\pm n}(t) & \rho^{12}_{\pm n}(t) \\ \rho^{21}_{\pm n}(t) & \rho^{22}_{\pm n}(t) \end{pmatrix} \quad ; \quad \rho^{11}_{\pm n}(t) + \rho^{22}_{\pm n}(t) = 1 \quad \Rightarrow \quad \rho_{\pm n}(t) = \frac{1}{2}(I + \vec{\rho}_{\pm n}(t) \cdot \vec{\sigma}) \tag{7c}$$

where we have introduced the Pauli spin matrix vector $\vec{\sigma}$ and a time evolving *density matrix vector* $\vec{\rho}_{\pm n}(t)$ defined by

$$\vec{\sigma} = (\sigma_x \ , \ \sigma_y \ , \ \sigma_z) \qquad ; \qquad \vec{\rho}_{\pm n}(t) = ( \ \rho^x_{\pm n}(t) \ , \ \rho^y_{\pm n}(t) \ , \ \rho^z_{\pm n}(t) \ )$$

$$\rho^x_{\pm n}(t) = \rho^{12}_{\pm n}(t) + \rho^{21}_{\pm n}(t) \quad ; \quad \rho^y_{\pm n}(t) = i \ ( \ \rho^{12}_{\pm n}(t) - \rho^{21}_{\pm n}(t) \ ) \quad ; \quad \rho^z_{\pm n}(t) = \rho^{11}_{\pm n}(t) - \rho^{22}_{\pm n}(t) \tag{7d}$$

Substituting the density matrix elements determined in equation (7*b*) into equation (7*d*), we obtain the components and length of the density matrix vector in explicit form

$$\rho^x_{\pm n}(t) = 0 \qquad ; \qquad \rho^y_{\pm n}(t) = -\sin(2R_{\pm n}t) \qquad ; \qquad \rho^z_{\pm n}(t) = \cos(2R_{\pm n}t)$$

$$\vec{\rho}_{\pm n}(t) = ( \ 0 \ , -\sin(2R_{\pm n}t) \ , \ \cos(2R_{\pm n}t) \ ) \qquad ; \qquad |\vec{\rho}_{\pm n}(t)| = 1 \tag{7e}$$

which shows that the density matrix vector $\vec{\rho}_{\pm n}(t)$ has unit length ( $|\vec{\rho}_{\pm n}(t)| = 1$). According to the specification in plane polar coordinates in equation (7*e*), we interpret the density matrix vector $\vec{\rho}_{\pm n}(t)$ as the radius vector of a *circle* of unit radius ($r = 1$) in the $yz$-plane. The time evolution of the density matrix vector thus describes the trajectory of a spectrum of state points specified by the coupled qubit state vectors $|\psi_{\pm n}\rangle$ , $|\phi_{\pm n}\rangle$ on the circumference of a circle of unit radius in the $yz$-plane. The time evolving density operator representation thus reveals that the geometric configuration of the state space of the rotating photospin is a circle of unit radius in the $yz$-plane, which agrees precisely with the geometric configuration determined in the description based on a Bloch sphere in [15] where, in non-resonant ($\delta \neq 0$) atom-field interaction, effective Rabi state transition oscillations occur only between the bare atom-field state $|\pm n\rangle$ (coinciding here with $|\psi_{\pm n}\rangle$) and a superposition of the bare states $|\pm n\rangle$ and $|\mp n \pm 1\rangle$ (coinciding here with $|\phi_{\pm n}\rangle$), with the time evolution similarly following a circular path of unit radius in the $yz$-plane. We note at this stage that in the density operator representation of quantum states in standard quantum optics, a density matrix vector of unit radius is generally called a Bloch vector, defined as the radius vector of a corresponding sphere of unit radius in a two-dimensional state space called the Bloch sphere [8 , 15 , 19 , 20].

We easily establish that the geometric configuration of the quantum state space of the rotating photospin is unchanged under resonance condition $\delta = 0$ , $c_{\pm n} = 0$ , $s_{\pm n} = 1$, remaining a circle of unit radius in the $yz$-plane specified by a time evolving density matrix vector of the same form in equation (7*e*), but with the Rabi frequency defined in equation (4*c*) modified according to

$$\delta = 0 \quad ; \quad c_{\pm n} = 0 \quad ; \quad s_{\pm n} = 1 \ : \quad R_{\pm n} = 2g\sqrt{n + \frac{1}{2} \pm \frac{1}{2}}$$

$$\vec{\rho}^{\,res}_{\pm n}(t) = ( \ 0 \ , -\sin(2R_{\pm n}t) \ , \ \cos(2R_{\pm n}t) \ ) \qquad ; \qquad |\vec{\rho}^{\,res}_{\pm n}(t)| = 1 \tag{7f}$$

which shows that the time evolving density matrix $\vec{\rho}^{\,res}_{\pm n}(t)$ at resonance ($\delta = 0$) is a unit radius vector of a circle in the $yz$-plane, thus revealing that the geometric configuration of the quantum state space of the photospin is unchanged in a transition from non-resonant to resonant dynamics, remaining a circle of unit radius in the $yz$-plane. The geometric property that the density matrix vector has unit length means that the photospin is in a pure state.

We state here that, apart from specifying the geometric configuration of the quantum state space and providing a simpler scheme for studying dynamical evolution of quantum entanglement, entropy and related statistical thermodynamic properties of the rotating photospin, the time evolving density operator $\hat{\rho}_{\pm n}(t)$ is useful in determining mean values, especially correlation functions, of the composite atom-field operators $\hat{O}_{af}$ by taking the trace according to

$$\overline{O}_{af}(t) = Tr \ \hat{\rho}_{\pm n}(t)\hat{O}_{af} \tag{7g}$$

which is easily evaluated using the density operator expressed in explicit form in equation (7$b$). Dynamical properties of the individual atom and field mode can be determined using their respective time evolving reduced density operators derivable by taking partial trace of the photospin density operator $\hat{\rho}_{\pm n}(t)$ as appropriate. In such a case, the definitions of the qubit state vectors $|\psi_{\pm n}$ , $|\phi_{\pm n}\rangle$ in equation (4$c$) are substituted into equation (7$b$) to express $\hat{\rho}_{\pm n}(t)$ in terms of the bare atom-field state vectors $|\pm n\rangle$ and $|\mp n \pm 1\rangle$ for ease of evaluation of the partial trace with respect to the atomic or field mode state vectors.

### 2.1.1 Field mode in initial vacuum state

Let us now consider some special features of the dynamics which arise when the field mode starts off in an initial vacuum (0-photon) state $|0\rangle$, with the atom in the spin-up or spin-down state. The composite atom-field initial state is then $|\psi_{\pm 0}\rangle = |\pm 0\rangle$.

For the rotating photospin generated in a Jaynes-Cummings interaction starting with the field mode in the vacuum state $|0\rangle$ and the atom in spin-up state $|+\rangle$ or spin-down state $|-\rangle$, we set $n = 0$ in equations (4$c$) , (6$e$) to obtain the following interaction parameters and state vectors describing the dynamics of a rotating photospin starting from an atom-field initial 0-photon spin-up or spin-down state $|\psi_{\pm 0}\rangle = |\pm 0\rangle$ in the form

$$n = 0$$

$$R_{\pm 0} = \frac{1}{2}\sqrt{16g^2(\frac{1}{2} \pm \frac{1}{2}) + \delta^2} \quad ; \quad R_{+0} = \frac{1}{2}\sqrt{16g^2 + \delta^2} \; ; \quad R_{-0} = \frac{1}{2}\delta \; ; \quad \delta = \omega_0 - \omega$$

$$c_{\pm 0} = \frac{\delta}{2R_{\pm 0}} \; ; \quad c_{+0} = \frac{\delta}{2R_{+0}} \; ; \quad c_{-0} = 1 \quad ; \quad s_{\pm 0} = \frac{2g\sqrt{\frac{1}{2} \pm \frac{1}{2}}}{R_{\pm 0}} \; ; \quad s_{+0} = \frac{2g}{R_{+0}} \; ; \quad s_{-0} = 0 \qquad (8a)$$

$$|\psi_{\pm 0}\rangle = |\pm 0\rangle \; ; \quad |\psi_{+0}\rangle = |+0\rangle \; ; \quad |\psi_{-0}\rangle = |-0\rangle \quad ; \quad |\phi_{\pm 0}\rangle = \pm c_{\pm 0}|\pm 0\rangle + s_{\pm 0}|\mp 0 \pm 1\rangle$$

$$|\phi_{+0}\rangle = c_{+0}|+0\rangle + s_{+0}|-1\rangle \quad ; \quad |\phi_{-0}\rangle = -|-0\rangle \; = -|\psi_{-0}\rangle \qquad (8b)$$

$$|\psi_{\pm 0}\rangle = |\pm 0\rangle \; : \quad |\Psi_{\pm 0}(t)\rangle = e^{\mp\frac{i}{2}\omega t}\left( \cos(R_{\pm 0}t)|\psi_{\pm 0}\rangle - i\sin(R_{\pm 0}t)|\phi_{\pm 0}\rangle \right) \qquad (8c)$$

$$|\psi_{+0}\rangle = |+0\rangle \; : \quad |\Psi_{+0}(t)\rangle = e^{-\frac{i}{2}\omega t}\left( \cos(R_{+0}t)|\psi_{+0}\rangle - i\sin(R_{+0}t)|\phi_{+0}\rangle \right)$$

$$P^{ea}_{+0}(t) = s^2_{+0}\sin^2(R_{+0}t) \qquad ; \qquad P^{in}_{+0}(t) = \cos^2(R_{+0}t) + c^2_{+0}\sin^2(R_{+0}t) \qquad (8d)$$

$$|\psi_{-0}\rangle = |-0\rangle \; : \quad |\phi_{-0}\rangle = -|\psi_{-0}\rangle \; ; \quad |\Psi_{-0}(t)\rangle = e^{\frac{i}{2}\omega_0 t}|\psi_{-0}\rangle \quad \Rightarrow \quad |\Psi_{-0}(t)\rangle = |0\rangle \; e^{\frac{i}{2}\omega_0 t}|-\rangle$$

$$P^{ea}_{-0}(t) = 0 \quad ; \quad P^{in}_{-0}(t) = 1 \qquad (8e)$$

The time evolving rotating photospin state vectors in equations (8$d$) , (8$e$) reveal interesting physical phenomena in the dynamics generated through the Jaynes-Cummings interaction mechanism starting with the field mode in the vacuum state $|0\rangle$ and the atom in either spin-up (excited) state $|+\rangle$ or spin-down (ground) state $|-\rangle$.

According to equation (8$d$), the time evolving *entangled* state vector $|\Psi_{+0}(t)\rangle$ describes a phenomenon in which the *atom in spin-up state* $|+\rangle$ entering the electromagnetic cavity *sees* the *rotating positive frequency field mode* in the vacuum state $|0\rangle$ and *spontaneously emits a positive energy photon*, thereby triggering Rabi oscillations at frequency $R_{+0}$ between qubit states $|\psi_{+0}\rangle$ , $|\phi_{+0}\rangle$, with respective qubit state probabilities $P^{\psi}_{+0}(t) = \cos^2(R_{+0}t)$ , $P^{\phi}_{+0}(t) = \sin^2(R_{+0}t)$. The oscillatory time evolving probability to be in the initial state $P^{in}_{+0}(t)$ and probability of spontaneous positive energy photon emission by the atom $P^{ea}_{+0}(t)$ in equation $(8d)$ confirm the existence of Rabi oscillations triggered by the spontaneous positive energy photon emission by the spin-up (excited) atom.

On the other hand, the time evolving *separable* state vector $|\Psi_{-0}(t)\rangle$ in equation $(8e)$, describes a phenomenon in which the *atom in spin-down state* $|-\rangle$ entering the electromagnetic cavity *does-not-see* the *rotating positive frequency field mode* in the vacuum state $|0\rangle$ and *propagates as a free plane wave* without coupling to the field mode. The cavity thus contains a system of non-interacting $(g = 0)$ free rotating field mode in the vacuum state $|0\rangle$ and free atom in spin-down (ground) state $|-\rangle$, with the corresponding Jaynes-Cummings Hamiltonian $H$ now reduced to the free evolution form $g = 0$ , $H \rightarrow H_0 = \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\omega_0 s_z$ generating dynamical evolution from the composite atom-field initial state $|\psi_{-0}\rangle$ according to (expanding operator exponentials as appropriate)

$$g = 0 \ , \quad H \rightarrow H_0 = \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\omega_0 s_z \ : \quad |\Psi_{-0}(t)\rangle = e^{-\frac{i}{\hbar}H_0 t}|\psi_{-0}\rangle \ = e^{-i\omega t\hat{a}^\dagger\hat{a}}|0\rangle \ e^{-i\omega_0 t s_z}|-\rangle$$

$$e^{-i\omega t\hat{a}^\dagger\hat{a}}|0\rangle = |0\rangle \ ; \quad s_z|-\rangle = -\frac{1}{2}|-\rangle \ ; \quad e^{-i\omega_0 t s_z}|-\rangle = e^{\frac{i}{2}\omega_0 t}|-\rangle \quad \Rightarrow \quad |\Psi_{-0}(t)\rangle = e^{\frac{i}{2}\omega_0 t}|\psi_{-0}\rangle \qquad (8f)$$

which provides the result in equation $(8e)$. The vanishing probability of spontaneous positive energy photon emission $P^{ea}_{-0}(t) = 0$ and the unit value probability to be in the initial state $P^{in}_{-0}(t) = 1$ in equation $(8e)$ confirm the free evolution of the non-interacting rotating field mode in vacuum state $|0\rangle$ and atom in spin-down state $|-\rangle$ collectively specified by the composite atom-field 0-photon spin-down state vector $|\psi_{-0}\rangle = |0\rangle|-\rangle$. The explanation we have given here explicitly in equation $(8f)$ accounts for the inclusion of the 0-photon spin-down state $|\psi_{-0}\rangle = |-0\rangle \equiv |0g\rangle$ as an uncoupled state eigenvector in defining the completeness relation for the state eigenvectors determined through diagonalization of the Jaynes-Cummings Hamiltonian in standard quantum optics literature [12 , 21].

An important physical property which emerges in atom-field dynamics starting with the field mode in the vacuum state $|0\rangle$ is that, in the Jaynes-Cummings interaction which generates a rotating photospin, only the atom entering the cavity in a spin-up (excited) state $|+\rangle$ $(|e\rangle)$ couples to the rotating positive frequency field mode in the vacuum state $|0\rangle$, while the atom entering the cavity in a spin-down (ground) state $|-\rangle$ $(|g\rangle)$ does-not-couple to the rotating positive frequency field mode in the vacuum state $|0\rangle$ and moves freely as a plane wave inside the cavity.

But something still remains curious, seeking clarification and possible physical interpretation. In the rotating photospin dynamics starting from the 0-photon spin-down initial state $|\psi_{-0}\rangle$, the bare atom-field state probabilities take the expected constant values $P^{in}_{-0}(t) = 1$ , $P^{ea}_{-0}(t) = 0$ specifying that the plane wave time evolving qubit state vector $|\Psi_{-0}(t)\rangle = e^{\frac{i}{2}\omega_0 t}|\psi_{-0}\rangle$ determined in equation $(8e)$ describes transition *without-energy-exchange* from state $|\psi_{-0}\rangle$ to state $|\phi_{-0}\rangle = -|\psi_{-0}\rangle$, *yet* equations $(6e)$ , $(8a)$ yield periodically time evolving qubit state probabilities $P^{\psi}_{-0}(t)$ , $P^{\phi}_{-0}(t)$ determined in the form

$$P^{\psi}_{-0}(t) = \cos^2(\frac{1}{2}\delta t) \qquad ; \qquad P^{\phi}_{-0}(t) = \sin^2(\frac{1}{2}\delta t) \qquad ; \qquad \delta = \omega_0 - \omega \qquad (8g)$$

which reveals that the time evolving qubit state vector $|\Psi_{-0}(t)\rangle$ obtained from equation $(8c)$ in the qubit state vector basis $\{\ |\psi_{-0}\rangle\ ,\ |\phi_{-0}\rangle\ \}$ in the form

$$|\Psi_{-0}(t)\rangle = e^{\frac{i}{2}\omega t}(\cos(\frac{1}{2}\delta t)|\psi_{-0}\rangle - i\sin(\frac{1}{2}\delta t)|\phi_{-0}\rangle) \qquad (8h)$$

effectively describes Rabi oscillations at frequency $R_{-0} = \frac{1}{2}\delta$ between the qubit states $|\psi_{-0}\rangle$ and $|\phi_{-0}\rangle = -|\psi_{-0}\rangle$, thus fully accounting for the corresponding periodically time evolving qubit state probabilities $P^{\psi}_{-0}(t)$ , $P^{\phi}_{-0}(t)$ determined in equation $(8g)$. In this respect, we may interpret the transition $|\psi_{-0}\rangle \rightarrow |\phi_{-0}\rangle = -|\psi_{-0}\rangle$ to be equivalent to an odd-parity or a $\pi$-phase transformation which does not involve energy exchange. This dynamical property is hidden in the full plane wave representation of the time evolving state vector $|\Psi_{-0}(t)\rangle = e^{\frac{i}{2}\omega_0 t}|\psi_{-0}\rangle$ as determined in equation $(8e)$. We note that at resonance $\omega = \omega_0$ , $\delta = 0$, the time evolving state vector $|\Psi_{-0}(t)\rangle$ in equation $(8h)$ reduces to the free evolution plane wave form.

## 2.2 Interacting rotating photospins : Jaynes-Cummings optical lattice

We have now determined the basic algebraic and dynamical properties of the rotating photospin generated in the Jaynes-Cummings interaction. General statistical properties and fundamental quantum mechanical phenomena characterizing the internal dynamics of the photospin can easily be determined using the time evolving state vector or density operator which we have evaluated explicitly in the general treatment in the previous section. The next important challenge, which we address in this section, is how to build models of interacting photospin systems to provide foundations for studying general dynamical properties and devising practical applications of interacting photospins in the design and implementation of quantum information processing, quantum computation and all related quantum technologies.

Motivated by the dynamical property that a photospin behaves as a photon-carrying two-state physical entity specified by two qubit state vectors and a corresponding qubit state transition operator, with algebraic properties exactly similar to the algebraic properties of a two-state atomic spin (spin-$\frac{1}{2}$ particle), we develop a model of photospin interactions similar to the standard model of interacting atomic spins occupying sites on a linear crystal lattice in a solid. Hence, considering that in a Jaynes-Cummings interaction, a rotating photospin is generated in an optical cavity containing a single quantized rotating positive frequency electromagnetic field mode coupled to a single two-level atom, we introduce a linear chain of coupled optical cavities each carrying a rotating photospin, thus forming a *Jaynes-Cummings optical lattice*. We identify each optical cavity as an optical lattice site and therefore define a general Jaynes-Cummings optical lattice as a regular pattern of coupled arrays of optical cavities, which are the lattice sites. Like the atomic spins to which they are algebraically equivalent, photospins in a Jaynes-Cummings optical lattice interact directly with one another by coupling through their qubit state transition operators $\hat{\mathcal{E}}_{\pm}$.

We consider a simple Jaynes-Cummings optical lattice composed of a linear array of $S$ coupled optical cavities, each defined as a lattice site. Applying the physical property determined in section 2.1.1 above that an atom in a spin-up state $|+\rangle$ entering an electromagnetic cavity generally activates a Jaynes-Cummings interaction by coupling to the rotating positive frequency component of the field mode in any number state $|n\rangle$, including the vacuum state $|0\rangle$, we develop a model in which each site $i = 1, 2, ...., S$ in a Jaynes-Cummings optical lattice is an optical cavity containing an atom initially in a spin-up state $|+_i\rangle$ coupled to an electromagnetic field mode initially in a number state $|n_i\rangle$ in a Jaynes-Cummings interaction, which forms a rotating photospin from the initial atom-field state $|\psi_{+_i n_i}\rangle = |+_i\, n_i\rangle$.

At each site $i = 1, 2, ..., S$, we denote atomic spin state transition angular frequency and operators by $\omega_{0i}$ , $s_{zi}$ , $s_{-i}$ , $s_{+i}$ , field mode angular frequency and state annihilation , creation operators by $\omega_i$ , $\hat{a}_i$ , $\hat{a}_i^{\dagger}$ and the atom-field coupling constant by $g_i$, with the Jaynes-Cummings interaction frequency detuning $\delta_i$ and dimensionless frequency detuning parameter $\alpha_i$ defined according to equation $(3b)$ in the form

$$\delta_i = \omega_{0i} - \omega_i \qquad ; \qquad \alpha_i = \frac{\delta_i}{2g_i} \tag{9a}$$

The $i$-th site of the Jaynes-Cummings optical lattice contains a rotating photospin specified by two qubit state vectors $|\psi_{+_i n_i}\rangle$ , $|\phi_{+_i n_i}\rangle$, a qubit state transition operator $\hat{\mathcal{E}}_{+_i}$, identity operator $\hat{\mathcal{I}}_{+_i}$, excitation number operator $\hat{N}_{+_i}$ and Hamiltonian $H_{+_i}$ defined according to equations $(4c)$ , $(5a)$ , $(5d)$ in the form

$$|\psi_{+_i n_i}\rangle = |+_i \, n_i\rangle \quad ; \quad |\phi_{+_i n_i}\rangle = c_{+_i n_i}|+_i \, n_i\rangle + s_{+_i n_i}|-_i \, n_i+1\rangle \quad ; \quad A_{+_i n_i} = \sqrt{n_i + 1 + \frac{1}{4}\alpha_i^2}$$

$$c_{+_i n_i} = \frac{\delta_i}{2R_{+_i n_i}} \quad ; \quad s_{+_i n_i} = \frac{2g\sqrt{n_i+1}}{R_{+_i n_i}} \quad ; \quad R_{+_i n_i} = 2gA_{+_i n_i} \qquad (9b)$$

$$\hat{A}_i = \alpha s_{iz} + \hat{a}_i s_{i+} + \hat{a}_i^{\dagger} s_{i-} \; : \quad \hat{\mathcal{E}}_{+_i} = \frac{\hat{A}_i}{A_{+_i n_i}} \quad ; \quad \hat{\mathcal{I}}_{+_i} = \frac{\hat{A}_i^2}{A_{+_i n_i}^2} \quad \Rightarrow \quad \hat{\mathcal{E}}_{+_i}^2 = \hat{\mathcal{I}}_{+_i} \qquad (9c)$$

$$\hat{N}_{+_i} = (n_i + 1)\,\hat{\mathcal{I}}_{+_i} \quad ; \quad H_{+_i} = \hbar\omega_i\left(n_i + \frac{1}{2}\right)\hat{\mathcal{I}}_{+_i} + \hbar R_{+_i n_i}\hat{\mathcal{E}}_{+_i} \qquad (9d)$$

General algebraic properties of the qubit state transition operator $\hat{\mathcal{E}}_{+_i}$ follow from equation $(5c)$ in the form

$$\hat{\mathcal{E}}_{+_i}^2 = \hat{\mathcal{I}}_{+_i} \quad ; \quad \hat{\mathcal{E}}_{+_i}^{2k} = \hat{\mathcal{I}}_{+_i} \quad ; \quad \hat{\mathcal{E}}_{+_i}^{2k+1} = \hat{\mathcal{E}}_{+_i} \quad ; \quad e^{\pm\theta\hat{\mathcal{I}}_{+_i}} = e^{\pm\theta}\hat{\mathcal{I}}_{+_i} \quad ; \quad e^{\pm i\theta\hat{\mathcal{I}}_{+_i}} = e^{\pm i\theta}\hat{\mathcal{I}}_{+_i}$$

$$e^{\pm\theta\hat{\mathcal{E}}_{+_i}} = \cosh\theta\,\hat{\mathcal{I}}_{+_i} \pm \sinh\theta\,\hat{\mathcal{E}}_{+_i} \quad ; \quad e^{\pm i\theta\hat{\mathcal{E}}_{+_i}} = \cos\theta\,\hat{\mathcal{I}}_{+_i} \pm i\sin\theta\,\hat{\mathcal{E}}_{+_i} \qquad (9e)$$

State transition algebraic operations on the rotating photospin qubit state vectors $|\psi_{+_i n_i}\rangle$ , $|\phi_{+_i n_i}\rangle$ are generated by the qubit state transition operator $\hat{\mathcal{E}}_{+_i}$ and identity operator $\hat{\mathcal{I}}_{+_i}$ according to equation $(5b)$ in the form

$$\hat{\mathcal{E}}_{+_i}\,|\psi_{+_i n_i}\rangle = |\phi_{+_i n_i}\rangle \; ; \quad \hat{\mathcal{E}}_{+_i}\,|\phi_{+_i n_i}\rangle = |\psi_{+_i n_i}\rangle \; ; \quad \hat{\mathcal{I}}_{+_i}\,|\psi_{+_i n_i}\rangle = |\psi_{+_i n_i}\rangle \; ; \quad \hat{\mathcal{I}}_{+_i}\,|\phi_{+_i n_i}\rangle = |\phi_{+_i n_i}\rangle \quad (9f)$$

The state eigenvectors $|\Psi^{\pm}_{+_i n_i}\rangle$ determined as superpositions of the qubit state vectors $|\psi_{+_i n_i}\rangle$ , $|\phi_{+_i n_i}\rangle$ follow from equation $(4f)$ in the form

$$|\Psi^{+}_{+_i n_i}\rangle = \frac{1}{\sqrt{2(1 + c_{+_i n_i})}}(\,|\psi_{+_i n_i}\rangle + |\phi_{+_i n_i}\rangle\,) \quad ; \quad |\Psi^{-}_{+_i n_i}\rangle = \frac{1}{\sqrt{2(1 - c_{+_i n_i})}}(\,|\psi_{+_i n_i}\rangle - |\phi_{+_i n_i}\rangle\,) \quad (9g)$$

which satisfy eigenvalue equations generated by the qubit state transition operator $\hat{\mathcal{E}}_{+_i}$ and Hamiltonian $H_{+_i}$ from equation $(9d)$ in the form

$$\hat{\mathcal{E}}_{+_i}\,|\Psi^{\pm}_{+_i n_i}\rangle = \pm|\Psi^{\pm}_{+_i n_i}\rangle \quad ; \quad \hat{\mathcal{I}}_{+_i}\,|\Psi^{\pm}_{+_i n_i}\rangle = |\Psi^{\pm}_{+_i n_i}\rangle$$

$$H_{+_i}\,|\Psi^{\pm}_{+_i n_i}\rangle = E^{\pm}_{+_i n_i}\,|\Psi^{\pm}_{+_i n_i}\rangle \quad ; \quad E^{\pm}_{+_i n_i} = \hbar\omega_i(n_i + \frac{1}{2}) \pm \hbar R_{+_i n_i} \qquad (9h)$$

We observe that the qubit state vectors $|\psi_{+_i n_i}\rangle$ , $|\phi_{+_i n_i}\rangle$ defined in equation $(9b)$ are nonorthonormal, while the state eigenvectors $|\Psi^{+}_{+_i n_i}\rangle$ , $|\Psi^{-}_{+_i n_i}\rangle$ defined in equation $(9g)$ are orthonormal, satisfying the respective nonorthonormality or orthonormality relations

$$\langle\psi_{+_i n_i}|\psi_{+_i n_i}\rangle = 1 \; ; \quad \langle\phi_{+_i n_i}|\phi_{+_i n_i}\rangle = 1 \; ; \quad \langle\psi_{+_i n_i}|\phi_{+_i n_i}\rangle = c_{+_i n_i} \quad ; \quad \langle\phi_{+_i n_i}|\psi_{+_i n_i}\rangle = c_{+_i n_i} \qquad (9i)$$

$$\langle\Psi^{+}_{+in_i}|\Psi^{+}_{+in_i}\rangle = 1 \quad ; \quad \langle\Psi^{-}_{+in_i}|\Psi^{-}_{+in_i}\rangle = 1 \quad ; \quad \langle\Psi^{+}_{+in_i}|\Psi^{-}_{+in_i}\rangle = 0 \quad ; \quad \langle\Psi^{-}_{+in_i}|\Psi^{+}_{+in_i}\rangle = 0 \tag{9j}$$

The time evolution operator $U_{+_i}(t)$ generated by the Hamiltonian $H_{+_i}$ and the time evolving rotating photospin state vectors $|\Psi_{+in_i}(t)\rangle$ , $|\Phi_{+in_i}(t)\rangle$ generated from the respective stationary qubit state vector $|\psi_{+in_i}\rangle$ , $|\phi_{+in_i}\rangle$ by the time evolution operator $U_{+_i}(t)$ follow from equations $(6a)$-$(6c)$ , $(6g)$ in the respective final forms

$$U_{+_i}(t) = e^{-\frac{i}{\hbar}H_{+_i}t} \quad : \quad U_{+_i}(t) = e^{-i\omega_i t\left(n+\frac{1}{2}\right)}\left(\cos(R_{+in_i}t)\ \hat{\mathcal{I}}_{+_i} - i\sin(R_{+in_i}t)\ \hat{\mathcal{E}}_{+_i}\right) \tag{9k}$$

$$|\Psi_{+in_i}(t)\rangle = U_{+_i}(t)|\psi_{+in_i}\rangle \ :$$
$$|\Psi_{+in_i}(t)\rangle = e^{-i\omega_i(n_i+\frac{1}{2})t}\left(\ \cos(R_{+in_i}t)|\psi_{+in_i}\rangle - i\sin(R_{+in_i}t)|\phi_{+in_i}\rangle\ \right) \tag{9l}$$

$$|\Phi_{+in_i}(t)\rangle = U_{+_i}(t)|\phi_{+in_i}\rangle \ :$$
$$|\Phi_{+in_i}(t)\rangle = e^{-i\omega_i(n_i+\frac{1}{2})t}\left(\ \cos(R_{+in_i}t)|\phi_{+in_i}\rangle - i\sin(R_{+in_i}t)|\psi_{+in_i}\rangle\ \right) \tag{9m}$$

Applying the qubit state transition operator $\hat{\mathcal{E}}_{+_i}$ on the time evolving state vectors in equations $(9l)$ , $(9m)$ and using the qubit state transition algebraic operations from equation $(9f)$, we easily establish that the time evolving state vectors $|\Psi_{+in_i}(t)\rangle$ , $|\ \overline{\Phi}_{+in_i}(t)\rangle$ satisfy qubit state transition algebraic operations obtained as

$$\hat{\mathcal{E}}_{+_i}|\Psi_{+in_i}(t)\rangle = |\Phi_{+in_i}(t)\rangle \qquad ; \qquad \hat{\mathcal{E}}_{+_i}|\Phi_{+in_i}(t)\rangle = |\Psi_{+in_i}(t)\rangle \tag{9n}$$

The rotating photospins in different lattice sites $i, j$ interact by coupling through their qubit state transition operators $\hat{\mathcal{E}}_{+_i}$ , $\hat{\mathcal{E}}_{+_j}$, yielding interaction Hamiltonian of the form

$$H_{+_i+_j} = \hbar g_{ij}\hat{\mathcal{E}}_{+_i}\hat{\mathcal{E}}_{+_j} \quad ; \quad i\ ,\ j\ = 1, 2, 3, ..., S \tag{10a}$$

The total Hamiltonian of $S$ interacting rotating photospins in the Jaynes-Cummings optical lattice is easily obtained in the form

$$H_{+} = \sum_{i=1}^{S} H_{+_i} + \sum_{i\neq j=1}^{S} \hbar g_{ij}\hat{\mathcal{E}}_{+_i}\hat{\mathcal{E}}_{+_j} \tag{10b}$$

where $H_{+_i}$ is the Hamiltonian of the rotating photospin at site $i = 1, 2, ..., S$, defined in equation $(9d)$.

If we consider only nearest-neighbor interactions, then we can set $j = i+1$ in equations $(10a)$ , $(10b)$ to obtain the total Hamiltonian for $S$ rotating photospins with only nearest-neighbor interactions in the form

$$\mathcal{H}_{+} = \sum_{i=1}^{S} H_{+_i} + \sum_{i=1}^{S-1} \hbar g_{ij}\hat{\mathcal{E}}_{+_i}\hat{\mathcal{E}}_{+_{i+1}} \tag{10c}$$

We observe that the Hamiltonian $H_{+}$ in equation $(10b)$ or $\mathcal{H}_{+}$ in equation $(10c)$ of interacting photospins, which effectively provides a model of direct interaction between the rotating photospins ( polariton qubits) formed from the basic initial state $|\psi_{+in_i}\rangle$ at each site $i = 1, 2, ..., S$ in a Jaynes-Cummings optical lattice, differs significantly from the familiar Jaynes-Cummings-Hubbard Hamiltonian usually constructed in standard quantum optics literature [13 , 14 , 22–25], which effectively describes the dynamics of strongly-coupled photons generated in the polariton-forming Jaynes-Cummings interaction at each site (QED microcavity). In other words, the Jaynes-Cummings-Hubbard models in [13 , 14 , 22–25] and similar works by others can be interpreted as modifications of the standard Bose-Hubbard model of interacting bosons by introducing a two-state atomic spin at each QED microcavity (optical lattice site) to produce strongly-coupling photons through nonlinearities developed in the ensuing polariton-forming Jaynes-Cummings interactions. These models do not identify polaritons with qubit states and state transition operators as we have done in the present work, thus making the significant difference in formulating models of interacting polaritons (rotating photospins) in [13 , 14 , 22–25] as compared to the model of directly interacting rotating photospins in the present work.

Substituting the photospin Hamiltonian $H_{+_i}$ from equation $(9d)$ into equations $(10b)$ , $(10c)$ and separating the free evolution component to express the total Hamiltonian of the $S$ interacting rotating photospins in the form

$$H_+ = H_{+0} + H_{+I} \ ; \quad H_{+0} = \sum_{i=1}^{S} \hbar\omega_i \left(n_i + \frac{1}{2}\right) \hat{\mathcal{I}}_{+_i} \ ; \quad H_{+I} = \sum_{i=1}^{S} \hbar R_{+_i n_i} \hat{\mathcal{E}}_{+_i} + \sum_{i \neq j=1}^{S} \hbar g_{ij} \hat{\mathcal{E}}_{+_i} \hat{\mathcal{E}}_{+_j} \quad (10d)$$

$$\mathcal{H}_+ = H_{+0} + \mathcal{H}_{+I} \quad ; \quad \mathcal{H}_{+I} = \sum_{i=1}^{S} \hbar R_{+_i n_i} \hat{\mathcal{E}}_{+_i} + \sum_{i=1}^{S-1} \hbar g_{ij} \hat{\mathcal{E}}_{+_i} \hat{\mathcal{E}}_{+_{i+1}} \quad (10e)$$

we notice that in the eigenstate basis $|\Psi^{\pm}_{+_i n_i}\rangle$ where the qubit state transition operators $\hat{\mathcal{E}}_{+_i}$ take values $\pm 1$ at each site according to the eigenvalue equations in equation $(9h)$, the photospin-photospin interaction Hamiltonian component $H_{+I}$ in equation $(10d)$, which couples photospins at various sites over the entire lattice, can be identified with the one-dimensional Curie-Weiss model [26], while the interaction Hamiltonian component $\mathcal{H}_{+I}$ in equation $(10e)$, which couples only nearest-neighbor photospins, can be identified with the one-dimensional Ising model [27 , 28] of interacting atomic spins in a linear crystal lattice, where, in contrast to the standard Curie-Weiss and Ising models, the driving field and coupling parameters $R_{+_i n_i}$ , $g_{ij}$ in equations $(10d)$ , $(10e)$ take general site-dependent forms due to the quantum nature of the rotating photospins generated through the Jaynes-Cummings interaction at each site.

The time evolution operator $U_+(t)$ generated by the total Hamiltonian $H_+$ of the $S$ interacting rotating photospins given in equation $(10b)$ is obtained in the form

$$U_+(t) = e^{-\frac{it}{\hbar} H_+} \qquad \Rightarrow \qquad U_+(t) = e^{-\frac{it}{\hbar}\left( \sum_{i=1}^{S} H_{+_i} + \sum_{i \neq j=1}^{S} \hbar g_{ij} \hat{\mathcal{E}}_{+_i} \hat{\mathcal{E}}_{+_j} \right)} \quad (10f)$$

which, due to the algebraic property that the qubit state transition operators $\hat{\mathcal{E}}_{+_i}$ at all lattice sites commute and therefore commute also with the respective Hamiltonians $H_{+_i}$ at all sites $i = 1, 2, ..., S$, can be factorized in the form

$$U_+(t) = e^{-\frac{it}{\hbar} \sum_{i=1}^{S} H_{+_i}} \ e^{-\frac{it}{\hbar} \sum_{i \neq j=1}^{S} \hbar g_{ij} \hat{\mathcal{E}}_{+_i} \hat{\mathcal{E}}_{+_j}} \quad (10g)$$

We note that the time evolution operator generated by the Hamiltonian $\mathcal{H}_+$ for nearest-neighbor interactions given in equation $(10c)$ is obtained by setting $j = i + 1$ in equation $(10f)$ or $(10g)$. The commutation of the qubit state transition operators at different sites means that the time evolution operator in equation $(10g)$ can be evaluated exactly, noting that the commuting qubit state transition operators $\hat{\mathcal{E}}_{+_i}$ , $\hat{\mathcal{E}}_{+_j}$ at different sites act independently to generate qubit state transitions according to the algebraic operations in equation $(9f)$.

To complete the specification of the interacting photospin system, we must explicitly define the initial state of the rotating photospin at each lattice site. We note that, at each site $i = 1, 2, ..., S$, a rotating photospin is specified by two qubit state vectors $|\psi_{+_i n_i}\rangle$ , $|\phi_{+_i n_i}\rangle$ defined in equation $(9b)$. But the qubit state vector $|\psi_{+_i n_i}\rangle = |+_i\rangle|n_i\rangle$ is a separable bare atom-field initial state vector which cannot effectively represent a rotating photospin state, noting that a rotating photospin is generally an entangled atom-field quasiparticle excitation state. Even though the qubit state vector $|\phi_{+in_i}\rangle$ as defined in equation ($9b$) is an entangled atom-field state vector, it may not be ideal as an initial rotating photospin state vector, since it does not represent the process of formation of a rotating photospin from the basic atom-field separable state. Hence, we consider that a suitable initial state of a rotating photospin in a lattice site can be determined as a superposition of the two basic qubit state vectors $|\psi_{+in_i}\rangle$ , $|\phi_{+in_i}\rangle$. In this respect, we take the dynamical property that an atom-field Jaynes-Cummings interaction generated from the initial state $|\psi_{+in_i}\rangle$ by the respective site Hamiltonian $H_{+_i}$ in equation ($9d$) evolves over time $t$ to an entangled qubit state $|\Psi_{+in_i}(t)\rangle$ determined through the time evolution operator $U_{+_i}(t)$ according to equation ($9k$), which specifies a rotating photospin qubit state at site $i = 1, 2, ..., S$ at any time $t$. We therefore identify the entangled qubit state vector $|\Psi_{+in_i}(\tau_i)\rangle$ evolved from the initial atom-field state $|\psi_{+in_i}\rangle$ according to equation ($9k$) over a fixed interaction time $\tau_i$ of the formation of a rotating photospin at site $i$ to be the appropriate initial state vector of the $i$-th photospin. Hence, we set $t = \tau_i$ in equation ($9k$) to determine the initial state vector $|\Psi_{+in_i}(\tau_i)\rangle$ of the rotating photospin at site $i$ in the form

$$|\Psi_{+in_i}(\tau_i)\rangle = \eta_i|\psi_{+in_i}\rangle - i\ \xi_i|\phi_{+in_i}\rangle \quad ; \quad \eta_i = e^{-i\omega_i(n_i+\frac{1}{2})\tau_i}\cos(R_{+in_i}\tau_i)$$

$$\xi_i = e^{-i\omega_i(n_i+\frac{1}{2})\tau_i}\sin(R_{+in_i}\tau_i) \quad ; \quad i = 1, 2, ..., S \tag{10h}$$

which is a superposition of the two qubit state vectors $|\psi_{+in_i}\rangle$ , $|\phi_{+in_i}\rangle$ of the photospin at site $i$. The common global phase factor $e^{-i\omega_i(n_i+\frac{1}{2})\tau_i}$ may be dropped.

In performing the algebraic operations to determine the dynamical evolution of the interacting photospins generated by the Hamiltonian $H_+$ in equation ($10b$) (or $\mathcal{H}_+$ in equation ($10c$)) through the time evolution operator $U_+(t)$ in equation ($10f$) , ($10g$), we apply the physical property that the qubit state vectors and state transition operators ( $|\psi_{+in_i}\rangle$ , $|\phi_{+in_i}\rangle$ , $\hat{\mathcal{E}}_{+_i}$ ) and ( $|\psi_{+jn_j}\rangle$ , $|\phi_{+jn_j}\rangle$ , $\hat{\mathcal{E}}_{+_j}$ ) specifying rotating photospins at two different lattice sites $i, j$ are independent such that the state transition operators $\hat{\mathcal{E}}_{+_i}$ , $\hat{\mathcal{E}}_{+_j}$ at different sites commute ( $[\hat{\mathcal{E}}_{+_i} , \hat{\mathcal{E}}_{+_j}] = 0$ ) and act independently only on the corresponding qubit state vectors ( $|\psi_{+in_i}\rangle$ , $|\phi_{+in_i}\rangle$ ) , ( $|\psi_{+jn_j}\rangle$ , $|\phi_{+jn_j}\rangle$ ), generating qubit state transitions according to the algebraic operations in equation ($9f$) at the respective lattice sites $i, j$.

### 2.2.1 Two interacting rotating photospins

As an illustration of the calculations, we consider the simplest case of two interacting rotating photospins in a Jaynes-Cummings optical lattice of only two sites, where we set $i = 1, 2$ in equation ($9d$) and $S = 2$ in equation ($10b$) to obtain the Hamiltonian $H_+$ of the two interacting photospins in the form

$$H_+ = H_{+_1} + H_{+_2} + \hbar g\hat{\mathcal{E}}_1\hat{\mathcal{E}}_2 \quad ; \quad H_{+_1} = \hbar\omega_1\left(n_1 + \frac{1}{2}\right)\hat{\mathcal{I}}_{+_1} + \hbar R_{+1n_1}\hat{\mathcal{E}}_{+_1}$$

$$H_{+_2} = \hbar\omega_2\left(n_2 + \frac{1}{2}\right)\hat{\mathcal{I}}_{+_2} + \hbar R_{+2n_2}\hat{\mathcal{E}}_{+_2} \tag{11a}$$

The initial state vectors $|\Psi_{+1n_1}(\tau_1)\rangle$ , $|\Psi_{+2n_2}(\tau_2)\rangle$ of the rotating photospins at sites $i = 1, 2$, respectively, are determined according to equation ($10h$) in the form

$$|\Psi_{+1n_1}(\tau_1)\rangle = \eta_1|\psi_{+1n_1}\rangle - i\ \xi_1|\phi_{+1n_1}\rangle \quad ; \quad |\Psi_{+2n_2}(\tau_2)\rangle = \eta_2|\psi_{+2n_2}\rangle - i\ \xi_2|\phi_{+2n_2}\rangle$$

$$\eta_i = e^{-i\omega_i(n_i+\frac{1}{2})\tau_i}\cos(R_{+in_i}\tau_i) \qquad \xi_i = e^{-i\omega_i(n_i+\frac{1}{2})\tau_i}\sin(R_{+in_i}\tau_i) \quad ; \quad i = 1, 2 \tag{11b}$$

such that the total initial state vector $|\Psi_{+\tau_1\tau_2}\rangle$ is obtained as a tensor product expressed here simply as

$$|\Psi_{+\tau_1\tau_2}\rangle = |\Psi_{+1n_1}(\tau_1)\rangle\ |\Psi_{+2n_2}(\tau_2)\rangle\ = (\ \eta_1|\psi_{+1n_1}\rangle - i\ \xi_1|\phi_{+1n_1}\rangle\ )(\ \eta_2|\psi_{+2n_2}\rangle - i\ \xi_2|\phi_{+2n_2}\rangle\ ) \qquad (11c)$$

which we expand and express in the form

$$|\Psi_{+\tau_1\tau_2}\rangle = |\Psi^{-}_{+\tau_1\tau_2}\rangle - i\ |\Phi^{+}_{+\tau_1\tau_2}\rangle$$

$$|\Psi^{-}_{+\tau_1\tau_2}\rangle = \eta_1\eta_2|\psi_{+1n_1}\rangle|\psi_{+2n_2}\rangle - \xi_1\xi_2|\phi_{+1n_1}\rangle|\phi_{+2n_2}\rangle$$
$$|\Phi^{+}_{+\tau_1\tau_2}\rangle = \eta_1\xi_2|\psi_{+1n_1}\rangle|\phi_{+2n_2}\rangle + \xi_1\eta_2|\phi_{+1n_1}\rangle|\psi_{+2n_2}\rangle \qquad (11d)$$

where we identify $|\Psi^{-}_{+\tau_1\tau_2}\rangle$ , $|\Phi^{+}_{+\tau_1\tau_2}\rangle$ as entangled nonorthogonal state vectors [ 29 , 30].

Dynamical evolution of the two interacting photospins is governed by the time evolution operator $U_+(t)$ generated by the Hamiltonian $H_+$ in equation $(11a)$ in the form

$$U_+(t) = e^{-\frac{it}{\hbar}H_{+1}}e^{-\frac{it}{\hbar}H_{+2}}e^{-igt\hat{\mathcal{E}}_{+1}\hat{\mathcal{E}}_{+2}} \qquad (11e)$$

where the factorization is allowed by the commutation property of the photospin qubit state transition operators according to $[\hat{\mathcal{E}}_{+1}\ ,\ \hat{\mathcal{E}}_{+2}] = 0$. The time evolution operator in this factorized form can be evaluated explicitly using equation $(9j)$ and the algebraic relation

$$(\ \hat{\mathcal{E}}_{+1}\hat{\mathcal{E}}_{+2}\ )^{2k} = \hat{\mathcal{I}}_{+1}\hat{\mathcal{I}}_{+2} \qquad ; \qquad (\ \hat{\mathcal{E}}_{+1}\hat{\mathcal{E}}_{+2}\ )^{2k+1} = \hat{\mathcal{E}}_{+1}\hat{\mathcal{E}}_{+2}$$
$$e^{-i\theta\hat{\mathcal{E}}_{+1}\hat{\mathcal{E}}_{+2}} = \cos\theta\ \hat{\mathcal{I}}_{+1}\hat{\mathcal{I}}_{+2} - i\sin\theta\ \hat{\mathcal{E}}_{+1}\hat{\mathcal{E}}_{+2} \qquad (11f)$$

giving

$$U_+(t) = U_{+1}(t)U_{+2}(t)U_{+12}(t)$$
$$U_{+1}(t) = e^{-\frac{it}{\hbar}H_{+1}}\ =\ e^{-i\omega_1 t\left(n_1+\frac{1}{2}\right)}\left(\cos(R_{+1n_1}t)\ \hat{\mathcal{I}}_{+1} - i\sin(R_{+1n_1}t)\ \hat{\mathcal{E}}_{+1}\right)$$
$$U_{+2}(t) = e^{-\frac{it}{\hbar}H_{+2}}\ =\ e^{-i\omega_2 t\left(n_2+\frac{1}{2}\right)}\left(\cos(R_{+2n_2}t)\ \hat{\mathcal{I}}_{+2} - i\sin(R_{+2n_2}t)\ \hat{\mathcal{E}}_{+2}\right)$$
$$U_{+12}(t) = e^{-igt\hat{\mathcal{E}}_{+1}\hat{\mathcal{E}}_{+2}}\ =\ \cos(gt)\ \hat{\mathcal{I}}_{+1}\hat{\mathcal{I}}_{+2} - i\sin(gt)\ \hat{\mathcal{E}}_{+1}\hat{\mathcal{E}}_{+2} \qquad (11g)$$

The general time evolving state vector $|\Psi_{+\tau_1\tau_2}(t)\rangle$ describing the dynamics of the two interacting rotating photospins is generated from the initial state vector $|\Psi_{+\tau_1\tau_2}\rangle$ defined in equations $(11c)$ , $(11d)$ through the time evolution operator $U_+(t)$ in equation $(11g)$ in the form

$$|\Psi_{+\tau_1\tau_2}(t)\rangle = U_+(t)|\Psi_{+\tau_1\tau_2}\rangle \quad \Rightarrow \quad |\Psi_{+\tau_1\tau_2}(t)\rangle = |\Psi^{-}_{+\tau_1\tau_2}(t)\rangle - i\ |\Phi^{+}_{+\tau_1\tau_2}(t)\rangle \qquad (12a)$$

where we use the definitions of $|\Psi^{-}_{+\tau_1\tau_2}\rangle$ , $|\Phi^{+}_{+\tau_1\tau_2}\rangle$ in equation $(11d)$ and apply the explicit form of the time evolution operator $U_+(t)$ in equation $(11g)$, together with the qubit state transition algebraic operations in equation $(9f)$, noting that the photospin qubit state transition and identity operators $\hat{\mathcal{E}}_{+i}$ , $\hat{\mathcal{I}}_{+i}$ at each site $i = 1, 2, ..., S$ act only on corresponding stationary qubit state vectors $|\psi_{+in_i}\rangle$ , $|\phi_{+in_i}\rangle$ specifying the rotating photospin at the respective sites, to evaluate $|\Psi^{-}_{+\tau_1\tau_2}(t)\rangle$ , $|\Phi^{+}_{+\tau_1\tau_2}(t)\rangle$ explicitly according to

$$|\Psi^{-}_{+\tau_1\tau_2}(t)\rangle = U_{+1}(t)U_{+2}(t)U_{+12}(t)|\Psi^{-}_{+\tau_1\tau_2}\rangle$$
$$U_{+12}(t)|\Psi^{-}_{+\tau_1\tau_2}\rangle = \eta_1\eta_2(\ \cos(gt)|\psi_{+1n_1}\rangle|\psi_{+2n_2}\rangle - i\sin(gt)|\phi_{+1n_1}\rangle|\phi_{+2n_2}\rangle\ ) -$$
$$\xi_1\xi_2(\ \cos(gt)|\phi_{+1n_1}\rangle|\phi_{+2n_2}\rangle - i\sin(gt)|\psi_{+1n_1}\rangle|\psi_{+2n_2}\rangle\ )$$

$$\Rightarrow \quad |\Psi^{-}_{+\tau_1\tau_2}(t)\rangle = \eta_1\eta_2(\ \cos(gt)|\Psi_{+1n_1}(t)\rangle|\Psi_{+2n_2}(t)\rangle - i\sin(gt)|\Phi_{+1n_1}(t)\rangle|\Phi_{+2n_2}(t)\rangle\ ) -$$
$$\xi_1\xi_2(\ \cos(gt)|\Phi_{+1n_1}(t)\rangle|\Phi_{+2n_2}(t)\rangle - i\sin(gt)|\Psi_{+1n_1}(t)\rangle|\Psi_{+2n_2}(t)\rangle\ ) \qquad (12b)$$
$$|\Phi^{+}_{+\tau_1\tau_2}(t)\rangle = U_{+_1}(t)U_{+_2}(t)U_{+_{12}}(t)|\Phi^{+}_{+\tau_1\tau_2}$$
$$U_{+_{12}}(t)|\Phi^{+}_{+\tau_1\tau_2} = \eta_1\xi_2(\ \cos(gt)|\psi_{+1n_1}\rangle|\phi_{+2n_2}\rangle - i\sin(gt)|\phi_{+1n_1}\rangle|\psi_{+2n_2}\rangle\ )+$$
$$\xi_1\eta_2(\ \cos(gt)|\phi_{+1n_1}\rangle|\psi_{+2n_2}\rangle - i\sin(gt)|\psi_{+1n_1}\rangle|\phi_{+2n_2}\rangle\ )$$
$$\Rightarrow \quad |\Phi^{+}_{+\tau_1\tau_2}(t)\rangle = \eta_1\xi_2(\ \cos(gt)|\Psi_{+1n_1}(t)\rangle|\Phi_{+2n_2}(t)\rangle - i\sin(gt)|\Phi_{+1n_1}(t)\rangle|\Psi_{+2n_2}(t)\rangle\ ) +$$
$$\xi_1\eta_2(\ \cos(gt)|\Phi_{+1n_1}(t)\rangle|\Psi_{+2n_2}(t)\rangle - i\sin(gt)|\Psi_{+1n_1}(t)\rangle|\Phi_{+2n_2}(t)\rangle\ ) \qquad (12c)$$

where the respective time evolving qubit state vectors $|\Psi_{+1n_1}(t)\rangle$ , $|\Phi_{+1n_1}(t)\rangle$ and $|\Psi_{+2n_2}(t)\rangle$ , $|\Phi_{+2n_2}(t)\rangle$ of the photospins at sites $i = 1, 2$, generated here according to $|\Psi_{+in_i}(t)\rangle = U_{+_i}(t)|\psi_{+in_i}\rangle$ and $|\Phi_{+in_i}(t)\rangle = U_{+_i}(t)|\phi_{+in_i}\rangle$, are determined explicitly by setting $i = 1, 2$ in the general forms in equations $(9k)$ , $(9l)$.

Substituting $|\Psi^{-}_{+\tau_1\tau_2}(t)\rangle$ and $|\Phi^{+}_{+\tau_1\tau_2}(t)\rangle$ from equations $(12b)$ , $(12c)$ into equation $(12a)$ provides the desired general time evolving state vector $|\Psi_{+\tau_1\tau_2}(t)\rangle$ of the two interacting rotating photospins. It is clear from the explicit forms of $|\Psi^{-}_{+\tau_1\tau_2}(t)\rangle$ , $|\Phi^{+}_{+\tau_1\tau_2}(t)\rangle$ in equations $(12b)$ , $(12c)$ that the general time evolving interacting photospin state vector $|\Psi_{+\tau_1\tau_2}(t)\rangle$ determined in equation $(12a)$ is an exact entangled two-photospin state vector. The exact time evolving entangled state vector $|\Psi_{+\tau_1\tau_2}(t)\rangle$ provides an accurate description of the internal dynamics of the two coupled rotating photospins in the two-site Jaynes-Cummings optical lattice. General statistical properties, including the dynamical evolution of the quantum entanglement of the system, can be determined accurately using the time evolving density operator $\hat{\rho}_{+\tau_1\tau_2}(t) = |\Psi_{+\tau_1\tau_2}(t)\rangle\langle\Psi_{+\tau_1\tau_2}(t)|$ which is easily evaluated using the explicit results in equations $(12a)$ , $(12b)$ , $(12c)$. The entanglement property is particularly useful for the design and implementation of practical applications of the interacting photospin-photospin system in quantum information processing and related quantum technologies.

# 3 Antipolariton qubits

An antipolariton qubit is formed in an atom-field anti-Jaynes-Cummings interaction. The antipolariton qubit Hamiltonian is therefore obtained through a redefinition of the generating anti-Jaynes-Cummings Hamiltonian $\overline{H}$ in equation $(1e)$ by introducing an antipolariton qubit excitation number operator $\hat{\overline{N}}$ obtained as the sum of the quantized field mode and atomic spin excitation number operators in antinormal order form $\hat{\overline{N}} = \hat{a}\hat{a}^{\dagger} + s_{-}s_{+}$, where $\hat{a}$ , $\hat{a}^{\dagger}$ , $s_{-}$ , $s_{+}$ are the basic field mode and atomic spin operators.

Adding and subtracting an atomic spin antinormal order term $\hbar\omega s_{-}s_{+}$ in equation $(1e)$ and reorganizing using the algebraic relation

$$s_{-}s_{+} = \frac{1}{2} - s_z \qquad (13a)$$

to introduce the excitation number operator $\hat{\overline{N}}$, we redefine the anti-Jaynes-Cummings Hamiltonian $\overline{H}$ as an antipolariton qubit Hamiltonian in the form

$$\overline{H} = \hbar\omega\hat{\overline{N}} + 2\hbar g(\overline{\alpha}s_z + \hat{a}s_{-} + \hat{a}^{\dagger}s_{+}\ ) - \frac{1}{2}\hbar\omega \quad ; \quad \hat{\overline{N}} = \hat{a}\hat{a}^{\dagger} + s_{-}s_{+} \quad ; \quad \overline{\alpha} = \frac{\overline{\delta}}{2g} \quad ; \quad \overline{\delta} = \omega_0 + \omega \quad (13b)$$

where $\overline{\delta}$ is the frequency-detuning parameter arising in the anti-Jaynes-Cummings interaction mechanism. Noting that the interaction component of the Hamiltonian $\overline{H}$ in equation (13*b*) generates state transitions according to equation (2*e*), we introduce an *antipolariton qubit state transition operator* $\hat{\overline{A}}$ defined by

$$\hat{\overline{A}} = \overline{\alpha} s_z + \hat{a} s_- + \hat{a}^\dagger s_+ \tag{13c}$$

which on squaring and applying standard atom-field operator algebraic relations provides the antipolariton qubit excitation number operator $\hat{\overline{N}}$ defined in equation (13*b*) according to

$$\hat{\overline{A}}^{\,2} = \hat{\overline{N}} + \frac{1}{4}\,\overline{\alpha}^2 - 1 \qquad \Rightarrow \qquad \hat{\overline{N}} = \hat{\overline{A}}^{\,2} - \frac{1}{4}\,\overline{\alpha}^2 + 1 \tag{13d}$$

Substituting the state transition operator $\hat{\overline{A}}$ from equation (13*c*) and the excitation number operator $\hat{\overline{N}}$ from equation (13*d*) into equation (13*b*) provides the antipolariton qubit Hamiltonian $\overline{H}$ in the appropriate form

$$\overline{H} = \hbar(\ \omega \hat{\overline{A}}^{\,2} + 2g\hat{\overline{A}}\ ) - \frac{1}{4}\hbar\omega\overline{\alpha}^2 + \frac{1}{2}\hbar\omega \tag{13e}$$

The excitation number operator $\hat{\overline{N}}$ generates the antipolariton qubit $U(1)$-symmetry operator $\overline{U}(\theta)$ obtained together with the hermitian conjugate (noting $\hat{N}^\dagger = \hat{N}$ , $\hat{\overline{N}}^\dagger = \hat{\overline{N}}$) as

$$\overline{U}(\theta) = e^{-i\theta\hat{\overline{N}}} \qquad ; \qquad \overline{U}^\dagger(\theta) = e^{i\theta\hat{\overline{N}}} \tag{13f}$$

Setting $\theta = n\pi$, $n = 0, 1, 2, 3, ...$ in equation (13*f*) provides the antipolariton qubit $Z_2$-symmetry operator $\overline{U}_n(\pi)$ and parity-symmetry operator $\hat{\overline{\Pi}}$ in the form

$$\overline{U}_n(\pi) = e^{-in\pi\hat{\overline{N}}} \ = \ (\ \hat{\overline{\Pi}}\ )^n \quad ; \quad n = 0, 1, 2, ... \qquad ; \qquad \hat{\overline{\Pi}} = e^{-i\pi\hat{\overline{N}}} \tag{13g}$$

To determine the qubit state vectors, we consider the basic definition that an antipolariton qubit is formed in an anti-Jaynes-Cummings interaction starting with the field mode in a number state $|n\rangle$ and the atom in either spin-up state $|+\rangle$ or spin-down state $|-\rangle$, such that the composite atom-field initial state is the $n$-photon spin-up state $|\psi_{+n}\rangle$ or spin-down state $|\psi_{-n}\rangle$, which we denote collectively by $|\psi_{\pm n}\rangle = |\pm n\rangle$ as defined in equation (2*c*) for a consolidated description of the antipolariton qubit dynamics starting with the atom initially in either spin-up (excited) or spin-down (ground) state.

Applying the qubit state transition operator $\hat{\overline{A}} = \overline{\alpha} s_z + \hat{a} s_- + \hat{a}^\dagger s_+$ from equation (13*c*) on the composite atom-field initial $n$-photon spin-up and spin-down state vectors $|\psi_{+n}\rangle = |+n\rangle$ , $|\psi_{-n}\rangle = |-n\rangle$ and using the algebraic operations in equations (2*e*) and (2*f*), we obtain

$$\hat{\overline{A}}|\psi_{+n}\rangle = \frac{1}{2}\overline{\alpha}|\psi_{+n}\rangle + \sqrt{n + \frac{1}{2} - \frac{1}{2}}\ |-n-1\rangle \quad ; \quad \hat{\overline{A}}|\psi_{-n}\rangle = -\frac{1}{2}\overline{\alpha}|\psi_{-n}\rangle + \sqrt{n + \frac{1}{2} + \frac{1}{2}}\ |+n+1\rangle \tag{14a}$$

which we express in a convenient consolidated form

$$\hat{\overline{A}}|\psi_{\pm n}\rangle = \pm\frac{1}{2}\overline{\alpha}|\psi_{\pm n}\rangle + \sqrt{n + \frac{1}{2} \mp \frac{1}{2}}\ |\mp n \mp 1\rangle \tag{14b}$$

Reorganizing the r.h.s of equation (14*b*) provides the antipolariton qubit state vectors $|\psi_{\pm n}\rangle$ , $|\,\overline{\phi}_{\pm n}\rangle$ defined by

$$|\psi_{\pm n}\rangle = |\pm n\rangle \quad ; \quad |\,\overline{\phi}_{\pm n}\rangle = \pm\bar{c}_{\pm n}|\pm n\rangle + \bar{s}_{\pm n}|\mp n \mp 1\rangle \quad ; \quad \overline{A}_{\pm n} = \sqrt{(n+\frac{1}{2}\mp\frac{1}{2}) + \frac{1}{4}\overline{\alpha}^2}$$

$$\bar{c}_{\pm n} = \frac{\overline{\delta}}{2\overline{R}_{\pm n}} \quad ; \quad \bar{s}_{\pm n} = \frac{2g\sqrt{n+\frac{1}{2}\mp\frac{1}{2}}}{\overline{R}_{\pm n}} \quad ; \quad \overline{R}_{\pm n} = 2g\overline{A}_{\pm n} \tag{14c}$$

satisfying qubit state transition algebraic operations

$$\hat{\overline{A}}|\psi_{\pm n}\rangle = \overline{A}_{\pm n}|\,\overline{\phi}_{\pm n}\rangle \; ; \; \hat{\overline{A}}|\,\overline{\phi}_{\pm n}\rangle = \overline{A}_{\pm n}|\psi_{\pm n}\rangle \; ; \; \hat{\overline{A}}^2|\psi_{\pm n}\rangle = \overline{A}^2_{\pm n}|\psi_{\pm n}\rangle \; ; \; \hat{\overline{A}}^2|\,\overline{\phi}_{\pm n}\rangle = \overline{A}^2_{\pm n}|\,\overline{\phi}_{\pm n}\rangle \tag{14d}$$

The qubit state vectors $|\psi_{\pm n}\rangle$ , $|\,\overline{\phi}_{\pm n}\rangle$ are nonorthonormal, satisfying nonorthonormality relations

$$\langle\psi_{\pm n}|\psi_{\pm n}\rangle = 1 \quad ; \quad \langle\,\overline{\phi}_{\pm n}|\,\overline{\phi}_{\pm n}\rangle = 1 \quad ; \quad \langle\psi_{\pm n}|\,\overline{\phi}_{\pm n}\rangle = \pm\bar{c}_{\pm n} \quad ; \quad \langle\,\overline{\phi}_{\pm n}|\psi_{\pm n}\rangle = \pm\bar{c}_{\pm n} \tag{14e}$$

Antipolariton state eigenvectors $|\,\overline{\Psi}^{\,+}_{\pm n}\rangle$ , $|\,\overline{\Psi}^{\,-}_{\pm n}\rangle$ are obtained as simple superpositions of the qubit state vectors in the form

$$|\,\overline{\Psi}^{\,+}_{\pm n}\rangle = \frac{1}{\sqrt{2(1\pm\bar{c}_{\pm n})}}(\,|\psi_{\pm n}\rangle + |\,\overline{\phi}_{\pm n}\rangle\,) \quad ; \quad |\,\overline{\Psi}^{\,-}_{\pm n}\rangle = \frac{1}{\sqrt{2(1\mp\bar{c}_{\pm n})}}(\,|\psi_{\pm n}\rangle - |\,\overline{\phi}_{\pm n}\rangle\,) \tag{14f}$$

which satisfy eigenvalue equations generated by the antipolariton qubit state transition operator $\hat{\overline{A}}$ in equation (13$c$) and Hamiltonian $\overline{H}$ in equation (13$e$) in the form

$$\hat{\overline{A}}\,|\,\overline{\Psi}^{\,\pm}_{\pm n}\rangle = \pm\overline{A}_{\pm n}\,|\,\overline{\Psi}^{\,\pm}_{\pm n}\rangle \quad ; \quad \overline{H}\,|\,\overline{\Psi}^{\,\pm}_{\pm n}\rangle = \overline{E}^{\,\pm}_{\pm n}\,|\,\overline{\Psi}^{\,\pm}_{\pm n}\rangle \quad ; \quad \overline{E}^{\,\pm}_{\pm n} = \hbar\omega(n+1\mp\frac{1}{2}) \pm \hbar\overline{R}_{\pm n} \tag{14g}$$

We observe that so far, the antipolariton state eigenvectors $|\,\overline{\Psi}^{\,\pm}_{\pm n}\rangle$ and corresponding energy eigenvalues $\overline{E}^{\,\pm}_{\pm n}$, which we have obtained here in equations (14$f$) , (14$g$) and earlier in [1], do not have comparisons in standard quantum optics where the anti-Jaynes-Cummings Hamiltonian has generally been considered non-diagonalizable, until the present author applied algebraic normal and antinormal operator ordering to construct a well defined conserved excitation number operator in [13] and later in [1].

The state eigenvectors $|\,\overline{\Psi}^{\,+}_{\pm n}\rangle$ , $|\,\overline{\Psi}^{\,-}_{\pm n}\rangle$ determined in equation (14$f$) are orthonormal, satisfying orthonormality relations

$$\langle\,\overline{\Psi}^{\,+}_{\pm n}|\,\overline{\Psi}^{\,+}_{\pm n}\rangle = 1 \quad ; \quad \langle\,\overline{\Psi}^{\,-}_{\pm n}|\,\overline{\Psi}^{\,-}_{\pm n}\rangle = 1 \quad ; \quad \langle\,\overline{\Psi}^{\,+}_{\pm n}|\,\overline{\Psi}^{\,-}_{\pm n}\rangle = 0 \quad ; \quad \langle\,\overline{\Psi}^{\,-}_{\pm n}|\,\overline{\Psi}^{\,+}_{\pm n}\rangle = 0 \tag{14h}$$

## 3.1 Antirotating photospins

We now introduce a *photospin* arising as an antipolariton qubit specified by a *normalized* qubit state transition operator and the qubit state vectors. Squaring the normalized state transition operator provides a corresponding *qubit state identity operator*. All the dynamical and symmetry operators of the photospin are defined in terms of its normalized qubit state transition and identity operators. It emerges that the algebraic properties of the photospin within the two-dimensional state space spanned by its two qubit state vectors are similar to the algebraic properties of a two-state atomic spin in the two-dimensional state space spanned by its spin-up and spin-down qubit state vectors. The identification *photospin* originates from this algebraic property. We interpret a photospin as a quantized photon-carrying two-state quasiparticle with algebraic and dynamical properties precisely similar to the algebraic and dynamical properties of a two-state atomic spin. We characterize the photospin arising as an antipolariton qubit formed in an anti-Jaynes-Cummings interaction where the atomic spin couples to the *antirotating negative frequency component* of the field mode from the initial $n$-photon spin-up or spin-down state $|\psi_{\pm n}\rangle = | \pm n\rangle$ as an *antirotating photospin.*

The antirotating photospin is specified by the qubit state vectors $|\psi_{\pm n}\rangle$ , $|\overline{\phi}_{\pm n}\rangle$ obtained in equation (14$c$). The photospin state transition operator $\hat{\overline{\mathcal{E}}}_{\pm}$ and corresponding state identity operator $\hat{\overline{\mathcal{I}}}_{\pm}$ are obtained through normalization of the antipolariton qubit state transition operator $\hat{\overline{A}}$ in equation (13$c$) based on the qubit state transition algebraic operations in equation (14$d$) in the form

$$\hat{\overline{\mathcal{E}}}_{\pm} = \frac{\hat{\overline{A}}}{\overline{A}_{\pm n}} \qquad ; \qquad \hat{\overline{\mathcal{E}}}_{\pm}^{\,2} = \hat{\overline{\mathcal{I}}}_{\pm} \qquad \Rightarrow \qquad \hat{\overline{\mathcal{I}}}_{\pm} = \frac{\hat{\overline{A}}^{\,2}}{\overline{A}_{\pm n}^{\,2}} \tag{15$a$}$$

which generate state transition algebraic operations on the photospin qubit state vectors $|\psi_{\pm n}\rangle$ , $|\overline{\phi}_{\pm n}\rangle$ derived easily from the corresponding antipolariton qubit state transition algebraic operations in equation (14$d$) in the form

$$\hat{\overline{\mathcal{E}}}_{\pm} \, |\psi_{\pm n}\rangle = |\overline{\phi}_{\pm n}\rangle \; ; \quad \hat{\overline{\mathcal{E}}}_{\pm} \, |\overline{\phi}_{\pm n}\rangle = |\psi_{\pm n}\rangle \qquad ; \qquad \hat{\overline{\mathcal{I}}}_{\pm} \, |\psi_{\pm n}\rangle = |\psi_{\pm n}\rangle \; ; \quad \hat{\overline{\mathcal{I}}}_{\pm} \, |\overline{\phi}_{\pm n}\rangle = |\overline{\phi}_{\pm n}\rangle \tag{15$b$}$$

General algebraic properties of the antirotating photospin qubit state transition operator $\hat{\overline{\mathcal{E}}}_{\pm}$ are easily determined using equation (15$a$) in the form ($k = 0, 1, 2, ...$)

$$\hat{\overline{\mathcal{E}}}_{\pm}^{\,2} = \hat{\overline{\mathcal{I}}}_{\pm} \quad ; \quad \hat{\overline{\mathcal{E}}}_{\pm}^{\,2k} = \hat{\overline{\mathcal{I}}}_{\pm} \quad ; \quad \hat{\overline{\mathcal{E}}}_{\pm}^{\,2k+1} = \hat{\overline{\mathcal{E}}}_{\pm} \quad ; \quad e^{\pm\theta \, \hat{\overline{\mathcal{I}}}_{\pm}} = e^{\pm\theta} \, \hat{\overline{\mathcal{I}}}_{\pm} \quad ; \quad e^{\pm i\theta \, \hat{\overline{\mathcal{I}}}_{\pm}} = e^{\pm i\theta} \, \hat{\overline{\mathcal{I}}}_{\pm}$$

$$e^{\pm\theta \, \hat{\overline{\mathcal{E}}}_{\pm}} = \cosh\theta \, \hat{\overline{\mathcal{I}}}_{\pm} \pm \sinh\theta \, \hat{\overline{\mathcal{E}}}_{\pm} \qquad ; \qquad e^{\pm i\theta \, \hat{\overline{\mathcal{E}}}_{\pm}} = \cos\theta \, \hat{\overline{\mathcal{I}}}_{\pm} \pm i \sin\theta \, \hat{\overline{\mathcal{E}}}_{\pm} \tag{15$c$}$$

where we have applied exponential expansion with separated even and odd power terms, which are expressed as hyperbolic or trigonometric functions as appropriate.

Substituting $\hat{\overline{A}} = \overline{A}_{\pm n}\hat{\overline{\mathcal{E}}}_{\pm}$ , $\hat{\overline{A}}^{\,2} = \overline{A}_{\pm n}^{\,2}\hat{\mathcal{I}}_{\pm}$ from equation (15$a$) into equations (13$d$) , (13$e$) , (13$f$) and (13$g$), using $\overline{R}_{\pm n} = 2g\overline{A}_{\pm n}$ and the exponentiation of the identity operator $\hat{\overline{\mathcal{I}}}_{\pm}$ given in equation (15$c$) as appropriate, we easily determine the antirotating photospin excitation number operator $\hat{\overline{N}}_{\pm}$ , Hamiltonian $\overline{H}_{\pm}$ , $U(1)$-symmetry operator $\overline{U}_{\pm}(\theta)$ and parity-symmetry operator $\hat{\overline{\Pi}}_{\pm}$ in the form ($n = 0, 1, 2, ...$)

$$\hat{\overline{N}}_{\pm} = \left(n + 1 + \frac{1}{2} \mp \frac{1}{2}\right) \, \hat{\overline{\mathcal{I}}}_{\pm} \qquad ; \qquad \overline{H}_{\pm} = \hbar\omega \left(n + 1 \mp \frac{1}{2}\right) \hat{\overline{\mathcal{I}}}_{\pm} + \hbar\overline{R}_{\pm n}\hat{\overline{\mathcal{E}}}_{\pm} \tag{15$d$}$$

$$\overline{U}_{\pm}(\theta) = e^{\pm i\theta\left(n+1+\frac{1}{2}\mp\frac{1}{2}\right)}\hat{\overline{\mathcal{I}}}_{\pm} \; ; \quad \hat{\overline{\Pi}}_{\pm} = e^{\pm i\pi\left(n+1+\frac{1}{2}\mp\frac{1}{2}\right)}\hat{\overline{\mathcal{I}}}_{\pm} = \pm\hat{\overline{\mathcal{I}}}_{\pm} \;\Rightarrow\; \hat{\overline{\Pi}}_{+} = \pm\hat{\overline{\mathcal{I}}}_{+} \; ; \quad \hat{\overline{\Pi}}_{-} = \pm\hat{\overline{\mathcal{I}}}_{-} \tag{15$e$}$$

where we have specified how the parity-symmetry operator $\hat{\overline{\Pi}}_{\pm}$ applies. We note that the antirotating photospin $Z_2$-symmetry operator $\overline{U}_{n\pm}(\pi)$ as defined in equation (13$g$) is now just equal to the parity-symmetry operator $\hat{\overline{\Pi}}_{\pm}$ obtained in equation (15$e$).

Noting that the parity-symmetry operator $\hat{\overline{\Pi}}_{\pm}$ obtained in equation (15$e$) is proportional to the identity operator $\hat{\overline{\mathcal{I}}}_{\pm}$, we introduce a *parity-symmetry number* $p$ taking even or odd integer values $2k$

, $2k+1$ , $k=0,1,2,...$, to express the antirotating photospin parity-symmetry operator obtained in equation (15$e$) in the form

$$\hat{\overline{\Pi}}_{\pm} = (-1)^p \, \hat{\overline{\mathcal{I}}}_{\pm} \;\Rightarrow\; \hat{\overline{\mathcal{I}}}_{\pm} = (-1)^p \, \hat{\overline{\Pi}}_{\pm} \; ; \quad p = \begin{cases} 2k \; , \; k=0,1,2,... \; ; \; (-1)^p = +1 : \text{ even parity} \\ 2k+1 \; , \; k=0,1,2,...; \; (-1)^p = -1 : \text{odd parity} \end{cases} \tag{15f}$$

which we use to introduce the parity-symmetry operator $\hat{\overline{\Pi}}_{\pm}$ and number $p$ into the specification of the antirotating photospin excitation number operator $\hat{\overline{N}}_{\pm}$ and Hamiltonian $\overline{H}_{\pm}$ in equation (15$d$) according to

$$\hat{\overline{\mathcal{I}}}_{\pm} = (-1)^p \, \hat{\overline{\Pi}}_{\pm} \qquad \Rightarrow \qquad \hat{\overline{N}}_{\pm} = \left(p+1+\frac{1}{2}\mp\frac{1}{2}\right) \, (-1)^p \, \hat{\overline{\Pi}}_{\pm}$$

$$\overline{H}_{\pm} = \hbar\omega\left(p+1\mp\frac{1}{2}\right) \, (-1)^p \, \hat{\overline{\Pi}}_{\pm} + \hbar\overline{R}_{\pm n}\hat{\overline{\mathcal{E}}}_{\pm} \; ; \quad p = \begin{cases} 2n \; , \; n=0,1,2,... : \text{ even parity states} \\ 2n+1, \; n=0,1,2,... : \text{odd parity states} \end{cases} \tag{15g}$$

The antirotating photospin state eigenvectors $|\,\overline{\Psi}^{\pm}_{\pm n}\rangle$ are determined as superpositions of the qubit state vectors $|\psi_{\pm n}\rangle$ , $|\,\overline{\phi}_{\pm n}\rangle$ in equation (14$f$). Applying the photospin state transition operator $\hat{\overline{\mathcal{E}}}_{\pm}$ and Hamiltonian $\overline{H}_{\pm}$ in equation (15$d$) on the state eigenvectors in equation (14$f$) and using the qubit state transition algebraic operations in equation (15$b$) provides eigenvalue equations in the form

$$\hat{\overline{\mathcal{E}}}_{+} \,|\,\overline{\Psi}^{\pm}_{+n}\rangle = \pm\,|\,\overline{\Psi}^{\pm}_{+n}\rangle \; ; \quad \hat{\overline{\mathcal{E}}}_{-} \,|\,\overline{\Psi}^{\pm}_{-n}\rangle = \pm\,|\,\overline{\Psi}^{\pm}_{-n}\rangle \; ; \quad \hat{\overline{\mathcal{I}}}_{+} \,|\,\overline{\Psi}^{\pm}_{+n}\rangle = |\,\overline{\Psi}^{\pm}_{+n}\rangle \; ; \quad \hat{\overline{\mathcal{I}}}_{-} \,|\,\overline{\Psi}^{\pm}_{-n}\rangle = |\,\overline{\Psi}^{\pm}_{-n}\rangle$$

$$\overline{H}_{\pm} \,|\,\overline{\Psi}^{\pm}_{\pm n}\rangle = \overline{E}^{\pm}_{\pm n} \,|\,\overline{\Psi}^{\pm}_{\pm n}\rangle \qquad ; \qquad \overline{E}^{\pm}_{\pm n} = \hbar\omega(n+1\mp\frac{1}{2}) \pm \hbar\overline{R}_{\pm n} \tag{15h}$$

which yield the energy eigenvalues $\overline{E}^{\pm}_{\pm n}$ of the Hamiltonian $\overline{H}_{\pm}$ and reveal that the antirotating photospin state transition operator $\hat{\overline{\mathcal{E}}}_{\pm}$ has eigenvalues $\pm 1$.

The photospin Hamiltonian $\overline{H}_{\pm}$ in equation (15$d$) generates time evolution operator $\overline{U}_{\pm}(t)$ obtained in the form (noting $\hat{\overline{\mathcal{E}}}_{\pm}\hat{\overline{\mathcal{I}}}_{\pm} = \hat{\overline{\mathcal{I}}}_{\pm}\hat{\overline{\mathcal{E}}}_{\pm} = \hat{\overline{\mathcal{I}}}_{\pm}$ )

$$\overline{U}_{\pm}(t) = e^{-\frac{i}{\hbar}\overline{H}_{\pm}t} = e^{-i\omega t\left(n+1\mp\frac{1}{2}\right)\hat{\overline{\mathcal{I}}}_{\pm}} e^{-it\overline{R}_{\pm n}\hat{\overline{\mathcal{E}}}_{\pm}} \tag{16a}$$

which we apply the algebraic properties in equation (15$c$) to evaluate in explicit form

$$\overline{U}_{\pm}(t) = e^{-i\omega\left(n+1\mp\frac{1}{2}\right)t}\left(\cos(\,\overline{R}_{\pm n}t)\,\hat{\overline{\mathcal{I}}}_{\pm} - i\sin(\,\overline{R}_{\pm n}t)\,\hat{\overline{\mathcal{E}}}_{\pm}\right) \tag{16b}$$

The time evolving antirotating photospin state vector $|\,\overline{\Psi}_{\pm n}(t)\rangle$ is generated from the initial $n$-photon spin-up or spin-down state vector $|\psi_{\pm n}\rangle$ by applying the time evolution operator $\overline{U}_{\pm}(t)$ from equation (16$b$) and using the state transition algebraic operations from equation (15$b$) to obtain

$$|\,\overline{\Psi}_{\pm n}(t)\rangle = \overline{U}_{\pm}(t)|\psi_{\pm n}\rangle \; ; \quad |\,\overline{\Psi}_{\pm n}(t)\rangle = e^{-i\omega(n+1\mp\frac{1}{2})t}\left(\;\cos(\,\overline{R}_{\pm n}t)|\psi_{\pm n}\rangle - i\sin(\,\overline{R}_{\pm n}t)|\,\overline{\phi}_{\pm n}\rangle\;\right) \tag{16c}$$

which describes Rabi oscillations at frequency $\overline{R}_{\pm n}$ between the stationary qubit states $|\psi_{\pm n}\rangle$ and $|\,\overline{\phi}_{\pm n}\rangle$. Expressing the imaginary number in the polar form $-i = e^{-\frac{i}{2}\pi} = e^{-\frac{i}{4}\pi}e^{-\frac{i}{4}\pi}$ as in equation (6$d$), we rewrite the time evolving state vector $|\,\overline{\Psi}_{\pm n}(t)\rangle$ in equation (16$c$) in the standard form

$$|\,\overline{\Psi}_{\pm n}(t)\rangle = e^{-i(\omega(n+1\mp\frac{1}{2})t+\frac{1}{4}\pi)}\left(\;\cos(\overline{R}_{\pm n}t)e^{\frac{i}{4}\pi}\,|\psi_{\pm n}\rangle + \sin(\overline{R}_{\pm n}t)e^{-\frac{i}{4}\pi}\,|\,\overline{\phi}_{\pm n}\rangle\;\right) \tag{16d}$$

where we now identify the coefficient $\cos(\overline{R}_{\pm n}t)e^{\frac{i}{4}\pi}$ as the probability amplitude to be in the qubit state $|\psi_{\pm n}\rangle$ and the coefficient $\sin(\overline{R}_{\pm n}t)e^{-\frac{i}{4}\pi}$ as the probability amplitude to be in the qubit state $|\ \overline{\phi}_{\pm n}\rangle$. The corresponding probabilities $\overline{P}^{\psi}_{\pm n}(t)$ , $\overline{P}^{\overline{\phi}}_{\pm n}(t)$ to be in the respective qubit states $|\psi_{\pm n}\rangle$ , $|\ \overline{\phi}_{\pm n}\rangle$ are obtained as

$$\overline{P}^{\ \psi}_{\pm n}(t) = |\cos(\ \overline{R}_{\pm n}t)e^{\frac{i}{4}\pi}\ |^2\ = \cos^2(\ \overline{R}_{\pm n}t) \quad ; \quad \overline{P}^{\ \overline{\phi}}_{\pm n}(t) = |\sin(\ \overline{R}_{\pm n}t)e^{-\frac{i}{4}\pi}\ |^2\ = \sin^2(\ \overline{R}_{\pm n}t) \quad (16e)$$

which satisfy the standard probability relation $\overline{P}^{\ \psi}_{\pm n}(t) + \overline{P}^{\ \overline{\phi}}_{\pm n}(t) = 1$.

We complete the specification of the dynamical evolution of the antirotating photospin by applying the time evolution operator $\overline{U}_{\pm}(t)$ from equation (16*b*) on the entangled qubit state vector $|\ \overline{\phi}_{\pm n}\rangle$ and using the qubit state transition algebraic operations in equation (15*b*) to determine the corresponding time evolving qubit state vector $|\ \overline{\Phi}_{\pm n}(t)\rangle$ in the form

$$|\ \overline{\Phi}_{\pm n}(t)\rangle = \overline{U}_{\pm}(t)|\ \overline{\phi}_{\pm n}\rangle \ ; \quad |\ \overline{\Phi}_{\pm n}(t)\rangle = e^{-i\omega(n+1\mp\frac{1}{2})t}\Big(\ \cos(\ \overline{R}_{\pm n}t)|\phi_{\pm n}\rangle - i\sin(\ \overline{R}_{\pm n}t)|\ \overline{\psi}_{\pm n}\rangle\ \Big) \quad (16f)$$

Applying the antirotating photospin state transition operator $\hat{\overline{\mathcal{E}}}_{\pm}$ on the time evolving qubit state vectors $|\ \overline{\Psi}_{\pm n}(t)\rangle$ , $|\ \overline{\Phi}_{\pm n}(t)\rangle$ in equations (16*c*) , (16*f*) and using the qubit state transition algebraic operations from equation (15*b*) provides the time evolving qubit state transition algebraic operations in the form

$$\hat{\overline{\mathcal{E}}}_{\pm}|\ \overline{\Psi}_{\pm n}(t)\rangle = |\ \overline{\Phi}_{\pm n}(t)\rangle \quad ; \quad \hat{\overline{\mathcal{E}}}_{\pm}|\ \overline{\Phi}_{\pm n}(t)\rangle = |\ \overline{\Psi}_{\pm n}(t)\rangle \quad (16g)$$

It follows from the form of the time evolving qubit state vector $|\ \overline{\Psi}_{\pm n}(t)\rangle$ in equation (16*d*) that the coupled qubit state vectors $|\psi_{\pm n}\rangle$ , $|\ \overline{\phi}_{\pm n}\rangle$ undergo reversible transitions into each other in Rabi oscillations at frequency $\overline{R}_{\pm n}$. The time evolving qubit state vector $|\ \overline{\Psi}_{\pm n}(t)\rangle$ periodically evolves into the qubit state $|\psi_{\pm n}\rangle$ or $|\ \overline{\phi}_{\pm n}\rangle$ at respective times $\tau_k = \frac{k}{\overline{R}_{\pm n}}\pi$ or $\tau_k = \frac{2k+1}{2\overline{R}_{\pm n}}\pi$ , $k = 0, 1, 2, ....$

We note that, due to the nonorthogonality of the qubit state vectors $|\psi_{\pm n}\rangle$ , $|\ \overline{\phi}_{\pm n}\rangle$ determined by frequency detuning $\overline{\delta} = \omega_0 + \omega$ according to equations (14*c*) , (14*e*), the periodically time evolving qubit state probabilities $\overline{P}^{\ \psi}_{\pm n}(t)$ , $\overline{P}^{\ \overline{\phi}}_{\pm n}(t)$ in equation (16*e*) are generally different from the corresponding *state transition* probabilities $\overline{P}^{\ \psi\ \overline{\phi}}_{\pm n}(t)$ , $\overline{P}^{\ \overline{\phi}\ \psi}_{\pm n}(t)$ obtained according to standard definition in the explicit forms (using equation (14*e*) )

$$\overline{P}^{\ \psi\ \overline{\phi}}_{\pm n}(t) = |\langle\ \overline{\phi}_{\pm n}|\ \overline{\Psi}_{\pm n}(t)\rangle|^2\ = \bar{c}^{\,2}_{\pm n}\cos^2(\overline{R}_{\pm n}t) + \sin^2(\overline{R}_{\pm n}t)$$

$$\overline{P}^{\ \overline{\phi}\ \psi}_{\pm n}(t) = |\langle\psi_{\pm n}|\ \overline{\Psi}_{\pm n}(t)\rangle|^2\ = \cos^2(\overline{R}_{\pm n}t) + \bar{c}^{\,2}_{\pm n}\sin^2(\overline{R}_{\pm n}t) \quad (16h)$$

which we use equation (16*f*) to express in terms of the qubit state probabilities $\overline{P}^{\ \psi}_{\pm n}(t)$ , $\overline{P}^{\ \overline{\phi}}_{\pm n}(t)$ in the form

$$\overline{P}^{\ \psi\ \overline{\phi}}_{\pm n}(t) = \bar{c}^{\,2}_{\pm n}\overline{P}^{\ \psi}_{\pm n}(t) + \overline{P}^{\ \overline{\phi}}_{\pm n}(t) \quad ; \quad \overline{P}^{\ \overline{\phi}\ \psi}_{\pm n}(t) = \overline{P}^{\ \psi}_{\pm n}(t) + \bar{c}^{\,2}_{\pm n}\overline{P}^{\ \overline{\phi}}_{\pm n}(t) \quad (16i)$$

We emphasize here that the state transition probabilities determined in equation (16*h*) , (16*i*) do not satisfy the standard probability normalization relation, since their sum is greater than the expected unit value according to

$$\overline{P}^{\ \psi\ \overline{\phi}}_{\pm n}(t) + \overline{P}^{\ \overline{\phi}\ \psi}_{\pm n}(t) = 1 + \bar{c}^{2}_{\pm n} \quad \Rightarrow \quad \overline{P}^{\ \psi\ \overline{\phi}}_{\pm n}(t) + \overline{P}^{\ \overline{\phi}\ \psi}_{\pm n}(t) > 1 \quad (16j)$$

meaning that the transition probabilities $\overline{P}^{\ \psi\ \overline{\phi}}_{\pm n}(t)$ , $\overline{P}^{\ \overline{\phi}\ \psi}_{\pm n}(t)$ determined according to standard definition as squares of absolute values of state transition probability amplitudes in equation (16*h*) do not satisfy the definition of state probabilities for the coupled nonorthogonal qubit state vectors $|\psi_{\pm n}\rangle$ , $|\ \overline{\phi}_{\pm n}\rangle$.

We gain detailed insight into the internal dynamics of the antirotating photospin by noting that the anti-Jaynes-Cummings interaction mechanism generates the photospin through the coupling of the atomic spin to the antirotating negative frequency component of the quantized electromagnetic field mode. The transitions between the qubit states in an antirotating photospin are therefore driven by positive and negative energy photon emission-absorption processes. To understand this clearly, we substitute the definitions of the qubit state vectors $|\psi_{\pm n}\rangle$ , $|\ \overline{\phi}_{\pm n}\rangle$ from equation (14*c*) into equation (16*d*), reorganize and introduce a polar form

$$(\cos(\ \overline{R}_{\pm n}t)\mp i\bar{c}_{\pm n}\sin(\ \overline{R}_{\pm n}t)\ ) = \sqrt{\overline{P}^{in}_{\pm n}(t)}e^{\mp i\overline{\vartheta}_{\pm}(t)}\ ;\ \tan\ \overline{\vartheta}_{\pm}(t) = \bar{c}_{\pm n}\tan(\ \overline{R}_{\pm n}t)\ ;\ \sin(\ \overline{R}_{\pm n}t) = \sqrt{\overline{P}^{ea}_{\pm n}(t)}$$

$$\overline{P}^{in}_{\pm n}(t) = \cos^2(\ \overline{R}_{\pm n}t) + \bar{c}^2_{\pm n}\sin^2(\ \overline{R}_{\pm n}t) \qquad ; \qquad \overline{P}^{ea}_{\pm n}(t) = \sin^2(\ \overline{R}_{\pm n}t) \tag{16k}$$

to express the time evolving qubit state vector in the more transparent bare atom-field state basis $\{|\pm n\rangle\ ,\ |\mp n\mp 1\rangle\}$ in the form

$$|\ \overline{\Psi}_{\pm n}(t)\rangle = e^{-i(\omega(n+1\mp\frac{1}{2})t\pm\frac{1}{2}\overline{\varphi}_{\pm}(t)\ )}\left(\sqrt{\overline{P}^{in}_{\pm n}(t)}\ e^{\mp\frac{i}{2}\ \overline{\varphi}_{\pm}(t)}\ |\pm n\rangle + \sqrt{\overline{P}^{ea}_{\pm n}(t)}\ e^{\pm\frac{i}{2}\overline{\varphi}_{\pm}(t)}\ |\mp n\mp 1\rangle\right)$$

$$\overline{\varphi}_{\pm n}(t) = \overline{\vartheta}_{\pm n}(t)\mp\frac{1}{2}\pi \tag{16l}$$

where $\overline{P}^{in}_{\pm n}(t)$ is the probability to be in the initial state $|\pm n\rangle$ and $\overline{P}^{ea}_{\pm n}(t)$ is the probability to be in the photon emission-absorption state $|\mp n\mp 1\rangle$. The form of the time evolving qubit state vector $|\ \overline{\Psi}_{\pm n}(t)\rangle$ in the bare atom-field state basis in equation (16*l*) now reveals that, in a process starting from the $n$-photon spin-up state $|+n\rangle$ where the atom begins in an excited state $|+\rangle$, the excited atom emits a positive energy photon, triggering the antirotating negative frequency field mode to absorb a negative energy photon, causing a transition $|+n\rangle\ \rightarrow\ |-n-1\rangle$, while in a process starting from the $n$-photon spin-down state $|-n\rangle$ where the atom begins in a ground state $|-\rangle$, the antirotating negative frequency field mode emits a negative energy photon, triggering the atom to absorb a positive energy photon, causing a transition $|-n\rangle\ \rightarrow\ |+n+1\rangle$, thus accounting for the dynamical evolution which couples the bare atom-field states $|\pm n\rangle$ and $|\mp n\mp 1\rangle$ as described by the time evolving qubit state vector $|\ \overline{\Psi}_{\pm n}(t)\rangle$ in equation (16*l*). These state transitions driven by emission-absorption of positive and negative energy photons in an antirotating photospin (or an antipolariton qubit) generated in an anti-Jaynes-Cummings interaction mechanism are identified as *blue-sideband transitions* specified by frequency detuning $\overline{\delta} = \omega_0+\omega$ [16]. The mathematical property that the bare atom-field state probabilities $\overline{P}^{in}_{\pm n}(t)$ , $\overline{P}^{ea}_{\pm n}(t)$ as determined in equation (16*k*) cannot simultaneously take alternate values 0 and 1 means that, starting from the initial state $|\pm n\rangle$ the state vector $|\ \overline{\Psi}_{\pm n}(t)\rangle$ in equation (16*l*) cannot evolve to the photon emission-absorption state $|\mp n\mp 1\rangle$ and vice-versa. Instead, exact Rabi oscillations of the time evolving qubit state vector $|\ \overline{\Psi}_{\pm n}(t)\rangle$ occur only between the initial state $|\pm n\rangle$ and a superposition state $\overline{\beta}_{in}|\pm n\rangle + \overline{\beta}_{ea}|\mp n\mp 1\rangle$, which then agrees precisely with the exact Rabi oscillations between the antirotating photospin qubit states $|\psi_{\pm n}\rangle$ , $|\ \overline{\phi}_{\pm n}\rangle$ described by the time evolving qubit state vector determined directly in equation (16*c*) , (16*d*). This reaffirms the interpretation that the qubit state vectors $|\psi_{\pm n}\rangle = |\pm n\rangle$ , $|\ \overline{\phi}_{\pm n}\rangle = \pm\bar{c}_{\pm n}|\pm n\rangle + \bar{s}_{\pm n}|\mp n\mp 1\rangle$ as defined in equation (14*c*) are the natural state vectors of an antirotating photospin. Hence, the general dynamical evolution of the antirotating photospin is described by the time evolving qubit state vector $|\ \overline{\Psi}_{\pm n}(t)\rangle$ as determined in equation (16*c*) , (16*d*).

Finally, we note that the time evolving photospin state eigenvectors $|\ \overline{\Psi}^{\pm}_{\pm n}(t)\rangle$ are determined by applying the time evolution operator as defined in the form $\overline{U}_{\pm}(t) = e^{-\frac{i}{\hbar}\overline{H}_{\pm}t}$ in equation (16$a$) on the state eigenvectors $|\ \overline{\Psi}^{\pm}_{\pm n}\rangle$ in equation (14$f$) and using the eigenvalue equations (15$h$) giving the explicit form

$$|\ \overline{\Psi}^{\pm}_{\pm n}(t)\rangle = \overline{U}_{\pm}(t)|\ \overline{\Psi}^{\pm}_{\pm n}\rangle \ = \ e^{-\frac{i}{\hbar}\overline{E}^{\pm}_{\pm n}t}|\ \overline{\Psi}^{\pm}_{\pm n}\rangle \tag{16m}$$

which can be used where necessary.

For a comprehensive study of the distribution of states and general statistical properties of the antipolariton qubit within a geometrical frame specified by two coupled qubit state vectors $|\psi_{\pm n}\rangle$ , $|\ \overline{\phi}_{\pm n}\rangle$, we introduce the time evolving density operator $\hat{\overline{\rho}}_{\pm n}(t)$ defined by

$$\hat{\overline{\rho}}_{\pm n}(t) = |\ \overline{\Psi}_{\pm n}(t)\rangle\langle\ \overline{\Psi}_{\pm n}(t)| \tag{17a}$$

which on substituting the time evolving qubit state vector $|\ \overline{\Psi}_{\pm n}(t)\rangle$ from equation (16$c$) (or (16$e$)) and expanding takes the explicit form

$$\hat{\overline{\rho}}_{\pm n}(t) = \overline{\rho}^{\,11}_{\pm n}(t)|\psi_{\pm n}\rangle\langle\psi_{\pm n}| + \overline{\rho}^{\,12}_{\pm n}(t)|\psi_{\pm n}\rangle\langle\ \overline{\phi}_{\pm n}| + \overline{\rho}^{\,21}_{\pm n}(t)|\ \overline{\phi}_{\pm n}\rangle\langle\psi_{\pm n}| + \overline{\rho}^{\,22}_{\pm n}(t)|\ \overline{\phi}_{\pm n}\rangle\langle\ \overline{\phi}_{\pm n}|$$

$$\overline{\rho}^{\,11}_{\pm n}(t) = \cos^2(\ \overline{R}_{\pm n}t) \quad ; \quad \overline{\rho}^{\,12}_{\pm n}(t) = \frac{i}{2}\sin(\ 2\overline{R}_{\pm n}t) \quad ; \quad \overline{\rho}^{\,21}_{\pm n}(t) = -\frac{i}{2}\sin(\ 2\overline{R}_{\pm n}t)$$

$$\overline{\rho}^{\,22}_{\pm n}(t) = \sin^2(\ \overline{R}_{\pm n}t) \tag{17b}$$

We interpret the density operator coefficients $\overline{\rho}^{\,ij}_{\pm n}(t)$ , $i,j = 1,2$ as elements of a $2\times 2$ density matrix $\overline{\rho}_{\pm n}(t)$, which we express in terms of the standard $2\times 2$ Pauli spin matrices $I$ , $\sigma_x$ , $\sigma_y$ , $\sigma_z$ in the form

$$\overline{\rho}_{\pm n}(t) = \begin{pmatrix} \overline{\rho}^{\,11}_{\pm n}(t) & \overline{\rho}^{\,12}_{\pm n}(t) \\ \overline{\rho}^{\,21}_{\pm n}(t) & \overline{\rho}^{\,22}_{\pm n}(t) \end{pmatrix} \quad ; \quad \overline{\rho}^{\,11}_{\pm n}(t) + \overline{\rho}^{\,22}_{\pm n}(t) = 1 \quad \Rightarrow \quad \overline{\rho}_{\pm n}(t) = \frac{1}{2}(I + \vec{\overline{\rho}}_{\pm n}(t)\cdot\vec{\sigma}) \tag{17c}$$

where we have introduced the Pauli spin matrix vector $\vec{\sigma}$ and a time evolving density matrix vector $\vec{\overline{\rho}}_{\pm n}(t)$ defined by

$$\vec{\sigma} = (\sigma_x\ ,\ \sigma_y\ ,\ \sigma_z) \quad ; \quad \vec{\overline{\rho}}_{\pm n}(t) = (\ \overline{\rho}^{\,x}_{\pm n}(t)\ ,\ \overline{\rho}^{\,y}_{\pm n}(t)\ ,\ \overline{\rho}^{\,z}_{\pm n}(t)\ )$$

$$\overline{\rho}^{\,x}_{\pm n}(t) = \overline{\rho}^{\,12}_{\pm n}(t) + \overline{\rho}^{\,21}_{\pm n}(t) \quad ; \quad \overline{\rho}^{\,y}_{\pm n}(t) = i\ (\ \overline{\rho}^{\,12}_{\pm n}(t) - \overline{\rho}^{\,21}_{\pm n}(t)\ ) \quad ; \quad \overline{\rho}^{\,z}_{\pm n}(t) = \overline{\rho}^{\,11}_{\pm n}(t) - \overline{\rho}^{\,22}_{\pm n}(t) \tag{17d}$$

Substituting the density matrix elements determined in equation (17$b$) into equation (17$d$), we obtain the components and length of the density matrix vector in explicit form

$$\overline{\rho}^{\,x}_{\pm n}(t) = 0 \quad ; \quad \overline{\rho}^{\,y}_{\pm n}(t) = -\sin(\ 2\overline{R}_{\pm n}t) \quad ; \quad \overline{\rho}^{\,z}_{\pm n}(t) = \cos(\ 2\overline{R}_{\pm n}t)$$

$$\vec{\overline{\rho}}_{\pm n}(t) = (\ 0\ , -\sin(\ 2\overline{R}_{\pm n}t)\ ,\ \cos(\ 2\overline{R}_{\pm n}t)\ ) \quad ; \quad |\vec{\overline{\rho}}_{\pm n}(t)| = 1 \tag{17e}$$

which shows that the density matrix vector $\vec{\overline{\rho}}_{\pm n}(t)$ has unit length ( $|\vec{\overline{\rho}}_{\pm n}(t)| = 1$). According to the specification in plane polar coordinates in equation (17$e$), we interpret the density matrix vector $\vec{\overline{\rho}}_{\pm n}(t)$ as the radius vector of a *circle* of unit radius ($r = 1$) in the $yz$-plane. The time evolution of the density matrix vector thus describes the trajectory of a spectrum of state points specified by the coupled qubit state vectors $|\psi_{\pm n}\rangle$ , $|\ \overline{\phi}_{\pm n}\rangle$ on the circumference of a circle of unit radius in the $yz$-plane. The geometric property that the density matrix vector has unit length means that the antirotating photospin is in a pure state.

We note here that, apart from specifying the geometric configuration of the quantum state space and providing a simpler scheme for studying dynamical evolution of quantum entanglement, entropy and related statistical thermodynamic properties of the antirotating photospin, the time evolving density operator $\hat{\overline{\rho}}_{\pm n}(t)$ is useful in determining mean values, especially correlation functions, of the composite atom-field operators $\hat{O}_{af}$ by taking the trace according to

$$\overline{O}_{af}(t) = Tr\ \hat{\overline{\rho}}_{\pm n}(t)\hat{O}_{af} \qquad (17f)$$

which is easily evaluated using the density operator expressed in explicit form in equation (17$b$). Dynamical properties of the individual atom and field mode can be determined using their respective time evolving reduced density operators derivable by taking partial trace of the photospin density operator $\hat{\overline{\rho}}_{\pm n}(t)$ as appropriate. In such a case, the definitions of the qubit state vectors $|\psi_{\pm n}$ , $|\ \overline{\phi}_{\pm n}\rangle$ in equation (14$c$) are substituted into equation (17$b$) to express $\hat{\overline{\rho}}_{\pm n}(t)$ in terms of the bare atom-field state vectors $|\pm n\rangle$ and $|\mp n \mp 1\rangle$ for ease of evaluation of the partial trace with respect to the atomic or field mode state vectors.

### 3.1.1 Field mode in initial vacuum state

We now consider some special features of the dynamics which arise when the field mode starts off in an initial vacuum (0-photon) state $|0\rangle$, with the atom in the spin-up or spin-down state. The composite atom-field initial state is then $|\psi_{\pm 0}\rangle = |\pm 0\rangle$.

Setting $n = 0$ in equations (14$c$) , (16$c$) we obtain the following interaction parameters and state vectors describing the dynamics of an antirotating photospin starting from an atom-field initial 0-photon spin-up or spin-down state $|\psi_{\pm 0}\rangle = |\pm 0\rangle$ in the form

$$n = 0$$

$$\overline{R}_{\pm 0} = \frac{1}{2}\sqrt{16g^2(\frac{1}{2} \mp \frac{1}{2}) + \overline{\delta}^2} \quad ; \quad \overline{R}_{+0} = \frac{1}{2}\overline{\delta} \quad ; \quad \overline{R}_{-0} = \frac{1}{2}\sqrt{16g^2 + \overline{\delta}^2} \quad ; \quad \overline{\delta} = \omega_0 + \omega$$

$$\bar{c}_{\pm 0} = \frac{\overline{\delta}}{2\overline{R}_{\pm 0}} \ ; \quad \bar{c}_{+0} = 1\ ; \quad \bar{c}_{-0} = \frac{\overline{\delta}}{2\overline{R}_{-0}} \quad ; \quad \bar{s}_{\pm 0} = \frac{2g\sqrt{\frac{1}{2} \mp \frac{1}{2}}}{\overline{R}_{\pm 0}} \ ; \quad \bar{s}_{+0} = 0\ ; \quad \bar{s}_{-0} = \frac{2g}{\overline{R}_{-0}} \qquad (18a)$$

$$|\ \overline{\phi}_{\pm 0}\rangle = \pm\bar{c}_{\pm 0}|\pm 0\rangle + \bar{s}_{\pm 0}|\mp 0 \mp 1\rangle$$

$$|\ \overline{\phi}_{+0}\rangle = |+0\rangle \ = \ |\psi_{+0}\rangle \quad ; \quad |\ \overline{\phi}_{-0}\rangle = -\bar{c}_{-0}|-0\rangle + \bar{s}_{-0}|+1\rangle \qquad (18b)$$

$$|\psi_{\pm 0}\rangle = |\pm 0\rangle \ : \quad |\ \overline{\Psi}_{\pm 0}(t)\rangle = e^{-i\omega(1 \mp \frac{1}{2})t}\Big(\ \cos(\ \overline{R}_{\pm 0}t)|\psi_{\pm 0}\rangle - i\sin(\ \overline{R}_{\pm 0}t)|\ \overline{\phi}_{\pm 0}\rangle\ \Big) \qquad (18c)$$

$$|\psi_{+0}\rangle = |+0\rangle \quad ; \quad |\ \overline{\phi}_{+0}\rangle = |\psi_{+0}\rangle \quad ; \quad |\ \overline{\Psi}_{+0}(t)\rangle = e^{-\frac{i}{2}(\omega_0 + 2\omega)t}|\psi_{+0}\rangle$$

$$\Rightarrow \quad |\ \overline{\Psi}_{+0}(t)\rangle = e^{-i\omega t}|0\rangle\ e^{-\frac{i}{2}\omega_0 t}|+\rangle \quad ; \quad \overline{P}^{\,in}_{+0}(t) = 1 \quad ; \quad \overline{P}^{\,ea}_{+0}(t) = 0 \qquad (18d)$$

$$|\psi_{-0}\rangle = |-0\rangle \ : \quad |\ \overline{\Psi}_{-0}(t)\rangle = e^{-\frac{3i}{2}\omega t}\Big(\ \cos(\ \overline{R}_{-0}t)|\psi_{-0}\rangle - i\sin(\ \overline{R}_{-0}t)|\ \overline{\phi}_{-0}\rangle\ \Big)$$

$$\overline{P}^{\,in}_{-0}(t) = \cos^2(\ \overline{R}_{-0}t) + \bar{c}^2_{-0}\sin^2(\ \overline{R}_{-0}t) \quad ; \quad \overline{P}^{\,ea}_{-0}(t) = \bar{s}^2_{-0}\sin^2(\ \overline{R}_{-0}t) \qquad (18e)$$

The time evolving antirotating photospin state vectors in equations (18$d$) , (18$e$) reveal interesting physical phenomena in the dynamics generated through the anti-Jaynes-Cummings interaction mechanism starting with the field mode in the vacuum state $|0\rangle$ and the atom in either spin-up (excited) state $|+\rangle$ or spin-down (ground) state $|-\rangle$.

According to equation (18$d$), the time evolving *separable* state vector $|\overline{\Psi}_{+0}(t)\rangle$ describes a phenomenon in which the *atom in spin-up state* $|+\rangle$ entering the electromagnetic cavity *does-not-see* the *antirotating negative frequency field mode* in the vacuum state $|0\rangle$ and *propagates as a free plane wave* without coupling to the field mode. The cavity thus contains a system of non-interacting ($g=0$) free antirotating field mode in the vacuum state $|0\rangle$ and free atom in spin-up (excited) state $|+\rangle$, with the corresponding anti-Jaynes-Cummings Hamiltonian $\overline{H}$ now reduced to the free evolution form $g=0$ , $\overline{H} \rightarrow \overline{H}_0 = \hbar\omega\hat{a}\hat{a}^\dagger + \hbar\omega_0 s_z$ generating dynamical evolution from the composite atom-field initial state $|\psi_{+0}\rangle$ according to (using $\hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a}+1$)

$$g=0\ ,\ \overline{H} \rightarrow \overline{H}_0 = \hbar\omega\hat{a}\hat{a}^\dagger + \hbar\omega_0 s_z\ :\quad |\Psi_{+0}(t)\rangle = e^{-\frac{i}{\hbar}\overline{H}_0 t}|\psi_{+0}\rangle\ = e^{-i\omega t}e^{-i\omega t\hat{a}^\dagger\hat{a}}|0\rangle\ e^{-i\omega_0 t s_z}|+\rangle$$

$$e^{-i\omega t\hat{a}^\dagger\hat{a}}|0\rangle = |0\rangle\ ;\ s_z|+\rangle = \frac{1}{2}|+\rangle\ ;\quad e^{-i\omega_0 t s_z}|+\rangle = e^{-\frac{i}{2}\omega_0 t}|+\rangle\ \Rightarrow\quad |\Psi_{+0}(t)\rangle = e^{-\frac{i}{2}(\omega_0+2\omega)t}|\psi_{+0}\rangle \quad (18f)$$

which provides the result in equation (18$d$). The vanishing probability of spontaneous negative energy photon emission by the antirotating field mode $\overline{P}^{\,ea}_{+0}(t)=0$ and the unit value probability to be in the initial state $\overline{P}^{\,in}_{+0}(t)=1$ in equation (18$d$) confirm the free evolution of the non-interacting antirotating field mode in vacuum state $|0\rangle$ and atom in spin-up state $|+\rangle$ collectively specified by the composite atom-field 0-photon spin-up state vector $|\psi_{+0}\rangle = |0\rangle|+\rangle$.

On the other hand, the time evolving *entangled* state vector $|\overline{\Psi}_{-0}(t)\rangle$ in equation (18$e$) describes a phenomenon in which the *atom in spin-down state* $|-\rangle$ entering the electromagnetic cavity *sees* the *antirotating negative frequency field mode* in the vacuum state $|0\rangle$, which *spontaneously emits a negative energy photon*, thereby triggering Rabi oscillations at frequency $\overline{R}_{-0}$ between qubit states $|\psi_{-0}\rangle$ and $|\overline{\phi}_{-0}\rangle$, with the respective qubit state probabilities $\overline{P}^{\,\psi}_{-0}(t) = \cos^2(\overline{R}_{-0}t)$ , $\overline{P}^{\,\overline{\phi}}_{-0}(t) = \sin^2(\overline{R}_{-0}t)$. The oscillatory time evolving probability to be in the initial state $\overline{P}^{\,in}_{-0}(t)$ and probability of spontaneous negative energy photon emission by the antirotating field mode $\overline{P}^{\,ea}_{-0}(t)$ in equation (18$e$) confirm the existence of Rabi oscillations triggered by the spontaneous negative energy photon emission by the antirotating field mode.

An important physical property which emerges in atom-field dynamics starting with the field mode in the vacuum state $|0\rangle$ is that, in the anti-Jaynes-Cummings interaction which generates an antirotating photospin, only the atom entering the cavity in a spin-down (ground) state $|-\rangle$ ($|g\rangle$) couples to the antirotating negative frequency field mode in the vacuum state $|0\rangle$, while the atom entering the cavity in a spin-up (excited) state $|+\rangle$ ($|e\rangle$) does not couple to the antirotating negative frequency field mode in the vacuum state $|0\rangle$ and moves freely as a plane wave inside the cavity.

But something still remains curious, seeking clarification and possible physical interpretation. In the antirotating photospin dynamics starting from the 0-photon spin-up initial state $|\psi_{+0}\rangle$, the bare atom-field state probabilities take the expected constant values $\overline{P}^{\,in}_{+0}(t)=1$ , $\overline{P}^{\,ea}_{+0}(t)=0$ specifying that the plane wave time evolving qubit state vector $|\overline{\Psi}_{+0}(t)\rangle = e^{-\frac{i}{2}(\omega_0+2\omega)t}|\psi_{+0}\rangle$ determined in equation (18$d$) describes transition *without-energy-exchange* from state $|\psi_{+0}\rangle$ to state $|\overline{\phi}_{+0}\rangle = |\psi_{+0}\rangle$, *yet* equations (16$d$) , (18$a$) yield periodically time evolving qubit state probabilities $\overline{P}^{\,\psi}_{+0}(t)$ , $\overline{P}^{\,\phi}_{-0}(t)$ determined in the form

$$\overline{P}^{\,\psi}_{+0}(t) = \cos^2(\frac{1}{2}\overline{\delta}t) \qquad ; \qquad \overline{P}^{\,\overline{\phi}}_{+0}(t) = \sin^2(\frac{1}{2}\overline{\delta}t) \qquad ; \qquad \overline{\delta} = \omega_0+\omega \qquad (18g)$$

which reveals that the time evolving qubit state vector $|\ \overline{\Psi}_{+0}(t)\rangle$ obtained from equation ($18c$) in the qubit state vector basis $\{\ |\psi_{+0}\rangle\ ,\ |\ \overline{\phi}_{+0}\rangle\ \}$ in the form

$$|\ \overline{\Psi}_{+0}(t)\rangle = e^{-\frac{i}{2}\omega t}(\cos(\frac{1}{2}\overline{\delta}t)|\psi_{+0}\rangle - i\sin(\frac{1}{2}\overline{\delta}t)|\ \overline{\phi}_{+0}\rangle) \tag{18h}$$

effectively describes some 'hidden' Rabi oscillations at frequency $\overline{R}_{+0} = \frac{1}{2}\overline{\delta}$ between the qubit states $|\psi_{+0}\rangle$ and $|\phi_{+0}\rangle = |\psi_{+0}\rangle$, thus fully accounting for the corresponding periodically time evolving qubit state probabilities $\overline{P}^{\,\psi}_{+0}(t)$ , $\overline{P}^{\,\overline{\phi}}_{+0}(t)$ determined in equation ($18g$). In this case, we may interpret the transition $|\psi_{+0}\rangle \rightarrow |\ \overline{\phi}_{+0}\rangle = |\psi_{+0}\rangle$ to be equivalent to an even-parity or a $2\pi$-phase transformation which does not involve energy exchange. This shuttle dynamical property is highly hidden in the full plane wave representation of the time evolving state vector $|\ \overline{\Psi}_{+0}(t)\rangle = e^{-\frac{i}{2}(\omega_0+2\omega)t}|\psi_{+0}\rangle$ as determined in equation ($18d$).

## 3.2 Interacting antirotating photospins : anti-Jaynes-Cummings optical lattice

We have now determined the basic algebraic and dynamical properties of the antirotating photospin generated in the anti-Jaynes-Cummings interaction. General statistical properties and fundamental quantum mechanical phenomena characterizing the internal dynamics of the photospin can easily be determined using the time evolving state vector or density operator which we have evaluated explicitly in the general treatment in the previous section. The next important challenge, which we address in this section, is how to build models of interacting photospin systems to provide foundations for studying general dynamical properties and devising practical applications of interacting photospins in the design and implementation of quantum information processing, quantum computation and all related quantum technologies.

Motivated by the dynamical property that a photospin behaves as a photon-carrying two-state physical entity specified by two qubit state vectors and a corresponding qubit state transition operator, with algebraic properties exactly similar to the algebraic properties of a two-state atomic spin (spin-$\frac{1}{2}$ particle), we develop a model of photospin interactions similar to the standard model of interacting atomic spins occupying sites on a linear crystal lattice in a solid. Hence, considering that in an anti-Jaynes-Cummings interaction, an antirotating photospin is generated in an optical cavity containing a single quantized antirotating negative frequency electromagnetic field mode coupled to a single two-level atom, we introduce a linear chain of coupled optical cavities each carrying an antirotating photospin, thus forming an *anti-Jaynes-Cummings optical lattice*. We identify each optical cavity as an optical lattice site and therefore define a general anti-Jaynes-Cummings optical lattice as a regular pattern of coupled arrays of optical cavities, which are the lattice sites. Like the atomic spins to which they are algebraically equivalent, photospins in an anti-Jaynes-Cummings optical lattice interact directly with one another by coupling through their qubit state transition operators $\hat{\overline{\mathcal{E}}}_{\pm}$.

We consider a simple anti-Jaynes-Cummings optical lattice composed of a linear array of $S$ coupled optical cavities, each defined as a lattice site. Applying the physical property determined in section 3.1.1 above that an atom in a spin-down state $|-\rangle$ entering an electromagnetic cavity generally activates an anti-Jaynes-Cummings interaction by coupling to the antirotating negative component of the field mode in any number state $|n\rangle$, including the vacuum state $|0\rangle$, we develop a model in which each site $i = 1, 2, ...., S$ in an anti-Jaynes-Cummings optical lattice is an optical cavity containing an atom initially in a spin-down state $|-_i\rangle$ coupled to an electromagnetic field mode initially in a number state $|n_i\rangle$ in an anti-Jaynes-Cummings interaction, which forms an antirotating photospin from the initial atom-field state $|\psi_{-in_i}\rangle = |-_i\, n_i\rangle$.

At each site $i = 1, 2, ..., S$, we denote atomic spin state transition angular frequency and operators by $\omega_{0i}$ , $s_{zi}$ , $s_{-i}$ , $s_{+i}$ , field mode angular frequency and state annihilation , creation operators by $\omega_i$ , $\hat{a}_i$ , $\hat{a}_i^\dagger$ and the atom-field coupling constant by $g_i$, with the anti-Jaynes-Cummings interaction frequency detuning $\delta_i$ and dimensionless frequency detuning parameter $\alpha_i$ defined according to equation $(13b)$ in the form

$$\overline{\delta}_i = \omega_{0i} + \omega_i \qquad ; \qquad \overline{\alpha}_i = \frac{\overline{\delta}_i}{2g_i} \tag{19a}$$

The $i$-th site of the anti-Jaynes-Cummings optical lattice contains an antirotating photospin specified by two qubit state vectors $|\psi_{-in_i}\rangle$ , $|\,\overline{\phi}_{-in_i}\rangle$, a qubit state transition operator $\hat{\overline{\mathcal{E}}}_{-_i}$, identity operator $\hat{\overline{\mathcal{I}}}_{-_i}$, excitation number operator $\hat{\overline{N}}_{-_i}$ and Hamiltonian $\overline{H}_{-_i}$ defined according to equations $(13c)$ , $(14c)$ , $(15a)$ , $(15d)$ in the form

$$|\psi_{-in_i}\rangle = |-_i\, n_i\rangle \ ; \quad |\,\overline{\phi}_{-in_i}\rangle = -\bar{c}_{-in_i}|-_i\, n_i\rangle + \bar{s}_{-in_i}|+_i\, n_i+1\rangle \ ; \quad \overline{A}_{-in_i} = \sqrt{(n_i+1) + \frac{1}{4}\overline{\alpha}^2_{-i}}$$

$$\bar{c}_{-in_i} = \frac{\overline{\delta}_{-i}}{2\overline{R}_{-in_i}} \qquad ; \qquad \bar{s}_{-in_i} = \frac{2g\sqrt{n_i+1}}{\overline{R}_{-in_i}} \qquad ; \qquad \overline{R}_{-in_i} = 2g\overline{A}_{-in_i} \tag{19b}$$

$$\hat{\overline{A}}_i = \overline{\alpha} s_{zi} + \hat{a}_i s_{-i} + \hat{a}_i^\dagger s_{+i} \ : \quad \hat{\overline{\mathcal{E}}}_{-_i} = \frac{\hat{\overline{A}}_i}{\overline{A}_{-in_i}} \qquad ; \qquad \hat{\overline{\mathcal{E}}}_{-_i}^{\,2} = \hat{\overline{\mathcal{I}}}_{-_i} \qquad \Rightarrow \qquad \hat{\overline{\mathcal{I}}}_{-_i} = \frac{\hat{\overline{A}}_i^{\,2}}{\overline{A}^{\,2}_{-in_i}} \tag{19c}$$

$$\hat{\overline{N}}_{-_i} = (n_i+2)\,\hat{\overline{\mathcal{I}}}_{-_i} \qquad ; \qquad \overline{H}_{-_i} = \hbar\omega_i\left(n_i + \frac{3}{2}\right)\hat{\overline{\mathcal{I}}}_{-_i} + \hbar\overline{R}_{-in_i}\hat{\overline{\mathcal{E}}}_{-_i} \tag{19d}$$

General algebraic properties of the qubit state transition operator $\hat{\mathcal{E}}_{-_i}$ follow from equation $(15c)$ in the form

$$\hat{\overline{\mathcal{E}}}_{-_i}^{\,2} = \hat{\overline{\mathcal{I}}}_{-_i} \quad ; \quad \hat{\overline{\mathcal{E}}}_{-_i}^{\,2k} = \hat{\overline{\mathcal{I}}}_{-_i} \quad ; \quad \hat{\overline{\mathcal{E}}}_{-_i}^{\,2k+1} = \hat{\overline{\mathcal{E}}}_{-_i} \quad ; \quad e^{\pm\theta\,\hat{\overline{\mathcal{I}}}_{-_i}} = e^{\pm\theta}\,\hat{\overline{\mathcal{I}}}_{-_i} \quad ; \quad e^{\pm i\theta\,\hat{\overline{\mathcal{I}}}_{-_i}} = e^{\pm i\theta}\,\hat{\overline{\mathcal{I}}}_{-_i}$$

$$e^{\pm\theta\,\hat{\overline{\mathcal{E}}}_{-_i}} = \cosh\theta\,\hat{\overline{\mathcal{I}}}_{-_i} \pm \sinh\theta\,\hat{\overline{\mathcal{E}}}_{-_i} \qquad ; \qquad e^{\pm i\theta\,\hat{\overline{\mathcal{E}}}_{-_i}} = \cos\theta\,\hat{\overline{\mathcal{I}}}_{-_i} \pm i\sin\theta\,\hat{\overline{\mathcal{E}}}_{-_i} \tag{19e}$$

State transition algebraic operations on the antirotating photospin qubit state vectors $|\psi_{-in_i}\rangle$ , $|\,\overline{\phi}_{-in_i}\rangle$ are generated by the qubit state transition operator $\hat{\mathcal{E}}_{-_i}$ and identity operator $\hat{\mathcal{I}}_{-_i}$ according to equation $(15b)$ in the form

$$\hat{\overline{\mathcal{E}}}_{-_i}\,|\psi_{-in_i}\rangle = |\,\overline{\phi}_{-in_i}\rangle \qquad ; \qquad \hat{\overline{\mathcal{E}}}_{-_i}\,|\,\overline{\phi}_{-in_i}\rangle = |\psi_{-in_i}\rangle$$

$$\hat{\overline{\mathcal{I}}}_{-_i}\,|\psi_{-in_i}\rangle = |\psi_{-in_i}\rangle \qquad ; \qquad \hat{\overline{\mathcal{I}}}_{-_i}\,|\,\overline{\phi}_{-in_i}\rangle = |\,\overline{\phi}_{-in_i}\rangle \tag{19f}$$

The state eigenvectors $|\,\overline{\Psi}^{\,\pm}_{-in_i}\rangle$ determined as superpositions of the qubit state vectors $|\psi_{-in_i}\rangle$ , $|\,\overline{\phi}_{-in_i}\rangle$ follow from equation $(14f)$ in the form

$$|\,\overline{\Psi}^{\,+}_{-in_i}\rangle = \frac{1}{\sqrt{2(1-\bar{c}_{-in_i})}}(\,|\psi_{-in_i}\rangle + |\,\overline{\phi}_{-in_i}\rangle\,) \quad ; \quad |\,\overline{\Psi}^{\,-}_{-in_i}\rangle = \frac{1}{\sqrt{2(1+\bar{c}_{-in_i})}}(\,|\psi_{-in_i}\rangle - |\,\overline{\phi}_{-in}\rangle\,) \tag{19g}$$

which satisfy eigenvalue equations generated by the qubit state transition operator $\hat{\overline{\mathcal{E}}}_{-_i}$ and Hamiltonian $\overline{H}_{-_i}$ according to equation $(15h)$ in the form

$$\hat{\overline{\mathcal{E}}}_{-_i} \mid \overline{\Psi}^{\pm}_{-_in_i}\rangle = \pm \mid \overline{\Psi}^{\pm}_{-_in_i}\rangle \qquad ; \qquad \hat{\overline{\mathcal{I}}}_{-_i} \mid \overline{\Psi}^{\pm}_{-_in_i}\rangle = \mid \overline{\Psi}^{\pm}_{-_in_i}\rangle$$

$$\overline{H}_{-_i} \mid \overline{\Psi}^{\pm}_{-_in_i}\rangle = \overline{E}^{\pm}_{-_in_i} \mid \overline{\Psi}^{\pm}_{-_in_i}\rangle \qquad ; \qquad \overline{E}^{\pm}_{-_in_i} = \hbar\omega(n_i + \frac{3}{2}) \pm \hbar\overline{R}_{-_in_i} \qquad (19h)$$

We observe that the qubit state vectors $|\psi_{-_in_i}\rangle$ , $|\overline{\phi}_{-_in_i}\rangle$ defined in equation $(19b)$ are nonorthonormal, while the state eigenvectors $\mid \overline{\Psi}^{+}_{-_in_i}\rangle$ , $\mid \overline{\Psi}^{-}_{-_in_i}\rangle$ defined in equation $(19g)$ are orthonormal, satisfying the respective nonorthonormality or orthonormality relations

$$\langle\psi_{-_in_i}|\psi_{-_in_i}\rangle = 1 \; ; \quad \langle\phi_{-_in_i}|\phi_{-_in_i}\rangle = 1 \; ; \quad \langle\psi_{-_in_i}|\phi_{-_in_i}\rangle = -\bar{c}_{-_in_i} \; ; \quad \langle\phi_{-_in_i}|\psi_{-_in_i}\rangle = -\bar{c}_{-_in_i} \qquad (19i)$$

$$\langle \overline{\Psi}^{+}_{-_in_i}| \overline{\Psi}^{+}_{-_in_i}\rangle = 1 \quad ; \quad \langle \overline{\Psi}^{-}_{-_in_i}| \overline{\Psi}^{-}_{-_in_i}\rangle = 1 \quad ; \quad \langle \overline{\Psi}^{+}_{-_in_i}| \overline{\Psi}^{-}_{-_in_i}\rangle = 0 \quad ; \quad \langle \overline{\Psi}^{-}_{-_in_i}| \overline{\Psi}^{+}_{-_in_i}\rangle = 0 \quad (19j)$$

The time evolution operator $\overline{U}_{-_i}(t)$ generated by the Hamiltonian $\overline{H}_{-_i}$ and the time evolving antirotating photospin qubit state vectors $\mid \overline{\Psi}_{-_in_i}(t)\rangle$ and $\mid \overline{\Phi}_{-_in_i}(t)\rangle$ generated from the respective initial qubit state vectors $|\psi_{-_in_i}\rangle$ , $\mid \overline{\psi}_{-_in_i}\rangle$ by the time evolution operator $\overline{U}_{-_i}(t)$ follow from equations $(16a)$-$(16c)$ , $(16f)$ in the respective final forms

$$\overline{U}_{-_i}(t) = e^{-\frac{it}{\hbar}\overline{H}_{-_i}} \; : \quad \overline{U}_{-_i}(t) = e^{-i\omega_i\left(n_i+\frac{3}{2}\right)t}\left(\cos(\,\overline{R}_{-_in_i}t)\;\hat{\overline{\mathcal{I}}}_{-_i} - i\sin(\,\overline{R}_{-_in_i}t)\;\hat{\overline{\mathcal{E}}}_{-_i}\right) \qquad (19k)$$

$$\mid \overline{\Psi}_{-_in_i}(t)\rangle = \overline{U}_{-_i}(t)|\psi_{-_in_i}\rangle$$
$$\mid \overline{\Psi}_{-_in_i}(t)\rangle = e^{-i\omega_i(n+\frac{3}{2})t}\Big(\; \cos(\,\overline{R}_{-_in_i}t)|\psi_{-_in_i}\rangle - i\sin(\,\overline{R}_{-_in_i}t)|\,\overline{\phi}_{-_in_i}\rangle\;\Big) \qquad (19l)$$

$$\mid \overline{\Phi}_{-_in_i}(t)\rangle = \overline{U}_{-_i}(t)|\phi_{-_in_i}\rangle$$
$$\mid \overline{\Phi}_{-_in_i}(t)\rangle = e^{-i\omega_i(n+\frac{3}{2})t}\Big(\; \cos(\,\overline{R}_{-_in_i}t)|\,\overline{\phi}_{-_in_i}\rangle - i\sin(\,\overline{R}_{-_in_i}t)|\psi_{-_in_i}\rangle\;\Big) \qquad (19m)$$

Applying the qubit state transition operator $\hat{\overline{\mathcal{E}}}_{-_i}$ on the time evolving state vectors in equations $(19l)$ , $(19m)$ and using the qubit state transition algebraic operations from equation $(19f)$, we easily establish that the time evolving state vectors $\mid \overline{\Psi}_{-_in_i}(t)\rangle$ , $\mid \overline{\Phi}_{-_in_i}(t)\rangle$ satisfy qubit state transition algebraic operations obtained as

$$\hat{\overline{\mathcal{E}}}_{-_i}|\,\overline{\Psi}_{-_in_i}(t)\rangle = |\,\overline{\Phi}_{-_in_i}(t)\rangle \qquad ; \qquad \hat{\overline{\mathcal{E}}}_{-_i}|\,\overline{\Phi}_{-_in_i}(t)\rangle = |\,\overline{\Psi}_{-_in_i}(t)\rangle \qquad (19n)$$

The antirotating photospins in different lattice sites $i, j$ interact by coupling through their qubit state transition operators $\hat{\overline{\mathcal{E}}}_{-_i}$ , $\hat{\overline{\mathcal{E}}}_{-_j}$, yielding interaction Hamiltonian of the form

$$\overline{H}_{ij} = \hbar g_{ij}\hat{\overline{\mathcal{E}}}_{-_i}\hat{\overline{\mathcal{E}}}_{-_j} \qquad ; \qquad i\;,\;j\; = 1, 2, 3, ..., S \qquad (20a)$$

The total Hamiltonian of $S$ interacting antirotating photospins in the anti-Jaynes-Cummings optical lattice is easily obtained in the form

$$\overline{H}_{-} = \sum_{i=1}^{S}\overline{H}_{-_i} + \sum_{i\neq j=1}^{S}\hbar g_{ij}\hat{\overline{\mathcal{E}}}_{-_i}\hat{\overline{\mathcal{E}}}_{-_j} \qquad ; \qquad i\;,\;j\; = 1, 2, 3, ..., S \qquad (20b)$$

where $\overline{H}_{-_i}$ is the Hamiltonian of the antirotating photospin at site $i = 1, 2, ..., S$, defined in equation $(19d)$.

If we consider only nearest-neighbor interactions, then we can set $j = i + 1$ in equations $(20a)$ , $(20b)$ to obtain the total Hamiltonian for $S$ antirotating photospins with only nearest-neighbor interactions in the form

$$\overline{\mathcal{H}}_{-} = \sum_{i=1}^{S} \overline{H}_{-_i} + \sum_{i=1}^{S-1} \hbar g_{ij} \hat{\overline{\mathcal{E}}}_{-_i} \hat{\overline{\mathcal{E}}}_{-_{i+1}} \qquad ; \qquad i \ , \ j \ = 1, 2, 3, ..., S \tag{20c}$$

Substituting the photospin Hamiltonian $\overline{H}_{-_i}$ from equation $(19d)$ into equations $(20b)$ , $(20c)$ and separating the free evolution component to express the total Hamiltonian of the $S$ interacting antirotating photospins in the form

$$\overline{H}_{-} = \overline{H}_{-0} + \overline{H}_{-I} \qquad ; \qquad \overline{H}_{-0} = \sum_{i=1}^{S} \hbar\omega_i \left( n_i + \frac{3}{2} \right) \hat{\overline{\mathcal{I}}}_{-_i}$$

$$\overline{H}_{-I} = \sum_{i=1}^{S} \hbar \overline{R}_{-_{in_i}} \hat{\overline{\mathcal{E}}}_{-_i} + \sum_{i \neq j=1}^{S} \hbar g_{ij} \hat{\overline{\mathcal{E}}}_{-_i} \hat{\overline{\mathcal{E}}}_{-_j} \qquad ; \qquad i \ , \ j \ = 1, 2, 3, ..., S \tag{20d}$$

$$\overline{\mathcal{H}}_{-} = \overline{H}_{-0} + \overline{\mathcal{H}}_{-I} \quad ; \quad \overline{\mathcal{H}}_{-I} = \sum_{i=1}^{S} \hbar \overline{R}_{-_{in_i}} \hat{\overline{\mathcal{E}}}_{-_i} + \sum_{i \neq j=1}^{S-1} \hbar g_{ij} \hat{\overline{\mathcal{E}}}_{-_i} \hat{\overline{\mathcal{E}}}_{-_{i+1}} \qquad ; \qquad i \ , \ j \ = 1, 2, 3, ..., S \tag{20e}$$

where we now notice that in the eigenstate basis $| \ \overline{\Psi}^{\pm}_{-_{in_i}} \rangle$ where the qubit state transition operators $\hat{\overline{\mathcal{E}}}_{-_i}$ take values $\pm 1$ at each site according to the eigenvalue equations in equation $(19h)$, the photospin-photospin interaction Hamiltonian component $\overline{H}_{-I}$ in equation $(20d)$, which couples photospins at various sites over the entire lattice, can be identified with the one-dimensional Curie-Weiss model [26], while the interaction Hamiltonian component $\overline{\mathcal{H}}_{-I}$ in equation $(20e)$, which couples only nearest-neighbor photospins, can be identified with the one-dimensional Ising model [27 , 28] of interacting atomic spins in a linear crystal lattice, where, in contrast to the standard Curie-Weiss and Ising models, the driving field and coupling parameters $R_{-_{in_i}}$ , $g_{ij}$ in equations $(20d)$ , $(20e)$ take general site-dependent forms due to the quantum nature of the antirotating photospins generated through the anti-Jaynes-Cummings interaction at each site.

The time evolution operator $\overline{U}_{-}(t)$ generated by the total Hamiltonian $\overline{H}_{-}$ of the $S$ interacting antirotating photospins given in equation $(20b)$ is obtained in the form

$$\overline{U}_{-}(t) = e^{-\frac{it}{\hbar}\overline{H}_{-}} \qquad \Rightarrow \qquad \overline{U}_{-}(t) = e^{-\frac{it}{\hbar}\left( \ \sum_{i=1}^{S} \overline{H}_{-_i} + \sum_{i \neq j=1}^{S} \hbar g_{ij} \hat{\overline{\mathcal{E}}}_{-_i} \hat{\overline{\mathcal{E}}}_{-_j} \ \right)} \tag{20f}$$

which, due to the algebraic property that the qubit state transition operators $\hat{\overline{\mathcal{E}}}_{-_i}$ at all lattice sites commute and therefore commute also with the respective Hamiltonians $\overline{H}_{-_i}$ at all sites $i = 1, 2, ..., S$, can be factorized in the form

$$\overline{U}_{-}(t) = e^{-\frac{it}{\hbar}\sum_{i=1}^{S} \overline{H}_{-_i}} \ e^{-\frac{it}{\hbar}\sum_{i \neq j=1}^{S} \hbar g_{ij} \hat{\overline{\mathcal{E}}}_{-_i} \hat{\overline{\mathcal{E}}}_{-_j}} \tag{20g}$$

We note that the time evolution operator generated by the Hamiltonian $\mathcal{H}_{-}$ for nearest-neighbor interactions given in equation $(20c)$ is obtained by setting $j = i + 1$ in equation $(20f)$ or $(20g)$. The commutation of the qubit state transition operators at different sites means that the time evolution operator in equation (20$g$) can be evaluated exactly, noting that the commuting qubit state transition operators $\hat{\overline{\mathcal{E}}}_{-_i}$ , $\hat{\overline{\mathcal{E}}}_{-_j}$ at different sites act independently to generate qubit state transitions according to the algebraic operations in equation (19$f$).

We now complete the specification of the interacting photospin system by explicitly defining the initial state of the antirotating photospin at each lattice site. We note that, at each site $i = 1, 2, ..., S$, an antirotating photospin is specified by two qubit state vectors $|\psi_{-in_i}\rangle$ , $|\ \overline{\phi}_{-in_i}\rangle$ defined in equation (19$b$). But the qubit state vector $|\psi_{-in_i}\rangle = |-_i\rangle|n_i\rangle$ is a separable bare atom-field initial state vector which cannot effectively represent an antirotating photospin state, noting that an antirotating photospin is generally an entangled atom-field quasiparticle excitation state. Even though the qubit state vector $|\ \overline{\phi}_{-in_i}\rangle$ as defined in equation (19$b$) is an entangled atom-field state vector, it may not be ideal as an initial antirotating photospin state vector, since it does not represent the process of formation of an antirotating photospin from the basic atom-field separable state. Hence, we consider that a suitable initial state of an antirotating photospin in a lattice site can be determined as a superposition of the two basic qubit state vectors $|\psi_{-in_i}\rangle$ , $|\ \overline{\phi}_{-in_i}\rangle$. In this respect, we take the dynamical property that an atom-field anti-Jaynes-Cummings interaction generated from the initial state $|\psi_{-in_i}\rangle$ by the respective site Hamiltonian $\overline{H}_{-_i}$ in equation (19$d$) evolves over time $t$ to an entangled qubit state $|\ \overline{\Psi}_{-in_i}(t)\rangle$ determined through the time evolution operator $\overline{U}_{-_i}(t)$ according to equation (19$l$), which specifies an antirotating photospin qubit state at site $i = 1, 2, ..., S$ at any time $t$. We therefore identify the entangled qubit state vector $|\ \overline{\Psi}_{-in_i}(\tau_i)\rangle$ evolved from the initial atom-field state $|\psi_{-in_i}\rangle$ according to equation (19$l$) over a fixed interaction time $\tau_i$ of the formation of an antirotating photospin at site $i$ to be the appropriate initial state vector of the $i$-th photospin. Hence, we set $t = \tau_i$ in equation (19$l$) to determine the initial state vector $|\ \overline{\Psi}_{-in_i}(\tau_i)\rangle$ of the antirotating photospin at site $i$ in the form

$$|\ \overline{\Psi}_{-in_i}(\tau_i)\rangle = \overline{\eta}_i|\psi_{-in_i}\rangle - i\ \overline{\xi}_i|\ \overline{\phi}_{-in_i}\rangle \qquad ; \qquad \overline{\eta}_i = e^{-i\omega_i(n_i+\frac{3}{2})\tau_i}\cos(\ \overline{R}_{-in_i}\tau_i)$$

$$\overline{\xi}_i = e^{-i\omega_i(n_i+\frac{3}{2})\tau_i}\sin(\overline{R}_{-in_i}\tau_i) \qquad ; \qquad i = 1, 2, ..., S \tag{20$h$}$$

In performing the algebraic operations to determine the dynamical evolution of the interacting antirotating photospins generated by the Hamiltonian $\overline{H}_-$ in equation (20$b$) (or $\overline{\mathcal{H}}_-$ in equation (20$c$)) through the time evolution operator $\overline{U}_-(t)$ in equation (20$f$) , (20$g$), we apply the physical property that the qubit state vectors and state transition operators ( $|\psi_{-in_i}\rangle$ , $|\ \overline{\phi}_{-in_i}\rangle$ , $\hat{\overline{\mathcal{E}}}_{-_i}$) and ( $|\psi_{-jn_j}\rangle$ , $|\ \overline{\phi}_{-jn_j}\rangle$ , $\hat{\overline{\mathcal{E}}}_{+_j}$ ) specifying antirotating photospins at two different lattice sites $i, j$ are independent such that the state transition operators $\hat{\overline{\mathcal{E}}}_{-_i}$ , $\hat{\overline{\mathcal{E}}}_{-_j}$ at different sites commute ( $[\hat{\overline{\mathcal{E}}}_{-_i}\ ,\ \hat{\overline{\mathcal{E}}}_{-_j}] = 0$ ) and act independently only on the corresponding qubit state vectors ( $|\psi_{-in_i}\rangle$ , $|\ \overline{\phi}_{-in_i}\rangle$ ) , ( $|\psi_{-jn_j}\rangle$ , $|\ \overline{\phi}_{-jn_j}\rangle$ ), generating qubit state transitions according to the algebraic operations in equation (19$f$) at the respective lattice sites $i, j$.

### 3.2.1 Two interacting antirotating photospins

We illustrate these calculations by considering the simple model of two interacting antirotating photospins in a two-site ($S = 2$) anti-Jaynes-Cummings optical lattice. Setting $i = 1, 2$ in equation (19$d$) and $S = 2$ in equation (20$b$) provides the Hamiltonian $\overline{H}_-$ of the two interacting photospins in the form

$$\overline{H}_- = \overline{H}_{-_1} + \overline{H}_{-_2} + \hbar g\hat{\overline{\mathcal{E}}}_1\hat{\overline{\mathcal{E}}}_2 \qquad ; \qquad \overline{H}_{-_1} = \hbar\omega_1\left(n_1 + \frac{3}{2}\right)\ \hat{\overline{\mathcal{I}}}_{-_1} + \hbar\overline{R}_{-1n_1}\hat{\overline{\mathcal{E}}}_{-_1}$$

$$\overline{H}_{-_2} = \hbar\omega_2\left(n_1 + \frac{3}{2}\right)\ \hat{\overline{\mathcal{I}}}_{-_2} + \hbar\overline{R}_{-2n_2}\hat{\overline{\mathcal{E}}}_{-_2} \tag{21$a$}$$

which generates time evolution operator $\overline{U}_{-}(t)$ obtained from equation ($20g$) in the factorized form

$$\overline{U}_{-}(t) = \overline{U}_{-1}(t)\overline{U}_{-2}(t)\overline{U}_{-12}(t)$$

$$\overline{U}_{-1}(t) = e^{-\frac{it}{\hbar}\overline{H}_{-1}} \quad : \quad \overline{U}_{-1}(t) = e^{-i\omega_1\left(n_1+\frac{3}{2}\right)t}\left(\cos(\ \overline{R}_{-1n_1}t)\ \hat{\overline{\mathcal{I}}}_{-1} - i\sin(\ \overline{R}_{-1n_1}t)\ \hat{\overline{\mathcal{E}}}_{-1}\right)$$

$$\overline{U}_{-2}(t) = e^{-\frac{it}{\hbar}\overline{H}_{-2}} \quad : \quad \overline{U}_{-2}(t) = e^{-i\omega_2\left(n_2+\frac{3}{2}\right)t}\left(\cos(\ \overline{R}_{-2n_2}t)\ \hat{\overline{\mathcal{I}}}_{-2} - i\sin(\ \overline{R}_{-2n_2}t)\ \hat{\overline{\mathcal{E}}}_{-2}\right)$$

$$\overline{U}_{-12}(t) = e^{-igt\hat{\overline{\mathcal{E}}}_1\hat{\overline{\mathcal{E}}}_2} \quad : \quad \overline{U}_{-12}(t) = \cos(gt)\ \hat{\overline{\mathcal{I}}}_1\hat{\overline{\mathcal{I}}}_2 - i\sin(gt)\ \hat{\overline{\mathcal{E}}}_1\hat{\overline{\mathcal{E}}}_2 \tag{21b}$$

where in evaluating $\overline{U}_{-12}(t)$, we have used the property that the qubit state transition operators are independent, satisfying commutation relation $[\hat{\overline{\mathcal{E}}}_{-1}, \hat{\overline{\mathcal{E}}}_{-2}] = 0$ which gives general algebraic relations

$$(\ \hat{\overline{\mathcal{E}}}_{-1}\hat{\overline{\mathcal{E}}}_{-2}\ )^{2k} = \hat{\overline{\mathcal{I}}}_{-1}\hat{\overline{\mathcal{I}}}_{-2} \qquad ; \qquad (\ \hat{\overline{\mathcal{E}}}_{-1}\hat{\overline{\mathcal{E}}}_{-2}\ )^{2k+1} = \hat{\overline{\mathcal{E}}}_{-1}\hat{\overline{\mathcal{E}}}_{-2}$$

$$e^{-i\theta\hat{\overline{\mathcal{E}}}_{-1}\hat{\overline{\mathcal{E}}}_{-2}} = \cos\theta\ \hat{\overline{\mathcal{I}}}_{-1}\hat{\overline{\mathcal{I}}}_{-2} - i\sin\theta\ \hat{\mathcal{E}}_{+1}\hat{\mathcal{E}}_{+2} \tag{21c}$$

The initial state vectors $|\ \overline{\Psi}_{-1n_1}(\tau_1)\rangle$ , $|\ \overline{\Psi}_{-2n_2}(\tau_2)\rangle$ of the antirotating photospins at sites $i = 1, 2$, respectively, are determined according to equation ($20h$) in the form

$$|\ \overline{\Psi}_{-1n_1}(\tau_1)\rangle = \overline{\eta}_1|\psi_{-1n_1}\rangle - i\ \overline{\xi}_1|\ \overline{\phi}_{-1n_1}\rangle \qquad ; \qquad |\ \overline{\Psi}_{-2n_2}(\tau_2)\rangle = \overline{\eta}_2|\psi_{-2n_2}\rangle - i\ \overline{\xi}_2|\ \overline{\phi}_{-2n_2}\rangle$$

$$\overline{\eta}_i = e^{-i\omega_i(n_i+\frac{3}{2})\tau_i}\cos(\ \overline{R}_{-in_i}\tau_i) \qquad ; \qquad \overline{\xi}_i = e^{-i\omega_i(n_i+\frac{3}{2})\tau_i}\sin(\overline{R}_{-in_i}\tau_i) \qquad ; \qquad i = 1, 2 \tag{21d}$$

such that the total initial state vector $|\ \overline{\Psi}_{-\tau_1\tau_2}\rangle$ is obtained as a tensor product expressed here simply as

$$|\ \overline{\Psi}_{-\tau_1\tau_2}\rangle = |\ \overline{\Psi}_{-1n_1}(\tau_1)\rangle\ |\ \overline{\Psi}_{-2n_2}(\tau_2)\rangle\ = |\ \overline{\Psi}^{\,-}_{-\tau_1\tau_2}\rangle - i\ |\ \overline{\Phi}^{\,+}_{-\tau_1\tau_2}\rangle$$

$$|\ \overline{\Psi}^{\,-}_{-\tau_1\tau_2}\rangle = \overline{\eta}_1\overline{\eta}_2|\psi_{-1n_1}\rangle|\psi_{-2n_2}\rangle - \overline{\xi}_1\overline{\xi}_2|\ \overline{\phi}_{-1n_1}\rangle|\ \overline{\phi}_{-2n_2}\rangle$$

$$|\ \overline{\Phi}^{\,+}_{-\tau_1\tau_2}\rangle = \overline{\eta}_1\overline{\xi}_2|\psi_{-1n_1}\rangle\ |\overline{\phi}_{-2n_2}\rangle + \overline{\xi}_1\overline{\eta}_2|\ \overline{\phi}_{-1n_1}\rangle|\psi_{-2n_2}\rangle \tag{21e}$$

where we identify $|\ \overline{\Psi}^{\,-}_{-\tau_1\tau_2}\rangle$ , $|\ \overline{\Phi}^{\,+}_{-\tau_1\tau_2}\rangle$ as entangled nonorthogonal state vectors [ 29 , 30].

Applying the time evolution operator $\overline{U}_{-}(t)$ in equation ($21b$) on the initial state vector $|\ \overline{\Psi}_{-\tau_1\tau_2}\rangle$ in equation ($21e$) provides the general time evolving state vector $|\ \overline{\Psi}_{-\tau_1\tau_2}(t)\rangle$ describing the dynamical evolution of the two interacting antirotating photospins, which we follow the corresponding procedure presented in detail in equations ($12a$)-($12c$) to determine explicitly in the final form

$$|\ \overline{\Psi}_{-\tau_1\tau_2}(t)\rangle = \overline{U}_{-}(t)|\ \overline{\Psi}_{-\tau_1\tau_2}(t)\rangle \quad \Rightarrow \quad |\ \overline{\Psi}_{-\tau_1\tau_2}(t)\rangle = |\ \overline{\Psi}^{\,-}_{-\tau_1\tau_2}(t)\rangle - i\ |\ \overline{\Phi}^{\,+}_{-\tau_1\tau_2}(t)\rangle \tag{21f}$$

where we have evaluated

$$|\ \overline{\Psi}^{\,-}_{-\tau_1\tau_2}(t)\rangle = \overline{\eta}_1\overline{\eta}_2(\ \cos(gt)|\ \overline{\Psi}_{-1n_1}(t)\rangle|\ \overline{\Psi}_{-2n_2}(t)\rangle - i\ \sin(gt)|\ \overline{\Phi}_{-1n_1}(t)\rangle\ |\overline{\Phi}_{-2n_2}(t)\rangle\ ) -$$
$$\overline{\xi}_1\overline{\xi}_2(\ \cos(gt)|\ \overline{\Phi}_{-1n_1}(t)\rangle|\ \overline{\Phi}_{-2n_2}(t)\rangle - i\ \sin(gt)|\ \overline{\Psi}_{-1n_1}(t)\rangle|\ \overline{\Psi}_{-2n_2}(t)\rangle\ )$$

$$|\ \overline{\Phi}^{\,+}_{-\tau_1\tau_2}(t)\rangle = \overline{\eta}_1\overline{\xi}_2(\ \cos(gt)|\ \overline{\Psi}_{-1n_1}(t)\rangle|\ \overline{\Phi}_{-2n_2}(t)\rangle - i\ \sin(gt)|\ \overline{\Phi}_{-1n_1}(t)\rangle|\ \overline{\Psi}_{-2n_2}(t)\rangle\ ) +$$
$$\overline{\xi}_1\overline{\eta}_2(\ \cos(gt)|\ \overline{\Phi}_{-1n_1}(t)\rangle|\ \overline{\Psi}_{-2n_2}(t)\rangle - i\ \sin(gt)|\ \overline{\Psi}_{-1n_1}(t)\rangle|\ \overline{\Phi}_{-2n_2}(t)\rangle\ ) \tag{21g}$$

with the respective time evolving qubit state vectors $|\,\overline{\Psi}_{-1n_1}(t)\rangle$ , $|\,\overline{\Phi}_{-1n_1}(t)\rangle)$ and $|\,\overline{\Psi}_{-2n_2}(t)\rangle$ , $|\,\overline{\Phi}_{-2n_2}(t)\rangle)$ of the photospins at sites $i=1,2$, generated here according to $|\,\overline{\Psi}_{-in_i}(t)\rangle=\overline{U}_{-i}(t)|\psi_{-in_i}\rangle$ and $|\,\overline{\Phi}_{-in_i}(t)\rangle=\overline{U}_{-i}(t)|\,\overline{\phi}_{-in_i}\rangle$ determined explicitly by setting $i=1,2$ in the general forms in equations $(19l)$ , $(19m)$.

Substituting $|\,\overline{\Psi}^{\,-}_{-\tau_1\tau_2}(t)\rangle$ and $|\,\overline{\Phi}^{\,+}_{-\tau_1\tau_2}(t)\rangle$ from equation $(21g)$ into equation $(21f)$ provides the desired general time evolving state vector $|\,\overline{\Psi}_{-\tau_1\tau_2}(t)\rangle$ of the two interacting antirotating photospins. It is clear from the explicit forms of $|\,\overline{\Psi}^{\,-}_{-\tau_1\tau_2}(t)\rangle$ , $|\,\overline{\Phi}^{\,+}_{-\tau_1\tau_2}(t)\rangle$ in equation $(21g)$ that the general time evolving interacting photospin state vector $|\,\overline{\Psi}_{-\tau_1\tau_2}(t)\rangle$ determined in equation $(21f)$ is an exact entangled two-photospin state vector. The exact time evolving entangled state vector $|\,\overline{\Psi}_{-\tau_1\tau_2}(t)\rangle$ provides an accurate description of the internal dynamics of the two coupled antirotating photospins in the two-site anti-Jaynes-Cummings optical lattice. General statistical properties, including the dynamical evolution of the quantum entanglement of the system, can be determined accurately using the time evolving density operator $\hat{\overline{\rho}}_{+\tau_1\tau_2}(t)=|\,\overline{\Psi}_{-\tau_1\tau_2}(t)\rangle\langle\,\overline{\Psi}_{-\tau_1\tau_2}(t)|$, which is easily evaluated using the explicit results in equations $(21f)$ , $(21g)$. The entanglement property is particularly useful for the design and implementation of practical applications of the interacting photospin-photospin system in quantum information processing and related quantum technologies.

# 4 Conclusion

We have established that the quantum Rabi model of a two-level atom interacting with a single quantized electromagnetic mode can be interpreted as a model of photospins, defined as quantized photon-carrying two-state quasiparticle excitation modes formed in the Jaynes-Cummings and anti-Jaynes-Cummings interaction mechanisms. A photospin is specified by Hamiltonian, qubit state vectors, state eigenvectors, energy eigenvalues, conserved qubit state transition, identity, excitation number, $U(1)$-symmetry and $Z_2$/parity-symmetry operators defined within a two-dimensional state space spanned by the two nonorthonormal qubit state vectors. We have identified a photospin formed in the Jaynes-Cummings interaction where the atom couples to the rotating positive frequency field mode component as a rotating photospin and a photospin formed in the anti-Jaynes-Cummings interaction where the atom couples to the antirotating negative frequency field mode component as an antirotaing photospin. A photospin has algebraic properties exactly the same as the algebraic properties of an atomic spin (spin-$\frac{1}{2}$ particle) specified by two spin-up and spin-down qubit state vectors. The standard algebraic properties provide a simple procedure for determining exact time evolving qubit state vectors and density operators, which describe exact Rabi oscillations between the qubit state vectors on a circular trajectory of unit radius in the $yz$-plane. Rotating and antirotating photospins have similar dynamical properties, differing only in two fundamental features : (i) the internal dynamics of a rotating photospin is characterized by *red-sideband* state transitions due to the emission-absorption of positive energy photons by both atom and rotating positive frequency field mode component, while the internal dynamics of an antirotating photospin is characterized by *blue-sideband* state transitions due to the emission-absorption of positive energy photons by the atom and negative energy photons by the antirotating negative frequency field mode component (ii) in an interaction starting with the field mode initially in the vacuum state $|0\rangle$, the atom entering the electromagnetic cavity in a spin-up (excited) state $|+\rangle$ $(|e\rangle)$ couples only to the rotating positive frequency field mode component, thus activating only the Jaynes-Cummings interaction characterized by spontaneous emission of positive energy photons by the atom, triggering spontaneous red-sideband state transitions $|+0\rangle\rightarrow|-1\rangle$, while the atom entering the electromagnetic cavity in a spin-down (ground) state $|+\rangle$ $(|g\rangle)$ couples only to the antirotating negative frequency field mode component, thus activating only the anti-Jaynes-Cummings interaction characterized by spontaneous emission of negative energy photons by the field mode, triggering spontaneous blue-sideband state transitions $|-0\rangle \rightarrow |+1\rangle$. In general, in an interpretation of the quantum Rabi interaction where the quantized electromagnetic field mode is decomposed into rotating positive frequency and antirotating negative frequency components (a) *the rotating positive frequency component* is generally compatible with the *atomic spin-up state* and couples with it in a Jaynes-Cummings interaction mechanism with the field mode in any general number state $|n\rangle$, including the vacuum state $|0\rangle$ ; the atomic spin-down state does-not-couple to this rotating positive frequency component if the field mode is in the vacuum state $|0\rangle$, but couples to it only if it is in a nonzero number state $|n\rangle, n = 1, 2, ...$, a process characterized as conditional dynamics in [21], (b) *the antirotating negative frequency component* is generally compatible with the *atomic spin-down state* and couples with it in an anti-Jaynes-Cummings interaction mechanism with the field mode in any general number state $|n\rangle$, including the vacuum state $|0\rangle$ ; the atomic spin-up state does-not-couple to this antirotating negative frequency component if the field mode is in the vacuum state $|0\rangle$, but couples to it only if it is in a nonzero number state $|n\rangle, n = 1, 2, ...$, which again constitutes conditional dynamics.

Taking advantage of the equivalence of the algebraic properties of a photospin to the algebraic properties of an atomic spin (spin-$\frac{1}{2}$ particle), we have formulated exactly solvable models of interacting photospins on Jaynes-Cummings and anti-Jaynes-Cummings optical lattices, which in the state eigenvector basis where the state transition operators have eigenvalues $\pm 1$, may be interpreted as equivalent to one-dimensional Curie-Weiss or Ising models of interacting spins on linear crystal lattices in solids specified as appropriate. An exactly evaluated time evolving state vector of two interacting rotating photospins or antirotating photospins takes the form of entangled nonorthogonal state vectors, which has great potential for practical applications in the design and implementation of quantum information processing, quantum computation, quantum teleportation and communication, quantum state tomography and related quantum technologies.

Comprehensive studies of the dynamical properties and practical applications of individual photospins or systems of interacting photospins are underway and will be reported later.

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