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Abstract

$SU(N)$ symmetry groups are useful in formulating gauge theories of elementary particle interactions in quantum field theory. Gauge bosons and particle states are associated with the symmetry group generators. The accuracy of the physical structure and predictions of the gauge theory thus depends on the accuracy of the group generators. In this article, we present an accurate mathematical method for determining all generators of an $SU(N)$ symmetry group for any $N \geq 2$. Group generators are characterized as diagonal or non-diagonal symmetric and antisymmetric partners. There are $N(N-1)$ *traceless* non-diagonal symmetric and antisymmetric generators, $(N-1)$ *traceless* diagonal antisymmetric generators and $(N-1)$ *non-traceless* diagonal symmetric generators. An $SU(N)$ symmetry group is therefore specified by a total of N^2-1 standard *traceless* non-diagonal and diagonal symmetric and antisymmetric generators and $(N-1)$ *non-traceless* diagonal symmetric generators. The procedure is particularly effective in enumerating the correct generators of $SU(N)$ groups used in formulating various models of gauge theories of elementary particle interactions driven by fundamental forces of nature. As simple illustrations, we have applied the procedure to determine the generators of the $SU(2)$, $SU(3)$, $SU(4)$, $SU(5)$, $SU(6)$ and $SU(7)$ symmetry groups. While the $SU(2)$ and $SU(3)$ symmetry group generators have been used as the building blocks of the largely successful Standard Model (SM) of quantum field theory, the determination of the correct forms of the complete set of generators of the $SU(5)$ and $SU(6)$ groups, comprising the standard $5^2-1=24$ or $6^2-1=35$ traceless non-diagonal and diagonal symmetric and antisymmetric generators and the other emerging $5-1=4$ or $6-1=5$ non-traceless diagonal symmetric generators, respectively, reveals that the physical structure and predictions of the $SU(5)$ and $SU(6)$ models of the Grand Unified Theory needs a radical review, particularly with respect to the incorrect forms of the standard third, fourth and fifth traceless diagonal antisymmetric generators used in the current forms of the models.

1 Introduction

In my book, "Parametric Processes and Quantum States of Light" [1], I used a model of an N-level atom interacting with an external electromagnetic field to provide a simple mathematical method for determining all the N^2-1 generators of an $SU(N)$ symmetry group. Considering the method to be basic and possibly known to everybody with a working knowledge of Lie groups and Lie algebras, I applied it to determine only the standard $SU(2)$ and $SU(3)$ symmetry group generators as examples. Since the $SU(N)$ symmetry group idea was only a diversionary spin-off from the dynamical structure of the N-level atom-field interaction, I suspended further study of its algebraic properties, assuming more elaborate algebraic methods have been developed by workers in the field for enumerating and characterizing generators of the $SU(N)$ symmetry groups for all $N \geq 2$. I realized only recently that such an algebraic method has never been developed and the $SU(N)$ generators for $N \geq 4$ presented in the literature on gauge theories of elementary particle interactions [2-10] have just been obtained through "intelligent pattern building" using the known $SU(2)$ and $SU(3)$ generators as the basic building blocks. For the $SU(5)$ and $SU(6)$ symmetry groups, the pattern building procedure is presented in detail in [2 , 9 , 10] where the $SU(2)$, $SU(3)$ or $SU(4)$ generators are fitted into 2×2 , 3×3 or 4×4 matrix blocks in a 5×5 or 6×6 matrix grid to construct the desired set of $SU(5)$ or $SU(6)$ generators as appropriate. The arbitrariness of the pattern building procedure is clearly captured in the assumption stated in [2] that "there exists an infinite number of ways in which to choose the matrices which constitute the $SU(5)$ symmetry group generators", meaning that there can be a number of different sets of generators of an $SU(N)$, $N \geq 4$, for formulating a grand unified theory. This challenging situation has given me the justification and necessary motivation to lift the well defined mathematical method for determining $SU(N)$ symmetry group generators from the book [1] and elaborate it in this article for the benefit of readers who may not be keen to access the book, which is focussed on quantum optics. Since elements of $SU(N)$ symmetry groups govern dynamical evolution of particle interactions driven by gauge fields, we consider the group generators to be defined within an appropriately specified quantum state space.

2 Quantum state space of an $SU(N)$ symmetry group

We define the quantum state space of an $SU(N)$ symmetry group as an N-dimensional (integer $N = 2,3,4,....$) state space specified by N unit state vectors $|n\rangle$, $n = 1,2,3,....,N$ defined as column matrices, i.e., $N \times 1$ matrices, with entries 0 in all rows except entry 1 in the n-th row according to

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix} \quad ; \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix} \quad ; \quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ . \\ . \\ . \\ 0 \end{pmatrix} \quad ; \quad .... \quad ; \quad |N-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ . \\ 1 \\ 0 \end{pmatrix} \quad ; \quad |N\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ . \\ 1 \end{pmatrix} \qquad (1a)$$

These unit state vectors may specify either the energy level or spin angular momentum spectrum of a system of interacting particles and fields. To describe state transitions characterizing the dynamics of particles driven by external fields or particle-particle interactions mediated by gauge fields, the N-state quantum space is decomposed into $\frac{1}{2}N(N-1)$ 2-state subspaces nm specified by two unit state vectors $|n\rangle$, $|m\rangle$ coupled by transitions $n \leftrightarrow m$, $n = 1,2,....,m-1$, $m = 2,3,....,N$. A two-state subspace specified by two unit state vectors $|n\rangle$, $|m\rangle$ within the N-state quantum space is identified as a basic $SU(2)$ symmetry space. The subspace state vectors $|n\rangle, |m\rangle$ are orthonormal, satisfying the orthonormalization relation

$$\langle n|m\rangle = \delta_{nm} \qquad (1b)$$

Within the two-state $|n\rangle$, $|m\rangle$ subspace, the basic $SU(2)$ group generators I_{nm} , σ^z_{nm} , σ^x_{nm} , σ^y_{nm} are $N \times N$ basic matrices obtained as diagonal or non-diagonal symmetric and antisymmetric tensor products of the unit state vectors $|n\rangle$, $|m\rangle$.

We identify I_{nm} , σ^z_{nm} as the diagonal symmetric and antisymmetric generators obtained in the respective unit state vector tensor product forms

$$I_{nm} = |n\rangle\langle n| + |m\rangle\langle m| \qquad ; \qquad \sigma^z_{nm} = |n\rangle\langle n| - |m\rangle\langle m| \qquad (1c)$$

and σ^x_{nm} , σ^y_{nm} as the non-diagonal symmetric and antisymmetric generators obtained in the respective unit state vector tensor product forms

$$\sigma^x_{nm} = |n\rangle\langle m| + |m\rangle\langle n| \qquad ; \qquad \sigma^y_{nm} = -i(|n\rangle\langle m| - |m\rangle\langle n|) \qquad (1d)$$

where $n = 1,2,....,m-1$, $m = 2,3,....,N$ and the indices x,y,z denote components in the Cartesian coordinate axes as usual. The imaginary number factor $-i$ in the definition of the non-diagonal antisymmetric generator σ^y_{nm} effects the algebraic property that the symmetry group generators are interpreted as Hermitian quantum operators. We observe that the non-diagonal symmetric and antisymmetric generators σ^x_{nm} , σ^y_{nm} and the diagonal antisymmetric generator σ^z_{nm} are *traceless*, but the diagonal symmetric generator I_{nm} is *non-traceless.*

Applying the orthonormalization relation in equation $(1b)$, we identify the basic non-diagonal symmetric and antisymmetric $SU(2)$ generators σ^x_{nm} , σ^y_{nm} defined in equation $(1d)$ as state transition operators generating state algebraic operations

$$\sigma^x_{nm}|n\rangle = |m\rangle \quad ; \quad \sigma^x_{nm}|m\rangle = |n\rangle \quad ; \quad \sigma^y_{nm}|n\rangle = i|m\rangle \quad ; \quad \sigma^y_{nm}|m\rangle = -i|n\rangle \qquad (1e)$$

and the basic diagonal symmetric and antisymmetric $SU(2)$ generators I_{nm} , σ^z_{nm} defined in equation $(1c)$ as state identity and eigenvalue operators, respectively, generating state algebraic operations

$$I_{nm}|n\rangle = |n\rangle \quad ; \quad I_{nm}|m\rangle = |m\rangle \quad ; \quad \sigma^z_{nm}|n\rangle = |n\rangle \quad ; \quad \sigma^z_{nm}|m\rangle = -|m\rangle \qquad (1f)$$

Even though the standard $SU(N)$ generators are normally considered traceless, we shall maintain identification of the non-traceless identity matrix I_{nm} as the symmetric counterpart of the familiar traceless diagonal antisymmetric generator σ^z_{nm} for consistent characterization of the general $SU(N)$ symmetry group generators as *diagonal or non-diagonal symmetric and antisymmetric partners.*

2.1 Quantum state transitions: random and focal state transitions

The general dynamics of a system of particles and gauge fields in an N-state quantum space is characterized by transitions among the quantum states. As we have explained above, the N-state quantum space is decomposed into $\frac{1}{2}N(N-1)$ two-state subspaces, specified by paired state numbers nm, within which transitions occur, such that each transition couples only two states $|n\rangle$, $|m\rangle$ at a time. The decomposition of the N-state quantum space into $\frac{1}{2}N(N-1)$ two-state subspaces is determined by a rich spectrum of state transition processes, which we classify in two types, namely *random state* and *focal state* transition processes.

A transition from a given state into *any one* of the other $(N-1)$ states constitutes a *random state transition process.* It follows from the probabilistic nature of quantum mechanics that, unless determined otherwise by a specified interaction mechanism, quantum state transitions may generally be classified as random state transition processes.

The probabilistic nature of quantum mechanics also means that it is possible for a number of transitions to converge at a given state. A number of transitions from various states into a given state constitute a *focal state transition process.* A focal state transition process is specified by a *focal state*, which we define as a state into which transitions from a specified number of different states converge. The general N-state quantum space contains $(N-1)$ focal states $|m\rangle$, $m = 2, 3, ..., N$. There are $(m-1)$ transitions from $(m-1)$ different states $|n\rangle$, $n = 1, 2, ..., (m-1)$ into a focal state $|m\rangle$. This means that there are $(m-1)$ two-state subspaces sharing a common focal state $|m\rangle$ such that transitions from the different states $|n\rangle$ defined within each of the $(m-1)$ two-state subspaces all end up in the focal state $|m\rangle$, as illustrated here in the general form

$$1 \leq n \leq m-1 \ ; \ 2 \leq m \leq N \ : \quad |1\rangle \rightarrow |m\rangle \ ; \quad |2\rangle \rightarrow |m\rangle \ ; \quad |3\rangle \rightarrow |m\rangle \ ; \quad .... \ ; \quad |m-1\rangle \rightarrow |m\rangle$$

which can be put in a better diagrammatic form with all arrows from the various states $|1\rangle$, $|2\rangle$, ..., $|m-1\rangle$ terminating at the focal state $|m\rangle$. We arrive at a physical interpretation that a focal state transition process within an N-state quantum space is equivalent to a stream of electromagnetic radiation propagating from $(m-1)$ different levels $|n\rangle$, $n = 1, 2, ..., m-1$, into a focal level $|m\rangle$, $m = 2, 3, ..., N$ or a stream of light rays from various sources converging at a focal point in classical geometrical optics. A focal state in the quantum state space thus corresponds to a focal point in classical geometrical optics.

To provide a group theoretic interpretation, we define a focal state transition process specified by a focal state $|m\rangle$ as a collection of $(m-1)$ transitions into the focal state $|m\rangle$. Noting that each of the $(m-1)$ transitions occurs within a two-state subspace, we arrive at the algebraic property that the quantum space within which a focal state transition process occurs is composed of $(m-1)$ two-state subspaces connected to the focal state $|m\rangle$. Let us call the quantum space within which a focal state transition process occurs a *focal state transition space.* A focal state transition space specified by a focal state $|m\rangle$ is thus composed of $(m-1)$ two-state subspaces connected to the focal state $|m\rangle$. The physical property that there are $(N-1)$ focal states each specifying a focal state transition space means that there are $(N-1)$ focal state transition spaces in the general N-state quantum space. We can now provide a group theoretic interpretation that a focal state transition space corresponds to a Cartan subspace defined by a Cartan subalgebra and the number $(N-1)$ of the focal state transition spaces corresponds to the rank of the underlying $SU(N)$ symmetry group.

According to the group theoretic interpretation provided above, the classification of state transitions in an N-state quantum space as random or focal state transition processes provides a physical basis for determining the associated $SU(N)$ symmetry group generators, which we clarify below.

2.2 Determining $SU(N)$ generators in a quantum state space

Noting that elements of $SU(N)$ symmetry group govern the dynamical evolution of particle and field interactions, we consider that the group generators are associated with the random and focal state transition processes within the N-state quantum space. The decomposition of the N-state quantum space into $\frac{1}{2}N(N-1)$ two-state subspaces $|n\rangle$, $|m\rangle$ means that the generators of the underlying $SU(N)$ symmetry group are composed of the basic $SU(2)$ group generators I_{nm} , σ^z_{nm} , σ^x_{nm} , σ^y_{nm} determined within each of the $\frac{1}{2}N(N-1)$

two-state subspaces as diagonal or non-diagonal symmetric and antisymmetric tensor products of the unit state vectors $|n\rangle$, $|m\rangle$ according to equations $(1c)$, $(1d)$.

Due to their algebraic property as state transition operators generating the random and focal state transition processes within the N-state quantum space, *all* the $\frac{1}{2}N(N-1)$ non-diagonal symmetric $SU(2)$ generators σ^x_{nm} and *all* the $\frac{1}{2}N(N-1)$ non-diagonal antisymmetric $SU(2)$ generators σ^y_{nm} defined in equation $(1d)$ form a set of $\frac{1}{2}N(N-1)+\frac{1}{2}N(N-1)=N(N-1)$ *traceless* non-diagonal symmetric and antisymmetric generators $\lambda_j=\sigma^x_{nm}$, σ^y_{nm} of the $SU(N)$ group.

The algebraic property that the basic diagonal symmetric and antisymmetric $SU(2)$ generators I_{nm} , σ^z_{nm} are the respective state identity and eigenvalue operators within each of the $\frac{1}{2}N(N-1)$ two-state subspaces $|n\rangle$, $|m\rangle$, leaving the state vectors unchanged or only changing the sign of a state vector according to equations $(1e)$, $(1f)$ means that the basic $SU(2)$ generators I_{nm} , σ^z_{nm} do not separately constitute the expected effective diagonal symmetric and antisymmetric $SU(N)$ generators. We must now take into account the underlying dynamical property that there are focal state transition processes specified by $(N-1)$ focal states $|m\rangle$, $m=2,3,....,N$ within the N-state quantum space. It follows from the group theoretic interpretation provided above that an effective diagonal symmetric or antisymmetric $SU(N)$ generator is determined within each of the $(N-1)$ focal state transition spaces. The algebraic property that each focal state transition space is composed of $(m-1)$ two-state subspaces connected to the focal state $|m\rangle$ means that there are $(N-1)$ *resultant* traceless diagonal antisymmetric $SU(N)$ generators Λ_{m-1} composed as normalized sums of the basic traceless diagonal antisymmetric $SU(2)$ generators σ^z_{nm} from each of the $(m-1)$ two-state subspaces connected to a focal state $|m\rangle$ according to the composition formula

$$\Lambda_{m-1}=\sqrt{\frac{2}{m(m-1)}}\sum_{n=1}^{m-1}\sigma^z_{nm} \quad ; \quad n=1,2,....,m-1 \quad , \quad m=2,3,....,N \qquad (1g)$$

and $(N-1)$ *resultant* non-traceless diagonal symmetric $SU(N)$ generators $\overline{\Lambda}_{m-1}$ composed as normalized sums of the basic non-traceless diagonal symmetric $SU(2)$ generators I_{nm} according to the composition formula

$$\overline{\Lambda}_{m-1}=\sqrt{\frac{2}{m(m-1)}}\sum_{n=1}^{m-1}I_{nm} \quad ; \quad n=1,2,....,m-1 \quad , \quad m=2,3,....,N \qquad (1h)$$

We characterize the non-traceless diagonal symmetric generators $\overline{\Lambda}_{m-1}$ determined through the formula in equation $(1h)$ as the symmetric counterparts of the standard traceless diagonal antisymmetric generators Λ_{m-1} determined through the formula in equation $(1g)$.

In summary, an $SU(N)$ symmetry group has $N(N-1)$ traceless non-diagonal symmetric and antisymmetric generators $\lambda_j=\sigma^x_{nm}$, σ^y_{nm} determined using equation $(1d)$, $(N-1)$ traceless diagonal antisymmetric generators Λ_{m-1} determined using equations $(1c)$, $(1g)$ and $(N-1)$ non-traceless diagonal symmetric generators $\overline{\Lambda}_{m-1}$ determined using equations $(1c)$, $(1h)$, giving a total of $N(N-1)+(N-1)=N^2-1$ traceless non-diagonal and diagonal symmetric and antisymmetric generators and $(N-1)$ non-traceless diagonal symmetric generators.

In specifying *all* the $N(N-1)+2(N-1)=(N+2)(N-1)=N^2+N-2$ traceless and non-traceless $SU(N)$ generators, we consider it necessary to distinguish the notation for the non-diagonal generators obtained as the basic $SU(2)$ generators σ^x_{nm} , σ^y_{nm} from the notation for the $2(N-1)$ diagonal generators obtained as normalized sums of the basic diagonal generators σ^z_{nm} , I_{nm} from $(m-1)$ two-state subspaces connected to a focal state $|m\rangle$. We have therefore introduced a revised notation, denoting the $N(N-1)$ traceless non-diagonal symmetric and antisymmetric generators σ^x_{nm} , σ^y_{nm} obtained using equation $(1d)$ by the usual Gell-Mann symbols λ_j, $j=1,2,....,N(N-1)$, the $(N-1)$ traceless diagonal antisymmetric generators obtained as normalized sums of the basic traceless diagonal antisymmetric generators σ^z_{nm} using equations $(1c)$, $(1g)$ by the upper case symbols Λ_k , $k=1,2,....,N-1$ and the $(N-1)$ non-traceless diagonal symmetric generators obtained as normalized sums of the basic non-traceless diagonal generators I_{nm} using equations $(1c)$, $(1h)$ by the upper case symbols $\overline{\Lambda}_k$, $k=1,2,....,N-1$. We observe that only the $N(N-1)+(N-1)=N^2-1$ traceless non-diagonal and diagonal symmetric and antisymmetric generators λ_j , Λ_k , $j=1,2,....$, $N(N-1)$, $k=1,2,....$, $(N-1)$ are generally known to be the standard form of the full set of generators of an $SU(N)$ symmetry group [2-12]. The $(N-1)$ non-traceless diagonal symmetric generators $\overline{\Lambda}_k$, $k=1,2,....$, $(N-1)$ emerged for the first time in [1] and have been elaborated in the present article as the symmetric counterparts of the standard traceless diagonal antisymmetric generators Λ_k , $k=1,2,....$, $(N-1)$. We have not found a valid mathematical condition or algebraic property to justify ignoring or discarding the unfamiliar non-traceless diagonal symmetric matrices $\overline{\Lambda}_k$ from the general set of $SU(N)$ symmetry group generators, even though we are yet to determine their full algebraic properties and possible physical significance in $SU(N)$ gauge theory models.

We now illustrate the mathematical procedure by using the unit state vector definitions in equation $(1a)$, then applying the state vector tensor product relations in equations $(1c)$, $(1d)$ and the formulae in equations $(1g)$, $(1h)$ to determine generators of the $SU(N)$ groups for $N = 2-7$ as examples below. We have presented the generators in explicit detail including their definitions in terms of the basic $SU(2)$ generators to enhance clarity, which some readers may find unnecessary.

2.2.1 $SU(2)$ generators

$$N = 2 \ : \quad n = 1 \quad ; \quad m = 2 \ : \qquad |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad ; \qquad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The $2^2 - 1 = 3$ traceless non-diagonal and diagonal symmetric and antisymmetric $SU(2)$ generators are obtained as

$$\lambda_1 = \sigma^x_{12} \ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad ; \qquad \lambda_2 = \sigma^y_{12} \ = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad ; \qquad \Lambda_1 = \sigma^z_{12} \ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2a}$$

The $2 - 1 = 1$ non-traceless diagonal symmetric $SU(2)$ generator is obtained as

$$\overline{\Lambda}_1 = I_{12} \ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2b}$$

2.2.2 $SU(3)$ generators

$$N = 3 \ : \quad n = 1, 2 \quad ; \quad m = 2, 3 \ : \qquad |1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad ; \qquad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad ; \qquad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The $3^2 - 1 = 8$ traceless non-diagonal and diagonal symmetric and antisymmetric $SU(3)$ generators are obtained as

$$\lambda_1 = \sigma^x_{12} \ = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad ; \qquad \lambda_2 = \sigma^y_{12} \ = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad ; \qquad \lambda_3 = \sigma^x_{13} \ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \sigma^y_{13} \ = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad ; \qquad \lambda_5 = \sigma^x_{23} \ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad ; \qquad \lambda_6 = \sigma^y_{23} \ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\Lambda_1 = \sigma^z_{12} \ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad ; \qquad \Lambda_2 = \frac{1}{\sqrt{3}}(\sigma^z_{13} + \sigma^z_{23}) \ = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{3a}$$

The $3 - 1 = 2$ non-traceless diagonal symmetric $SU(3)$ generators are obtained as

$$\overline{\Lambda}_1 = I_{12} \ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad ; \qquad \overline{\Lambda}_2 = \frac{1}{\sqrt{3}}(I_{13} + I_{23}) \ = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{3b}$$

2.2.3 $SU(4)$ generators

$$N = 4 \ : \quad n = 1, 2, 3 \quad ; \quad m = 2, 3, 4$$

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad ; \qquad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad ; \qquad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad ; \qquad |4\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The $4^2 - 1 = 15$ traceless non-diagonal and diagonal symmetric and antisymmetric $SU(4)$ generators are obtained as

$$\lambda_1 = \sigma^x_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad ; \quad \lambda_2 = \sigma^y_{12} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad ; \quad \lambda_3 = \sigma^x_{13} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \sigma^y_{13} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad ; \quad \lambda_5 = \sigma^x_{14} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad ; \quad \lambda_6 = \sigma^y_{14} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_7 = \sigma^x_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad ; \quad \lambda_8 = \sigma^y_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad ; \quad \lambda_9 = \sigma^x_{24} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\lambda_{10} = \sigma^y_{24} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \quad ; \quad \lambda_{11} = \sigma^x_{34} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad ; \quad \lambda_{12} = \sigma^y_{34} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

$$\Lambda_1 = \sigma^z_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad ; \quad \Lambda_2 = \frac{1}{\sqrt{3}}(\sigma^z_{13} + \sigma^z_{23}) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Lambda_3 = \frac{1}{\sqrt{6}}(\sigma^z_{14} + \sigma^z_{24} + \sigma^z_{34}) = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \tag{4a}$$

The $4 - 1 = 3$ non-traceless diagonal symmetric $SU(4)$ generators are obtained as

$$\overline{\Lambda}_1 = I_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad ; \quad \overline{\Lambda}_2 = \frac{1}{\sqrt{3}}(I_{13} + I_{23}) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\overline{\Lambda}_3 = \frac{1}{\sqrt{6}}(I_{14} + I_{24} + I_{34}) = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \tag{4b}$$

Taking into account the revised notation we have introduced, denoting the $4(4-1) = 12$ traceless non-diagonal symmetric and antisymmetric generators by λ_j, $j = 1, 2, ..., N(N-1)$ and the $(4-1) = 3$ traceless diagonal antisymmetric generators by Λ_k, $k = 1, 2, ..., N-1$, we observe that the full set of 15 $(4^2 - 1)$ standard traceless non-diagonal and diagonal symmetric and antisymmetric $SU(4)$ generators we have determined above in equation $(4a)$ agree exactly with the corresponding $SU(4)$ generators obtained in [2 , 6 , 12] through an intelligent pattern building by fitting (embedding) $SU(2)$ or $SU(3)$ generators as appropriate into a 4×4 matrix grid. The pattern building procedure thus works accurately for $SU(4)$.

2.2.4 $SU(5)$ generators

$$N = 5 \ : \quad n = 1, 2, 3, 4 \quad ; \quad m = 2, 3, 4, 5$$

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad ; \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad ; \quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad ; \quad |4\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad ; \quad |5\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The $5^2-1=24$ traceless non-diagonal and diagonal symmetric and antisymmetric $SU(5)$ generators are obtained as

$$\lambda_1=\sigma^x_{12} = \begin{pmatrix} 0&1&0&0&0\\ 1&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix} \; ; \; \lambda_2=\sigma^y_{12} = \begin{pmatrix} 0&-i&0&0&0\\ i&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix} \; ; \; \lambda_3=\sigma^x_{13} = \begin{pmatrix} 0&0&1&0&0\\ 0&0&0&0&0\\ 1&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}$$

$$\lambda_4=\sigma^y_{13} = \begin{pmatrix} 0&0&-i&0&0\\ 0&0&0&0&0\\ i&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix} \; ; \; \lambda_5=\sigma^x_{14} = \begin{pmatrix} 0&0&0&1&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 1&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix} \; ; \; \lambda_6=\sigma^y_{14} = \begin{pmatrix} 0&0&0&-i&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ i&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}$$

$$\lambda_7=\sigma^x_{15} = \begin{pmatrix} 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 1&0&0&0&0 \end{pmatrix} \; ; \; \lambda_8=\sigma^y_{15} = \begin{pmatrix} 0&0&0&0&-i\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ i&0&0&0&0 \end{pmatrix} \; ; \; \lambda_9=\sigma^x_{23} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&1&0&0\\ 0&1&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}$$

$$\lambda_{10}=\sigma^y_{23} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&-i&0&0\\ 0&i&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix} \; ; \; \lambda_{11}=\sigma^x_{24} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&0 \end{pmatrix} \; ; \; \lambda_{12}=\sigma^y_{24} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&-i&0\\ 0&0&0&0&0\\ 0&i&0&0&0\\ 0&0&0&0&0 \end{pmatrix}$$

$$\lambda_{13}=\sigma^x_{25} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&1&0&0&0 \end{pmatrix} \; ; \; \lambda_{14}=\sigma^y_{25} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&-i\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&i&0&0&0 \end{pmatrix} \; ; \; \lambda_{15}=\sigma^x_{34} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&1&0\\ 0&0&1&0&0\\ 0&0&0&0&0 \end{pmatrix}$$

$$\lambda_{16}=\sigma^y_{34} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&-i&0\\ 0&0&i&0&0\\ 0&0&0&0&0 \end{pmatrix} \; ; \; \lambda_{17}=\sigma^x_{35} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&1&0&0 \end{pmatrix} \; ; \; \lambda_{18}=\sigma^y_{35} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&-i\\ 0&0&0&0&0\\ 0&0&i&0&0 \end{pmatrix}$$

$$\lambda_{19}=\sigma^x_{45} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&1\\ 0&0&0&1&0 \end{pmatrix} \; ; \; \lambda_{20}=\sigma^y_{45} = \begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&-i\\ 0&0&0&i&0 \end{pmatrix}$$

$$\Lambda_1=\sigma^z_{12} = \begin{pmatrix} 1&0&0&0&0\\ 0&-1&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix} \quad ; \quad \Lambda_2=\frac{1}{\sqrt{3}}(\sigma^z_{13}+\sigma^z_{23}) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&-2&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}$$

$$\Lambda_3=\frac{1}{\sqrt{6}}(\sigma^z_{14}+\sigma^z_{24}+\sigma^z_{34}) = \frac{1}{\sqrt{6}} \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&-3&0\\ 0&0&0&0&0 \end{pmatrix}$$

$$\Lambda_4 = \frac{1}{\sqrt{10}}(\sigma^z_{15} + \sigma^z_{25} + \sigma^z_{35} + \sigma^z_{45}) \ = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{pmatrix} \tag{5a}$$

The $5 - 1 = 4$ non-traceless diagonal symmetric $SU(5)$ generators are obtained as

$$\overline{\Lambda}_1 = I_{12} \ = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad ; \quad \overline{\Lambda}_2 = \frac{1}{\sqrt{3}}(I_{13} + I_{23}) \ = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\overline{\Lambda}_3 = \frac{1}{\sqrt{6}}(I_{14} + I_{24} + I_{34}) \ = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\overline{\Lambda}_4 = \frac{1}{\sqrt{10}}(I_{15} + I_{25} + I_{35} + I_{45}) \ = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \tag{5b}$$

We observe that, taking account of the revised notation in the present work, all the $5(5-1) = 20$ traceless non-diagonal symmetric and antisymmetric generators $\lambda_1, \lambda_2, ..., \lambda_{20}$ we have determined above in equation ($5a$) agree exactly with the corresponding traceless non-diagonal symmetric and antisymmetric $SU(5)$ generators obtained through the pattern building procedure in [2 , 5 , 7 , 8]. Of the $5 - 1 = 4$ traceless diagonal antisymmetric generators $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ determined above in equation ($5a$), only the first two, Λ_1 , Λ_2, have been determined accurately through the pattern building procedure in [2 , 5 , 7 , 8], while the third and fourth two traceless diagonal antisymmetric $SU(5)$ generators denoted there by L^{11} , L^{12} are completely different from the respective correct forms Λ_3, Λ_4 determined in equation ($5a$). In particular, we identify L^{11} in [2 , 5 , 7 , 8] as the basic diagonal generator σ^z_{45}, which is one of the four components of Λ_4 as determined above in the present work. We note that the form and normalization factors of the two traceless diagonal antisymmetric generators L^{11} , L^{12} determined in [2 , 5 , 7 , 8] as $L^{11} = \text{diag}(0, 0, 0, 1, -1)$, $L^{12} = \frac{1}{\sqrt{15}}\text{diag}(-2, -2, -2, 3, 3)$ do not agree with the correct form of the corresponding generators Λ_3 , Λ_4 obtained above in equation ($5a$). We observe that the normalization factor $\frac{1}{\sqrt{15}}$ of L^{12} belongs to the fifth traceless diagonal antisymmetric generator Λ_5 of the higher $SU(N)$, $N \geq 6$ groups, which we demonstrate using $SU(6)$, $SU(7)$ groups below for clarity.

An important physical consequence which emerges from the incorrect forms of the two traceless diagonal antisymmetric $SU(5)$ generators L^{11} , L^{12} determined and used in [2 , 5 , 7 , 8] and other related work is that the formulation, interpretation and predictions of the $SU(5)$ Grand Unified Theory have to be reviewed to take account of the correct traceless diagonal antisymmetric $SU(5)$ generators Λ_3 , Λ_4, particularly noting that in the 5-representation of fermions as defined in [2 , 5 , 7 , 8], the charge operator Q determined there as a linear combination of the generators L^{11} and L^{12} in the form $Q = \frac{1}{2}\left(L^{12} + \sqrt{\frac{5}{3}}L^{12}\right)$ is completely specified by the correct traceless diagonal antisymmetric generator Λ_3 obtained here in equation ($5a$) in the form

$$\Lambda_3 = \sqrt{\frac{3}{2}} \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \Rightarrow \qquad Q = -\sqrt{\frac{2}{3}}\ \Lambda_3 \tag{5c}$$

In general, the identification of various types of elementary particle states, fermions or bosons, with the $SU(5)$ generators, together with the value of the weak-interaction angle parameter $\sin^2\theta_W$ predicted within the $SU(5)$ grand unified theory, may change radically, especially if there is physical meaning attached to the algebraic properties of the generators as traceless or non-traceless, diagonal or non-diagonal, symmetric or antisymmetric, which would broaden the representation scheme to include the four non-traceless diagonal symmetric generators $\overline{\Lambda}_k$ listed separately in equation $(5b)$.

To clarify the observation we made above that the normalization factor $\frac{1}{\sqrt{15}}$ of L^{12} determined in [2 , 5 , 7 , 8] belongs to the fifth diagonal generator Λ_5 of the higher $SU(N)$, $N \geq 6$, groups, we determine the $SU(6)$, $SU(7)$ generators as examples and list only the respective diagonal generators Λ_k , $\overline{\Lambda}_k$ below.

2.2.5 $SU(6)$ generators

$$N = 6 \ : \quad n = 1,2,3,4,5 \quad ; \quad m = 2,3,4,5,6$$

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad ; \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad ; \quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad ; \quad |4\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad ; \quad |5\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad ; \quad |6\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Applying the general state vector tensor products in equations $(1c)$, $(1d)$) using the 6 unit state vectors defined above and the mathematical formulae in equations $(1g)$, $(1h)$, all the $6^2 - 1 = 35$ traceless non-diagonal and diagonal symmetric and antisymmetric $SU(6)$ generators λ_1 , $\lambda_2, ...,$ λ_{30} , Λ_1 , $\Lambda_2, ...,$ Λ_5 and the $6 - 1 = 5$ non-traceless diagonal symmetric $SU(6)$ generators $\overline{\Lambda}_1$, $\overline{\Lambda}_2, ...,$ $\overline{\Lambda}_5$ are easily obtained. For the specific purpose of comparing the normalization factors of the fifth traceless diagonal antisymmetric generator Λ_5 of $SU(6)$ with that of the fourth diagonal generator L^{12} of $SU(5)$ determined in [2 , 5 , 7 , 8] and generally used in the current models of the $SU(5)$ grand unified theory, we list only the 5 traceless diagonal antisymmetric $SU(6)$ generators Λ_1 , $\Lambda_2, ...,$ Λ_5 and their counterpart non-traceless diagonal symmetric generators $\overline{\Lambda}_1$, $\overline{\Lambda}_2, ...,$ $\overline{\Lambda}_5$ here.

The $6 - 1 = 5$ traceless diagonal antisymmetric $SU(6)$ generators are obtained as

$$\Lambda_1 = \sigma^z_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad ; \quad \Lambda_2 = \frac{1}{\sqrt{3}}(\sigma^z_{13} + \sigma^z_{23}) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Lambda_3 = \frac{1}{\sqrt{6}}(\sigma^z_{14} + \sigma^z_{24} + \sigma^z_{34}) = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Lambda_4 = \frac{1}{\sqrt{10}}(\sigma^z_{15} + \sigma^z_{25} + \sigma^z_{35} + \sigma^z_{45}) = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Lambda_5 = \frac{1}{\sqrt{15}}(\sigma^z_{16} + \sigma^z_{26} + \sigma^z_{36} + \sigma^z_{46} + \sigma^z_{56}) = \frac{1}{\sqrt{15}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 \end{pmatrix} \qquad (6a)$$

The $6-1=5$ non-traceless diagonal symmetric $SU(6)$ generators are obtained as

$$\overline{\Lambda}_1 = I_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad ; \quad \overline{\Lambda}_2 = \frac{1}{\sqrt{3}}(I_{13} + I_{23}) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\overline{\Lambda}_3 = \frac{1}{\sqrt{6}}(I_{14} + I_{24} + I_{34}) = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\overline{\Lambda}_4 = \frac{1}{\sqrt{10}}(I_{15} + I_{25} + I_{35} + I_{45}) = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\overline{\Lambda}_5 = \frac{1}{\sqrt{15}}(I_{16} + I_{26} + I_{36} + I_{46} + I_{56}) = \frac{1}{\sqrt{15}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix} \qquad (6b)$$

It is clear in equation $(6a)$ that the normalization factor $\frac{1}{\sqrt{15}}$ belongs to the fifth traceless diagonal antisymmetric generator Λ_5 of the $SU(6)$ group, revealing that this normalization factor is not appropriate for an $SU(5)$ generator as determined for L^{12} in [2 , 5 , 7 , 8]. We observe that in a formulation of $SU(6)$ Grand Unified Theory in [9], all the $6(6-1)=30$ traceless non-diagonal symmetric and antisymmetric $SU(6)$ generators corresponding to λ_1 , $\lambda_2, ..., \lambda_{30}$ and the first two diagonal antisymmetric generators corresponding to Λ_1 , Λ_2 (see equation $(6a)$) in our revised notation have been determined correctly, while the last three diagonal antisymmetric generators corresponding to Λ_3 , Λ_4 , Λ_5 are incorrect as compared to the correct forms determined here in equation $(6a)$.

2.2.6 $SU(7)$ generators

$$N = 7 \;:\quad n = 1,2,3,4,5,6 \quad ; \quad m = 2,3,4,5,6,7$$

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} ; \; |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} ; \; |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} ; \; |4\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} ; \; |5\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} ; \; |6\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} ; \; |7\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Applying the general state vector tensor products in equations $(1c)$, $(1d)$) using the 7 unit state vectors defined above and the mathematical formulae in equations $(1g)$, $(1h)$, all the $7^2-1=48$ traceless non-diagonal and diagonal symmetric and antisymmetric $SU(7)$ generators λ_1 , $\lambda_2, ..., \lambda_{42}$, Λ_1 , $\Lambda_2, ..., \Lambda_6$ and the $7-1=6$ non-traceless diagonal symmetric $SU(7)$ generators $\overline{\Lambda}_1$, $\overline{\Lambda}_2, ..., \overline{\Lambda}_6$ are easily obtained. Here again, we list only the 6 traceless diagonal antisymmetric $SU(7)$ generators Λ_1 , $\Lambda_2, ..., \Lambda_6$ and their counterpart non-traceless diagonal symmetric generators $\overline{\Lambda}_1$, $\overline{\Lambda}_2, ..., \overline{\Lambda}_6$ for the specific purpose of comparing the normalization factors of the fifth traceless diagonal antisymmetric generator Λ_5 of $SU(7)$ with that of the fourth diagonal generator L^{12} of $SU(5)$ determined in [2 , 5 , 7 , 8].

The $7-1=6$ traceless diagonal antisymmetric $SU(7)$ generators are obtained as

$$\Lambda_1 = \sigma^z_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad ; \quad \Lambda_2 = \frac{1}{\sqrt{3}}(\sigma^z_{13}+\sigma^z_{23}) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Lambda_3 = \frac{1}{\sqrt{6}}(\sigma^z_{14}+\sigma^z_{24}+\sigma^z_{34}) = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Lambda_4 = \frac{1}{\sqrt{10}}(\sigma^z_{15}+\sigma^z_{25}+\sigma^z_{35}+\sigma^z_{45}) = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Lambda_5 = \frac{1}{\sqrt{15}}(\sigma^z_{16}+\sigma^z_{26}+\sigma^z_{36}+\sigma^z_{46}+\sigma^z_{56}) = \frac{1}{\sqrt{15}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Lambda_6 = \frac{1}{\sqrt{21}}(\sigma^z_{17}+\sigma^z_{27}+\sigma^z_{37}+\sigma^z_{47}+\sigma^z_{57}+\sigma^z_{67}) = \frac{1}{\sqrt{21}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -6 \end{pmatrix} \qquad (7a)$$

The $7-1=6$ non-traceless diagonal symmetric $SU(7)$ generators are obtained as

$$\overline{\Lambda}_1 = I_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad ; \quad \overline{\Lambda}_2 = \frac{1}{\sqrt{3}}(I_{13}+I_{23}) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\overline{\Lambda}_3 = \frac{1}{\sqrt{6}}(I_{14}+I_{24}+I_{34}) = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\overline{\Lambda}_4 = \frac{1}{\sqrt{10}}(I_{15} + I_{25} + I_{35} + I_{45}) \ = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\overline{\Lambda}_5 = \frac{1}{\sqrt{15}}(I_{16} + I_{26} + I_{36} + I_{46} + I_{56}) \ = \frac{1}{\sqrt{15}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\overline{\Lambda}_6 = \frac{1}{\sqrt{21}}(I_{17} + I_{27} + I_{37} + I_{47} + I_{57} + I_{67}) \ = \frac{1}{\sqrt{21}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix} \qquad (7b)$$

Here again, it is clear in equation $(7a)$ that the normalization factor $\frac{1}{\sqrt{15}}$ belongs to the fifth traceless diagonal antisymmetric generator Λ_5 of the $SU(7)$ group, once again revealing that this normalization factor is not appropriate for an $SU(5)$ generator as determined for L^{12} in [2 , 5 , 7 , 8].

We note that all the $N(N-1) + N - 1 = N^2 - 1$ traceless non-diagonal and diagonal symmetric and antisymmetric $SU(N)$ generators λ_1 , $\lambda_2, ..., \lambda_{N(N-1)}$, Λ_1 , $\Lambda_2, ..., \Lambda_{N-1}$ obtained above for $N = 2, 3, 4, 5, 6, 7$ satisfy the standard $SU(N)$ generator normalization conditions $(i, j = 1, 2, ..., N(N-1)$; $k, l = 1, 2, ..., (N-1))$

$$\text{Tr}\lambda_j = 0 \quad ; \quad \text{Tr}\Lambda_k = 0 \quad ; \quad \text{Tr}\lambda_i\lambda_j = 2\delta_{ij} \quad ; \quad \text{Tr}\Lambda_k\Lambda_l = 2\delta_{kl} \quad ; \quad \text{Tr}\lambda_j\Lambda_k = 0 \qquad (8)$$

which have generally been applied in the various pattern building procedures for determining $SU(N)$ generators in [2-12].

It has emerged that all $(N-1)^2 - 1$ standard traceless non-diagonal and diagonal symmetric and antisymmetric generators of a lower $SU(N-1)$ symmetry group are contained in the corresponding set of $N^2 - 1$ standard traceless non-diagonal and diagonal symmetric and antisymmetric generators of the one-step higher $SU(N)$ symmetry group, noting that in this case, each $(N-1) \times (N-1)$ $SU(N-1)$ generator is extended to a corresponding $N \times N$ $SU(N)$ generator by simply adding a column of entries 0 to the right and a row of entries 0 at the bottom as clearly evident in the set of equations $(2a)$-$(7b)$. In addition to the $(N-1)^2 - 1$ generators which can as well be determined through 0-column and 0-row extensions of the corresponding generators of a one-step lower $SU(N-1)$ symmetry group, the $SU(N)$ group has an extra $2N-1$ distinct traceless non-diagonal and diagonal symmetric and antisymmetric generators, making a total of $(N-1)^2 - 1 + (2N-1) = N^2 - 1$ as expected. We observe that this algebraic property that all generators of a lower $SU(N-1)$ symmetry group are contained in the full set of generators of a one-step higher $SU(N)$ symmetry group, which remains valid on including the non-traceless diagonal symmetric generators of each symmetry group, has been applied to determine $SU(6)$ generators in [9 , 10], but the procedure does not provide a mathematical formula for determining the standard traceless diagonal antisymmetric generators and it has generally failed to determine the correct forms of the last $(N-1) - 2 = N - 3$ standard traceless diagonal antisymmetric generators for $N \geq 5$.

3 Conclusion

The work in this article, which is an elaboration of earlier work in the author's book [1] published in 2014, provides an accurate mathematical method which completely solves the problem of determining the correct generators of an $SU(N)$ symmetry group for any $N \geq 2$. The generators are obtained as symmetric and antisymmetric tensor products of unit state vectors $|n\rangle$, $n = 1, 2, ..., N$ defined in an N-state quantum space decomposed into $\frac{1}{2}N(N-1)$ two-state subspaces. Classifying the generators as traceless or non-traceless diagonal or non-diagonal symmetric and antisymmetric partners yields the usual $N^2 - 1$ standard traceless diagonal and non-diagonal symmetric and antisymmetric generators, together with an additional $(N-1)$ non-traceless diagonal symmetric generators, arising as the symmetric counterparts of the standard traceless diagonal antisymmetric generators. The $SU(N)$ generators are physically associated with state transitions classified as random and focal state transition processes within the N-state quantum space. While the traceless non-diagonal symmetric and antisymmetric state transition operators defined within each of the $\frac{1}{2}N(N-1)$ two-state subspaces are identified as $SU(N)$ generators, the group theoretic interpretation of a focal state transition space as a Cartan space defined by a Cartan subalgebra of an underlying $SU(N)$ Lie algebra leads to a precise mathematical composition formula for determining the standard traceless diagonal antisymmetric generators and their non-traceless diagonal symmetric counterparts. The determination of the correct sets of $SU(N)$ generators achieved through the accurate mathematical method presented in this article provides an algebraic platform for a thorough review of current models of $SU(N)$ Grand Unified Theories of elementary particle interactions, including the latest model of $SU(5)$ Grand Unified Theory Without Proton Decay [3 , 4].

Finally, we observe that the phenomenon of a focal state transition process composed of a collection of $(m-1)$ transitions, equivalent to a stream of (single mode) electromagnetic radiation from $(m-1)$ different sources, propagating into a common focal state $|m\rangle$ is an important physical property which brings a focal state in an N-state quantum space into direct correspondence with a focal point into which a stream of light rays from various sources converge in classical geometrical optics. There are $(N-1)$ focal states in an N-state quantum space.

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