A note on momentum and energy conservation in dynamics under arbitrary forces in classical Newtonian and relativistic mechanics

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In teaching mechanics course to first year undergraduate students, I have come across a simple, but very important, gap in the formulation of the principle of conservation of momentum and energy in the dynamics of a body of mass m, velocity $\mathbf{v}$ and linear momentum $\mathbf{p}$ governed by Newton's equation of motion

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} \tag{1a}$$

where, at position (displacement) vector $\mathbf{r}$, the velocity and linear momentum at time t are obtained according to the usual definitions

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \quad ; \quad \mathbf{p} = m\mathbf{v} \tag{1b}$$

The force $\mathbf{F}$ in equation $(1a)$ can take any general form, zero or non-zero static (constant or position-dependent), time-dependent, velocity-dependent or a combination of various types of forces, including frictional (dissipative) forces.

We identify Newton's equation $(1a)$ as the linear momentum transfer equation expressible in explicit form

$$d\mathbf{p} = \mathbf{F}dt \tag{1c}$$

Taking the dot product of equation $(1a)$ with the velocity $\mathbf{v}$, substituting $\mathbf{p} = m\mathbf{v}$ from equation $(1b)$ and reorganizing

$$\mathbf{v} \cdot \frac{d\mathbf{p}}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) \quad ; \quad v^2 = \mathbf{v} \cdot \mathbf{v} \tag{1d}$$

we introduce the kinetic energy T according to the usual definition

$$T = \frac{1}{2}mv^2 \tag{1e}$$

to transform Newton's equation into the energy transfer equation

$$\frac{dT}{dt} = \mathbf{F} \cdot \mathbf{v} \tag{1f}$$

which provides the rate of change of kinetic energy with time under the action of an external force $\mathbf{F}$, where $\mathbf{F} \cdot \mathbf{v}$ is the energy due to work done by the force per unit time, normally defined as *power*. The energy transfer equation $(1f)$ can be expressed in explicit form

$$dT = \mathbf{F} \cdot \mathbf{v}dt \tag{1g}$$

We observe that the r.h.s of equation $(1g)$ is expressible as $\mathbf{F} \cdot d\mathbf{r}$ only if the force $\mathbf{F}$ is independent of time t and velocity $\mathbf{v}$. This observation specifies the gap in the usual formulation of the energy conservation principle, where the assumption $\mathbf{F} \cdot \mathbf{v}dt = \mathbf{F} \cdot d\mathbf{r}$ [1 , 2 , 3] automatically excludes effects of time-dependent forces and velocity-dependent frictional forces. We address this problem below.

In classical mechanics textbooks or lecture notes [1 , 2 , 3], the standard formulation of the momentum and energy conservation principles is based on dynamics in *conservative force fields*, always excluding motion under time-dependent and frictional forces, which are usually classified under *non-conservative force fields*. However, using Newton's equation $(1a)$, we demonstrate in this note that application of a simple mathematical relation in basic calculus or standard integration of equations $(1c)$, $(1g)$ followed by differentiation of the general result with respect to time, generalizes the momentum and energy conservation principles to include dynamics under time-dependent and frictional forces.

From basic calculus, we rewrite $\mathbf{F}$, $\mathbf{F}\cdot\mathbf{v}$ on the r.h.s of equations $(1a)$, $(1f)$ in equivalent time derivative forms using the mathematical relations

$$\mathbf{F} = \frac{d}{dt}\int_0^t \mathbf{F}dt' \qquad ; \qquad \mathbf{F}\cdot\mathbf{v} = \frac{d}{dt}\int_0^t \mathbf{F}\cdot\mathbf{v}dt' \tag{2a}$$

where we identify the *impulse* $\mathbf{I}$ and total work W done by the force over the time duration t obtained as usual in the respective forms

$$\mathbf{I} = \int_0^t \mathbf{F}dt' \qquad ; \qquad W = \int_0^t \mathbf{F}\cdot\mathbf{v}dt' \tag{2b}$$

which we can substitute back into equation $(2a)$ to relate force to the impulse generated $\mathbf{I}$ and power $\mathbf{F}\cdot\mathbf{v}$ to the work done by the force over the time duration t according to

$$\mathbf{F} = \frac{d\mathbf{I}}{dt} \qquad ; \qquad \mathbf{F}\cdot\mathbf{v} = \frac{dW}{dt} \tag{2c}$$

These are standard results in basic mechanics, yet they do not seem to have been considered in formulating the linear momentum and energy conservation principles.

Substituting $\mathbf{F}$, $\mathbf{F}\cdot\mathbf{v}$ from equation $(2a)$ into equations $(1a)$, $(1f)$ as appropriate, we express the linear momentum and energy transfer equations in the respective conservation forms

$$\frac{d}{dt}\left(\mathbf{p} - \int_0^t \mathbf{F}dt'\right) = 0 \qquad ; \qquad \frac{d}{dt}\left(T - \int_0^t \mathbf{F}\cdot\mathbf{v}dt'\right) = 0 \tag{2d}$$

As stated above, these conservation equations can be derived directly by first integrating Newton's and energy transfer equations expressed in the alternative forms $(1c)$, $(1g)$ to obtain

$$\mathbf{p} - \mathbf{p}_0 = \int_0^t \mathbf{F}dt' \quad \Rightarrow \quad \left(\mathbf{p} - \int_0^t \mathbf{F}dt'\right) = \mathbf{p}_0 \quad ; \quad \mathbf{p}_0 = m\mathbf{v}_0$$

$$T - T_0 = \int_0^t \mathbf{F}\cdot\mathbf{v}dt' \quad \Rightarrow \quad \left(T - \int_0^t \mathbf{F}\cdot\mathbf{v}dt'\right) = T_0 \quad ; \quad T_0 = \frac{1}{2}mv_0^2 \tag{2e}$$

where $\mathbf{v}_0$, $\mathbf{p}_0$, T_0 are the initial velocity, linear momentum and kinetic energy, respectively. Differentiating the results in equation $(2e)$ with respect to time t provides the respective conservation equations obtained in equation $(2d)$.

Introducing the transferred linear momentum $\mathbf{p}_I$ and the transferred energy U, each related to the respective impulse generated $\mathbf{I}$ and work done W by the force according to

$$\mathbf{p}_I = -\mathbf{I} \ = -\int_0^t \mathbf{F}dt' \qquad ; \qquad U = -W \ = -\int_0^t \mathbf{F}\cdot\mathbf{v}dt' \tag{2f}$$

we obtain the instantaneous total linear momentum $\mathbf{P}$ and total energy E in the form

$$\mathbf{P} = \mathbf{p} + \mathbf{p}_I \ = \ \mathbf{p} - \int_0^t \mathbf{F}dt' \qquad ; \qquad E = T + U \ = \ T - \int_0^t \mathbf{F}\cdot\mathbf{v}dt' \tag{2g}$$

which we substitute into equation $(2d)$ as appropriate to obtain the linear momentum conservation equation in the final form

$$\mathbf{P} = \mathbf{p} + \mathbf{p}_I : \qquad \frac{d\mathbf{P}}{dt} = 0 \qquad \Rightarrow \qquad \mathbf{p} + \mathbf{p}_I = \mathbf{p}_0 \tag{2h}$$

and the energy conservation equation in the final form

$$E = T + U : \qquad \frac{dE}{dt} = 0 \qquad \Rightarrow \qquad T + U = T_0 \tag{2i}$$

where the initial linear momentum $\mathbf{p}_0$ and initial kinetic energy T_0 are defined in equation $(2e)$.

For completeness, we take the cross product of Newton's equation $(1a)$ with the position vector $\mathbf{r}$ to introduce orbital angular momentum $\mathbf{L}$ and torque $\mathbf{N}$ characterizing rotational dynamics according to the corresponding Newton's equation of motion

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \quad ; \quad \mathbf{N} = \mathbf{r} \times \mathbf{F} : \qquad \frac{d\mathbf{L}}{dt} = \mathbf{N} \tag{3a}$$

expressible in explicit orbital angular momentum transfer form

$$d\mathbf{L} = \mathbf{N}dt \tag{3b}$$

The orbital angular momentum conservation equation is easily determined in the form

$$\frac{d}{dt}\left(\mathbf{L} - \int_0^t \mathbf{N}dt'\right) = 0 \qquad \Rightarrow \qquad \mathbf{L} - \int_0^t \mathbf{N}dt' = \mathbf{L}_0 \qquad ; \qquad \mathbf{L}_0 = \mathbf{r}_0 \times \mathbf{p}_0 \tag{3c}$$

The fundamental physical property which arises in the above derivations is that the momentum and energy conservation relations in equations $(2h)$, $(2i)$, $(3c)$ apply to dynamics generated by external force(s) of general form, including time-dependent and frictional forces usually considered non-conservative. The generalization of momentum and energy conservation principles governing dynamics under arbitrary external forces is the main result of this note.

To gain a clear understanding, let us focus attention on the energy conservation principle. According to the definition in equation $(2f)$, the transferred energy U arising from the work done by the external force $\mathbf{F}$ over the time duration t can take any form, either being stored in the body as potential (or other form of internal) energy or being extracted from the body in various forms, including energy dissipated as heat, depending on the nature of the external force. It then follows from the definition of the instantaneous total energy E in equation $(2g)$ that the generalized energy conservation principle in equation $(2i)$ means that *the sum of the instantaneous kinetic energy and energy transferred in various forms as work done by the external force(s) remains constant, equal to the initial kinetic energy.* We demonstrate the general energy conservation principle by considering dynamics under three types of forces, the first being the familiar position-dependent force derivable from a field potential, the second an arbitrarily defined time-dependent force and the third one being the familiar velocity-dependent frictional force.

(i) Position-dependent force

To obtain the familiar form of the energy conservation principle as derived in standard classical mechanics textbooks or lecture notes [1 , 2 , 3], we consider an external force $\mathbf{F}$ derivable from a position-dependent field potential $V(\mathbf{r})$, commonly understood to constitute a *conservative force field*, where the force is obtained as a potential gradient in the form

$$\mathbf{F} = -\nabla V(\mathbf{r}) \tag{4a}$$

which we substitute into the definition of transferred energy U in equation $(2f)$ to obtain

$$U = \int_0^t \nabla V(\mathbf{r}) \cdot \mathbf{v}dt' = \int_{\mathbf{r}_0}^{\mathbf{r}} \nabla V(\mathbf{r}) \cdot d\mathbf{r}' \qquad \Rightarrow \qquad U = V(\mathbf{r}) - V(\mathbf{r}_0) \tag{4b}$$

where $V(\mathbf{r}_0)$ is the potential energy at initial position $\mathbf{r}_0$. Substituting U from equation $(4b)$ into equation $(2i)$ provides the energy conservation relation in a conservative force field in the familiar form [1 , 2 , 3]

$$T + V(\mathbf{r}) = T_0 + V(\mathbf{r}_0) \;=\; \text{constant} \tag{4c}$$

In the second and third examples of time-dependent and frictional forces usually classified under *non-conservative force fields*, we consider one-dimensional motion, which can easily be generalized to three-dimensional forms. We choose the time-dependent force in an arbitrary form $F = \alpha t^2 + \beta t$ and the frictional force in the common velocity-dependent form $F = -\alpha v$, where α, β are constants, while for one-dimensional motion v is the speed of the body.

(ii) Time-dependent force: $F = \alpha t^2 + \beta t$

We obtain the speed v from Newton's equation of motion in the form

$$\frac{dv}{dt} = \frac{1}{m}F \qquad \Rightarrow \qquad v = v_0 + \frac{1}{m}\int_0^t (\alpha t'^2 + \beta t')dt' \; = \; v_0 + \frac{1}{m}\left(\frac{\alpha}{3}t^3 + \frac{\beta}{2}t^2 \right) \tag{5a}$$

where v_0 is the initial speed. The kinetic energy is obtained using the speed v from equation $(5a)$ in the final form

$$T = \frac{1}{2}mv^2 \; = \; T_0 + v_0\left(\frac{\alpha}{3}t^3 + \frac{\beta}{2}t^2 \right) + \frac{1}{2m}\left(\frac{\alpha}{3}t^3 + \frac{\beta}{2}t^2 \right)^2 \tag{5b}$$

where T_0 is the initial kinetic energy. The transferred energy U as defined in equation $(2f)$ is evaluated using $F = \alpha t^2 + \beta t$ and v from equation $(5a)$ in the final form

$$U = -\int_0^t (\alpha t'^2 + \beta t')(v_0 + \frac{1}{m}(\frac{\alpha}{3}t^3 + \frac{\beta}{2}t^2)dt' \; = -v_0\left(\frac{\alpha}{3}t^3 + \frac{\beta}{2}t^2 \right) - \frac{1}{m}\left(\frac{\alpha^2}{18}t^6 + \frac{\alpha\beta}{6}t^5 + \frac{\beta^2}{8}t^4 \right)$$

$$\Rightarrow \qquad U = -v_0\left(\frac{\alpha}{3}t^3 + \frac{\beta}{2}t^2 \right) - \frac{1}{2m}\left(\frac{\alpha}{3}t^3 + \frac{\beta}{2}t^2 \right)^2 \tag{5c}$$

Using the kinetic energy T from equation $(5b)$ and the transferred energy U from equation $(5c)$, we obtain the instantaneous total energy E at any time t as

$$E = T + U = T_0 \tag{5d}$$

in agreement with the energy conservation relation in equation $(2i)$.

(iii) Frictional force : $F = -\alpha v$

We obtain the speed v from Newton's equation of motion in the form

$$\frac{dv}{dt} = \frac{1}{m}F \; = -\frac{\alpha}{m}v \qquad \Rightarrow \qquad v = v_0 e^{-\frac{\alpha}{m}t} \tag{5e}$$

where v_0 is the initial speed. The kinetic energy is obtained using the speed v from equation $(5e)$ in the final form

$$T = \frac{1}{2}mv^2 \; = \; T_0 e^{-\frac{2\alpha}{m}t} \tag{5f}$$

The transferred energy U as defined in equation $(2f)$ is evaluated using $F = -\alpha v$ and v from equation $(5e)$ or kinetic energy T from equation $(5f)$ in the final form

$$U = \alpha\int_0^t v^2 dt' \; = \frac{2\alpha}{m}\int_0^t T dt' \qquad \Rightarrow \qquad U = T_0\left(1 - e^{-\frac{2\alpha}{m}t} \right) \tag{5g}$$

Using the kinetic energy T from equation $(5f)$ and the transferred energy U from equation $(5g)$, we obtain the instantaneous total energy E at any time t as

$$E = T + U = T_0 \tag{5h}$$

in agreement with the energy conservation relation in equation $(2i)$. It is evident in equation $(5g)$ that the transferred energy U generated as work done by the frictional force does not dissipate to zero, but settles down to the initial kinetic energy T_0 in the long time limit.

The validity of the momentum and energy conservation principles in dynamics under arbitrary forces means that the physical property of conservative and non-conservative force fields has to be re-interpreted. The demonstration of energy conservation under a position-dependent force in example (i) leads to the physical interpretation that in a conservative force field, *all* the *energy arising as work* done by the external force(s) is *stored* in the system as potential energy or an appropriate form of internal energy, while the demonstration of energy conservation under time-dependent and frictional forces in examples (ii) and (iii) leads to the physical interpretation that in a non-conservative force field, *all* the *energy arising as work* done by the external force(s) is *transformed into various forms of energy released by the system*, except for some particular types of forces where part of the energy may be stored in the system over some time periods.

Noting that the momentum and energy conservation principles occur separately in equations $(2h)$, $(2i)$, having been determined separately from equations $(1a)$, $(1f)$, respectively, we now proceed to unify the two conservation principles by making an advancement from the Newtonian classical mechanics to the corresponding relativistic classical mechanics to determine a unified relativistic energy-momentum conservation principle, from which the Newtonian conservation principles in equations $(2h)$, $(2i)$ can be determined as the slow motion non-relativistic limit. In this note, we do not include the conservation of orbital angular momentum in the generalization to relativistic mechanics.

Relativistic mechanics is formulated within a four-dimensional spacetime frame specified by a spacetime coordinate (displacement) four-vector X defined in contravariant or covariant form in a cartesian coordinate system as

$$X^\mu = (X^0 \ , \ X^1 \ , \ X^2 \ , \ X^3) = (ct \ , \ \mathbf{r}) \qquad ; \qquad X_\mu = (X_0 \ , \ X_1 \ , \ X_2 \ , \ X_3) = (ct \ , \ -\mathbf{r})$$

$$X^0 = X_0 = ct \ , \ X^1 = x \ , \ X^2 = y \ , \ X^3 = z \quad ; \quad X_1 = -X^1 \ , \ X_2 = -X^2 \ , \ X_3 = -X^3 \qquad (6a)$$

with corresponding velocity four-vector V obtained as

$$V^\mu = \frac{dX^\mu}{dt} = (c \ , \ \mathbf{v}) \qquad ; \qquad V_\mu = \frac{dX_\mu}{dt} = (c \ , \ -\mathbf{v}) \qquad (6b)$$

and spacetime derivative four-vector ∂ defined by

$$\partial_\mu = \frac{\partial}{\partial X^\mu} = (\frac{1}{c}\frac{\partial}{\partial t} \ , \ \nabla) \qquad ; \qquad \partial^\mu = \frac{\partial}{\partial X_\mu} = (\frac{1}{c}\frac{\partial}{\partial t} \ , \ -\nabla) \qquad (6c)$$

where c is the speed of light. In this note, we denote four-vector and tensor components using Greek symbols taking values $\mu \ , \ \nu = 0 \ , \ 1, \ 2, \ 3$.

To develop the relativistic mechanics in a form consistent with the corresponding Newtonian mechanics in the slow motion limit under a given force, we consider a general force field created by a fundamental physical property of matter, normally defined as charge ξ, which generates the field potential four-vector A specified as usual by temporal (scalar potential ϕ) and spatial (vector potential $\mathbf{A}$) components in the form

$$A^\mu = (A^0 \ , \ \mathbf{A}) = (\phi \ , \ \mathbf{A}) \quad ; \qquad A_\mu = (A_0 \ , \ -\mathbf{A}) = (\phi \ , \ -\mathbf{A}) \ ; \quad A^0 = A_0 = \phi \ ; \quad \mathbf{A} = (A_x \ , \ A_y \ , \ A_z) \ (6d)$$

The generalized curl of the field potential four-vector provides the field intensity (field strength) tensor $F^{\mu\nu}$ obtained in the form

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \qquad ; \qquad \mu \ , \ \nu = 0 \ , \ 1, \ 2, \ 3 \qquad (6e)$$

which can be evaluated explicitly and decomposed into *Lorentz-boost and Lorentz-rotation* components in the form

$$F^{\mu\nu} = \mathbf{K}\cdot\mathbf{f} + \mathbf{L}\cdot\mathbf{d} \qquad (6f)$$

where $\mathbf{K} = (K_x \ , \ K_y \ , \ K_z)$, $\mathbf{L} = (L_x \ , \ L_y \ , \ L_z)$ are the respective Lorentz-boost and Lorentz-rotation symmetry generating matrices, revealing that in relativistic mechanics, the force field is specified by a *Lorentz-boost* field intensity $\mathbf{f}$ and a *Lorentz-rotation* field intensity $\mathbf{d}$ obtained as

$$\mathbf{f} = -\nabla A^0 - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \qquad ; \qquad \mathbf{d} = \nabla\times\mathbf{A} \qquad (6g)$$

We note that in cases of dynamics in force fields where the Lorentz-rotation field intensity component $\mathbf{d}$ does not arise in a manifest form, we may follow the definition in equation $(6g)$ to set the vector potential $\mathbf{A}$ equal to the gradient of a scalar φ according to

$$\mathbf{A} = \nabla\varphi \qquad \Rightarrow \qquad \mathbf{f} = -\nabla(\ A^0 + \frac{1}{c}\frac{\partial\varphi}{\partial t} \ ) \qquad ; \qquad \mathbf{d} = \nabla\times\nabla\varphi \ = 0 \qquad (6h)$$

The linear momentum four-vector P of a moving body of mass m and velocity four-vector V is obtained in contravariant or covariant form as

$$P^\mu = mV^\mu = (mc\ ,\ \mathbf{p}) \qquad ; \qquad P_\mu = mV_\mu = (mc\ ,\ -\mathbf{p}) \qquad ; \qquad \mathbf{p} = m\mathbf{v} \tag{7a}$$

which on introducing the relativistic energy $\mathcal{E}$ according to Einstein's energy-mass equivalence relation

$$\mathcal{E} = mc^2 \tag{7b}$$

is normally interpreted as the energy-momentum four-vector defined in the form

$$P^\mu = \left(\frac{\mathcal{E}}{c}\ ,\ \mathbf{p} \right) \qquad ; \qquad P_\mu = \left(\frac{\mathcal{E}}{c}\ ,\ -\mathbf{p} \right) \tag{7c}$$

For completeness, we observe that in relativistic mechanics, the mass m and time duration dt measured in an inertial frame co-moving with a body is related to the rest mass m_0 and *proper* time duration $d\tau$ measured in the rest frame in the respective forms

$$m = \gamma m_0 \qquad ; \qquad dt = \gamma d\tau \qquad ; \qquad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{7d}$$

The equation of motion of a body of mass m, charge ξ, velocity four-vector V and energy-momentum four-vector P in a force field specified by field intensity tensor $F^{\mu\nu}$ is obtained as

$$\frac{dP^\mu}{dt} = \frac{\xi}{c} F^{\mu\nu} V_\nu \tag{7e}$$

which on introducing a *power-force* four-vector F^μ defined by

$$F^\mu = \frac{\xi}{c} F^{\mu\nu} V_\nu \tag{7f}$$

takes the generalized Newtonian form

$$\frac{dP^\mu}{dt} = F^\mu \tag{7g}$$

which unifies equations $(1a)$ and $(1f)$ of Newtonian classical mechanics. We observe that the relativistic equation of motion $(7g)$ is derived in terms of the proper time τ in [2], where the power-force four-vector F^μ defined here in equation $(7f)$ is determined as the *Minkowski force*.

Setting $\mu = 0, 1, 2, 3$ in equation $(7f)$ and applying Einstein's summation convention over the repeated index $\nu = 0, 1, 2, 3$ in usual manner, we determine the power-force four-vector F^μ in explicit form

$$F^\mu = \left(\frac{\xi \mathbf{f} \cdot \mathbf{v}}{c}\ ,\ \xi (\ \mathbf{f} + \frac{\mathbf{v}}{c} \times \mathbf{d}\ ) \right) \ = \ \left(\frac{\mathbf{F} \cdot \mathbf{v}}{c}\ ,\ \mathbf{F} \right) \quad ; \quad \mathbf{F} = \xi (\ \mathbf{f} + \frac{\mathbf{v}}{c} \times \mathbf{d}\ ) \quad ; \quad \mathbf{F} \cdot \mathbf{v} = \xi \mathbf{f} \cdot \mathbf{v} \tag{7h}$$

Note that in a force field where the vector potential $\mathbf{A}$ is derivable as the gradient of a scalar function as defined in equation $(6h)$, the power-force four-vector takes the same form, with the force $\mathbf{F}$ appropriately determined according to

$$\mathbf{A} = \nabla \varphi \qquad \Rightarrow \qquad F^\mu = \left(\frac{\mathbf{F} \cdot \mathbf{v}}{c}\ ,\ \mathbf{F} \right) \quad ; \qquad \mathbf{F} = \xi \mathbf{f} \tag{7i}$$

The arbitrary nature of the fundamental physical property of matter ξ which we have defined as charge provides the flexibility to use the relativistic equation of motion $(7g)$ to describe dynamics conserving linear momentum and energy in any force field where the force driving the dynamics is derivable from a spacetime coordinate-dependent potential four-vector. For dynamics in an electromagnetic field, we identify the arbitrary charge ξ as the electric charge q, such that the Lorentz-boost field intensity component $\mathbf{f}$ is the electric field intensity $\mathbf{E}$, the Lorentz-rotation field intensity component $\mathbf{d}$ is the magnetic field intensity (magnetic induction) $\mathbf{B}$ and the force $\mathbf{F} = \xi(\ \mathbf{f} + \frac{\mathbf{v}}{c} \times \mathbf{d}\ )$ is the Lorentz force $\mathbf{F}_L = q(\ \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\ )$. Similarly, for dynamics in a gravitational field, we identify the arbitrary charge ξ as the gravitational mass m_g, commonly interpreted as equivalent to inertial mass m, such that the Lorentz-boost field intensity component $\mathbf{f}$ is the *gravitoelectric* field intensity (gravitational acceleration) $\mathbf{g}$, the Lorentz-rotation field intensity component $\mathbf{d}$ is the *gravitomagnetic* field intensity $\mathbf{D}$ and the force $\mathbf{F} = \xi(\mathbf{f} + \frac{\mathbf{v}}{c} \times \mathbf{d})$ is the *gravito-Lorentz* force $\mathbf{F}_{gL} = m_g(\mathbf{g} + \frac{\mathbf{v}}{c} \times \mathbf{D})$. We observe that a gravitational force field defined within an inertial spacetime frame arises as *gravitoelectromagnetic* field through linearization of Einstein's general relativity field equations, where the nature and form of gravitational charge m_g and gravitoelectromagnetic field intensities $\mathbf{g}$, $\mathbf{D}$ are clearly defined [4]. We note that, depending on the form of the vector potential $\mathbf{A}$, equations $(7h)$, $(7i)$ can provide any type of force derivable from a field potential in Newtonian or relativistic mechanics with appropriate specification of the arbitrary charge ξ. Only frictional forces which cannot be derived from spacetime coordinate-dependent field potentials are not provided by the definition of force in the set of equations $(6d)$-$(6h)$, $(7f)$. This means that, as in the Newtonian case, frictional forces can be specified separately in appropriate form, including derivation from dissipation function [2], then added to the equation of motion $(7g)$.

Applying either of the equivalent mathematical procedures presented in equations $(2a)$, $(2e)$ to the relativistic equation of motion $(2g)$, we obtain a general energy-momentum conservation equation in the form

$$\frac{d}{dt}(P^\mu - I^\mu) = 0 \qquad ; \qquad I^\mu = \int_0^t F^\mu dt' \tag{8a}$$

after introducing an impulse four-vector I^μ obtained as

$$I^\mu = \int_0^t F^\mu dt' \; = \left(\frac{W}{c} \; , \; \mathbf{I} \right) \quad ; \quad W = \int_0^t \mathbf{F} \cdot \mathbf{v} dt' \quad ; \quad \mathbf{I} = \int_0^t \mathbf{F} dt' \quad ; \quad \mathbf{F} = \xi(\mathbf{f} + \frac{\mathbf{v}}{c} \times \mathbf{d}) \tag{8b}$$

where W is the work done and $\mathbf{I}$ is the impulse generated by the force $\mathbf{F}$. Introducing transferred energy-momentum four-vector P_I^μ and instantaneous total energy-momentum four-vector $\mathcal{P}^\mu$ obtained as

$$P_I^\mu = -I^\mu \; = \left(\frac{U}{c} \; , \; \mathbf{p}_I \right) \qquad ; \qquad \mathcal{P}^\mu = P^\mu + P_I^\mu \tag{8c}$$

where $U = -W$ is the transferred energy and $\mathbf{p}_I = -\mathbf{I}$ is the transferred linear momentum as defined earlier in equation $(2f)$, we express the general energy-momentum conservation relation in the form

$$\mathcal{P}^\mu = P^\mu + P_I^\mu : \qquad \frac{d\mathcal{P}^\mu}{dt} = 0 \qquad \Rightarrow \qquad \mathcal{P}^\mu = P^\mu + P_I^\mu = P_0^\mu \tag{8d}$$

where P_0^μ is the energy-momentum four-vector in the rest frame obtained as

$$P_0^\mu = \left(\frac{\mathcal{E}_0}{c} \; , \; 0 \right) \qquad ; \qquad \mathcal{E}_0 = m_0 c^2 \tag{8e}$$

Note that in this relativistic case, we have assumed that the body is initially at rest, i.e., the motion starts off from a rest frame, in contrast to the Newtonian case where we can assume the body to be initially at rest or moving with an initial velocity. This situation does not pose any challenge, since in relativistic mechanics, a state of uniform motion with an initial velocity can be transformed into a state of rest in a rest frame.

Using the energy-momentum and transferred energy-momentum four-vectors P^μ , P_I^μ as defined in equations $(7c)$, $(8c)$, we determine the instantaneous total energy-momentum four-vector $\mathcal{P}^\mu$ defined in equation $(8c)$ in the explicit form

$$\mathcal{P}^\mu = \left(\frac{\mathcal{E} + U}{c} \; , \; \mathbf{p} + \mathbf{p}_I \right) \tag{8f}$$

Expressing the general energy-momentum conservation relation in equation $(8d)$ in the equivalent invariance form

$$\mathcal{P}_\mu \mathcal{P}^\mu = P_{0\mu} P_0^\mu \tag{8g}$$

and using P_0^μ , $\mathcal{P}^\mu$ from equations $(8e)$, $(8f)$, noting their covariant forms $P_{0\mu}$, $\mathcal{P}_\mu$, we obtain a general relativistic energy conservation relation in the form

$$(\mathcal{E} + U)^2 = m_0^2 c^4 + c^2 (\mathbf{p} + \mathbf{p}_I)^2 \tag{8h}$$

which includes the effects of forces driving the dynamics of a body in a force field, excluding effects of frictional forces which cannot be derived from spacetime coordinate-dependent field potential four-vectors specified in the present formulation. We recall that in the Newtonian case where the external forces can be specified as desired without reference to their origin, the formulation of the general energy conservation relation easily includes effects of frictional forces.

We observe that closing the gaps by including the effects of forces of arbitrary nature in the formulation of the momentum and energy or energy-momentum conservation relations reduces or, in some cases, eliminates discrepancies between theoretical predictions and experimental observations.

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