Quadrature fluctuation energy, effective Hamiltonians, quasi-particle modes and quantum phase transitions in the Rabi and Dicke models

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12 August 2017

Abstract

The main purpose of this article is to provide a simple physically motivated exact analytical procedure for determining effective Rabi and Dicke Hamiltonians, which have so far posed serious challenges in studies of quantum phase transition phenomena in the Rabi and Dicke models, where only the low-energy components have been obtained through approximate unitary transformation, or perturbation methods to determine the critical coupling constant for quantum phase transition. The full effective Hamiltonian automatically reveals the expected scaling of field mode energy and quadrature components near the critical quantum phase transition point and is easily reorganized as a sum of correlated field and quasi-particle spinor modes. A fully quantized Lipkin-Meshkov-Glick Hamiltonian emerges as an effective form of the Dicke Hamiltonian.

1 Introduction

The dynamics of a system of two-level atoms interacting with quantized electromagnetic field modes in the Rabi and Dicke models is characterized by a number of fundamental quantum mechanical properties such as squeezing, entanglement, population collapses and revivals, fractional revivals, quantum phase transitions and related universal scaling phenomena, etc [1-9]. In the present article, we focus attention on determination of effective Hamiltonians, which has proved to be a formidable task in studies of quantum phase transition phenomena in the Rabi and Dicke models [5-10]. Only the leading order or at best, next-to-leading order, effective low-energy components of the Hamiltonians have been determined through approximate unitary transformation, perturbation or Holstein-Primakoff bosonization methods under specified approximation limits. We address the challenge of determining the full effective Rabi and Dicke Hamiltonians in this article by identifying the basic interaction mechanisms and applying the physical interpretation that the atom-field coupling through quadrature components generates quadrature fluctuation energy which drives the internal dynamics of the system. Simple reorganization of the Hamiltonians to introduce the field mode and atomic spin quadrature fluctuation energy components allows application of unitary squeeze operator transformations to determine the full effective Hamiltonian. It is best to present the Rabi and Dicke models separately, starting with the Rabi model to develop the basic interaction mechanisms, then applying the same procedure to the more general Dicke model.

2 The Rabi model

The quantum Rabi model describes the dynamics of a quantized electromagnetic field mode interacting with a two-level atom generated by Hamiltonian [1 , 4 , 5]

$$H_R = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \hbar\omega_0 s_z + \hbar g(\hat{a} + \hat{a}^\dagger)(s_+ + s_-) \tag{1a}$$

where ω , $\hat{a}$, $\hat{a}^\dagger$ are the quantized field mode angular frequency, annihilation and creation operators, while ω_0 , s_z , s_+ , s_- are the atomic state transition angular frequency and operators, noting $\sigma_x = s_+ + s_-$,

$\sigma_y = -i(s_+ - s_-)$. We substitute $s_z = s_+ s_- - \frac{1}{2}$ to express the Rabi Hamiltonian in the form

$$H_R = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \hbar\omega_0\left(s_+ s_- - \frac{1}{2}\right) + \hbar g(\hat{a} + \hat{a}^\dagger)(s_+ + s_-) \tag{1b}$$

which displays an important algebraic symmetry of the field mode and atomic spin operators, showing that the ground state energy of the field mode is $\frac{1}{2}\hbar\omega$, while the ground state energy of the atomic spin is $-\frac{1}{2}\hbar\omega_0$. We use the Rabi Hamiltonian in the forms in equation (1a) , (1b) alternately in this work.

The field mode operators $\hat{a}$, $\hat{a}^\dagger$, $\hat{a}^\dagger\hat{a}$ and atomic spin operators s_- , s_+ , $s_z = s_+ s_- - \frac{1}{2}$ satisfy respective bosonic and spinor algebraic relations

$$[\ \hat{a}\ ,\ \hat{a}^\dagger\ ] = 1 \quad ; \quad [\ \hat{a}^\dagger\hat{a}\ ,\ \hat{a}^\dagger\ ] = \hat{a}^\dagger \quad ; \quad [\ \hat{a}^\dagger\hat{a}\ ,\ \hat{a}\ ] = -\hat{a}$$

$$[\ s_+\ ,\ s_-\ ] = 2s_z \quad ; \quad [\ s_z\ ,\ s_+\ ] = s_+ \quad ; \quad [\ s_z\ ,\ s_-\ ] = -s_- \tag{1c}$$

To gain insight into the internal dynamics of the Rabi model, we adopt a physical interpretation that the dynamics is generated through two alternate interaction mechanisms, one in which the field mode drives the atomic spin dynamics and the other in which the atomic spin drives the field mode dynamics. The Rabi Hamiltonian H_R can therefore be reorganized in two alternative forms

$$H_{aR} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \left\{\hbar\omega_0\left(s_+ s_- - \frac{1}{2}\right) + \hbar g(\hat{a} + \hat{a}^\dagger)(s_+ + s_-)\right\} \tag{1d}$$

or

$$H_{fR} = \hbar\omega_0\left(s_+ s_- - \frac{1}{2}\right) + \left\{\hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \hbar g(\hat{a} + \hat{a}^\dagger)(s_+ + s_-)\right\} \tag{1e}$$

where in the form H_{aR} in equation (1d), the second component is interpreted as the interactive Rabi Hamiltonian for an interaction mechanism in which the atomic spin dynamics is driven by the field mode, while in the form H_{fR} in equation (1e), the second component is interpreted as the interactive Rabi Hamiltonian for an interaction mechanism in which the field mode dynamics is driven by the atomic spin. In general, we identify H_{aR} in equation (1d) as the Rabi Hamiltonian for field mode driven atomic spin dynamics and H_{fR} in equation (1e) as the Rabi Hamiltonian for atomic spin driven field mode dynamics.

2.1 Field mode and atomic spin quadrature fluctuation energy

An important dynamical property which we notice immediately in equation (1a) and its equivalent forms (1b) , (1d) , (1e) is that the interaction component $\hbar g(\hat{a} + \hat{a}^\dagger)(s_+ + s_-)$ of the Rabi Hamiltonian H_R is generated by the coupling of the field mode quadrature component $\hat{x} = \hat{a} + \hat{a}^\dagger$ to the atomic spin quadrature component $\sigma_x = s_+ + s_-$, which leads to a physical interpretation that the internal dynamics of the Rabi system is driven by field mode and atomic spin quadrature fluctuations. To determine the quadrature fluctuation energy, we express the coupled quadrature interaction component in the equivalent alternative forms

$$\hbar g(\hat{a} + \hat{a}^\dagger)(s_+ + s_-) = \hbar\omega_0\left(\ 2\frac{g}{\omega_0}(\hat{a} + \hat{a}^\dagger)\frac{(s_+ + s_-)}{2}\ \right) \ = \ \hbar\omega\left(\ 2\frac{g}{\omega}(s_+ + s_-)\frac{(\hat{a} + \hat{a}^\dagger)}{2}\ \right) \tag{2a}$$

and complete the square in the respective forms of the Rabi interaction according to

$$\hbar g(\hat{a} + \hat{a}^\dagger)(s_+ + s_-) = \hbar\omega_0\left\{\ \left(\frac{g}{\omega_0}(\hat{a} + \hat{a}^\dagger) + \frac{(s_+ + s_-)}{2}\right)^2 - \frac{g^2}{\omega_0^2}(\hat{a} + \hat{a}^\dagger)^2 - \frac{(s_+ + s_-)^2}{4}\ \right\} \tag{2c}$$

or

$$\hbar g(\hat{a} + \hat{a}^\dagger)(s_+ + s_-) = \hbar\omega\left\{\ \left(\frac{g}{\omega}(s_+ + s_-) + \frac{(\hat{a} + \hat{a}^\dagger)}{2}\right)^2 - \frac{g^2}{\omega^2}(s_+ + s_-)^2 - \frac{(\hat{a} + \hat{a}^\dagger)^2}{4}\ \right\} \tag{2d}$$

depending on the atom-field interaction mechanism generated by the Rabi Hamiltonian according to equations (1d) , (1e), respectively.

Substituting equations (2c) , (2d) into equations (1d) , (1e), respectively, reorganizing atomic spin and field mode quadrature fluctuation terms and using

$$s_+ + s_- = \sigma_x \quad ; \quad \sigma_x^2 = I \quad ; \quad \frac{(s_+ + s_-)^2}{4} = \frac{1}{4}I \equiv \frac{1}{4} \tag{2e}$$

we express the Rabi Hamiltonian H_{aR} generating the field mode driven atomic spin dynamics in the form

$$H_{aR} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} - \frac{1}{4}\lambda^2 (\hat{a} + \hat{a}^\dagger)^2 \right) + \hbar\omega_0 \left(s_z - \frac{1}{4} \right) + \frac{1}{4}\hbar\omega_0 \left(\sigma_x + \frac{2g}{\omega_0}(\hat{a} + \hat{a}^\dagger) \right)^2 \tag{2f}$$

and the Rabi Hamiltonian H_{fR} generating the atomic spin driven field mode dynamics in the form

$$H_{fR} = \hbar\omega_0 \left(s_z - \frac{1}{4}\lambda^2 \right) + \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} - \frac{1}{4}(\hat{a} + \hat{a}^\dagger)^2 \right) + \frac{1}{4}\hbar\omega \left(\hat{a} + \hat{a}^\dagger + \frac{2g}{\omega}\sigma_x \right)^2 \tag{2g}$$

where we have introduced a dimensionless parameter λ defined by

$$\lambda = \frac{2g}{\sqrt{\omega_0 \omega}} \qquad \Rightarrow \qquad \frac{g^2}{\omega_0 \omega} = \frac{1}{4}\lambda^2 \tag{2h}$$

to simplify comparison with other work in the literature [5 , 6].

We now see that the atom-field interaction mechanism generated by each of the Rabi Hamiltonians H_{aR} , H_{fR} in equations $(2f)$, $(2g)$ provides field mode or atomic spin quadrature fluctuation energy, which modifies the free evolution field mode or atomic spin component. We interpret the last component of each Hamiltonian as the quadrature fluctuation energy of a *composite atom-field quasi-particle spinor / bosonic mode* formed in the Rabi interaction to be defined later. It turns out that the field mode and atomic spin quadrature fluctuation energies drive squeezing, quantum phase transition and related dynamical effects in the Rabi system.

In the form of the Rabi Hamiltonian H_{aR} in equation $(2f)$, it is evident that in the field mode driven atomic spin dynamics, the transfer of the field mode quadrature fluctuation energy leaves the field mode in a (nonlinear) squeezed state. We identify an effective low-energy component H_{aRLE} of the Hamiltonian H_{aR} in equation $(2f)$ obtained here as

$$H_{aRLE} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} - \frac{1}{4}\lambda^2 (\hat{a} + \hat{a}^\dagger)^2 \right) + \hbar\omega_0 s_z \tag{3a}$$

which is exactly equal to the effective low-energy Hamiltonian determined in recent studies of quantum phase transition in the Rabi model in [5], noting

$$\langle d|H_{aRLE}|d\rangle = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} - \frac{1}{4}\lambda^2 (\hat{a} + \hat{a}^\dagger)^2 \right) - \frac{1}{2}\hbar\omega_0 \tag{3b}$$

where $|d\rangle$ is the atomic spin-down state vector. In contrast to the derivation in [5] where the effective low-energy Hamiltonian is determined as an approximation in the limit of an infinite atom-field frequency ratio, $\frac{\omega_0}{\omega} \rightarrow \infty$, the derivation in the present work based on determining the underlying quadrature fluctuation energy generated in the atom-field interaction is exact and independent of the atom-field frequency ratio $\frac{\omega_0}{\omega}$.

In the atomic spin driven field mode dynamics generated by the Rabi Hamiltonian H_{fR} in equation $(2g)$, the transfer of the atomic spin quadrature fluctuation energy evaluated explicitly using the algebraic property in equation $(2e)$ takes a trivial form which only shifts the atomic spin ground state energy. We identify the effective low-energy component H_{fRLE} of H_{fR} in equation $(2g)$ obtained here as

$$H_{fRLE} = \hbar\omega_0 \left(s_z - \frac{1}{4}\lambda^2 \right) + \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \tag{3c}$$

which on averaging in the field mode vacuum (ground) state $|0\rangle$ takes the form

$$\langle 0|H_{fRLE}|0\rangle = \hbar\omega_0 \left(s_z - \frac{1}{4}\lambda^2 \right) + \frac{1}{2}\hbar\omega \tag{3d}$$

agreeing exactly with the Rabi limit ($N = 1$, $J_z = s_z$, $(J_- + J_+)^2 = (s_- + s_+)^2 = 1$) of the corresponding low-energy Lipkin-Meshkov-Glick (LMG) Hamiltonian determined in a study of quantum phase transition in the Dicke model in [6].

We observe that in isolating the effective low-energy components of the Hamiltonians H_{aR} , H_{fR} in equations $(2f)$, $(2g)$ in the respective forms H_{aREF} , H_{fRLE} in equations $(3a)$, $(3c)$, we have left out a quadrature fluctuation component which cancels out in the calculations to determine the full effective Rabi Hamiltonian presented below.

We can now proceed to determine the full effective Rabi Hamiltonian. In this respect, we note that, according to the Hamiltonian form H_{aR} in equation $(2f)$, the transfer of field mode quadrature fluctuation energy driving the atomic spin dynamics transforms the free field mode component to a non-trivial squeezed form, which can be diagonalized to determine the effective form. On the other hand, the Hamiltonian form H_{fR} in equation $(2g)$ shows that the transfer of atomic spin quadrature fluctuation energy driving the field mode dynamics leaves the free atomic spin component in the same diagonal form, but only shifts the energy, noting $s_z - \frac{1}{4}\lambda^2 = s_+s_- - \frac{1}{2} - \frac{1}{4}\lambda^2$. We therefore use only the Hamiltonian form H_{aR} to determine an underlying effective Rabi Hamiltonian.

2.2 Effective Rabi Hamiltonian from H_{aR}

According to equation $(2f)$, the Hamiltonian form H_{aR} for the field mode driven atomic spin dynamics has a non-trivial squeezed field mode component, which we can use as a starting point to determine the underlying effective Hamiltonian by applying a factorization procedure.

Expressing the squeezed field mode component in the explicit degenerate parametric down-conversion form (writing $\frac{1}{4}\lambda^2 = \frac{1}{2}\ (\frac{1}{2}\lambda^2)$)

$$\hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2} - \frac{1}{4}\lambda^2(\hat{a}+\hat{a}^\dagger)^2\right) = \hbar\omega\left(\ (1-\frac{1}{2}\lambda^2)(\hat{a}^\dagger\hat{a}+\frac{1}{2}) - \frac{1}{2}\left(\frac{1}{2}\lambda^2\right)\ (\hat{a}^2+\hat{a}^{\dagger 2})\ \right) \tag{4a}$$

we use the relation

$$\left(1-\frac{1}{2}\lambda^2\right)^2 - \left(\frac{1}{2}\lambda^2\right)^2 = 1-\lambda^2 \qquad \Rightarrow \qquad \left(\frac{1-\frac{1}{2}\lambda^2}{\sqrt{1-\lambda^2}}\right)^2 - \left(\frac{\frac{1}{2}\lambda^2}{\sqrt{1-\lambda^2}}\right)^2 = 1 \tag{4b}$$

to introduce hyperbolic functions $\cosh\eta$, $\sinh\eta$ defined by

$$\cosh\eta = \frac{1-\frac{1}{2}\lambda^2}{\sqrt{1-\lambda^2}} \quad ; \quad \sinh\eta = \frac{\frac{1}{2}\lambda^2}{\sqrt{1-\lambda^2}} \quad ; \quad \cosh^2\eta - \sinh^2\eta = 1 \tag{4c}$$

giving a suitable form

$$\hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2} - \frac{1}{4}\lambda^2(\hat{a}+\hat{a}^\dagger)^2\right) = \hbar\omega\sqrt{1-\lambda^2}\left(\ \cosh\eta\ \hat{a}^\dagger\hat{a} - \frac{1}{2}\sinh\eta\ (\hat{a}^2+\hat{a}^{\dagger 2}) + \frac{1}{2}\cosh\eta\ \right) \tag{4d}$$

We obtain a factorization

$$\cosh\eta\ \hat{a}^\dagger\hat{a} - \frac{1}{2}\sinh\eta\ (\hat{a}^2+\hat{a}^{\dagger 2}) = \left(\cosh\frac{1}{2}\eta\ \hat{a}^\dagger - \sinh\frac{1}{2}\eta\ \hat{a}\right)\left(\cosh\frac{1}{2}\eta\ \hat{a} - \sinh\frac{1}{2}\eta\ \hat{a}^\dagger\right) - \sinh^2\frac{1}{2}\eta \tag{4e}$$

which can be generated through a squeeze operator $\mathcal{S}(\eta)$ using the general operator expansion relation

$$e^{\hat{P}}\hat{Q}e^{-\hat{P}} = \hat{Q} + \frac{1}{1!}[\hat{P},\hat{Q}] + \frac{1}{2!}[\hat{P},[\hat{P},\hat{Q}]] + \frac{1}{3!}[\hat{P},[\hat{P},[\hat{P},\hat{Q}]]] + ... \tag{4f}$$

by setting $\hat{Q} = \hat{a}$, $\hat{a}^\dagger$, $\mathcal{S}(\eta) = e^{-\hat{P}}$, $\mathcal{S}^\dagger(\eta) = e^{\hat{P}}$ according to

$$\mathcal{S}(\eta) = e^{\frac{1}{4}\eta(\hat{a}^{\dagger 2}-\hat{a}^2)} : \quad \mathcal{S}^\dagger(\eta)\hat{a}\mathcal{S}(\eta) = \cosh\frac{1}{2}\eta\ \hat{a} - \sinh\frac{1}{2}\eta\ \hat{a}^\dagger \quad ; \quad \mathcal{S}^\dagger(\eta)\hat{a}^\dagger\mathcal{S}(\eta) = \cosh\frac{1}{2}\eta\ \hat{a}^\dagger - \sinh\frac{1}{2}\eta\ \hat{a} \tag{4g}$$

Substituting equation $(4g)$ into equation $(4e)$, using the result in equation $(4d)$ and evaluating a term $\frac{1}{2}\cosh\eta - \sinh^2\frac{1}{2}\eta = \frac{1}{2}$, we obtain the diagonal form

$$\hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2} - \frac{1}{4}\lambda^2(\hat{a}+\hat{a}^\dagger)^2\right) = \mathcal{S}^\dagger(\eta)\ \hbar\omega\sqrt{1-\lambda^2}\left(\ \hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\ \mathcal{S}(\eta) \tag{4h}$$

We use equation $(4c)$ to determine the squeeze parameter η in the form

$$\tanh\eta = \frac{\frac{1}{2}\lambda^2}{1-\frac{1}{2}\lambda^2} \qquad \Rightarrow \qquad e^{-2\eta} = 1-\lambda^2 \quad ; \quad \eta = -\frac{1}{2}\ln(1-\lambda^2) \tag{4i}$$

Substituting equation $(4h)$ into equation $(2f)$ and applying the squeeze operator unitarity property $\mathcal{S}^\dagger(\eta)\mathcal{S}(\eta) = \mathcal{S}(\eta)\mathcal{S}^\dagger(\eta) = 1$ to express

$$\frac{1}{4}\hbar\omega_0\left(\sigma_x + \frac{2g}{\omega_0}(\hat{a}+\hat{a}^\dagger)\right)^2 = \mathcal{S}^\dagger(\eta)\left\{\mathcal{S}(\eta)\frac{1}{4}\hbar\omega_0\left(\sigma_x + \frac{2g}{\omega_0}(\hat{a}+\hat{a}^\dagger)\right)^2\mathcal{S}^\dagger(\eta)\right\}\mathcal{S}(\eta) \tag{5a}$$

we obtain the Rabi Hamiltonian H_{aR} in the diagonal form

$$H_{aR} = \mathcal{S}^\dagger(\eta)\mathcal{H}_{aR}\mathcal{S}(\eta) \qquad \Rightarrow \qquad \mathcal{S}(\eta)H_{aR}\mathcal{S}^\dagger(\eta) = \mathcal{H}_{aR} \tag{5b}$$

where $\mathcal{H}_{aR}$ is the desired underlying effective Rabi Hamiltonian for the field mode driven atomic spin dynamics obtained in the form

$$\mathcal{H}_{aR} = \hbar\omega\sqrt{1-\lambda^2}\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right) + \hbar\omega_0\left(s_z - \frac{1}{4}\right) + \frac{1}{4}\hbar\omega_0\left(\sigma_x + \frac{2g}{\omega_0}(\ \hat{a}(\eta)+\hat{a}^\dagger(\eta)\ )\right)^2 \tag{5c}$$

after introducing (reversed) squeezed field mode operators $\hat{a}(\eta)$, $\hat{a}^\dagger(\eta)$ obtained as

$$\hat{a}(\eta) = \mathcal{S}(\eta)\hat{a}\mathcal{S}^\dagger(\eta) = \cosh\frac{1}{2}\eta\ \hat{a} + \sinh\frac{1}{2}\eta\ \hat{a}^\dagger \qquad ; \qquad \hat{a}^\dagger(\eta) = \mathcal{S}(\eta)\hat{a}^\dagger\mathcal{S}^\dagger(\eta) = \cosh\frac{1}{2}\eta\ \hat{a}^\dagger + \sinh\frac{1}{2}\eta\ \hat{a} \tag{5d}$$

Using equation $(5d)$ and applying the squeeze parameter definition from equation $(4i)$, we obtain

$$\hat{a}(\eta)+\hat{a}^\dagger(\eta) = e^{\frac{1}{2}\eta}(\hat{a}+\hat{a}^\dagger) \ = \ \frac{\hat{a}+\hat{a}^\dagger}{(1-\lambda^2)^{\frac{1}{4}}} \tag{5e}$$

Substituting $\hat{a}(\eta)+\hat{a}^\dagger(\eta)$ from equation $(5e)$ into equation $(5c)$ and expanding the squared component provides the underlying effective Rabi Hamiltonian $\mathcal{H}_{aR}$ in the final form (reintroducing $s_z = s_+s_- - \frac{1}{2}$, $\sigma_x = s_- + s_+$)

$$\mathcal{H}_{aR} = \hbar\omega\sqrt{1-\lambda^2}\left(\ \hat{a}^\dagger\hat{a}+\frac{1}{2}\right) + \hbar\omega_0\left(s_+s_- - \frac{1}{2}\right) + \frac{\hbar g}{(1-\lambda^2)^{\frac{1}{4}}}(\hat{a}+\hat{a}^\dagger)(s_-+s_+) + \frac{\hbar g^2}{\omega_0\sqrt{1-\lambda^2}}(\hat{a}+\hat{a}^\dagger)^2 \tag{5f}$$

Comparing equations $(5f)$ and $(1a)$, $(1b)$ reveals that the underlying effective Rabi Hamiltonian $\mathcal{H}_{aR}$ in equation $(5a)$ is just the original Rabi Hamiltonian H_R in equation $(1a)$, $(1b)$, but with the free field mode component scaled by a factor $\sqrt{1-\lambda^2}$, field mode quadrature component scaled by a factor $(1-\lambda^2)^{-\frac{1}{4}}$ (see equation $(5e)$) and an additional nonlinear field mode quadrature fluctuation component.

According to equation $(5b)$, the effective Rabi Hamiltonian $\mathcal{H}_{aR}$ in equation $(5f)$ is related to the corresponding Hamiltonian H_{aR} for the field mode driven atomic spin dynamics by a unitary transformation generated by a field mode squeeze operator $\mathcal{S}(\eta)$. In addition, we easily establish that the effective Hamiltonian $\mathcal{H}_{aR}$ is also directly related to the original Rabi Hamiltonian H_R in equation $(1a)$, $(1b)$ by a similar field mode squeezing transformation. Reorganizing the effective Hamiltonian $\mathcal{H}_{aR}$ in equation $(5f)$ and introducing a dimensionless parameter ξ defined by

$$\xi = \frac{\lambda}{\sqrt{\lambda^2-1}} \qquad \Rightarrow \qquad \xi^2 = \frac{\lambda^2}{\lambda^2-1} = -\frac{\lambda^2}{1-\lambda^2} \tag{6a}$$

we obtain the form

$$\mathcal{H}_{aR} = \hbar\omega\sqrt{1-\lambda^2}\left(\ \hat{a}^\dagger\hat{a}+\frac{1}{2}-\frac{1}{4}\xi^2(\hat{a}+\hat{a}^\dagger)^2\right) + \hbar\omega_0\left(s_+s_- - \frac{1}{2}\right) + \frac{\hbar g}{(1-\lambda^2)^{\frac{1}{4}}}(\hat{a}+\hat{a}^\dagger)(s_-+s_+) \tag{6b}$$

with squeezed field mode component similar to that in H_{aR} in equation $(2f)$. Applying the factorization procedure developed above then gives $(s_+s_- - \frac{1}{2} = s_z)$, $s_- + s_+ = \sigma_x)$

$$\mathcal{H}_{aR} = \mathcal{R}^\dagger(\beta)\left\{\hbar\omega\sqrt{1-\lambda^2}\sqrt{1-\xi^2}\left(\ \hat{a}^\dagger\hat{a}+\frac{1}{2}\right) + \hbar\omega_0 s_z + \frac{\hbar g}{(1-\lambda^2)^{\frac{1}{4}}(1-\xi^2)^{\frac{1}{4}}}(\hat{a}+\hat{a}^\dagger)\sigma_x\right\}\mathcal{R}(\beta) \tag{6c}$$

where the squeeze operator $\mathcal{R}(\beta)$ is defined by

$$\mathcal{R}(\beta) = e^{\frac{1}{4}\beta(\hat{a}^{\dagger 2}-\hat{a}^2)} \qquad ; \qquad \tanh\beta = \frac{\frac{1}{2}\xi^2}{1-\frac{1}{2}\xi^2} \qquad \Rightarrow \qquad e^{-2\beta} = 1-\xi^2 \quad ; \quad \beta = -\frac{1}{2}\ln(1-\xi^2) \tag{6d}$$

Using ξ from equation ($6a$) provides

$$\sqrt{1-\lambda^2}\sqrt{1-\xi^2}=1 \qquad ; \qquad \frac{1}{(1-\lambda^2)^{\frac{1}{4}}(1-\xi^2)^{\frac{1}{4}}}=1 \tag{6e}$$

which we substitute into equation ($6c$) to obtain the expected unitary transformation

$$\mathcal{H}_{aR}=\mathcal{R}^{\dagger}(\beta)\ H_R\ \mathcal{R}(\beta) \qquad \Rightarrow \qquad \mathcal{R}(\beta)\ \mathcal{H}_{aR}\ \mathcal{R}^{\dagger}(\beta)=H_R \tag{6f}$$

where H_R is the original Rabi Hamiltonian in equation ($1a$). Hence, $\mathcal{R}(\beta)$ defined in equation ($6d$) is an alternative squeeze operator compared to $\mathcal{S}(\eta)$ defined in equation ($4g$) for transforming the original and the underlying effective Rabi Hamiltonians into each other. Substituting ξ from equation ($6a$) into equation ($6d$) reveals that the alternative squeeze parameter β is related to the original squeeze parameter η in equation ($4i$) according to $\beta=-\eta$, thus giving squeeze operator relation $\mathcal{R}(\beta)=\mathcal{S}^{-1}(\eta)$.

2.2.1 Quasi-particle spinor modes in the effective Rabi interaction

Based on the nonlinear form of the effective Rabi Hamiltonian $\mathcal{H}_{aR}$ in equation ($5f$), we can provide two alternative physical interpretations of the Rabi dynamics. Following a model of Dicke Hamiltonian for studying quantum phase transitions in [10], we provide a physical interpretation that the effective Hamiltonian $\mathcal{H}_{aR}$ in equation ($5f$) is the Rabi Hamiltonian for nonlinear quantized light (field mode) interacting with a single two-level atom (atomic spin).

In an alternative interpretation, we recall the earlier physical interpretation that the transfer of the field mode quadrature fluctuation energy drives the atomic spin to a composite atom-field quasi-particle spinor mode. Hence, in the effective Rabi Hamiltonian $\mathcal{H}_{aR}$ in equation ($5f$), we introduce a quasi-particle spinor mode specified by hermitian conjugate *effective spinor operators* S_- , S_+, which are atomic spin state lowering and raising operators displaced by the effective field mode quadrature component obtained as

$$S_-=s_-+\frac{g}{\omega_0(1-\lambda^2)^{\frac{1}{4}}}(\hat{a}+\hat{a}^{\dagger}) \qquad ; \qquad S_+=s_++\frac{g}{\omega_0(1-\lambda^2)^{\frac{1}{4}}}(\hat{a}+\hat{a}^{\dagger}) \tag{7a}$$

Taking the normal order product S_+S_-, multiplying by $\hbar\omega_0$ and subtracting $\frac{1}{2}\hbar\omega_0$ provides the quasi-particle spinor mode Hamiltonian $\mathcal{H}_S$ in the form

$$\mathcal{H}_S=\hbar\omega_0\left(\ S_+S_--\frac{1}{2}\ \right) \ = \ \hbar\omega_0\left(s_+s_--\frac{1}{2}\right)+\frac{\hbar g}{(1-\lambda^2)^{\frac{1}{4}}}(\hat{a}+\hat{a}^{\dagger})(s_-+s_+)+\frac{\hbar g^2}{\omega_0\sqrt{1-\lambda^2}}(\hat{a}+\hat{a}^{\dagger})^2 \tag{7b}$$

which we substitute into equation ($5f$) to express the effective Rabi Hamiltonian $\mathcal{H}_{aR}$ as a sum of correlated effective squeezed field and quasi-particle spinor mode components in the form

$$\mathcal{H}_{aR}=\hbar\omega\sqrt{1-\lambda^2}\left(\ \hat{a}^{\dagger}\hat{a}+\frac{1}{2}\right)+\hbar\omega_0\left(\ S_+S_--\frac{1}{2}\ \right) \tag{7c}$$

The correlation property follows from the algebraic property that the two components of the effective Hamiltonian $\mathcal{H}_{aR}$ in equation ($7c$) do not commute.

3 The Dicke Hamiltonian

The standard Dicke model describes the interaction between a single quantized field mode and a collection of N identical, but distinguishable two-level atoms generated by the basic Dicke Hamiltonian [6-9]

$$H_D=\hbar\omega\left(\hat{a}^{\dagger}\hat{a}+\frac{1}{2}\right)+\hbar\omega_0J_z+\frac{g}{\sqrt{N}}(\hat{a}+\hat{a}^{\dagger})(J_++J_-) \tag{8a}$$

after introducing the collective total spin angular momentum operators J_z , J_+ , J_- for the N atomic spins obtained as

$$J_z=\sum_{j=1}^{N}s_j^z \quad ; \quad J_+=\sum_{j=1}^{N}s_j^+ \quad ; \quad J_-=\sum_{j=1}^{N}s_j^- \quad ; \quad \Sigma_x=J_++J_- \quad ; \quad \Sigma_y=-i(J_+-J_-) \tag{8b}$$

where the single atomic spin operators s_j^z , s_j^+ , s_j^- in standard notation have usual meanings as defined in the Rabi model above. We observe that the Rabi Hamiltonian H_R in equation ($1a$) is obtained as the simplest form of the Dicke Hamiltonian H_D in equation ($8a$) for a single quantized field mode interacting with a single atomic spin ($N = 1$).

The collective atomic spin angular momentum operators in the Dicke model satisfy spinor algebraic relations

$$[\, J_+ \,,\, J_- \,] = 2J_z \qquad ; \qquad [\, J_z \,,\, J_+ \,] = J_+ \qquad ; \qquad [\, J_z \,,\, J_- \,] = -J_- \tag{8c}$$

exactly the same as the single atomic spin operator algebraic relations governing the Rabi dynamics in equation ($1c$).

As in the Rabi model, we specify the interaction mechanisms driving the internal dynamics of the Dicke model by reorganizing the Hamiltonian H_D in equation ($8a$) in two alternative forms

$$H_{aD} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \left\{\hbar\omega_0 J_z + \frac{\hbar g}{\sqrt{N}}(\hat{a} + \hat{a}^\dagger)(J_+ + J_-)\right\} \tag{8d}$$

$$H_{fD} = \hbar\omega_0 J_z + \left\{\hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \frac{\hbar g}{\sqrt{N}}(\hat{a} + \hat{a}^\dagger)(J_+ + J_-)\right\} \tag{8e}$$

where in the form H_{aD} in equation ($8d$) the second component is interpreted as the interactive Dicke Hamiltonian for an interaction mechanism in which the field mode drives the collective atomic spin dynamics, while in the form H_{fD} in equation ($8e$) the second component is interpreted as the interactive Dicke Hamiltonian for an interaction mechanism in which the collective atomic spin drives the field mode dynamics. We observe that in the study of quantum phase transition in the Dicke model in [6], the interaction mechanism in which the field mode drives the collective atomic spin dynamics (generated here by Hamiltonian H_{aD}) is characterized as the classical oscillator (low-frequency field mode) limit, while the interaction mechanism in which the collective atomic spin drives the field mode dynamics (generated here by Hamiltonian H_{fD}) is characterized as the fast oscillator (high-frequency field mode) limit.

Expressing the coupled quadrature interaction component $\frac{\hbar g}{\sqrt{N}}(\hat{a}+\hat{a}^\dagger)(J_+ + J_-)$ of the Dicke Hamiltonian in terms of the field mode and collective atomic spin quadrature fluctuation energy in the alternative forms

$$\hbar\frac{g}{\sqrt{N}}(\hat{a} + \hat{a}^\dagger)(J_+ + J_-) = \hbar\omega_0\left\{\left(\frac{g}{\omega_0}(\hat{a} + \hat{a}^\dagger) + \frac{(J_+ + J_-)}{2\sqrt{N}}\right)^2 - \frac{g^2}{\omega_0^2}(\hat{a} + \hat{a}^\dagger)^2 - \frac{(J_+ + J_-)^2}{4N}\right\} \tag{9a}$$

or

$$\hbar\frac{g}{\sqrt{N}}(\hat{a} + \hat{a}^\dagger)(J_+ + J_-) = \hbar\omega\left\{\left(\frac{g}{\omega}(J_+ + J_-) + \frac{(\hat{a} + \hat{a}^\dagger)}{2\sqrt{N}}\right)^2 - \frac{g^2}{\omega^2}(J_+ + J_-)^2 - \frac{(\hat{a} + \hat{a}^\dagger)^2}{4N}\right\} \tag{9b}$$

and using

$$J_+ + J_- = \Sigma_x \qquad ; \qquad (J_+ + J_-)^2 = \Sigma_x^2 \tag{9c}$$

we reorganize the Hamiltonian H_{aD} in equation ($8d$) for the field mode driven collective atomic spin dynamics in the form

$$H_{aD} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2} - \frac{1}{4}\lambda^2(\hat{a} + \hat{a}^\dagger)^2\right) + \hbar\omega_0\left(J_z - \frac{1}{4N}\Sigma_x^2\right) + \frac{1}{4N}\hbar\omega_0\left(\Sigma_x + \frac{2g\sqrt{N}}{\omega_0}(\hat{a} + \hat{a}^\dagger)\right)^2 \tag{9d}$$

and the Hamiltonian H_{fD} in equation ($8e$) for the collective atomic spin driven field mode dynamics in the form

$$H_{fD} = \hbar\omega_0\left(J_z - \frac{1}{4}\lambda^2\Sigma_x^2\right) + \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2} - \frac{1}{4N}(\hat{a} + \hat{a}^\dagger)^2\right) + \frac{1}{4N}\hbar\omega\left(\hat{a} + \hat{a}^\dagger + \frac{2g\sqrt{N}}{\omega}\Sigma_x\right)^2 \tag{9e}$$

In the Dicke Hamiltonian form H_{aD} in equation ($9d$), we identify the effective low-energy Hamiltonian H_{aDLE} for the field mode driven collective atomic spin dynamics obtained as

$$H_{aDLE} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2} - \frac{1}{4}\lambda^2(\hat{a} + \hat{a}^\dagger)^2\right) + \hbar\omega_0 J_z \tag{10a}$$

which agrees exactly with the effective low-energy Hamiltonian determined in a study of quantum phase transition in the Dicke model in [6], noting

$$\langle d|H_{aDLE}|d\rangle = \hbar\omega\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2} - \frac{1}{4}\lambda^{2}(\hat{a}+\hat{a}^{\dagger})^{2} \right) - j\hbar\omega_{0} \qquad ; \qquad j = \frac{N}{2} \tag{10b}$$

where $|d\rangle$ is the collective atomic spin ground state vector.

On the other hand, in the Dicke Hamiltonian form H_{fD} in equation (9e), we identify the effective low-energy Hamiltonian H_{fDLE} for the collective atomic spin driven field mode dynamics obtained as

$$H_{fDLE} = \hbar\omega_{0}\left(J_{z} - \frac{1}{4}\lambda^{2}\Sigma_{x}^{2}\right) + \hbar\omega\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right) \tag{10c}$$

We observe that in the study of quantum phase transition in the Dicke model in [6], an effective low-energy Hamiltonian was determined in the form of the first component of H_{fDLE} in equation (10c), which is identified as the Lipkin-Meshkov-Glick (LMG) Hamiltonian [11 , 12 , 13]. We note that in [6] the field mode operators have been eliminated through (path) integration, which we may also achieve in the present work by averaging H_{fDLE} in equation (10c) in the field mode vacuum (ground) state $|0\rangle$ to obtain

$$\langle 0|H_{fDLE}|0\rangle = \hbar\omega_{0}\left(J_{z} - \frac{1}{4}\lambda^{2}\Sigma_{x}^{2}\right) + \frac{1}{2}\hbar\omega \tag{10d}$$

giving the exact form determined in [6], where the collective atomic spin driven field mode dynamics as defined in the present work is characterized as the fast oscillator (high-frequency field mode) limit.

The dynamical property that the effective low-energy Hamiltonians H_{aRLE} , H_{fRLE} obtained here in equations (3a) , (3c) and in [5] in the Rabi model are exactly the same as the corresponding effective low-energy Hamiltonians H_{aDLE} , H_{fDLE} obtained here in equations (10a) , (10c) and in [6] in the Dicke model means that dynamics in both Rabi ($N = 1$) and Dicke models is generated by the same interaction mechanisms driven by field mode and (collective) atomic spin quadrature fluctuation energies.

3.1 Effective Dicke Hamiltonians

We now determine the effective Dicke Hamiltonians based on the interpretation that the internal dynamics of the Dicke model is driven by the field mode and collective atomic spin quadrature fluctuation energy. According to the forms in equations (9d) , (9e), the Dicke Hamiltonian H_{aD} or H_{fD} has a quadrature fluctuation generated squeezed field mode or collective atomic spin component which can be diagonalized through factorization equivalent to a squeeze operator generated unitary transformation to determine the corresponding effective Hamiltonian as we summarize below.

Noting that the Dicke Hamiltonian form H_{ad} in equation (9d) has exactly the same algebraic form as the corresponding Rabi Hamiltonian H_{aR} in equation (1d), we apply exactly the same factorization procedure developed in section 2.2 to determine an underlying effective Dicke Hamiltonian $\mathcal{H}_{aD}$ for the field mode driven collective atomic spin dynamics in the final form (reintroducing $\Sigma_{x} = J_{-} + J_{+}$)

$$\mathcal{H}_{aD} = \hbar\omega\sqrt{1-\lambda^{2}}\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right) + \hbar\omega_{0}J_{z} + \frac{\hbar g}{\sqrt{N}(1-\lambda^{2})^{\frac{1}{4}}}(\hat{a}+\hat{a}^{\dagger})(J_{-}+J_{+}) + \frac{\hbar g^{2}}{\omega_{0}\sqrt{1-\lambda^{2}}}(\hat{a}+\hat{a}^{\dagger})^{2} \tag{11a}$$

which is equivalently obtained through a unitary transformation of the Hamiltonian H_{aD} in equation (9d) using the field mode squeeze operator $\mathcal{S}(\eta)$ defined in equation (4g) according

$$H_{aD} = \mathcal{S}^{\dagger}(\eta)\mathcal{H}_{aD}\mathcal{S}(\eta) \qquad \Rightarrow \qquad \mathcal{S}(\eta)H_{aD}\mathcal{S}^{\dagger}(\eta) = \mathcal{H}_{aD} \tag{11b}$$

We observe that the effective Dicke Hamiltonian $\mathcal{H}_{aD}$ obtained in equation (11a) takes exactly the form of the Dicke Hamiltonian used in a detailed study of quantum phase transition of nonlinear light interacting with a finite number of two-level atoms in [10]. The effective Dicke Hamiltonian $\mathcal{H}_{aD}$ in equation (11a) may then be interpreted as the Dicke Hamiltonian of nonlinear light interacting with collective atomic spins. In contrast to the model in [10], the nonlinear term in the effective Dicke Hamiltonian $\mathcal{H}_{aD}$ obtained here in equation (11a) arises from the transfer of field mode quadrature fluctuation energy, which drives the collective atomic spin into an effective quasi-particle spinor mode.

Reorganizing equation (11a) in the form

$$\mathcal{H}_{aD} = \hbar\omega\sqrt{1-\lambda^2}\left(\ \hat{a}^\dagger\hat{a} + \frac{1}{2} - \frac{1}{4}\xi^2(\hat{a}+\hat{a}^\dagger)^2\right) + \hbar\omega_0 J_z + \frac{\hbar g}{(1-\lambda^2)^{\frac{1}{4}}}(\hat{a}+\hat{a}^\dagger)(J_- + J_+) \tag{11c}$$

where ξ is the dimensionless parameter defined in equation (6a), we apply exactly the procedure outlined in equations (6b)-(6f) for the corresponding effective Rabi Hamiltonian $\mathcal{H}_{aR}$ to establish that the effective Dicke Hamiltonian $\mathcal{H}_{aD}$ in equation (11a) is directly related to the original Dicke Hamiltonian H_D in equation (8a) through a unitary transformation generated by a field mode squeeze operator $\mathcal{R}(\beta)$ in the form

$$\mathcal{H}_{aD} = \mathcal{R}^\dagger(\beta)\ H_D\ \mathcal{R}(\beta) \qquad \Rightarrow \qquad \mathcal{R}(\beta)\ \mathcal{H}_{aD}\ \mathcal{R}^\dagger(\beta) = H_D \tag{11c}$$

where the squeeze operator $\mathcal{R}(\beta) = \mathcal{S}^{-1}(\eta)$ and corresponding squeeze parameter β are defined in equation (6d). A unitary transformation of the standard Dicke Hamiltonian H_D in equation (8a) generated by a field mode squeeze operator $\mathcal{R}(\xi) = \mathcal{S}^{-1}(\eta)$ thus provides the effective Dicke Hamiltonian $\mathcal{H}_{aD}$ in equation (11a). This squeezing phenomenon is associated with the transfer of field mode quadrature fluctuation energy which drives the collective atomic spin into an effective quasi-particle spinor mode as explained above.

We now determine the effective quasi-particle spinor mode in the field mode driven collective atomic spin dynamics generated by the effective Dicke Hamiltonian $\mathcal{H}_{aD}$. Using the algebraic relations

$$J^2 = J_x^2 + J_y^2 + J_z^2 \qquad ; \qquad J_+J_- = J^2 - J_z^2 + J_z \qquad ; \qquad J_-J_+ = J^2 - J_z^2 - J_z \tag{12a}$$

in equation (11a), we obtain a suitable form

$$\mathcal{H}_{aD} = \hbar\omega\sqrt{1-\lambda^2}\left(\ \hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \hbar\omega_0(\ J_+J_- - (J^2 - J_z^2)\ ) + \frac{\hbar g(\hat{a}+\hat{a}^\dagger)}{\sqrt{N}(1-\lambda^2)^{\frac{1}{4}}}(J_- + J_+) + \frac{\hbar g^2(\hat{a}+\hat{a}^\dagger)^2}{\omega_0\sqrt{1-\lambda^2}} \tag{12b}$$

Noting that the operator $(J^2 - J_z^2)$ commutes with the Hamiltonian in equation (12b), we may replace it with its (ground state) eigenvalue according to

$$[\ J^2 - J_z^2\ ,\ J_\mp\ ] = 0 \quad ; \quad [\ J^2 - J_z^2\ ,\ \mathcal{H}_{ad}\ ] = 0 \quad \Rightarrow \quad J^2 - J_z^2 \longrightarrow j(j+1) - j^2 = j \quad ; \quad J_z = J_+J_- - j \tag{12c}$$

to express the effective Dicke Hamiltonian $\mathcal{H}_{aD}$ in the form

$$\mathcal{H}_{aD} = \hbar\omega\sqrt{1-\lambda^2}\left(\ \hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \hbar\omega_0(\ J_+J_- - j\ ) + \frac{\hbar g(\hat{a}+\hat{a}^\dagger)}{\sqrt{N}(1-\lambda^2)^{\frac{1}{4}}}(J_- + J_+) + \frac{\hbar g^2(\hat{a}+\hat{a}^\dagger)^2}{\omega_0\sqrt{1-\lambda^2}} \tag{12d}$$

We specify the effective quasi-particle spinor mode by hermitian conjugate *effective quasi-particle spinor operators* $\mathcal{J}_-$, $\mathcal{J}_+$, which are the collective atomic spin state lowering and raising operators displaced by the effective field mode quadrature component obtained as

$$\mathcal{J}_- = J_- + \frac{g}{\omega_0\sqrt{N}(1-\lambda^2)^{\frac{1}{4}}}(\hat{a}+\hat{a}^\dagger) \qquad ; \qquad \mathcal{J}_+ = J_+ + \frac{g}{\omega_0\sqrt{N}(1-\lambda^2)^{\frac{1}{4}}}(\hat{a}+\hat{a}^\dagger) \tag{12e}$$

Taking the normal order product $\mathcal{J}_+\mathcal{J}_-$, multiplying by $\hbar\omega_0$ and subtracting $j\hbar\omega_0$ provides the collective quasi-particle spinor mode Hamiltonian $\mathcal{H}_J$ in the form

$$\mathcal{H}_J = \hbar\omega_0(\ \mathcal{J}_+\mathcal{J}_- - j\ )\ =\ \hbar\omega_0(\ J_+J_- - j\ ) + \frac{\hbar g}{\sqrt{N}(1-\lambda^2)^{\frac{1}{4}}}(\hat{a}+\hat{a}^\dagger)(J_- + J_+) + \frac{\hbar g^2}{\omega_0\sqrt{1-\lambda^2}}(\hat{a}+\hat{a}^\dagger)^2 \tag{12f}$$

which we substitute into equation (12d) to express the effective Dicke Hamiltonian $\mathcal{H}_{aD}$ as a sum of correlated effective field mode and quasi-particle spinor mode components in the form

$$\mathcal{H}_{aD} = \hbar\omega\sqrt{1-\lambda^2}\left(\ \hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + \hbar\omega_0(\ \mathcal{J}_+\mathcal{J}_- - j\ ) \tag{12g}$$

Finally, we present the procedure for determining the effective Dicke Hamiltonian for the collective atomic spin driven field mode dynamics generated by the Hamiltonian form H_{fD} in equation (9e). We notice that the transfer of the collective atomic spin quadrature fluctuation energy transforms the collective atomic spin component of $\mathcal{H}_{fD}$ into a nontrivial form which can be diagonalized through a unitary transformation.

The diagonalization may be achieved directly by applying a unitary transformation operator based on the algebraic properties of the collective atomic spin angular momentum operators J_- , J_+ , J_z or indirectly by first applying the Holstein-Primakoff transformation of the collective atomic spin angular momentum operators into equivalent bosonic operators [14 , 15 , 16], then determining an appropriate bosonic unitary transformation operator. In this work, we adopt the latter approach, since the algebraic properties of the basic bosonic operators are easier to handle compared to the algebraic properties of the corresponding collective atomic spin angular momentum operators.

The Holstein-Primakoff transformation of the collective atomic spin angular momentum operators J_- , J_+ to the corresponding bosonic annihilation and creation operators $\hat{b}$, $\hat{b}^\dagger$ takes the form [7 , 9 , 12-16]

$$J_- = \sqrt{N - \hat{b}^\dagger\hat{b}}\;\hat{b} \qquad ; \qquad J_+ = \hat{b}^\dagger\sqrt{N - \hat{b}^\dagger\hat{b}} \qquad ; \qquad [\,\hat{b}\,,\,\hat{b}^\dagger\,] = 1 \tag{13a}$$

For purposes of mathematical consistency, we deviate from the conventional approach [7 , 9 , 12 , 13 , 15 , 16] and obtain the Holstein-Primakoff transformation of the z-component collective atomic spin angular momentum operator J_z by directly substituting the transformations from equation (13a) into the relation $J_z = J_+J_- - j$ obtained in equation (12c), noting $j = \frac{1}{2}N$, giving

$$J_z = \hat{b}^\dagger\sqrt{N - \hat{b}^\dagger\hat{b}}\;\sqrt{N - \hat{b}^\dagger\hat{b}}\;\hat{b} - \frac{1}{2}N \qquad \Rightarrow \qquad J_z = N\left(\hat{b}^\dagger\sqrt{1 - \frac{\hat{b}^\dagger\hat{b}}{N}}\;\sqrt{1 - \frac{\hat{b}^\dagger\hat{b}}{N}}\;\hat{b} - \frac{1}{2}\right) \tag{13b}$$

For completeness, we apply the same approach to obtain the Holstein-Primakoff transformation of $\Sigma_x = J_+ + J_-$, $\Sigma_y = -i(J_+ - J_-)$ in the form

$$\Sigma_x = \sqrt{N}\left(\hat{b}^\dagger\sqrt{1 - \frac{\hat{b}^\dagger\hat{b}}{N}} + \sqrt{1 - \frac{\hat{b}^\dagger\hat{b}}{N}}\;\hat{b}\right) \qquad ; \qquad \Sigma_y = -i\sqrt{N}\left(\hat{b}^\dagger\sqrt{1 - \frac{\hat{b}^\dagger\hat{b}}{N}} - \sqrt{1 - \frac{\hat{b}^\dagger\hat{b}}{N}}\;\hat{b}\right) \tag{13c}$$

We observe that the form of the Holstein-Primakoff spin-boson operator transformation may differ in various works, e.g., [7 , 9 , 12-16], the present work included, in terms of the interchange or ordering of the collective atomic spin angular momentum operators J_- , J_+, giving instead the form $J_+ = \sqrt{N - \hat{b}^\dagger\hat{b}}\;\hat{b}$, $J_- = \hat{b}^\dagger\sqrt{N - \hat{b}^\dagger\hat{b}}$, $J_z = N(\frac{1}{2} - \hat{b}^\dagger\hat{b})$ adopted in the original work [14] and in [12 , 13].

The Holstein-Primakoff transformed Dicke Hamiltonian for collective atomic spin driven field mode dynamics is obtained by substituting the collective atomic spin angular momentum operators J_z , Σ_x from equations (13b) , (13c) into the Hamiltonian H_{fD} in equation (9e). Algebraic operations of the transformed Hamiltonian on specified bosonic state vectors are effected by expanding the square root operator appearing in equations (13b) , (13c) as appropriate [6 , 8 , 12 , 13]. Handling the expansion is a bit problematic for a finite number $N < \infty$ of atoms, but simplifies a great deal in the thermodynamic limit, specified by a very large number of atoms, $N \to \infty$, such that to a very good approximation, the square root operator can be set equal to unity in the thermodynamic limit of the Dicke interaction. Since the objective of the work in this article is only to determine effective Dicke Hamiltonians good enough to generate dynamics characterized by quantum phase transitions and related fundamental quantum mechanical phenomena, but not necessarily a comprehensive study of all the dynamical properties of the Dicke model, we specialize to the thermodynamic limit, specified by $N \to \infty$, in this article. Hence, we determine the effective Holstein-Primakoff transformed Dicke Hamiltonian for collective atomic spin driven field mode dynamics generated by the Dicke Hamiltonian H_{fD} in the thermodynamic limit.

In the thermodynamic limit where

$$N \to \infty \qquad ; \qquad \sqrt{1 - \frac{\hat{b}^\dagger\hat{b}}{N}} \approx 1 \tag{13d}$$

the Holtein-Primakoff transformation in equations (13a)-(13c) take much simpler forms

$$N \to \infty \quad \Rightarrow \quad J_- = \sqrt{N}\,\hat{b} \quad ; \quad J_+ = \sqrt{N}\,\hat{b}^\dagger \quad ; \quad J_z = N\left(\hat{b}^\dagger\hat{b} - \frac{1}{2}\right) \quad ; \quad \Sigma_x = \sqrt{N}\,(\hat{b} + \hat{b}^\dagger) \tag{13e}$$

Substituting J_z , Σ_x from equation (13e) into H_{fD} in equation (9e), we obtain the Holstein-Primakoff transformed Dicke Hamiltonian $\overline{H}_{fD}^{\infty}$ for collective atomic spin driven field mode dynamics in the thermodynamic limit in the form

$$\overline{H}_{fD}^{\infty} = N\hbar\omega_0 \left(\hat{b}^\dagger \hat{b} - \frac{1}{2} - \frac{1}{4}\lambda^2(\hat{b}+\hat{b}^\dagger)^2 \right) + \hbar\omega \left(\hat{a}^\dagger\hat{a} + \frac{1}{2} - \frac{(\hat{a}+\hat{a}^\dagger)^2}{4N} \right) + \frac{\hbar\omega}{4N} \left(\hat{a} + \hat{a}^\dagger + \frac{2gN}{\omega}(\hat{b}+\hat{b}^\dagger) \right)^2 \tag{13f}$$

We notice that the transfer of the collective atomic spin quadrature fluctuation energy (which is equivalent to Holstein-Primakoff bosonic mode quadrature fluctuation energy), modifies the free Holstein-Primakoff bosonic mode component of $\overline{H}_{fD}^{\infty}$ in equation $(13f)$ to a squeezed state form $N\hbar\omega_0 \left(\hat{b}^\dagger\hat{b} - \frac{1}{2} - \frac{1}{4}\lambda^2(\hat{b}+\hat{b}^\dagger)^2 \right)$ exactly the same form as the corresponding modified free field mode component $\hbar\omega \left(\hat{a}^\dagger\hat{a} - \frac{1}{2} - \frac{1}{4}\lambda^2(\hat{a}+\hat{a}^\dagger)^2 \right)$ of the Rabi and Dicke Hamiltonian forms H_{aR} , H_{aD} in equations $(2f)$, $(9d)$, respectively. Hence, noting that the interaction is characterized by the same dimensionless parameter λ defined in equation $(2h)$, we apply exactly the same factorization procedure developed in section 2.2 to determine a squeeze operator $B(\eta)$ of the Holstein-Primakoff bosonic mode annihilation and creation operators $\hat{b}$, $\hat{b}^\dagger$, to transform $\overline{H}_{fD}^{\infty}$ in equation $(13f)$ into the corresponding effective Hamiltonian in the thermodynamic limit according to the unitary transformations

$$B(\eta) = e^{\frac{1}{4}\eta(\hat{b}^{\dagger 2} - \hat{b}^2)} \quad ; \quad \tanh\eta = \frac{\frac{1}{2}\lambda^2}{1-\frac{1}{2}\lambda^2} \quad ; \quad \eta = -\frac{1}{2}\ln(1-\lambda^2) \tag{14a}$$

$$B^\dagger(\eta)\hat{b}B(\eta) = \cosh\frac{1}{2}\eta\ \hat{b} - \sinh\frac{1}{2}\eta\ \hat{b}^\dagger \quad ; \quad B^\dagger(\eta)\hat{b}^\dagger B(\eta) = \cosh\frac{1}{2}\eta\ \hat{b}^\dagger - \sinh\frac{1}{2}\eta\ \hat{b}$$
$$B(\eta)\hat{b}B^\dagger(\eta) = \cosh\frac{1}{2}\eta\ \hat{b} + \sinh\frac{1}{2}\eta\ \hat{b}^\dagger \quad ; \quad B(\eta)\hat{b}^\dagger B^\dagger(\eta) = \cosh\frac{1}{2}\eta\ \hat{b}^\dagger + \sinh\frac{1}{2}\eta\ \hat{b} \tag{14b}$$

$$B(\eta)(\hat{b}+\hat{b}^\dagger)B^\dagger(\eta) = e^{\frac{1}{2}\eta}(\hat{b}+\hat{b}^\dagger) \ = \ \frac{\hat{b}+\hat{b}^\dagger}{(1-\lambda^2)^{\frac{1}{4}}} \tag{14c}$$

$$B^\dagger(\eta)\hat{a}B(\eta) = B(\eta)\hat{a}B^\dagger(\eta) = \hat{a} \quad ; \quad B^\dagger(\eta)\hat{a}^\dagger B(\eta) = B(\eta)\hat{a}^\dagger B^\dagger(\eta) = \hat{a}^\dagger \tag{14d}$$

giving unitary squeezing transformations (noting $B^\dagger(\eta)B(\eta) = B(\eta)B^\dagger(\eta) = 1$)

$$N\hbar\omega_0 \left(\hat{b}^\dagger\hat{b} - \frac{1}{2} - \frac{1}{4}\lambda^2(\hat{b}+\hat{b}^\dagger)^2 \right) = B^\dagger(\eta)\ N\hbar\omega_0\sqrt{1-\lambda^2} \left(\hat{b}^\dagger\hat{b} - \frac{1}{2} \right) - \frac{1}{2}N\hbar\omega_0\lambda^2)B(\eta) \tag{14e}$$

$$\hbar\omega \left(\hat{a}^\dagger\hat{a} + \frac{1}{2} - \frac{(\hat{a}+\hat{a}^\dagger)^2}{4N} \right) = B^\dagger(\eta) \left\{ B(\eta)\hbar\omega \left(\hat{a}^\dagger\hat{a} + \frac{1}{2} - \frac{(\hat{a}+\hat{a}^\dagger)^2}{4N} \right) B^\dagger(\eta) \right\} B(\eta)$$
$$= \hbar\omega \left(\hat{a}^\dagger\hat{a} + \frac{1}{2} - \frac{(\hat{a}+\hat{a}^\dagger)^2}{4N} \right) \tag{14f}$$

$$\frac{\hbar\omega}{4N} \left(\hat{a}+\hat{a}^\dagger + \frac{2gN}{\omega}(\hat{b}+\hat{b}^\dagger) \right)^2 = B^\dagger(\eta) \left\{ B(\eta)\frac{\hbar\omega}{4N} \left(\hat{a}+\hat{a}^\dagger + \frac{2gN}{\omega}(\hat{b}+\hat{b}^\dagger) \right)^2 B^\dagger(\eta) \right\} B(\eta)$$
$$= \frac{\hbar\omega}{4N} \left(\hat{a}+\hat{a}^\dagger + \frac{2gN}{\omega(1-\lambda^2)^{\frac{1}{4}}}(\hat{b}+\hat{b}^\dagger) \right)^2 \tag{14g}$$

where in equation $(13e)$, we have reorganized

$$-\frac{1}{4}\lambda^2(\hat{b}+\hat{b}^\dagger)^2 = -\frac{1}{4}\lambda^2(\hat{b}^2+\hat{b}^{\dagger 2}) - \frac{1}{2}\lambda^2(\ (\hat{b}^\dagger\hat{b} - \frac{1}{2}\ ) + 1\ ) \tag{14h}$$

In the above, we applied the algebraic property that the field mode annihilation and creation operators $\hat{a}$, $\hat{a}^\dagger$ are independent of the Holstein-Primakoff bosonic mode annihilation and creation operators $\hat{b}$, $\hat{b}^\dagger$, such that the unitary transformation generated by the squeeze operator $B(\eta)$ defined in equation $(14a)$ does not affect $\hat{a}$, $\hat{a}^\dagger$ as presented in equation $(14d)$.

Substituting equations $(14e)$, $(14f)$, $(14g)$ into equation $(13f)$ provides the underlying effective Hamiltonian $\overline{\mathcal{H}}_{fD}^{\infty}$ through unitary transformations

$$\overline{H}_{fD}^{\infty} = B^\dagger(\eta)\overline{\mathcal{H}}_{fD}^{\infty}B(\eta) \qquad \Rightarrow \qquad \overline{\mathcal{H}}_{fD}^{\infty} = B^\dagger(\eta)\overline{H}_{fD}^{\infty}B(\eta) \tag{15a}$$

where the effective Dicke Hamiltonian $\overline{\mathcal{H}}_{fD}^{\infty}$ which generates the collective atomic spin driven field mode dynamics in the thermodynamic limit is obtained in the form

$$\overline{\mathcal{H}}_{fD}^{\infty} = N\hbar\omega_0\sqrt{1-\lambda^2}\left(\ \hat{b}^{\dagger}\hat{b} - \frac{1}{2}\right) - \frac{1}{2}N\hbar\omega_0\lambda^2 + \hbar\omega\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right) + \frac{\hbar g(\hat{a}+\hat{a}^{\dagger})(\hat{b}+\hat{b}^{\dagger})}{(1-\lambda^2)^{\frac{1}{4}}} + \frac{\hbar g^2 N(\hat{b}+\hat{b}^{\dagger})^2}{\omega\sqrt{1-\lambda^2}} \quad (15b)$$

Reorganizing this effective Hamiltonian in the form

$$\overline{\mathcal{H}}_{fD}^{\infty} = N\hbar\omega_0\sqrt{1-\lambda^2}\left(\ \hat{b}^{\dagger}\hat{b} - \frac{1}{2} - \frac{1}{4}\xi^2(\hat{b}+\hat{b}^{\dagger})^2\right) - \frac{1}{2}N\hbar\omega_0\lambda^2 + \hbar\omega\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right) + \frac{\hbar g(\hat{a}+\hat{a}^{\dagger})(\hat{b}+\hat{b}^{\dagger})}{(1-\lambda^2)^{\frac{1}{4}}} \quad (15c)$$

where ξ is the dimensionless parameter defined in equation ($6a$), we repeat the factorization procedure in section 2.2, going through the same steps as in equations ($6a$)-($6f$) to determine a thermodynamic Holstein-Primakoff bosonic mode squeeze operator $\mathcal{T}$ defined by

$$\mathcal{T}(\beta) = e^{\frac{1}{4}\beta(\hat{b}^{\dagger 2}-\hat{b}^2)} \qquad ; \qquad \tanh\beta = \frac{\frac{1}{2}\xi^2}{1-\frac{1}{2}\xi^2} \qquad \Rightarrow \qquad \beta = -\frac{1}{2}\ln(1-\xi^2) \quad (15d)$$

which relates the effective Hamiltonian $\overline{\mathcal{H}}_{fD}^{\infty}$ in equation ($15b$) (reorganized in a useful algebraically equivalent form in equation ($15c$)) to the thermodynamic ($N\to\infty$) Holstein-Primakoff transform H_D^{∞} of the original Dicke Hamiltonian H_D in equation ($8a$) through unitary transformations

$$\overline{\mathcal{H}}_{fD}^{\infty} = \mathcal{T}^{\dagger}(\beta)\ H_D^{\infty}\ \mathcal{T}(\beta) \qquad \Rightarrow \qquad \mathcal{T}(\beta)\ \overline{\mathcal{H}}_{fD}^{\infty}\ \mathcal{T}^{\dagger}(\beta) = H_D^{\infty} \quad (15e)$$

where the thermodynamic ($N\to\infty$) Holstein-Primakoff transform H_D^{∞} of the original Dicke Hamiltonian H_D has been determined in the form

$$H_D^{\infty} = \hbar\omega\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right) + N\hbar\omega_0\left(\ \hat{b}^{\dagger}\hat{b} - \frac{1}{2}\right) + \hbar g(\hat{a}+\hat{a}^{\dagger})(\hat{b}+\hat{b}^{\dagger}) - \frac{1}{2}N\hbar\omega_0\lambda^2 \quad (15f)$$

which is usually obtained through direct Holstein-Primakoff transformation of the Dicke Hamiltonian H_D in equation ($8a$) in the thermodynamic limit [7 , 9], except for the excess collective atomic spin quadrature fluctuation energy term $-\frac{1}{2}N\hbar\omega_0\lambda^2$ and the factor N on the free Holstein-Primakoff mode component $N\hbar\omega_0\left(\ \hat{b}^{\dagger}\hat{b} - \frac{1}{2}\right)$, which arises here in the J_z transformation according to equations ($13b$) , ($13e$). We note that, if we ignore the excess collective atomic spin quadrature fluctuation energy term $-\frac{1}{2}N\hbar\omega_0\lambda^2$, then both forms of the Holstein-Primakoff transformed Hamiltonians obtained here in equation [$15f$] and in [7 , 9] revert back to the same form of the original Dicke Hamiltonian H_D in equation ($8a$) on substituting the Holstein-Primakoff bosonic operators $\hat{b}$, $\hat{b}^{\dagger}$ with the corresponding collective atomic spin angular momentum operators J_- , J_+ using the thermodynamic limit transformations in equation ($13e$).

Transforming the effective Hamiltonian $\mathcal{H}_{fD}^{\infty}$ in equation ($15b$) back to the collective atomic spin angular momentum space by substituting the Holstein-Primakoff mode operators from equation ($13e$) provides the form

$$\overline{\mathcal{H}}_{fD}^{\infty} = \hbar\omega_0\sqrt{1-\lambda^2}\ J_z + \hbar\omega\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right) + \frac{\hbar g}{\sqrt{N}(1-\lambda^2)^{\frac{1}{4}}}(\hat{a}+\hat{a}^{\dagger})\Sigma_x + \frac{\hbar g^2}{\omega\sqrt{1-\lambda^2}}\Sigma_x^2 - \frac{1}{2}N\hbar\omega_0\lambda^2 \quad (16a)$$

For physical interpretation, we reorganize this effective Dicke Hamiltonian in two alternative forms. First, we introduce a quasi-particle bosonic mode specified by hermitian conjugate *effective bosonic operators* $\hat{\mathcal{A}}$, $\hat{\mathcal{A}}^{\dagger}$, which are the field mode annihilation and creation operators displaced by the effective collective atomic spin quadrature component obtained as

$$\hat{\mathcal{A}} = \hat{a} + \frac{g}{\omega\sqrt{N}(1-\lambda^2)^{\frac{1}{4}}}\Sigma_x \qquad ; \qquad \hat{\mathcal{A}}^{\dagger} = \hat{a}^{\dagger} + \frac{g}{\omega\sqrt{N}(1-\lambda^2)^{\frac{1}{4}}}\Sigma_x \quad (16b)$$

Taking the normal order product $\hat{\mathcal{A}}^{\dagger}\hat{\mathcal{A}}$, multiplying by $\hbar\omega$ and adding $\frac{1}{2}\hbar\omega$ provides the quasi-particle bosonic Hamiltonian $\mathcal{H}_B$ in the form

$$\mathcal{H}_B = \hbar\omega\left(\ \hat{\mathcal{A}}^{\dagger}\hat{\mathcal{A}} + \frac{1}{2}\ \right) \ = \ \hbar\omega\left(\ \hat{a}^{\dagger}\hat{a} + \frac{1}{2}\ \right) + \frac{\hbar g}{\sqrt{N}(1-\lambda^2)^{\frac{1}{4}}}(\hat{a}+\hat{a}^{\dagger})\Sigma_x + \frac{\hbar g^2}{\omega\sqrt{1-\lambda^2}}\Sigma_x^2 \quad (16c)$$

Substituting equation (16c) into equation (16a), we express the effective Dicke Hamiltonian $\mathcal{H}_{aD}^{\infty}$ for the collective atomic spin driven field mode dynamics as a sum of correlated effective collective atomic spin and quasi-particle bosonic mode components in the form

$$\mathcal{H}_{aD}^{\infty} = \hbar\omega_0\sqrt{1-\lambda^2}\ J_z + \hbar\omega\left(\ \hat{\mathcal{A}}^{\dagger}\hat{\mathcal{A}} + \frac{1}{2}\ \right) - \frac{1}{2}N\hbar\omega_0\lambda^2 \tag{16d}$$

This form of the effective Dicke Hamiltonian $\mathcal{H}_{fD}^{\infty}$ leads to the physical interpretation that quasi-particle bosonic modes specified by effective bosonic annihilation and creation operators $\hat{\mathcal{A}}$, $\hat{\mathcal{A}}^{\dagger}$ are formed in the interaction mechanism in which the collective atomic spin drives the field mode dynamics.

Alternatively, we reorganize the effective Hamiltonian $\mathcal{H}_{fD}^{\infty}$ in equation (16a) in the form

$$\overline{\mathcal{H}}_{fD}^{\infty} = \hbar\omega_0\sqrt{1-\lambda^2}\left(\ J_z + \frac{1}{4}\alpha^2\Sigma_x^2\right) + \hbar\omega\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right) + \frac{\hbar g}{\sqrt{N}(1-\lambda^2)^{\frac{1}{4}}}(\hat{a}+\hat{a}^{\dagger})\Sigma_x - \frac{1}{2}N\hbar\omega_0\lambda^2 \tag{16e}$$

or in the form

$$\overline{\mathcal{H}}_{fD}^{\infty} = \hbar\omega_0\sqrt{1-\lambda^2}\left(\ J_z - \frac{1}{4}\xi^2\Sigma_x^2\right) + \hbar\omega\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right) + \frac{\hbar g}{\sqrt{N}(1-\lambda^2)^{\frac{1}{4}}}(\hat{a}+\hat{a}^{\dagger})\Sigma_x - \frac{1}{2}N\hbar\omega_0\lambda^2 \tag{16f}$$

where the dimensionless parameter ξ in equation (16f) is defined in equation (6a), while the dimensionless parameter α in equation (16e) is defined by

$$\alpha^2 = \frac{\lambda^2}{1-\lambda^2} \qquad \Rightarrow \qquad \alpha^2 = -\xi^2 \qquad ; \qquad \xi^2 = \frac{\lambda^2}{\lambda^2-1} \tag{16g}$$

In the form obtained in equation (16e) or (16f), the effective Dicke Hamiltonian $\overline{\mathcal{H}}_{fD}^{\infty}$ for the collective atomic spin driven field mode dynamics may now be interpreted as an effective Hamiltonian for a fully quantized uniaxial Lipkin-Meshkov-Glick model in which the external magnetic field driving the N spin-$\frac{1}{2}$ particles [13] is replaced by a quantized electromagnetic field mode.

4 Critical coupling constant and quantum phase transition

It is evident from the form of the effective Rabi and Dicke Hamiltonians $\mathcal{H}_{aR}$, $\mathcal{H}_{aD}$, $\mathcal{H}_{fD}$ in equations (5f), (11a), (15b)/(16e) that the transfer of the field mode and collective atomic spin quadrature fluctuation energy driving the atom-field dynamics in the Rabi and Dicke models causes modification and scaling of the free field mode and collective atomic spin energy component by a factor $\sqrt{1-\lambda^2}$ and the corresponding quadrature component by a factor $(1-\lambda^2)^{-\frac{1}{4}}$, (see equations (5$e$) , (14$c$), noting that in (14$c$), $\hat{b}+\hat{b}^{\dagger} \equiv \frac{1}{\sqrt{N}}\Sigma_x$). It follows that the nature of the dynamics generated by the effective Rabi or Dicke Hamiltonian $\mathcal{H}_{aR}$, $\mathcal{H}_{aD}$, $\mathcal{H}_{fD}^{\infty}$ is determined by the effective interaction parameter $\sqrt{1-\lambda^2}$, which can take values ($\lambda = \frac{2g}{\sqrt{\omega_0\omega}}$)

$$\sqrt{1-\lambda^2} = \begin{cases} 0 < \sqrt{1-\lambda^2} < 1 \quad : \quad \lambda < 1 \qquad \Rightarrow \qquad g < \frac{1}{2}\sqrt{\omega_0\omega} \\ 0 \quad : \quad \lambda = \lambda_c = 1 \quad \Rightarrow \quad \lambda_c = \frac{2g_c}{\sqrt{\omega_0\omega}} = 1 \\ i\sqrt{\lambda^2-1} > 1 \quad : \quad \lambda > 1 \qquad \Rightarrow \qquad g > \frac{1}{2}\sqrt{\omega_0\omega} \end{cases} \tag{17a}$$

We identify a critical coupling constant g_c at which the interaction parameter $\sqrt{1-\lambda^2}$ vanishes, obtained as

$$\lambda = \lambda_c = 1 \qquad \Rightarrow \qquad g_c = \frac{1}{2}\sqrt{\omega_0\omega} \tag{17b}$$

which is exactly the value of the critical coupling constant determined in studies of quantum phase transition in the Rabi model [5] and the Dicke model [6 , 7]. It is clear in the effective Hamiltonians $\mathcal{H}_{aR}$, $\mathcal{H}_{aD}$, $\mathcal{H}_{fD}^{\infty}$ in equations (5f) , (11a) , (15b)/(16e) that the free field mode and collective atomic spin (equivalent to the free Holstein-Primakoff bosonic mode) energy components vanish, while either the field mode or the collective atomic spin (Holstein-Primakoff bosonic mode) quadrature component and the associated atom-field interaction energy diverge at the critical coupling $\lambda_c = 1$, $g_c = \frac{1}{2}\sqrt{\omega_0\omega}$, where quantum phase transition occurs in the Rabi and Dicke models. Normal phase dynamics occurs in the coupling regime $g < g_c$ ($\lambda < 1$), while in the coupling regime $g > g_c$ ($\lambda > 1$) the system is in the superradiance phase.

Substituting the critical coupling constant $g_c = \frac{1}{2}\sqrt{\omega_0 \omega}$ from equation $(17b)$ into the general definition of λ in equation $(2h)$, we rewrite

$$\lambda = \frac{g}{g_c} \qquad ; \qquad \sqrt{1-\lambda^2} = \frac{\sqrt{g_c + g}}{g_c}\sqrt{g_c - g} \;=\; \frac{\sqrt{g + g_c}}{g_c}\;|g - g_c|^{\frac{1}{2}}$$

$$\frac{1}{(1-\lambda^2)^{\frac{1}{4}}} = \left(\frac{g + g_c}{g_c^2}\right)^{-\frac{1}{4}} \; |g - g_c|^{-\frac{1}{4}} \tag{17c}$$

which we substitute into equations $(5e)$, $(5f)$, $(11a)$, $(14c)$, $(15b)/(16e)$ to determine the universal scaling properties, i.e., near the critical coupling constant g_c, the free field mode and collective atomic spin energy component vanishes as $|g - g_c|^{\frac{1}{2}}$, while the corresponding field mode or collective atomic spin (Holstein-Primakoff bosonic mode) quadrature component (length scale) diverges as $|g - g_c|^{-\frac{1}{4}}$, as established in studies of quantum phase transition in the Rabi and Dicke models in [5 , 7 , 8 , 12].

Taking the expectation values of the effective Rabi and Dicke Hamiltonians $\mathcal{H}_{aR}$, $\mathcal{H}_{aD}$, $\mathcal{H}_{fD}^{\infty}$ in equations $(5f)$, $(11a)$, $(15b)/(16e)$ with respect to the corresponding ground state eigenvectors provides the respective ground state energies $\mathcal{E}_{aR}^{g}$, $\mathcal{E}_{aD}^{g}$, $\mathcal{E}_{fD}^{\infty\, g}$ of the Rabi and Dicke systems in the form

$$\mathcal{E}_{aR}^{g} = \frac{1}{2}\hbar\omega\sqrt{1-\lambda^2} + \frac{\hbar g^2}{\omega_0\sqrt{1-\lambda^2}} - \frac{1}{2}\hbar\omega_0 \qquad ; \qquad \mathcal{E}_{aD}^{g} = \frac{1}{2}\hbar\omega\sqrt{1-\lambda^2} + \frac{\hbar g^2}{\omega_0\sqrt{1-\lambda^2}} - \frac{1}{2}N\hbar\omega_0$$

$$\mathcal{E}_{fD}^{\infty\, g} = \frac{1}{2}N\hbar\omega_0\sqrt{1-\lambda^2} + \frac{\hbar N g^2}{\omega\sqrt{1-\lambda^2}} + \frac{1}{2}\hbar\omega - \frac{1}{2}N\hbar\omega_0\lambda^2 \tag{17d}$$

each of which agreeing with the scaling property established in [8 , 12] that the ground state energy in the Dicke and Lipkin-Meshkov-Glick models is composed of regular and singular components. It follows from the ground state energy $\mathcal{E}_{aR}^{g}$ in equation $(17d)$ that the same scaling property applies to the Rabi model.

5 Conclusion

Applying the underlying dynamical property that the internal dynamics of the Rabi and Dicke models is driven by field mode and (collective) atomic spin quadrature fluctuation energy, we have determined exact effective Rabi and Dicke Hamiltonians. Two alternative interaction mechanisms, one in which the field mode drives the (collective) atomic spin dynamics and the other in which the (collective) atomic spin drives the field mode dynamics, have been identified, which yield the effective Hamiltonians in the expected complete forms. It has emerged that in the interaction mechanism in which the field mode drives the (collective) atomic spin dynamics, the transfer of the field mode quadrature fluctuation energy generates an effective quasi-particle spinor mode, leaving the field mode in an effective squeezed state, while in the interaction mechanism in which the (collective) atomic spin drives the field mode dynamics, the transfer of the (collective) atomic spin quadrature fluctuation energy generates an effective quasi-particle bosonic mode, leaving the (collective) atomic spin (or equivalent Holstein-Primakoff bosonic mode) in an effective squeezed state. As an alternative to the quasi-particle spinor mode interpretation, the effective Hamiltonians for the field mode driven (collective) atomic spin dynamics may equivalently be interpreted as the respective Rabi or Dicke model for nonlinear (squeezed or degenerate parametric down-conversion) light interacting with two-level atom(s), while as an alternative to the quasi-particle bosonic mode interpretation, the effective Dicke Hamiltonian for the collective atomic spin driven field mode dynamics may be interpreted as a fully quantized Lipkin-Meshkov-Glick model. It follows from the nonlinear forms that the effective Rabi and Dicke Hamiltonians generate dynamics characterized by quantum phase transition and universal scaling features at a critical coupling constant which we have determined in exact form, as well as squeezing, entanglement and related non-classical fundamental quantum mechanical properties such as population collapses, revivals and fractional revivals, which can be established in other work.

6 Acknowledgement

I thank Maseno University for providing facilities and a conducive work environment during the preparation of the manuscript.

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