Polariton and anti-polariton qubits in the Rabi model

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Abstract

This paper presents a precise algebraic and physical framework for studying the dynamics and practical applications of the quantum Rabi model. We redefine the quantum Rabi interaction in terms of polariton and anti-polariton qubits generated by the Jaynes-Cummings and anti-Jaynes-Cummings interactions, respectively. The formation of a polariton qubit involves absorption or emission of positive energy photon by the field mode, while the formation of an anti-polariton qubit involves absorption or emission of negative energy photon by the field mode, triggered by initial emission or absorption of positive energy photon by the atom. A polariton or anti-polariton qubit is a two-state quantized particle specified by two state vectors, Hamiltonian, conserved excitation number, identity, state transition, $U(1)$-symmetry, parity-symmetry, $SU(2)/U(1)$-symmetry and $SU(1,1)/U(1)$-symmetry operators. Superpositions of the qubit state vectors provide the eigenvectors and energy eigenvalues of the respective Hamiltonians. The polariton or anti-polariton qubit state transition operator defined within the two-dimensional subspace spanned by the qubit state vectors has algebraic properties equivalent to the algebraic properties of an atomic spin state transition operator (Pauli matrix) σ_x, leading to a *photospin* interpretation of a polariton or anti-polariton qubit. Dynamical evolution describing Rabi oscillations between qubit states is easily evaluated and basic features of the dynamics are determined explicitly. The similarity of polariton and anti-polariton qubits to the atomic spin qubits, i.e., the photospin picture, naturally leads to the introduction of a *quantum Rabi optical lattice* as a geometrical framework for studying the dynamics and physical properties of systems of interacting polariton and anti-polariton qubits.

KEYWORDS: Rabi model , Jaynes-Cummings and anti-Jaynes-Cummings interactions , polariton and anti-polariton qubits , conserved excitation number and symmetry operators , quantum Rabi optical lattice.

1 Introduction

The quantum Rabi model describes the dynamics of a quantized electromagnetic field mode interacting with a two-level atom generated by Hamiltonian [1-9]

$$H_R = \frac{1}{2}\hbar\omega\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right) + \hbar\omega_0 s_z + \hbar g(\hat{a} + \hat{a}^\dagger)(s_+ + s_-) \tag{1a}$$

where ω , $\hat{a}$, $\hat{a}^\dagger$ are the quantized field mode angular frequency, annihilation and creation operators, while ω_0 , s_z , s_+ , s_- are the atomic state transition angular frequency and operators. We have used $\sigma_x = s_- + s_+$ and expressed the free field mode Hamiltonian in appropriate symmetrized normal and anti-normal order form $\frac{1}{2}\hbar\omega(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger)$.

Exact analytical solutions of the eigenvalue equation for the Rabi Hamiltonian H_R in equation $(1a)$ have been obtained by Braak and others [1-5], but determining the general time evolving state vector to provide a comprehensive understanding of the dynamical properties of the quantum Rabi interaction still remains a formidable task [6 , 7]. The eigenvalue spectrum obtained in [1-5] is too complicated to give a clear picture of the nature of transitions which can occur between the internal states of the system.

In this paper, we seek to gain insight into the internal dynamics of the quantum Rabi model by symmetrizing the Hamiltonian H_R in equation $(1a)$ into its rotating and anti-rotating components. Collecting the normal and anti-normal order terms in equation (1), we express the Rabi Hamiltonian in the symmetrized form

$$H_R = \frac{1}{2}(\ H + \overline{H}\ ) \tag{1b}$$

where we have identified the normal order rotating component as the Jaynes-Cummings Hamiltonian H obtained as

$$H = \hbar(\ \omega \hat{a}^{\dagger}\hat{a} + \omega_0 s_z + 2g(\hat{a}s_+ + \hat{a}^{\dagger}s_-)\ ) \tag{1c}$$

and the anti-normal order anti-rotating component as the anti-Jaynes-Cummings Hamiltonian $\overline{H}$ obtained as

$$\overline{H} = \hbar(\ \omega \hat{a}\hat{a}^{\dagger} + \omega_0 s_z + 2g(\hat{a}s_- + \hat{a}^{\dagger}s_+)\ ) \tag{1d}$$

We observe that the algebraic property of operator ordering which distinguishes the rotating (Jaynes-Cummings) and anti-rotating (anti-Jaynes-Cummings) components H , $\overline{H}$ of the Rabi Hamiltonian H_R is not arbitrary, but has physical foundation. Noting that an electromagnetic field mode is composed of positive and negative frequency components [10], we provide a physical interpretation that the Jaynes-Cummings interaction represents the coupling of the atomic spin to the rotating *positive frequency* component of the field mode, while the anti-Jaynes-Cummings interaction represents the coupling of the atomic spin to the anti-rotating *negative frequency* component of the field mode. We note that dynamical effects arising from interactions involving negative frequency radiation have been observed in recent experiments [11 , 12].

Having overcome the long standing challenge on the existence of a conserved excitation number operator for the anti-Jaynes-Cummings interaction in a recent article [13], which we also elaborate in the present paper, we study the dynamics generated by the Jaynes-Cummings and anti-Jaynes-Cummings Hamiltonians H , $\overline{H}$ separately. The Jaynes-Cummings interaction forms a *polariton qubit* through emission or absorption of positive energy photon by the atom and absorption or emission of positive energy photon by the rotating positive frequency component of the field mode, while the anti-Jaynes-Cummings interaction forms an *anti-polariton qubit* through emission or absorption of positive energy photon by the atom and absorption or emission of negative energy photon by the anti-rotating negative frequency component of the field mode. We interpret a polariton or anti-polariton qubit as a two-state quantized particle specified by conserved excitation number, identity, state transition and symmetry operators, with an explicit eigenvalue spectrum. Polariton qubit dynamics is characterized by *red-sideband* state transitions generated by the Jaynes-Cummings interaction, while anti-polariton qubit dynamics is characterized by *blue-sideband* state transitions generated by the anti-Jaynes-Cummings interaction. Noting that the algebraic properties of polariton and anti-polariton qubits are similar to the algebraic properties of atomic spin qubits, we introduce a *quantum Rabi optical lattice* as a geometrical framework in which polariton and anti-polariton qubits are formed within micro-cavities interpreted as the lattice sites. Polariton and anti-polariton qubits in different lattice sites interact through their state transition operators.

We introduce the dynamical operators and prove conservation of excitation number operators of polariton (Jaynes-Cummings interaction) and anti-polariton (anti-Jaynes-Cummings interaction) qubits in section 2. The algebraic properties and operations of the state transition operators within the two-dimensional subspaces spanned by the respective polariton and anti-polariton qubit state vectors are presented in section 3, where eigenvectors, energy eigenvalues, dynamical evolution and state transition probabilities are determined explicitly. Section 4 contains the model of quantum Rabi optical lattice for systems of interacting polariton and anti-polariton qubits. We end with conclusions in section 5.

2 Dynamical operators of polaritons and anti-polaritons

As stated above, we interpret a polariton or an anti-polariton formed in a Jaynes-Cummings or anti-Jaynes-Cummings interaction, respectively, as a quantized particle specified by well-defined dynamical operators, namely Hamiltonian, excitation number, identity, state transition and symmetry operators. We establish that the excitation number and state transition operators are conserved and they generate corresponding $U(1)$, parity and $SU(2)/U(1)$ or $SU(1,1)/U(1)$ symmetry operators of the respective polariton and anti-polariton Hamiltonians.

2.1 Polariton and anti-polariton Hamiltonians

Polariton and anti-polariton Hamiltonians are obtained through appropriate redefinitions of the Jaynes-Cummings and anti-Jaynes-Cummings Hamiltonians, respectively. Adding and subtracting an atomic spin normal order term $\hbar\omega s_+ s_-$ in equation (1c) and anti-normal order term $\hbar\omega s_- s_+$ in equation (1d), then reorganizing using algebraic relations $s_+s_- = \frac{1}{2} + s_z$, $s_-s_+ = \frac{1}{2} - s_z$ ($\frac{1}{2} \equiv \frac{1}{2}I$) and factoring out $2g$, we obtain the polariton Hamiltonian in the standard form

$$H = \hbar\omega(\hat{a}^\dagger\hat{a} + s_+s_-) + 2\hbar g(\alpha s_z + \hat{a}s_+ + \hat{a}^\dagger s_-\ ) - \frac{1}{2}\hbar\omega \qquad ; \qquad \alpha = \frac{\delta}{2g} \qquad ; \qquad \delta = \omega_0 - \omega \tag{2a}$$

and the anti-polariton Hamiltonian in the form

$$\overline{H} = \hbar\omega(\hat{a}\hat{a}^\dagger + s_-s_+) + 2\hbar g(\overline{\alpha} s_z + \hat{a}s_- + \hat{a}^\dagger s_+\ ) - \frac{1}{2}\hbar\omega \qquad ; \qquad \overline{\alpha} = \frac{\overline{\delta}}{2g} \qquad ; \qquad \overline{\delta} = \omega_0 + \omega \tag{2b}$$

where we have introduced respective dimensionless frequency-detuning parameters α , $\overline{\alpha}$ defined as indicated.

2.2 Polariton and anti-polariton excitation number operators

We open this subsection by noting that a conserved excitation number operator for an anti-polariton (anti-Jaynes-Cummings interaction) has been discovered in a recent work by the present author [13], but for completeness and ease of reference, we repeat the calculations here.

In the polariton Hamiltonian H in equation $(2a)$, we identify the normally ordered *polariton excitation number operator* $\hat{N}$, while in the anti-polariton Hamiltonian $\overline{H}$ in equation $(2b)$, we identify the anti-normally ordered *anti-polariton excitation number operator* $\hat{\overline{N}}$ defined by

$$\hat{N} = \hat{a}^\dagger\hat{a} + s_+s_- \qquad ; \qquad \hat{\overline{N}} = \hat{a}\hat{a}^\dagger + s_-s_+ \tag{3a}$$

which we introduce in equations $(2a)$, $(2b)$ as appropriate to express the polariton and anti-polariton Hamiltonians in the form

$$H = \hbar\omega\hat{N} + 2\hbar g(\ \alpha s_z + \hat{a}s_+ + \hat{a}^\dagger s_-\ ) - \frac{1}{2}\hbar\omega \qquad ; \qquad \overline{H} = \hbar\omega\hat{\overline{N}} + 2\hbar g(\ \overline{\alpha} s_z + \hat{a}s_- + \hat{a}^\dagger s_+\ ) - \frac{1}{2}\hbar\omega \tag{3b}$$

Using standard atomic spin and field mode operator algebraic relations

$$[s_+, s_-] = 2s_z\ ; \quad [s_z, s_-] = -s_-\ ; \quad [s_z, s_+] = s_+\ ; \quad s_+s_- = \frac{1}{2} + s_z\ ; \quad s_-s_+ = \frac{1}{2} - s_z$$
$$[s_+s_-\ ,\ s_+] = s_+\ ; \quad [s_-s_+\ ,\ s_+] = -s_+\ ; \quad [s_+s_-\ ,\ s_-] = -s_-\ ; \quad [s_-s_+\ ,\ s_-] = s_-$$
$$\hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + 1 \quad ; \quad [\hat{a}^\dagger\hat{a}\ ,\ \hat{a}] = -\hat{a} \quad ; \quad [\hat{a}^\dagger\hat{a}\ ,\ \hat{a}^\dagger] = \hat{a}^\dagger \tag{3c}$$

we easily prove that the excitation number operators $\hat{N}$, $\hat{\overline{N}}$ in equation $(3a)$ commute with the respective polariton and anti-polariton Hamiltonians H , $\overline{H}$ in equation $(3b)$ according to

$$[\ \hat{N}\ ,\ H\ ] = 0 \qquad ; \qquad [\ \hat{\overline{N}}\ ,\ \overline{H}\ ] = 0 \tag{3d}$$

which proves the standard dynamical property that the polariton excitation number operator $\hat{N} = \hat{a}^\dagger\hat{a} + s_+s_-$ is conserved in the dynamics generated by the polariton Hamiltonian H and the *new* dynamical property that the anti-polariton excitation number operator $\hat{\overline{N}} = \hat{a}\hat{a}^\dagger + s_-s_+$ is conserved in the dynamics generated by the anti-polariton Hamiltonian $\overline{H}$. We easily establish that the anti-polariton excitation number operator $\hat{\overline{N}}$ is not conserved in the polariton dynamics, while the polariton excitation number operator $\hat{N}$ is not conserved in the anti-polariton dynamics according to the commutation relations

$$[\ \hat{\overline{N}}\ ,\ H\ ] \neq 0 \qquad ; \qquad [\ \hat{N}\ ,\ \overline{H}\ ] \neq 0 \tag{3e}$$

2.2.1 Polariton and anti-polariton $U(1)$-symmetry operators

The polariton excitation number operator $\hat{N} = \hat{a}^\dagger\hat{a} + s_+s_-$ generates a $U(1)$-symmetry operator $U(\theta)$ of the polariton Hamiltonian H obtained as

$$U(\theta) = e^{-i\theta\hat{N}} \qquad ; \qquad U^\dagger(\theta) = e^{i\theta\hat{N}} \tag{4a}$$

which according to the commutation relation in equation $(3d)$ satisfies $U(1)$-symmetry relation

$$[\ U(\theta)\ ,\ H\ ] = 0 \qquad \Rightarrow \qquad U^\dagger(\theta)\ H\ U(\theta) = H \tag{4b}$$

Similarly, the anti-polariton excitation number operator $\hat{\overline{N}} = \hat{a}\hat{a}^\dagger + s_- s_+$ generates a $U(1)$-symmetry operator $\overline{U}(\theta)$ of the anti-polariton Hamiltonian $\overline{H}$ obtained as

$$\overline{U}(\theta) = e^{-i\theta\hat{\overline{N}}} \qquad ; \qquad \overline{U}^\dagger(\theta) = e^{i\theta\hat{\overline{N}}} \tag{4c}$$

which according to the commutation relation in equation (3d) satisfies $U(1)$-symmetry relation

$$[\ \overline{U}(\theta)\ ,\ \overline{H}\ ] = 0 \qquad \Rightarrow \qquad \overline{U}^\dagger(\theta)\ \overline{H}\ \overline{U}(\theta) = \overline{H} \tag{4d}$$

It follows from the commutation relations in equation (3e) that $U(\theta)$ in equation (4a) is not a $U(1)$-symmetry operator of the anti-polariton Hamiltonian $\overline{H}$, while $\overline{U}(\theta)$ in equation (4c) is not a $U(1)$-symmetry operator of the polariton Hamiltonian H according to the relations

$$[\ U(\theta)\ ,\ \overline{H}\ ] \neq 0 \quad \Rightarrow \quad U^\dagger(\theta)\ \overline{H}\ U(\theta) \neq \overline{H} \qquad ; \qquad [\ \overline{U}(\theta)\ ,\ H\ ] \neq 0 \quad \Rightarrow \quad \overline{U}^\dagger(\theta)\ H\ \overline{U}(\theta) \neq H \tag{4e}$$

2.2.2 Polariton and anti-polariton parity-symmetry operator

As we have explained above, the commutation relations in equations (3d) and (3e) reveal that for general values of the symmetry transformation parameter θ, the operators $U(\theta)$, $\overline{U}(\theta)$ in equations (4a) , (4c) are only $U(1)$-symmetry operators of the respective polariton or anti-polariton Hamiltonian H or $\overline{H}$, but not symmetry operators of both H and $\overline{H}$. However, there exist a special set of values of θ for which both operators constitute a *common*-symmetry operator of both H and $\overline{H}$. Evaluating the symmetry transformations in equation (4e) in the explicit forms

$$U^\dagger(\theta)\overline{H}U(\theta) = \hbar\omega\hat{\overline{N}} + 2\hbar g(\ \overline{\alpha}s_z + e^{-2i\theta}\hat{a}s_- + e^{2i\theta}\hat{a}^\dagger s_+\ ) - \frac{1}{2}\hbar\omega$$
$$\overline{U}^\dagger(\theta)H\overline{U}(\theta) = \hbar\omega\hat{N} + 2\hbar g(\ \alpha s_z + e^{-2i\theta}\hat{a}s_+ + e^{2i\theta}\hat{a}^\dagger s_-\ ) - \frac{1}{2}\hbar\omega \tag{5a}$$

we can determine the special θ-values for a *common* symmetry operator of both polariton and anti-polariton Hamiltonians H , $\overline{H}$ by imposing the common-symmetry condition

$$e^{-2i\theta} = e^{2i\theta}\ =\ 1 \qquad \Rightarrow \qquad 2\theta = 2n\pi \quad ; \quad \theta = n\pi \quad ; \quad n = 1, 2, 3, ... \tag{5b}$$

where $n = 0$ defines the identity operator. Substituting $\theta = n\pi$ into equations (4a) , (4c), we obtain the common polariton and anti-polariton symmetry operator $\hat{\Pi}_n(\pi)$ in the form

$$\hat{\Pi}_n(\pi) = U(n\pi)\ =\ e^{-in\pi\hat{N}}\ =\ \overline{U}(n\pi)\ =\ e^{-in\pi\hat{\overline{N}}} \quad ; \quad n = 1, 2, 3, ... \tag{5c}$$

Expressing $\hat{\Pi}_n(\pi)$ in the form

$$\hat{\Pi}_n(\pi) = (e^{-i\pi\hat{N}}\ )^n\ =\ (e^{-i\pi\hat{\overline{N}}}\ )^n\ =\ (\hat{\Pi})^n \tag{5d}$$

we identify the standard polariton and anti-polariton parity-symmetry operator $\hat{\Pi}$ defined here by

$$\hat{\Pi} = e^{-i\pi\hat{N}}\ =\ e^{-i\pi\hat{\overline{N}}} \tag{5e}$$

Substituting $\hat{N} = \hat{a}^\dagger\hat{a} + s_+s_-$, $\hat{\overline{N}} = \hat{a}\hat{a}^\dagger + s_-s_+$ and using algebraic relations

$$\hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + 1 \qquad ; \qquad s_-s_+ = s_+s_- - 2s_z \qquad ; \qquad \hat{\overline{N}} = \hat{N} + 2s_-s_+ \tag{5f}$$

we obtain

$$e^{-i\pi\hat{\overline{N}}} = e^{-i\pi\hat{N}}e^{-2i\pi s_-s_+} \qquad ; \qquad e^{-2i\pi s_-s_+} = I \qquad \Rightarrow \qquad e^{-i\pi\hat{\overline{N}}} = e^{-i\pi\hat{N}} \tag{5g}$$

which establishes the common polariton and anti-polariton parity-symmetry operator relation in equation (5e).

It is easy to establish that the polariton and anti-polariton parity-symmetry operator $\hat{\Pi}$ is a symmetry operator of the Rabi Hamiltonian $H_R = \frac{1}{2}(H + \overline{H})$ in equation (1b) according to the symmetry transformation operations

$$\hat{\Pi}^\dagger\ H\ \hat{\Pi} = H \qquad ; \qquad \hat{\Pi}^\dagger\ \overline{H}\ \hat{\Pi} = \overline{H} \qquad ; \qquad \hat{\Pi}^\dagger\ H_R\ \hat{\Pi} = H_R \tag{5h}$$

We observe that the common-symmetry operator $\hat{\Pi}_n(\pi)$ and parity-symmetry operator $\hat{\Pi}$ in equations (5d) and (5e) constitute Z_2-symmetry operators of the polariton, anti-polariton and Rabi Hamiltonians H , $\overline{H}$, H_R.

2.3 Polariton and anti-polariton state transition operators

Noting that the interaction components of the Hamiltonians H , $\overline{H}$ in equation ($3d$) generate state transitions, we introduce a *polariton state transition operator* $\hat{A}$ and an *anti-polariton state transition operator* $\hat{\overline{A}}$ defined by

$$\hat{A} = \alpha s_z + \hat{a} s_+ + \hat{a}^\dagger s_- \qquad ; \qquad \hat{\overline{A}} = \overline{\alpha} s_z + \hat{a} s_- + \hat{a}^\dagger s_+ \tag{6a}$$

which on squaring and applying standard atom-field operator algebraic relations

$$\hat{a}\hat{a}^\dagger = \hat{a}^\dagger \hat{a} + 1 \; ; \quad s_z^2 = \frac{1}{4} \; ; \quad s_-^2 = s_+^2 = 0 \; ; \quad s_+ s_- + s_- s_+ = 1 \; ; \quad s_z s_+ + s_+ s_z = 0 \; ; \quad s_z s_- + s_- s_z = 0 \tag{6b}$$

provide the respective polariton and anti-polariton excitation number operators $\hat{N}$, $\hat{\overline{N}}$ defined in equation ($3c$) in the form

$$\hat{A}^2 = \hat{a}^\dagger \hat{a} + s_+ s_- + \frac{1}{4}\alpha^2 \qquad \Rightarrow \qquad \hat{A}^2 = \hat{N} + \frac{1}{4}\alpha^2$$
$$\hat{\overline{A}}^2 = \hat{a}\hat{a}^\dagger + s_- s_+ + \frac{1}{4}\,\overline{\alpha}^2 - 1 \qquad \Rightarrow \qquad \hat{\overline{A}}^2 = \hat{\overline{N}} + \frac{1}{4}\,\overline{\alpha}^2 - 1 \tag{6c}$$

Substituting $\hat{A}$, $\hat{\overline{A}}$ from equation ($6a$) and $\hat{N} = \hat{A}^2 - \frac{1}{4}\alpha^2$, $\hat{\overline{N}} = \hat{\overline{A}}^2 - \frac{1}{4}\overline{\alpha}^2 + 1$ from equation ($6c$) into equation ($3d$), we express the polariton and anti-polariton Hamiltonians in terms of the respective state transition operators in the form

$$H = \hbar(\,\omega \hat{A}^2 + 2g\hat{A}) - \frac{1}{4}\hbar\omega\alpha^2 - \frac{1}{2}\hbar\omega \qquad ; \qquad \overline{H} = \hbar(\,\omega \hat{\overline{A}}^2 + 2g\hat{\overline{A}}) - \frac{1}{4}\hbar\omega\overline{\alpha}^2 + \frac{1}{2}\hbar\omega \tag{6d}$$

Using equations ($6c$) and ($6d$) easily confirms the polariton and anti-polariton excitation number operator conservation or non-conservation commutation relations in equations ($3f$) , ($3g$).

Similarly, the state transition operators $\hat{A}$, $\hat{\overline{A}}$ are conserved in the dynamics generated by the respective Hamiltonians H , $\overline{H}$, but not in the dynamics generated by the other component Hamiltonian according to the commutation relations

$$[\,\hat{A}\,,\,H\,] = 0 \qquad ; \qquad [\,\hat{A}\,,\,\overline{H}\,] \neq 0 \qquad ; \qquad [\,\hat{\overline{A}}\,,\,\overline{H}\,] = 0 \qquad ; \qquad [\,\hat{\overline{A}}\,,\,H\,] \neq 0 \tag{6e}$$

We demonstrate below that the polariton state transition operator $\hat{A}$ generates *red-sideband* state transitions, while the anti-polariton state transition operator $\hat{\overline{A}}$ generates *blue-sideband* state transitions. Hence, the polariton Hamiltonian H in equation ($6d$) has a *red-sideband* eigenvalue spectrum, while the anti-polariton Hamiltonian $\overline{H}$ has a *blue-sideband* eigenvalue spectrum.

2.3.1 Polariton and anti-polariton $SU(2)/U(1)$, $SU(1,1)/U(1)$ symmetry operators

The conserved polariton state transition operator $\hat{A}$ generates an $SU(2)/U(1)$ coset symmetry operator $C(\eta)$ of the the polariton Hamiltonian H obtained as

$$C(\eta) = e^{-i\eta\hat{A}} \qquad ; \qquad C^\dagger(\eta) = e^{i\eta\hat{A}} \tag{7a}$$

which according to the commutation relation in equation ($6e$) satisfies symmetry relation

$$[C(\eta)\,,\,H] = 0 \qquad \Rightarrow \qquad C^\dagger(\eta)\;H\;C(\eta) = H \tag{7b}$$

Similarly, the conserved anti-polariton state transition operator $\hat{\overline{A}}$ generates an $SU(1,1)/U(1)$ coset symmetry operator $\overline{C}(\eta)$ of the the anti-polariton Hamiltonian $\overline{H}$ obtained as

$$\overline{C}(\overline{\eta}) = e^{-i\overline{\eta}\hat{\overline{A}}} \qquad ; \qquad \overline{C}^\dagger(\overline{\eta}) = e^{i\overline{\eta}\hat{\overline{A}}} \tag{7c}$$

which according to the commutation relation in equation ($6e$) satisfies symmetry relation

$$[\overline{C}(\overline{\eta})\,,\,\overline{H}] = 0 \qquad \Rightarrow \qquad \overline{C}^\dagger(\overline{\eta})\;\overline{H}\;\overline{C}(\overline{\eta}) = \overline{H} \tag{7d}$$

3 Polariton and anti-polariton qubits

We consider a minimal model in which polariton and anti-polariton qubits are formed in a quantum Rabi interaction between a two-level atom initially in a spin-up (excited) or spin-down (ground) state $|u\rangle$ or $|d\rangle$ and a single quantized electromagnetic field mode initially in a stationary number state $|n\rangle$. The initial quantum Rabi state space for polariton and anti-polariton qubit formation is therefore specified by only two n-photon spin-up and spin-down stationary state vectors $|nu\rangle$, $|nd\rangle$ defined in the usual separable product form

$$|nu\rangle = |n\rangle|u\rangle \quad ; \quad |nd\rangle = |n\rangle|d\rangle \quad ; \quad |u\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad ; \quad |d\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8a}$$

Since the n-photon spin-up and spin-down stationary state vectors $|nu\rangle$, $|nd\rangle$ are orthogonal and cannot be transformed into each other through any *composite* atom-field symmetry operator, we consider that they specify two disconnected subspaces of the quantum Rabi state space, namely, the *upper Rabi subspace* specified by the initial n-photon spin-up stationary state vector $|nu\rangle$ and the *lower Rabi subspace* specified by the initial n-photon spin-down stationary state vector $|nd\rangle$. The two Rabi subspaces are distinguished by characteristic physical features regarding the emission and absorption of photons in the formation of polariton or anti-polariton qubits.

In the *upper Rabi subspace* specified by $|nu\rangle$, dynamics begins with the atom in excited (spin-up) state $|u\rangle$ *emitting* a positive energy photon and the field mode in state $|n\rangle$ *absorbs* a *positive energy photon* in a transition to the $(n+1)$-photon spin-down state $|n+1d\rangle$ in a Jaynes-Cummings interaction forming a polariton qubit or a *negative energy photon* in a transition to the $(n-1)$-photon spin-down state $|n-1d\rangle$ in an anti-Jaynes-Cummings interaction forming an anti-polariton qubit. Defining *emission* as emission of positive energy photon, *absorption* as absorption of positive energy photon and *anti-absorption* as absorption of negative energy photon, we characterize the upper Rabi subspace by (initial) photon *emission by the atom* and *absorption or anti-absorption by the field mode* in polariton or anti-polariton qubit formation.

In the *lower Rabi subspace* specified by $|nd\rangle$, dynamics begins with the atom in ground (spin-down) state $|d\rangle$ *absorbing* a positive energy photon and the field mode in state $|n\rangle$ *emitting* a *positive energy photon* in a transition to the $(n-1)$-photon spin-up state $|n-1u\rangle$ in a Jaynes-Cummings interaction forming a polariton qubit or a *negative energy photon* in a transition to the $(n+1)$-photon spin-up state $|n+1u\rangle$ in an anti-Jaynes-Cummings interaction forming an anti-polariton qubit. Noting the definitions of emission , absorption given above and defining *anti-emission* as emission of negative energy photon, we characterize the lower Rabi subspace by (initial) photon *absorption by the atom* and *emission or anti-emission by the field mode* in polariton or anti-polariton qubit formation.

Algebraic operations on state vectors within the quantum Rabi state space are determined through the standard atomic and field mode state algebraic operations

$$s_+|d\rangle = |u\rangle \ ; \quad s_-|u\rangle = |d\rangle \ ; \quad s_+|u\rangle = 0 \ ; \quad s_-|d\rangle = 0 \ ; \quad s_z|u\rangle = \frac{1}{2}|u\rangle \ ; \quad s_z|d\rangle = -\frac{1}{2}|d\rangle$$

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \quad ; \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \tag{8b}$$

Composite atom-field state transition algebraic operations generated by the polariton or anti-polariton state transition operators $\hat{A}$, $\hat{\overline{A}}$ are characterized respectively by *red-shifted* or *blue-shifted* excitation numbers n_α , $n_{\overline{\alpha}}$ obtained as

$$n_\alpha = n + \frac{1}{4}\alpha^2 \quad ; \quad n_{\overline{\alpha}} = n + \frac{1}{4}\overline{\alpha}^2 \quad ; \quad n = 0, 1, 2, 3, ... \tag{8c}$$

3.1 Polariton and anti-polariton qubits in the upper Rabi subspace

In the upper Rabi subspace, interaction begins with the atom in an excited (spin-up) state $|u\rangle$ and the field mode in an initial number state $|n\rangle$, such that a polariton or anti-polariton qubit is formed in an initial n-photon spin-up state $|\psi_{nu}\rangle$ defined by

$$|\psi_{nu}\rangle = |nu\rangle \tag{9a}$$

A polariton qubit is formed in a de-excitation of the atom from the initial n-photon spin-up state $|\psi_{nu}\rangle$ to a *de-excitation polariton state* $|\phi_{nu}\rangle$. Using the standard algebraic operations in equation (8b), we establish that the polariton state transition operator $\hat{A}$ defined in equation (6a) generates reversible state transition algebraic operations between the initial and de-excitation state vectors $|\psi_{nu}\rangle$, $|\phi_{nu}\rangle$ obtained in the form

$$\hat{A}\ |\psi_{nu}\rangle = \sqrt{n_\alpha + 1}\ |\phi_{nu}\rangle \quad ; \quad \hat{A}\ |\phi_{nu}\rangle = \sqrt{n_\alpha + 1}\ |\psi_{nu}\rangle$$

$$\hat{A}^2\ |\psi_{nu}\rangle = (n_\alpha + 1)\ \ |\psi_{nu}\rangle \qquad ; \qquad \hat{A}^2\ |\phi_{nu}\rangle = (n_\alpha + 1)\ \ |\phi_{nu}\rangle \tag{9b}$$

where the de-excitation polariton state vector $|\phi_{nu}\rangle$ has been obtained in the form

$$|\phi_{nu}\rangle = c_{n+1}|nu\rangle + s_{n+1}|n+1d\rangle \tag{9c}$$

after reorganizing the results as appropriate to introduce dimensionless interaction parameters c_{n+1} , s_{n+1} and Rabi frequency R_{n+1} defined by

$$c_{n+1} = \frac{g\alpha}{R_{n+1}} = \frac{\delta}{2R_{n+1}} \quad ; \quad s_{n+1} = \frac{2g\sqrt{n+1}}{R_{n+1}} \quad ; \quad R_{n+1} = 2g\sqrt{n_\alpha + 1} = \frac{1}{2}\sqrt{16g^2(n+1) + \delta^2} \tag{9d}$$

It follows from the reversible state transition algebraic operations in equation (9b) that in the upper Rabi subspace, the de-excitation process forms a *de-excitation polariton qubit* specified by the normalized, but non-orthogonal state vectors $|\psi_{nu}\rangle$, $|\phi_{nu}\rangle$ defined in equations (9a) , (9c).

On the other hand, an anti-polariton is formed in a de-excitation of the atom from the initial n-photon spin-up state $|\psi_{nu}\rangle$ to a *de-excitation anti-polariton state* $|\ \overline{\phi}_{nu}\rangle$. We use the standard algebraic operations in equation (8b) to establish that the anti-polariton state transition operator $\hat{\overline{A}}$ defined in equation (6a) generates reversible state transition algebraic operations between the initial and de-exitation anti-polariton state vectors $|\psi_{nu}\rangle$, $|\ \overline{\phi}_{nu}\rangle$, $|\psi_{nu}\rangle$ obtained in the form

$$\hat{\overline{A}}\ |\psi_{nu}\rangle = \sqrt{n_{\overline{\alpha}}}\ |\ \overline{\phi}_{nu}\rangle \qquad ; \qquad \hat{\overline{A}}\ |\ \overline{\phi}_{nu}\rangle = \sqrt{n_{\overline{\alpha}}}\ |\psi_{nu}\rangle$$

$$\hat{\overline{A}}^2\ |\psi_{nu}\rangle = n_{\overline{\alpha}}\ |\psi_{nu}\rangle \qquad ; \qquad \hat{\overline{A}}^2\ |\ \overline{\phi}_{nu}\rangle = n_{\overline{\alpha}}\ |\ \overline{\phi}_{nu}\rangle \tag{10a}$$

where the de-excitation anti-polariton state vector $|\ \overline{\phi}_{nu}\rangle$ has been obtained in the form

$$|\ \overline{\phi}_{nu}\rangle = \bar{c}_n|nu\rangle + \bar{s}_n|n-1d\rangle \tag{10b}$$

after reorganizing the results to introduce dimensionless interaction parameters $\bar{c}_n$, $\bar{s}_n$ and Rabi frequency $\overline{R}_n$ defined by

$$\bar{c}_n = \frac{g\overline{\alpha}}{\overline{R}_n} = \frac{\overline{\delta}}{2\overline{R}_n} \quad ; \quad \bar{s}_n = \frac{2g\sqrt{n}}{\overline{R}_n} \quad ; \quad \overline{R}_n = 2g\sqrt{n_{\overline{\alpha}}} = \frac{1}{2}\sqrt{16g^2 n + \overline{\delta}^2} \tag{10c}$$

It follows from the reversible state transition algebraic operations in equation (10a) that in the upper Rabi subspace, the de-excitation process forms a *de-excitation anti-polariton qubit* specified by the normalized, but non-orthogonal state vectors $|\psi_{nu}\rangle$, $|\ \overline{\phi}_{nu}\rangle$ defined in equations (9a) , (10b).

3.1.1 The de-excitation polariton qubit subspace

According to the reversible state transition algebraic operations in equation (9b), we interpret a de-excitation polariton qubit as a two-state quantized particle specified by the two state vectors $|\psi_{nu}\rangle$, $|\phi_{nu}\rangle$ defined in equations (9a) , (9c). We therefore introduce a two-dimensional de-excitation polariton qubit subspace spanned by the normalized, but non-orthogonal qubit state vectors $|\psi_{nu}\rangle$, $|\phi_{nu}\rangle$ satisfying normalization and non-orthogonality relations

$$\langle\psi_{nu}|\psi_{nu}\rangle = 1 \quad ; \quad \langle\phi_{nu}|\phi_{nu}\rangle = 1 \quad ; \quad \langle\psi_{nu}|\phi_{nu}\rangle = c_{n+1} \quad ; \quad \langle\phi_{nu}|\psi_{nu}\rangle = c_{n+1} \tag{11}$$

The form of the state transition algebraic operations in equation (9b) suggests that we introduce *polariton qubit transition and identity operators* $\hat{\mathcal{E}}$, $\hat{\mathcal{I}}$ defined within the two-dimensional de-excitation polariton qubit subspace by

$$\hat{\mathcal{E}} = \frac{\hat{A}}{\sqrt{n_\alpha + 1}} \qquad ; \qquad \hat{\mathcal{E}}^2 = \hat{\mathcal{I}} \tag{12a}$$

Substituting equation (12a) into equation (9b), we determine the polariton qubit state transition algebraic operations in the form

$$\hat{\mathcal{E}}\ |\psi_{nu}\rangle = |\phi_{nu}\rangle \quad ; \quad \hat{\mathcal{E}}\ |\phi_{nu}\rangle = |\psi_{nu}\rangle \quad ; \quad \hat{\mathcal{I}}\ |\psi_{nu}\rangle = |\psi_{nu}\rangle \quad ; \quad \hat{\mathcal{I}}\ |\phi_{nu}\rangle = |\phi_{nu}\rangle \tag{12b}$$

The polariton qubit state transition operators have algebraic properties

$$\hat{\mathcal{E}}^\dagger = \hat{\mathcal{E}} \quad ; \quad \hat{\mathcal{E}}^2 = \hat{\mathcal{I}} \quad ; \quad \hat{\mathcal{E}}^{2k} = \hat{\mathcal{I}} \quad ; \quad \hat{\mathcal{E}}^{2k+1} = \hat{\mathcal{E}} \quad ; \quad k = 0, 1, 2, 3, ...$$

$$e^{\pm i\theta\hat{\mathcal{I}}} = e^{\pm i\theta}\hat{\mathcal{I}} \quad ; \quad e^{\pm i\theta\hat{\mathcal{E}}} = \cos\theta\ \hat{\mathcal{I}} \pm i\sin\theta\ \hat{\mathcal{E}} \tag{12c}$$

Substituting $\hat{A}^2$, $\hat{A}$ from equation (12a) into equation (6d) provides the polariton qubit Hamiltonian defined within the qubit subspace in the form

$$H = \hbar\omega\left(n + 1 + \frac{1}{4}\alpha^2\right)\hat{\mathcal{I}} + \hbar R_{n+1}\hat{\mathcal{E}} - \frac{1}{4}\hbar\omega\alpha^2 - \frac{1}{2}\hbar\omega \tag{12d}$$

where we have substituted the detuned excitation number n_α and Rabi frequency R_{n+1} defined in equations (8c) , (9e). Noting that the identity operator $\hat{\mathcal{I}}$ has unit eigenvalue according to equation (12b), we evaluate

$$\frac{1}{4}\hbar\omega\alpha^2\ \hat{\mathcal{I}} - \frac{1}{4}\hbar\omega\alpha^2 - \frac{1}{2}\hbar\omega = -\frac{1}{2}\hbar\omega\hat{\mathcal{I}} \tag{12e}$$

which we substitute back into equation (12d) to obtain the polariton qubit Hamiltonian H in the final form

$$H = \hbar\omega\left(\ n + \frac{1}{2}\ \right)\hat{\mathcal{I}} + \hbar R_{n+1}\hat{\mathcal{E}} \tag{12f}$$

The polariton qubit excitation number, $U(1)$-symmetry and parity-symmetry operators are obtained using equations (6c) , (4a) , (4c) , (5e) and applying equation (12c) in the form

$$\hat{N} = (n+1)\hat{\mathcal{I}} \quad ; \quad U(\theta) = e^{-i\theta(n+1)}\hat{\mathcal{I}} \quad ; \quad \hat{\Pi} = e^{-i\pi(n+1)}\hat{\mathcal{I}}\ =\ \pm\hat{\mathcal{I}} \tag{12g}$$

Finally, we apply the state transition algebraic operations in equation (12b) to establish that superpositions of the polariton qubit state vectors $|\psi_{nu}\rangle$, $|\phi_{nu}\rangle$ constitute eigenvectors $|\Psi^\pm_{nu}\rangle$ of the de-excitation polariton qubit identity and state transition operators $\hat{\mathcal{I}}$, $\hat{\mathcal{E}}$ obtained in the form

$$|\Psi^+_{nu}\rangle = \frac{1}{\sqrt{2}}(\ |\psi_{nu}\rangle + |\phi_{nu}\rangle\ ) \quad ; \quad |\Psi^-_{nu}\rangle = \frac{1}{\sqrt{2}}(\ |\psi_{nu}\rangle - |\phi_{nu}\rangle\ ) \tag{13a}$$

satisfying eigenvalue equations

$$\hat{\mathcal{I}}|\Psi^+_{nu}\rangle = |\Psi^+_{nu}\rangle \quad ; \quad \hat{\mathcal{I}}|\Psi^-_{nu}\rangle = |\Psi^-_{nu}\rangle \quad ; \quad \hat{\mathcal{E}}|\Psi^+_{nu}\rangle = |\Psi^+_{nu}\rangle \quad ; \quad \hat{\mathcal{E}}|\Psi^-_{nu}\rangle = -|\Psi^-_{nu}\rangle \tag{13b}$$

Apart from the standard identity operation $\hat{\mathcal{I}}|\Psi^\pm_{nu}\rangle = |\Psi^\pm_{nu}\rangle$, an important dynamical property which emerges in equation (13b) is that $|\Psi^+_{nu}\rangle$, $|\Psi^-_{nu}\rangle$ are eigenvectors of the polariton qubit state transition operator $\hat{\mathcal{E}}$ belonging to eigenvalues ± 1, respectively.

It follows from equations (12f) and (13b) that $|\Psi^\pm_{nu}\rangle$ are eigenvectors of the polariton qubit Hamiltonian H belonging to energy eigenvalues $E_{n\pm}$, respectively satisfying eigenvalue equations

$$H|\Psi^+_{nu}\rangle = E_{n+}|\Psi^+_{nu}\rangle \quad ; \quad E_{n+} = \hbar\omega\left(\ n + \frac{1}{2}\ \right) + \frac{1}{2}\hbar\sqrt{16g^2(n+1) + (\omega_0 - \omega)^2}$$

$$H|\Psi^-_{nu}\rangle = E_{n-}|\Psi^-_{nu}\rangle \quad ; \quad E_{n-} = \hbar\omega\left(\ n + \frac{1}{2}\ \right) - \frac{1}{2}\hbar\sqrt{16g^2(n+1) + (\omega_0 - \omega)^2} \tag{13c}$$

where we have substituted R_{n+1} from equation (9d), with $\delta = \omega_0 - \omega$ from equation (2a). Recalling the definitions of $|\psi_{nu}\rangle$, $|\phi_{nu}\rangle$ in equations (9a) , (9c) , (9d), we observe that the eigenvectors $|\Psi^\pm_{nu}\rangle$ in equation (13a), with the corresponding eigenvalues $E_{n\pm}$ in equation (13c), are precisely the eigenvectors and corresponding eigenvalues determined through diagonalization of the Jaynes-Cummings (polariton) Hamiltonian H in standard quantum optics literature [14 , 15 , 16], where we note that the symmetrization of the Rabi Hamiltonian H_R in equations (1a)-(1d) yields consistent coupling constant $2g$ instead of the commonly used g in defining the Jaynes-Cummings Hamiltonian such that $4g^2 \rightarrow 16g^2$ in all the results in this paper.

3.1.2 The de-excitation anti-polariton qubit subspace

Based on the reversible state transition algebraic operations in equation ($10a$), we interpret a de-excitation anti-polariton qubit as a two-state quantized particle specified by the two state vectors $|\psi_{nu}\rangle$, $|\overline{\phi}_{nu}\rangle$ defined in equations ($9a$) , ($10b$). We therefore introduce a two-dimensional de-excitation anti-polariton qubit subspace spanned by the normalized, but non-orthogonal qubit state vectors $|\psi_{nu}\rangle$, $|\overline{\phi}_{nu}\rangle$ satisfying normalization and non-orthogonality relations

$$\langle\psi_{nu}|\psi_{nu}\rangle = 1 \quad ; \quad \langle\overline{\phi}_{nu}|\overline{\phi}_{nu}\rangle = 1 \quad ; \quad \langle\psi_{nu}|\overline{\phi}_{nu}\rangle = \bar{c}_n \quad ; \quad \langle\overline{\phi}_{nu}|\psi_{nu}\rangle = \bar{c}_n \tag{14}$$

The form of the state transition algebraic operations in equation ($10a$) suggests that we introduce *anti-polariton qubit transition and identity operators* $\hat{\overline{\mathcal{E}}}$, $\hat{\overline{\mathcal{I}}}$ defined within the two-dimensional de-excitation anti-polariton qubit subspace in the form

$$\hat{\overline{\mathcal{E}}} = \frac{\hat{\overline{A}}}{\sqrt{n_{\overline{\alpha}}}} \qquad ; \qquad \hat{\overline{\mathcal{E}}}^2 = \hat{\overline{\mathcal{I}}} \tag{15a}$$

Substituting equation ($15a$) into equation ($10a$), we determine the anti-polariton qubit state transition algebraic operations in the form

$$\hat{\overline{\mathcal{E}}}\,|\psi_{nu}\rangle = |\overline{\phi}_{nu}\rangle \quad ; \quad \hat{\overline{\mathcal{E}}}\,|\overline{\phi}_{nu}\rangle = |\psi_{nu}\rangle \quad ; \quad \hat{\overline{\mathcal{I}}}\,|\psi_{nu}\rangle = |\psi_{nu}\rangle \quad ; \quad \hat{\overline{\mathcal{I}}}\,|\overline{\phi}_{nu}\rangle = |\overline{\phi}_{nu}\rangle \tag{15b}$$

The anti-polariton qubit state transition operators have algebraic properties

$$\hat{\overline{\mathcal{E}}}^\dagger = \hat{\overline{\mathcal{E}}} \quad ; \quad \hat{\overline{\mathcal{E}}}^2 = \hat{\overline{\mathcal{I}}} \quad ; \quad \hat{\overline{\mathcal{E}}}^{2k} = \hat{\overline{\mathcal{I}}} \quad ; \quad \hat{\overline{\mathcal{E}}}^{2k+1} = \hat{\overline{\mathcal{E}}} \quad ; \quad k = 0, 1, 2, 3, ...$$

$$e^{\pm i\theta\hat{\overline{\mathcal{I}}}} = e^{\pm i\theta}\hat{\overline{\mathcal{I}}} \qquad ; \qquad e^{\pm i\theta\hat{\overline{\mathcal{E}}}} = \cos\theta\,\hat{\overline{\mathcal{I}}} \pm i\sin\theta\,\hat{\overline{\mathcal{E}}} \tag{15c}$$

Substituting $\hat{\overline{A}}^2$, $\hat{\overline{A}}$ from equation ($15a$) into equation ($6d$) provides the anti-polariton qubit Hamiltonian defined within the qubit subspaces in the form

$$\overline{H} = \hbar\omega\left(n + \frac{1}{4}\overline{\alpha}^2\right)\hat{\overline{\mathcal{I}}} + \hbar\overline{R}_n\hat{\overline{\mathcal{E}}} - \frac{1}{4}\hbar\omega\overline{\alpha}^2 + \frac{1}{2}\hbar\omega \tag{15d}$$

where we have substituted the detuned excitation number $n_{\overline{\alpha}}$ and Rabi frequency $\overline{R}_n$ defined in equations ($8c$) , ($10d$). Noting that the identity operator $\hat{\overline{\mathcal{I}}}$ has unit eigenvalue according to equation ($15b$), we evaluate

$$\frac{1}{4}\hbar\omega\overline{\alpha}^2\,\hat{\overline{\mathcal{I}}} - \frac{1}{4}\hbar\omega\overline{\alpha}^2 + \frac{1}{2}\hbar\omega = \frac{1}{2}\hbar\omega\,\hat{\overline{\mathcal{I}}} \tag{15e}$$

which we substitute back into equation ($15d$) to obtain the anti-polariton qubit Hamiltonian $\overline{H}$ in the final form

$$\overline{H} = \hbar\omega\left(\,n + \frac{1}{2}\,\right)\hat{\overline{\mathcal{I}}} + \hbar\overline{R}_n\hat{\overline{\mathcal{E}}} \tag{15f}$$

The anti-polariton qubit excitation number, $U(1)$-symmetry and parity-symmetry operators are obtained using equations ($6c$) , ($4a$) , ($4c$) , ($5e$) and applying equations ($12e$) , ($12f$) in the form

$$\hat{\overline{N}} = (n+1)\hat{\overline{\mathcal{I}}} \quad ; \quad \overline{U}(\theta) = e^{-i\theta(n+1)}\hat{\overline{\mathcal{I}}} \quad ; \quad \hat{\overline{\Pi}} = e^{-i\pi(n+1)}\hat{\overline{\mathcal{I}}} \;=\; \pm\hat{\overline{\mathcal{I}}} \tag{15g}$$

Application of the state transition algebraic operations in equation ($15b$) reveals that superpositions of the anti-polariton qubit state vectors $|\psi_{nu}\rangle$, $|\overline{\phi}_{nu}\rangle$ constitute eigenvectors $|\overline{\Psi}^{\pm}_{nu}\rangle$ of the de-excitation anti-polariton qubit identity and state transition operators $\hat{\overline{\mathcal{I}}}$, $\hat{\overline{\mathcal{E}}}$ obtained in the form

$$|\overline{\Psi}^{+}_{nu}\rangle = \frac{1}{\sqrt{2}}(\,|\psi_{nu}\rangle + |\overline{\phi}_{nu}\rangle\,) \qquad ; \qquad |\overline{\Psi}^{-}_{nu}\rangle = \frac{1}{\sqrt{2}}(\,|\psi_{nu}\rangle - |\overline{\phi}_{nu}\rangle\,) \tag{16a}$$

satisfying eigenvalue equations

$$\hat{\overline{\mathcal{I}}}|\overline{\Psi}^{+}_{nu}\rangle = |\overline{\Psi}^{+}_{nu}\rangle \quad ; \quad \hat{\overline{\mathcal{I}}}|\overline{\Psi}^{-}_{nu}\rangle = |\overline{\Psi}^{-}_{nu}\rangle \quad ; \quad \hat{\overline{\mathcal{E}}}|\overline{\Psi}^{+}_{nu}\rangle = |\overline{\Psi}^{+}_{nu}\rangle \quad ; \quad \hat{\overline{\mathcal{E}}}|\overline{\Psi}^{-}_{nu}\rangle = -|\overline{\Psi}^{-}_{nu}\rangle \tag{16b}$$

It emerges in equation (16b) that $| \overline{\Psi}_{nu}^{+}\rangle$, $| \overline{\Psi}_{nu}^{-}\rangle$ are eigenvectors of the anti-polariton qubit state transition operator $\hat{\overline{\mathcal{E}}}$ belonging to eigenvalues ± 1, respectively.

It follows from equations (15f) and (16b) that $| \overline{\Psi}_{nu}^{\pm}\rangle$ are eigenvectors of the anti-polariton qubit Hamiltonian $\overline{H}$ belonging to energy eigenvalues $\overline{E}_{\pm}$, respectively satisfying eigenvalue equations

$$\overline{H} \; | \; \overline{\Psi}_{nu}^{+}\rangle = \overline{E}_{n+} \; | \; \overline{\Psi}_{nu}^{+}\rangle \qquad ; \qquad \overline{E}_{n+} = \hbar\omega \left(\; n + \frac{1}{2} \; \right) + \frac{1}{2}\hbar\sqrt{16g^2 n + (\omega_0 + \omega)^2}$$

$$\overline{H} \; | \; \overline{\Psi}_{nu}^{-}\rangle = \overline{E}_{n-} | \; \overline{\Psi}_{nu}^{-}\rangle \qquad ; \qquad \overline{E}_{n-} = \hbar\omega \left(\; n + \frac{1}{2} \; \right) - \frac{1}{2}\hbar\sqrt{16g^2 n + (\omega_0 + \omega)^2} \qquad (16c)$$

where we have substituted $\overline{R}_n$ from equation (10c), with $\overline{\delta} = \omega_0 + \omega$ from equation (2b). Noting the definitions of $|\psi_{nu}\rangle$, $| \; \overline{\phi}_{nu}\rangle$ in equations (9a) , (10b) , (10c), we observe that the anti-polariton qubit Hamiltonian eigenvectors $| \; \overline{\Psi}_{nu}^{\pm}\rangle$ in equation (16a), with the corresponding eigenvalues $\overline{E}_{n\pm}$ in equation (16c), are new, but can be determined through diagonalization of the anti-Jaynes-Cummings (anti-polariton) Hamiltonian $\overline{H}$ based on the existence of a conserved anti-polariton excitation number operator $\hat{\overline{N}}$, which we have established here in section 2.2.

3.1.3 Dynamical evolution of polariton and anti-polariton qubits

The polariton and anti-polariton qubit Hamiltonians H , $\overline{H}$ generate time evolution operators $U(t)$, $\overline{U}(t)$ obtained from the respective time-dependent Schroedinger equations

$$i\hbar\frac{\partial}{\partial t}|\Psi_{nu}\rangle = H|\Psi_{nu}\rangle \qquad ; \qquad i\hbar\frac{\partial}{\partial t}| \; \overline{\Psi}_{nu}\rangle = \overline{H} \; | \; \overline{\Psi}_{nu}\rangle \qquad (17a)$$

as

$$U(t) = e^{-\frac{i}{\hbar}Ht} \qquad ; \qquad \overline{U}(t) = e^{-\frac{i}{\hbar}\overline{H}t} \qquad (17b)$$

which on substituting H , $\overline{H}$ from equations (12f) , (15f) as appropriate and noting the commutation relations $[\hat{\mathcal{I}} \; , \; \hat{\mathcal{E}}] = 0$, $[\hat{\overline{\mathcal{I}}} \; , \; \hat{\overline{\mathcal{E}}}] = 0$, take the convenient factorized forms

$$U(t) = e^{-i\omega(\; n+\frac{1}{2} \;)\hat{\mathcal{I}}t} e^{-iR_{n+1}\hat{\mathcal{E}}t} \qquad ; \qquad \overline{U}(t) = e^{-i\omega(\; n+\frac{1}{2} \;)\hat{\overline{\mathcal{I}}}t} e^{-i\overline{R}_n\hat{\overline{\mathcal{E}}}t} \qquad (17c)$$

Using the respective general exponentiation results from equations (12c) , (15c) in equation (17c) as appropriate and noting $\hat{\mathcal{I}}\hat{\mathcal{E}} = \hat{\mathcal{E}}$, $\hat{\overline{\mathcal{I}}} \; \hat{\overline{\mathcal{E}}} = \hat{\overline{\mathcal{E}}}$, we evaluate the polariton qubit time evolution operator in the final form

$$U(t) = e^{-i\omega(\; n+\frac{1}{2} \;)t} \left(\cos(R_{n+1}t)\hat{\mathcal{I}} - i\sin(R_{n+1}t)\hat{\mathcal{E}}\right) \qquad (17d)$$

and the anti-polariton qubit time evolution operator in the final form

$$\overline{U}(t) = e^{-i\omega(\; n+\frac{1}{2} \;)t} \left(\cos(\overline{R}_n t)\hat{\overline{\mathcal{I}}} - i\sin(\overline{R}_n t)\hat{\overline{\mathcal{E}}}\right) \qquad (17e)$$

We use the respective time evolution operators $U(t)$, $\overline{U}(t)$ to determine the general time evolving polariton and anti-polariton qubit state vectors starting from the corresponding initial qubit state vectors. The general time evolving polariton qubit state vectors $|\Psi_{nu}(t)\rangle$, $|\Phi_{nu}(t)\rangle$ are generated from the respective initial qubit state vectors $|\psi_{nu}\rangle$, $|\phi_{nu}\rangle$, while the general time evolving anti-polariton qubit state vectors $| \; \overline{\Psi}_{nu}(t)\rangle$, $| \; \overline{\Psi}_{nu}(t)\rangle$ are generated from the respective initial qubit state vectors $|\psi_{nu}\rangle$, $| \; \overline{\phi}_{nu}\rangle$ through the respective time evolution operators according to

$$|\Psi_{nu}(t)\rangle = U(t)|\psi_{nu}\rangle \qquad ; \qquad |\Phi_{nu}(t)\rangle = U(t)|\phi_{nu}\rangle$$

$$| \; \overline{\Psi}_{nu}(t)\rangle = \overline{U}(t)|\psi_{nu}\rangle \qquad ; \qquad | \; \overline{\Phi}_{nu}(t)\rangle = \overline{U}(t)| \; \overline{\phi}_{nu}\rangle \qquad (17f)$$

Substituting $U(t)$, $\overline{U}(t)$ from equations (17d) , (17e) into equation (17f) and applying the respective state transition algebraic operations from equations (12b) , (15b) as appropriate, we obtain the general time evolving polariton qubit state vectors in the final form

$$|\Psi_{nu}(t)\rangle = e^{-i\omega(\; n+\frac{1}{2} \;)t} (\; \cos(R_{n+1}t)|\psi_{nu}\rangle - i\sin(R_{n+1}t)|\phi_{nu}\rangle \;)$$

$$|\Phi_{nu}(t)\rangle = e^{-i\omega(\ n+\frac{1}{2}\ )t}\left(\ \cos(R_{n+1}t)|\phi_{nu}\rangle - i\sin(R_{n+1}t)|\psi_{nu}\rangle\ \right) \qquad (17g)$$

and the general time evolving anti-polariton qubit state vectors in the final form

$$|\ \overline{\Psi}_{nu}(t)\rangle = e^{-i\omega(\ n+\frac{1}{2}\ )t}\left(\ \cos(\overline{R}_n t)|\psi_{nu}\rangle - i\sin(\overline{R}_n t)|\ \overline{\phi}_{nu}\rangle\ \right)$$

$$|\ \overline{\Phi}_{nu}(t)\rangle = e^{-i\omega(\ n+\frac{1}{2}\ )t}\left(\ \cos(\overline{R}_n t)|\ \overline{\phi}_{nu}\rangle - i\sin(\overline{R}_n t)|\psi_{nu}\rangle\ \right) \qquad (17h)$$

3.1.4 De-excitation, absorption and anti-absorption probabilities

We observe that the general time evolving polariton qubit state vectors $|\Psi_{nu}(t)\rangle$, $|\Phi_{nu}(t)\rangle$ in equation $(17g)$ describe Rabi oscillations at frequency R_{n+1} between the qubit states $|\psi_{nu}\rangle$ and $|\phi_{nu}\rangle$, while the general time evolving anti-polariton qubit state vectors $|\overline{\Psi}_{nu}(t)\rangle$, $|\overline{\Phi}_{nu}(t)\rangle$ in equation $(17h)$ describe Rabi oscillations at frequency $\overline{R}_n$ between the qubit states $|\psi_{nu}\rangle$ and $|\overline{\phi}_{nu}\rangle$.

Under dynamical evolution described by the respective general time evolving state vectors $|\Psi_{nu}(t)\rangle$, $|\ \overline{\Psi}_{nu}(t)\rangle$ in equations $(17g)$, $(17h)$, the probability $P_{dex}(t)$ of transition from the initial state $|\psi_{nu}\rangle$ to the de-excitation state $|\phi_{nu}\rangle$ in a de-excitation polariton qubit is obtained as

$$P_{dex}(t) = |\ \langle\phi_{nu}|\overline{\Psi}_{nu}(t)\rangle\ |^2 \qquad \Rightarrow \qquad P_{dex}(t) = \sin^2(R_{n+1}t) \qquad (18a)$$

while the probability $\overline{P}_{dex}(t)$ of transition from the initial state $|\psi_{nu}\rangle$ to the de-excitation state $|\ \overline{\phi}_{nu}\rangle$ in a de-excitation anti-polariton qubit is obtained as

$$\overline{P}_{dex}(t) = |\ \langle\ \overline{\phi}_{nu}|\overline{\Psi}_{nu}(t)\rangle\ |^2 \quad \Rightarrow \quad \overline{P}_{dex}(t) = \sin^2(\overline{R}_n t) \qquad (18b)$$

Since a de-excitation is characterized by emission of a positive frequency photon by the atom, we interpret the de-excitation probability $P_{dex}(t)$ or $\overline{P}_{dex}(t)$ as the probability of emission of a photon by the atom in polariton or anti-polariton qubit dynamics.

To gain more insight, we apply the interpretation that in the upper Rabi subspace, a de-excitation of the atom to the spin-down state $|d\rangle$, characterized by emission of a positive frequency photon, causes the rotating positive frequency field mode to absorb a positive energy photon in a transition from the n-photon spin-up state $|nu\rangle$ to the $(n+1)$-photon spin-down state $|n+1d\rangle$ in a polariton qubit or the anti-rotating negative frequency field mode to absorb a negative energy photon (anti-absorption) in a transition from the n-photon spin-up state $|nu\rangle$ to the $(n-1)$-photon spin-down state $|n-1d\rangle$ in an anti-polariton qubit. Substituting the stationary qubit state vectors $|\psi_{nu}\rangle$, $|\phi_{nu}\rangle$, $|\ \overline{\phi}_{nu}\rangle$ from equations $(9a)$, $(9d)$, $(10c)$ into equations $(17g)$, $(17h)$, respectively, we express the general time evolving polariton qubit state vectors in the form

$$|\Psi_{nu}(t)\rangle = e^{-i\omega(\ n+\frac{1}{2}\ )t}\left(\ (\cos(R_{n+1}t) - ic_{n+1}\sin(R_{n+1}t)\ )|nu\rangle - is_{n+1}\sin(R_{n+1}t)|n+1d\rangle\ \right)$$

$$|\Phi_{nu}(t)\rangle = e^{-i\omega(\ n+\frac{1}{2}\ )t}\left(\ (c_{n+1}\cos(R_{n+1}t) - i\sin(R_{n+1}t)\ )|nu\rangle + s_{n+1}\cos(R_{n+1}t)|n+1d\rangle\ \right) \qquad (18c)$$

which explicitly describe reversible time evolving transitions between the photon emission-absorption state vectors $|nu\rangle$, $|n+1d\rangle$ at Rabi frequency R_{n+1}, while the general time evolving anti-polariton qubit state vectors are expressed in the form

$$|\ \overline{\Psi}_{nu}(t)\rangle = e^{-i\omega(\ n+\frac{1}{2}\ )t}\left(\ (\cos(\overline{R}_n t) - i\bar{c}_n\sin(\overline{R}_n t)\ )|nu\rangle - i\bar{s}_n\sin(\overline{R}_n t)|n-1d\rangle\ \right)$$

$$|\ \overline{\Phi}_{nu}(t)\rangle = e^{-i\omega(\ n+\frac{1}{2}\ )t}\left(\ (\bar{c}_n\cos(\overline{R}_n t) - i\sin(\overline{R}_n t)\ )|nu\rangle + \bar{s}_n\cos(\overline{R}_n t)|n-1d\rangle\ \right) \qquad (18d)$$

which explicitly describe reversible time evolving transitions between the photon emission-anti-absorption state vectors $|nu\rangle$, $|n-1d\rangle$ at Rabi frequency $\overline{R}_n$, where we recall that *anti-absorption* means absorption of a negative energy photon by the anti-rotating negative frequency field mode.

We observe that the general time evolving polariton and anti-polariton qubit state vectors $|\Psi_{nu}(t)\rangle$, $|\Phi_{nu}(t)\rangle$, $|\ \overline{\Psi}_{nu}(t)\rangle$, $|\ \overline{\Phi}_{nu}(t)\rangle$ obtained in equations $(17g)$, $(17h)$, $(18c)$, $(18d)$ are entangled atom-field state vectors which preserve the normalization, non-orthogonality and state transition algebraic relations of the qubit state vectors in equations (11) , $(12b)$, (14) , $(15b)$, respectively in the form

$$\langle\Psi_{nu}(t)|\Psi_{nu}(t)\rangle = 1\ ; \quad \langle\Phi_{nu}(t)|\Phi_{nu}(t)\rangle = 1\ ; \quad \langle\Psi_{nu}(t)|\Phi_{nu}(t)\rangle = c_{n+1}\ ; \quad \langle\Phi_{nu}(t)|\Psi_{nu}(t)\rangle = c_{n+1}$$

$$\hat{\mathcal{E}}\ |\Psi_{nu}(t)\rangle = |\Phi_{nu}(t)\rangle \ ; \quad \hat{\mathcal{E}}\ |\Phi_{nu}(t)\rangle = |\Psi_{nu}(t)\rangle \ ; \quad \hat{I}\ |\Psi_{nu}(t)\rangle = |\Psi_{nu}(t)\rangle \ ; \quad \hat{I}\ |\Phi_{nu}(t)\rangle = |\Phi_{nu}(t)\rangle \qquad (18e)$$

$$\langle\ \overline{\Psi}_{nu}(t)|\ \overline{\Psi}_{nu}(t)\rangle = 1 \ ; \quad \langle\ \overline{\Phi}_{nu}(t)|\ \overline{\Phi}_{nu}(t)\rangle = 1 \ ; \quad \langle\ \overline{\Psi}_{nu}(t)|\ \overline{\Phi}_{nu}(t)\rangle = \bar{c}_n \ ; \quad \langle\ \overline{\Phi}_{nu}(t)|\ \overline{\Psi}_{nu}(t)\rangle = \bar{c}_n$$

$$\hat{\overline{\mathcal{E}}}\ |\Psi_{nu}(t)\rangle = |\ \overline{\Phi}_{nu}(t)\rangle \ ; \quad \hat{\overline{\mathcal{E}}}\ |\ \overline{\Phi}_{nu}(t)\rangle = |\Psi_{nu}(t)\rangle \ ; \quad \hat{\overline{I}}\ |\Psi_{nu}(t)\rangle = |\Psi_{nu}(t)\rangle \ ; \quad \hat{\overline{I}}\ |\ \overline{\Phi}_{nu}(t)\rangle = |\ \overline{\Phi}_{nu}(t)\rangle \qquad (18f)$$

In polariton qubit dynamics, the probability $P_{ab}(t)$ of absorption of a positive energy photon by the rotating positive frequency field mode in a transition from $|nu\rangle$ to $|n+1d\rangle$ is determined from the time evolving emission-absorption state vector $|\Psi_{nu}(t)\rangle$ in equation $(18c)$ according to

$$P_{ab}(t) = |\ \langle n+1d|\Psi_{nu}(t)\rangle\ |^2 \qquad \Rightarrow \qquad P_{ab}(t) = s_{n+1}^2 \sin^2(R_{n+1}t) \qquad (18e)$$

which on substituting s_{n+1} , R_{n+1} from equation $(9e)$ takes the explicit form

$$P_{ab}(t) = \frac{16g^2(n+1)}{16g^2(n+1)+(\omega_0-\omega)^2}\sin^2(\frac{1}{2}\sqrt{16g^2(n+1)+(\omega_0-\omega)^2}\ t) \qquad (18f)$$

accounting for *red-sideband* transitions generated by the polariton qubit Hamiltonian H.

In anti-polariton qubit dynamics, the probability of absorption of a negative energy photon by the anti-rotating negative frequency field mode in a transition from $|nu\rangle$ to $|n-1d\rangle$, which we call the *anti-absorption probability* $\overline{P}_{aab}(t)$, is determined from the time evolving emission-anti-absorption state vector $|\ \overline{\Psi}_{nu}(t)\rangle$ in equation $(18d)$ according to

$$\overline{P}_{aab}(t) = |\ \langle n-1d|\ \overline{\Psi}_{nu}(t)\rangle\ |^2 \qquad \Rightarrow \qquad \overline{P}_{aab}(t) = \bar{s}_n^2 \sin^2(\overline{R}_n t) \qquad (18g)$$

which on substituting $\bar{s}_n$, $\bar{R}_n$ from equation $(10d)$ takes the explicit form

$$\overline{P}_{aab}(t) = \frac{16g^2n}{16g^2n+(\omega_0+\omega)^2}\sin^2(\frac{1}{2}\sqrt{16g^2n+(\omega_0+\omega)^2}\ t) \qquad (18h)$$

accounting for *blue-sideband* transitions generated by the anti-polariton Hamiltonian $\overline{H}$.

Substituting equations $(18a)$, $(18b)$ into equations $(18e)$, $(18g)$ respectively, we express the absorption and anti-absorption probabilities $P_{ab}(t)$, $\overline{P}_{aab}(t)$ in terms of the respective de-excitation probabilities $P_{dex}(t)$, $\overline{P}_{dex}(t)$ in the form

$$P_{ab}(t) = s_{n+1}^2 P_{dex}(t) \qquad ; \qquad \overline{P}_{aab}(t) = \bar{s}_n^2 \overline{P}_{dex}(t) \qquad (19a)$$

which reveal that the probability of absorption or anti-absorption of a photon by the field mode is proportional to the probability of emission of a photon in a de-excitation of the atom in polariton or anti-polariton qubit dynamics.

To compare the de-excitation probabilities $P_{dex}(t)$, $\overline{P}_{dex}(t)$ determining the occurrence of red or blue sideband polariton or anti-polariton qubit state transitions, we substitute $\delta = \omega_0 - \omega$, $\overline{\delta} = \omega_0 + \omega$ from equations $(2a)$, $(2b)$ into R_{n+1} , $\overline{R}_n$ in equations $(9e)$, $(10d)$), respectively and rewrite

$$(\omega_0+\omega)^2 = (\omega_0-\omega)^2 + 4\omega_0\omega \quad ; \qquad 16g^2n = 16g^2(n+1) - 16g^2$$

$$R_{n+1} = \frac{1}{2}\sqrt{16g^2(n+1)+(\omega_0-\omega)^2} \quad ; \qquad \overline{R}_n = \frac{1}{2}\sqrt{16g^2(n+1)+(\omega_0-\omega)^2+4(\omega_0\omega-4g^2)} \qquad (19b)$$

giving the relation between the anti-polariton qubit Rabi oscillation frequency $\overline{R}_n$ and the polariton qubit Rabi oscillation frequency R_{n+1} in the form

$$\overline{R}_n = R_{n+1}\sqrt{1+\frac{\omega_0\omega-4g^2}{R_{n+1}^2}} \ = \ R_{n+1}\sqrt{1+\frac{\omega_0\omega}{R_{n+1}^2}\left(1-\frac{g^2}{g_c^2}\right)} \qquad ; \qquad \omega_0\omega - 4g^2 = \omega_0\omega\left(1-\frac{g^2}{g_c^2}\right) \quad (19c)$$

where we have introduced a *critical coupling constant* g_c obtained as

$$g_c = \frac{1}{2}\sqrt{\omega_0\omega} \qquad (19d)$$

Substituting $\overline{R}_n$ from equation (19c) into equation (18b), we express the de-excitation probability in anti-polariton qubit state transitions in the form

$$\overline{P}_{dex}(t) = \sin^2\left(R_{n+1}\sqrt{1 + \frac{\omega_0\omega}{R_{n+1}^2}\left(1 - \frac{g^2}{g_c^2}\right)}\; t\right) \tag{19e}$$

We identify $g_c = \frac{1}{2}\sqrt{\omega_0\omega}$ in equation (19d) as the critical coupling constant for quantum phase transition in the Rabi and Dicke models determined in elaborate calculations in [17 , 18]. In the present work, the critical coupling constant emerges easily as a dynamical parameter which determines the size of the Rabi oscillation frequency $\overline{R}_n$ of blue-sideband state transitions in an anti-polariton qubit relative to the Rabi oscillation frequency R_{n+1} of the red-sideband state transitions in a polariton qubit according to equation (19c). According to the form of state transition probabilities $P_{dex}(t)$, $\overline{P}_{dex}(t)$ in equations (18a) , (19e), the critical coupling constant provides a measure of the probability $\overline{P}_{dex}(t)$ of occurrence of blue-sideband state transitions generated by the anti-polariton (ant-Jaynes-Cummings interaction) Hamiltonian $\overline{H}$ relative to the probability $P_{dex}(t)$ of occurrence of red-sideband state transitions generated by the polariton (Jaynes-Cummings interaction) Hamiltonian H.

In the coupling regime where $g < g_c$, the de-excitation probability $\overline{P}_{dex}(t)$, which provides a measure of blue-sideband transitions between the anti-polariton qubit states, oscillates faster and on average is smaller than the corresponding de-excitation probability $P_{dex}(t)$, which provides a measure of red-sideband transitions between the polariton qubit states, such that dynamics in the $g < g_c$ regime is dominated by red-sideband transitions generated by the polariton qubit Hamiltonian H. On the other hand, in the coupling regime where $g > g_c$, the de-excitation probability $P_{dex}(t)$ for polariton qubit state transitions oscillates faster and on average is smaller than the corresponding de-excitation probability $\overline{P}_{dex}(t)$ for anti-polariton qubit state transitions such that dynamics in the $g > g_c$ regime is dominated by blue-sideband transitions generated by the anti-polariton qubit Hamiltonian $\overline{H}$. The critical coupling $g = g_c$ dynamics is characterized by equal de-excitation probabilities, $P^c_{dex}(t) = \overline{P}^c_{dex}(t)$, for polariton and anti-polariton qubit state transitions such that the $g = g_c$ dynamics is composed of equal red and blue sideband transitions generated by polariton and anti-polariton qubit Hamiltonians H^c , $\overline{H}^c$ obtained by setting $g = g_c$, $R^c_{n+1} = \overline{R}^c_n$ in equations (12f) , (15f), where

$$g^2 = g_c^2 \;=\; \frac{1}{4}\omega_0\omega \qquad \Rightarrow \qquad R^c_{n+1} = \overline{R}^c \;=\; \frac{1}{2}\sqrt{4\omega_0\omega(n+1) + (\omega_0 - \omega)^2} = \frac{1}{2}\sqrt{4\omega_0\omega n + (\omega_0 + \omega)^2} \tag{19f}$$

3.1.5 Spontaneous de-excitation, absorption and virtual anti-absorption

If the interaction begins with the atom in the spin-up (excited) state $|u\rangle$ and the field mode in the vacuum state $|0\rangle$, then polariton or anti-polariton qubit formation starts with *spontaneous de-excitation process* from the initial 0-photon spin-up state $|\psi_{0u}\rangle$ where the atom *spontaneously emits* a positive frequency photon, which causes spontaneous absorption of a positive energy photon by the rotating positive frequency field mode in a polariton qubit state transition from $|\psi_{0u}\rangle$ to $|\phi_{0u}\rangle$ or spontaneous absorption of a negative energy photon (spontaneous anti-absorption) by the anti-rotating negative frequency field mode in an anti-polariton qubit state transition from $|\psi_{0u}\rangle$ to $|\,\overline{\phi}_{0u}\rangle$, where the initial de-excitation polariton or anti-polariton qubit state vectors are specified by setting $n = 0$ in equations (9a) , (9d) , (9e) , (10c) , (10d) (noting $|-1d\rangle = 0$) in the form

$$|\psi_{0u}\rangle = |0u\rangle \qquad ; \qquad |\phi_{0u}\rangle = c_1|0u\rangle + s_1|1d\rangle \qquad ; \qquad |\,\overline{\phi}_{0u}\rangle = \bar{c}_0|0u\rangle \tag{20a}$$

with

$$c_1 = \frac{\omega_0 - \omega}{2R_1}\;;\quad s_1 = \frac{2g}{R_1}\;;\quad R_1 = \frac{1}{2}\sqrt{16g^2 + (\omega_0 - \omega)^2} \qquad ; \qquad \bar{c}_0 = 1\;;\quad \bar{s}_0 = 0\;;\quad \overline{R}_0 = \frac{1}{2}(\omega_0 + \omega) \tag{20b}$$

Setting $n = 0$ in equations (17g) , (18a) provides the general time evolving spontaneous de-excitation polariton qubit state vectors $|\Psi_{0u}(t)\rangle$, $|\Phi_{0u}(t)\rangle$, with corresponding spontaneous de-excitation and absorption probabilities $P^{spt}_{dex}(t)$, $P^{spt}_{ab}(t)$ in the form

$$|\Psi_{0u}(t)\rangle = e^{-\frac{i}{2}\omega t}\left(\,\cos(R_1 t)|\psi_{0u}\rangle - i\sin(R_1 t)|\phi_{0u}\rangle\,\right)\;;\quad P^{spt}_{dex}(t) = \sin^2 R_1 t\;;\quad P^{spt}_{ab}(t) = s_1^2 P^{spt}_{dex}(t)$$

$$|\Phi_{0u}(t)\rangle = e^{-\frac{i}{2}\omega t}\left(\,\cos(R_1 t)|\phi_{0u}\rangle - i\sin(R_1 t)|\psi_{0u}\rangle\,\right) \tag{20c}$$

which describe spontaneously generated Rabi oscillations at frequency R_1 between de-excitation polariton qubit states $|\psi_{0u}\rangle$, $|\phi_{0u}\rangle$ governed by spontaneous de-excitation probability $P^{spt}_{dex}(t)$, while setting $n = 0$ in equations (17h) , (18b) provides the general time evolving spontaneous de-excitation anti-polariton qubit state vectors $|\,\overline{\Psi}_{0u}(t)\rangle$, $|\,\overline{\Phi}_{0u}(t)\rangle$, with corresponding spontaneous de-excitation and anti-absorption probabilities $\overline{P}^{spt}_{dex}(t)$, $\overline{P}^{spt}_{aab}(t)$ in the form

$$|\,\overline{\phi}_{0u}\rangle = |\psi_{0u}\rangle \qquad ; \qquad |\,\overline{\Psi}_{0u}(t)\rangle = |\,\overline{\Phi}_{0u}(t)\rangle \;=\; e^{-\frac{i}{2}(2\omega+\omega_0)t}|\psi_{0u}\rangle$$

$$\overline{P}^{spt}_{dex}(t) = \sin^2\frac{1}{2}(\omega_0+\omega)t \qquad ; \qquad \overline{P}^{spt}_{aab}(t) = 0 \tag{20d}$$

where we have used equations (20a) , (20b) to obtain the final results in equation (20d).

An important physical property which emerges from equations (20c) , (20d) is that in a spontaneous de-excitation process starting with the atom in spin-up state $|u\rangle$ and the field mode in vacuum state $|0\rangle$, only the polariton qubit Hamiltonian H generates observable dynamical effects characterized by spontaneous emission and red-sideband transitions described by the general time evolving spontaneous de-excitation polariton qubit state vectors $|\Psi_{0u}(t)\rangle$, $|\Phi_{0u}(t)\rangle$ in equation (20c), while the anti-polariton qubit Hamiltonian $\overline{H}$ generates only *virtual transitions* between degenerate anti-polariton qubit states $|\psi_{0u}\rangle = |\,\overline{\phi}_{0u}\rangle$, signified by non-zero but fast oscillating spontaneous de-excitation probability $\overline{P}^{spt}_{dex}(t) = \sin^2\frac{1}{2}(\omega_0{+}\omega)t$ and vanishing spontaneous anti-absorption probability $\overline{P}^{spt}_{aab}(t) = 0$, the process being described by the degenerate time evolving plane wave spontaneous de-excitation anti-polariton qubit state vectors $|\,\overline{\Psi}_{0u}(t)\rangle = |\,\overline{\Phi}_{0u}(t)\rangle$ in equation (20d).

In summary, polariton or anti-polariton qubit dynamics in the upper Rabi subspace, starting with initial photon emission by the atom, is characterized by photon absorption or anti-absorption by the field mode. If the field mode is initially in the vacuum state, then the dynamics starting with spontaneous emission by the atom is dominated by red-sideband transitions generated by the polariton qubit Hamiltonian, while the anti-polariton qubit dynamics is suppressed into virtual transitions between degenerate qubit states described by plane waves.

3.2 Polariton and anti-polariton qubits in the lower Rabi subspace

We observe that the lower Rabi subspace where interaction begins with initial photon absorption by the atom has dynamical features running counter to the dynamical features characterizing interaction in the upper Rabi subspace, which we have described above. To determine the characteristic dynamical features of the interaction, we develop the algebraic framework within the lower Rabi subspace separately, even though at the expense of repeating the basic algebraic properties and operations. It is still necessary to follow the repeated algebraic operations, since they carry the underlying information on the characteristic features of the interaction.

In the lower Rabi subspace, interaction begins with the atom in an initial spin-down (ground) state $|d\rangle$ and the field mode in an initial number state $|n\rangle$, such that a polariton or anti-polariton qubit is formed in an initial n-photon spin-down state $|\psi_{nd}\rangle$ defined by

$$|\psi_{nd}\rangle = |nd\rangle \tag{21a}$$

A polariton qubit is formed in an excitation of the atom from the initial n-photon spin-down state $|\psi_{nd}\rangle$ to an *excitation polariton state* $|\phi_{nd}\rangle$, while an anti-polariton qubit is formed in an excitation of the atom from the initial n-photon spin-down state $|\psi_{nd}\rangle$ to an *excitation anti-polariton state* $|\,\overline{\phi}_{nd}\rangle$. Applying the polariton and anti-polariton state transition operators $\hat{A}$, $\hat{\overline{A}}$ defined in equation (6a) on the initial state $|\psi_{nd}\rangle$ and reorganizing, we obtain the polariton qubit state transition algebraic operations in the general form

$$\hat{A}\;|\psi_{nd}\rangle = \sqrt{n_\alpha}\;|\phi_{nd}\rangle \quad ; \quad \hat{A}\;|\phi_{nd}\rangle = \sqrt{n_\alpha}\;|\psi_{nd}\rangle \quad ; \quad \hat{A}^2\;|\psi_{nd}\rangle = n_\alpha\;|\psi_{nd}\rangle \quad ; \quad \hat{A}^2\;|\phi_{nd}\rangle = n_\alpha\;|\phi_{nd}\rangle$$

$$|\phi_{nd}\rangle = -c_n|nd\rangle + s_n|n-1u\rangle \qquad ; \qquad c_n = \frac{g\alpha}{R_n} \quad ; \quad s_n = \frac{2g\sqrt{n}}{R_n} \quad ; \quad R_n = 2g\sqrt{n_\alpha} \tag{21b}$$

and the anti-polariton qubit state transition algebraic operations in the general form

$$\hat{\overline{A}}\;|\psi_{nd}\rangle = \sqrt{n_{\overline{\alpha}}+1}\;|\,\overline{\phi}_{nd}\rangle \qquad ; \qquad \hat{\overline{A}}\;|\,\overline{\phi}_{nd}\rangle = \sqrt{n_{\overline{\alpha}}+1}\;|\psi_{nd}\rangle$$

$$\hat{\overline{A}}^2\;|\psi_{nd}\rangle = (n_{\overline{\alpha}}+1)\;\;|\psi_{nd}\rangle \qquad ; \qquad \hat{\overline{A}}^2\;|\,\overline{\phi}_{nd}\rangle = (n_{\overline{\alpha}}+1)\;\;|\,\overline{\phi}_{nd}\rangle$$

$$|\,\overline{\phi}_{nd}\rangle = -\bar{c}_{n+1}|nd\rangle + \bar{s}_{n+1}|n+1u\rangle \quad ; \quad \bar{c}_{n+1} = \frac{g\overline{\alpha}}{\overline{R}_{n+1}} \; ; \quad \bar{s}_{n+1} = \frac{2g\sqrt{n+1}}{\overline{R}_{n+1}} \; ; \quad \overline{R}_{n+1} = 2g\sqrt{n_{\overline{\alpha}}+1} \quad (21c)$$

Introducing the polariton and anti-polariton qubit transition and identity operators $\hat{\mathcal{E}}$, $\hat{\mathcal{I}}$ and $\hat{\overline{\mathcal{E}}}$, $\hat{\overline{\mathcal{I}}}$ defined within the respective two-dimensional qubit subspaces by

$$\hat{\mathcal{E}} = \frac{\hat{A}}{\sqrt{n_\alpha}} \quad ; \quad \hat{\mathcal{E}}^2 = \hat{\mathcal{I}} \quad ; \quad \hat{\overline{\mathcal{E}}} = \frac{\hat{\overline{A}}}{\sqrt{n_{\overline{\alpha}}+1}} \quad ; \quad \hat{\overline{\mathcal{E}}}^2 = \hat{\overline{\mathcal{I}}} \qquad (22a)$$

in equations $(21b)$, $(21c)$ as appropriate provides the polariton qubit state transition algebraic operations in the basic form

$$\hat{\mathcal{E}}\;|\psi_{nd}\rangle = |\phi_{nd}\rangle \quad ; \quad \hat{\mathcal{E}}\;|\phi_{nd}\rangle = |\psi_{nd}\rangle \quad ; \quad \hat{\mathcal{I}}\;|\psi_{nd}\rangle = |\psi_{nd}\rangle \quad ; \quad \hat{\mathcal{I}}\;|\phi_{nd}\rangle = |\phi_{nd}\rangle \qquad (22c)$$

and the anti-polariton qubit state transition algebraic operations in the basic form

$$\hat{\overline{\mathcal{E}}}\;|\psi_{nd}\rangle = |\,\overline{\phi}_{nd}\rangle \quad ; \quad \hat{\overline{\mathcal{E}}}\;|\,\overline{\phi}_{nd}\rangle = |\psi_{nd}\rangle \quad ; \quad \hat{\overline{\mathcal{I}}}\;|\psi_{nd}\rangle = |\psi_{nd}\rangle \quad ; \quad \hat{\overline{\mathcal{I}}}\;|\,\overline{\phi}_{nd}\rangle = |\,\overline{\phi}_{nd}\rangle \qquad (22d)$$

The qubit state transition operators $\hat{\mathcal{E}}$, $\hat{\overline{\mathcal{E}}}$ have the same algebraic properties obtained earlier in equations $(12c)$, $(15c)$, respectively.

Within the two-dimensional polariton and anti-polariton subspaces, the respective qubit state vectors $|\psi_{nd}\rangle$, $|\phi_{nd}\rangle$ and $|\psi_{nd}\rangle$, $|\,\overline{\phi}_{nd}\rangle$ are normalized, but non-orthogonal, satisfying normalization and non-orthogonality relations

$$\langle\psi_{nd}|\psi_{nd}\rangle = 1 \quad ; \quad \langle\phi_{nd}|\phi_{nd}\rangle = 1 \quad ; \quad \langle\psi_{nd}|\phi_{nd}\rangle = -c_n \quad ; \quad \langle\phi_{nd}|\psi_{nd}\rangle = -c_n \qquad (22e)$$

$$\langle\psi_{nd}|\psi_{nd}\rangle = 1 \quad ; \quad \langle\,\overline{\phi}_{nd}|\,\overline{\phi}_{nd}\rangle = 1 \quad ; \quad \langle\psi_{nd}|\,\overline{\phi}_{nd}\rangle = -\bar{c}_{n+1} \quad ; \quad \langle\,\overline{\phi}_{nd}|\psi_{nd}\rangle = -\bar{c}_{n+1} \qquad (22f)$$

The polariton qubit Hamiltonian, excitation number, $U(1)$-symmetry and parity-symmetry operators are obtained in the final form

$$H = \hbar\omega\left(\,n - \frac{1}{2}\,\right)\hat{\mathcal{I}} + \hbar R_n\;\hat{\mathcal{E}} \quad ; \quad \hat{N} = n\hat{\mathcal{I}} \quad ; \quad U(\theta) = e^{in\theta}\hat{\mathcal{I}} \quad ; \quad \hat{\Pi} = e^{in\pi}\hat{\mathcal{I}} \;=\; \pm\hat{\mathcal{I}} \qquad (22g)$$

while the anti-polariton qubit Hamiltonian, excitation number, $U(1)$-symmetry and parity-symmetry operators are obtained in the final form

$$\overline{H} = \hbar\omega\left(\,n + \frac{3}{2}\,\right)\hat{\overline{\mathcal{I}}} + \hbar\overline{R}_{n+1}\;\hat{\overline{\mathcal{E}}} \quad ; \quad \hat{\overline{N}} = (n+2)\hat{\overline{\mathcal{I}}} \; ; \quad \overline{U}(\theta) = e^{i(n+2)\theta}\hat{\overline{\mathcal{I}}} \; ; \quad \hat{\overline{\Pi}} = e^{i(n+2)\pi}\hat{\overline{\mathcal{I}}} = \pm\hat{\overline{\mathcal{I}}} \quad (22h)$$

where in each case, we have substituted the respective detuned excitation number n_α , $n_{\overline{\alpha}}$ and Rabi frequency R_n , $\overline{R}_{n+1}$ defined in equations $(8c)$, $(21b)$, $(21c)$.

Superpositions of the polariton qubit state vectors $|\psi_{nd}\rangle$, $|\phi_{nd}\rangle$ constitute eigenvectors $|\Psi^{\pm}_{nd}\rangle$ of the polariton qubit identity, state transition and Hamiltonian operators $\hat{\mathcal{I}}$, $\hat{\mathcal{E}}$, H obtained in the form

$$|\Psi^{+}_{nd}\rangle = \frac{1}{\sqrt{2}}(\;|\psi_{nd}\rangle + |\phi_{nd}\rangle\;) \quad ; \quad |\Psi^{-}_{nd}\rangle = \frac{1}{\sqrt{2}}(\;|\psi_{nd}\rangle - |\phi_{nd}\rangle\;) \qquad (23a)$$

satisfying eigenvalue equations

$$\hat{\mathcal{I}}|\Psi^{+}_{nd}\rangle = |\Psi^{+}_{nd}\rangle \quad ; \quad \hat{\mathcal{I}}|\Psi^{-}_{nd}\rangle = |\Psi^{-}_{nd}\rangle \quad ; \quad \hat{\mathcal{E}}|\Psi^{+}_{nd}\rangle = |\Psi^{+}_{nd}\rangle \quad ; \quad \hat{\mathcal{E}}|\Psi^{-}_{nd}\rangle = -|\Psi^{-}_{nd}\rangle \qquad (23b)$$

$$H|\Psi^{+}_{nd}\rangle = E_{n+}|\Psi^{+}_{nd}\rangle \quad ; \quad E_{n+} = \hbar\omega\left(\,n - \frac{1}{2}\,\right) + \frac{1}{2}\hbar\sqrt{16g^2n + (\omega_0-\omega)^2}$$

$$H|\Psi^{-}_{nd}\rangle = E_{n-}|\Psi^{-}_{nd}\rangle \quad ; \quad E_{n-} = \hbar\omega\left(\,n - \frac{1}{2}\,\right) - \frac{1}{2}\hbar\sqrt{16g^2n + (\omega_0-\omega)^2} \qquad (23c)$$

while superpositions of the anti-polariton qubit state vectors $|\psi_{nd}\rangle$, $|\,\overline{\phi}_{nd}\rangle$ constitute eigenvectors $|\,\overline{\Psi}^{\pm}_{nd}\rangle$ of the anti-polariton qubit identity, state transition and Hamiltonian operators $\hat{\overline{\mathcal{I}}}$, $\hat{\overline{\mathcal{E}}}$, $\overline{H}$ obtained in the form

$$|\,\overline{\Psi}^{+}_{nd}\rangle = \frac{1}{\sqrt{2}}(\;|\psi_{nd}\rangle + |\,\overline{\phi}_{nd}\rangle\;) \quad ; \quad |\,\overline{\Psi}^{-}_{nd}\rangle = \frac{1}{\sqrt{2}}(\;|\psi_{nd}\rangle - |\,\overline{\phi}_{nd}\rangle\;) \qquad (23d)$$

satisfying eigenvalue equations

$$\hat{\overline{\mathcal{I}}}|\,\overline{\Psi}_{nd}^{+}\rangle = |\,\overline{\Psi}_{nd}^{+}\rangle \quad ; \quad \hat{\overline{\mathcal{I}}}|\,\overline{\Psi}_{nd}^{-}\rangle = |\,\overline{\Psi}_{nd}^{-}\rangle \quad ; \quad \hat{\overline{\mathcal{E}}}|\,\overline{\Psi}_{nd}^{+}\rangle = |\,\overline{\Psi}_{nd}^{+}\rangle \quad ; \quad \hat{\overline{\mathcal{E}}}|\,\overline{\Psi}_{nd}^{-}\rangle = -|\,\overline{\Psi}_{nd}^{-}\rangle \quad (23e)$$

$$\overline{H}\,|\,\overline{\Psi}_{nd}^{+}\rangle = \overline{E}_{n+}\,|\,\overline{\Psi}_{nd}^{+}\rangle \quad ; \quad \overline{E}_{n+} = \hbar\omega\left(\,n+\frac{3}{2}\,\right) + \frac{1}{2}\hbar\sqrt{16g^2(n+1)+(\omega_0+\omega)^2}$$

$$\overline{H}\,|\,\overline{\Psi}_{nd}^{-}\rangle = \overline{E}_{n-}|\,\overline{\Psi}_{nd}^{-}\rangle \quad ; \quad \overline{E}_{n-} = \hbar\omega\left(\,n+\frac{3}{2}\,\right) - \frac{1}{2}\hbar\sqrt{16g^2(n+1)+(\omega_0+\omega)^2} \qquad (23f)$$

where recognize polariton qubit Hamiltonian energy eigenvalues $E_{n\pm}$ in equation $(23c)$ as the usual energy eigenvalues determined through diagonalization of the Jaynes-Cummings (polariton) Hamiltonian H in standard quantum optics literature [19 , 20 , 21], noting that the derivation of the Jaynes-Cummings Hamiltonian through symmetrization of the Rabi Hamiltonian H_R in equations $(1a)$-$(1d)$ yields consistent coupling constant $2g$ instead of the commonly used g such that $4g^2 \rightarrow 16g^2$. The anti-polariton qubit Hamiltonian energy eigenvalues $\overline{E}_{n\pm}$ in equation $(23f)$ are new, but can also be determined through diagonalization of the anti-Jaynes-Cummings (anti-polariton) Hamiltonian $\overline{H}$ based on the existence of a conserved anti-polariton excitation number operator $\hat{\overline{N}}$ as established here in section 2.2.

The polariton and anti-polariton qubit Hamiltonians H , $\overline{H}$ generate respective time evolution operators $U(t)$, $\overline{U}(t)$ obtained in final form

$$U(t) = e^{-\frac{i}{\hbar}Ht} \;\; : \qquad U(t) = e^{-i\omega(\,n-\frac{1}{2}\,)t}\left(\,\cos(R_n t)\hat{I} - i\sin(R_n t)\hat{\mathcal{E}}\,\right) \qquad (24a)$$

$$\overline{U}(t) = e^{-\frac{i}{\hbar}\overline{H}t} \;\; : \qquad \overline{U}(t) = e^{-i\omega(\,n+\frac{3}{2}\,)t}\left(\,\cos(\overline{R}_{n+1}t)\hat{\overline{I}} - i\sin(\overline{R}_{n+1}t)\hat{\overline{\mathcal{E}}}\,\right) \qquad (24b)$$

The time evolving excitation polariton qubit state vectors $|\Psi_{nd}(t)\rangle$, $|\Phi_{nd}(t)\rangle$ are generated from the initial qubit state vectors $|\psi_{nd}\rangle$, $|\phi_{nd}\rangle$ by the time evolution operator $U(t)$ in the final form

$$|\Psi_{nd}(t)\rangle = U(t)|\psi_{nd}\rangle \;\; : \qquad |\Psi_{nd}(t)\rangle = e^{-i\omega(\,n-\frac{1}{2}\,)t}\left(\,\cos(R_n t)|\psi_{nd}\rangle - i\sin(R_n t)|\phi_{nd}\rangle\,\right)$$

$$|\Phi_{nd}(t)\rangle = U(t)|\phi_{nd}\rangle \;\; : \qquad |\Phi_{nd}(t)\rangle = e^{-i\omega(\,n-\frac{1}{2}\,)t}\left(\,\cos(R_n t)|\phi_{nd}\rangle - i\sin(R_n t)|\psi_{nd}\rangle\,\right) \qquad (24c)$$

which describe Rabi oscillations at frequency R_n between the initial state $|\psi_{nd}\rangle$ and excitation state $|\phi_{nd}\rangle$ with excitation state transition probability $P_{ex}(t)$ obtained as

$$P_{ex}(t) = |\,\langle\phi_{nd}|\Psi_{nd}(t)\rangle\,|^2 \qquad \Rightarrow \qquad P_{ex}(t) = \sin^2(R_n t) \qquad (24d)$$

while the time evolving excitation anti-polariton qubit state vector $|\,\overline{\Psi}_{nd}(t)\rangle$, $|\,\overline{\Phi}_{nd}(t)\rangle$ are generated from the initial qubit state vectors $|\psi_{nd}\rangle$, $|\,\overline{\phi}_{nd}\rangle$ by the time evolution operator, $\overline{U}(t)$ in the final form

$$|\,\overline{\Psi}_{nd}(t)\rangle = \overline{U}_{nd}(t)|\psi_{nd}\rangle \;\; : \qquad |\,\overline{\Psi}_{nd}(t)\rangle = e^{-i\omega(\,n+\frac{3}{2}\,)t}\left(\,\cos(\overline{R}_{n+1}t)|\psi_{nd}\rangle - i\sin(\overline{R}_{n+1}t)|\,\overline{\phi}_{nd}\rangle\,\right)$$

$$|\,\overline{\Phi}_{nd}(t)\rangle = \overline{U}_{nd}(t)|\,\overline{\phi}_{nd}\rangle \;\; : \qquad |\,\overline{\Phi}_{nd}(t)\rangle = e^{-i\omega(\,n+\frac{3}{2}\,)t}\left(\,\cos(\overline{R}_{n+1}t)|\phi_{nd}\rangle - i\sin(\overline{R}_{n+1}t)|\,\overline{\psi}_{nd}\rangle\,\right) \qquad (24e)$$

which describe Rabi oscillations at frequency $\overline{R}_{n+1}$ between the initial state $|\psi_{nd}\rangle$ and excitation state $|\overline{\phi}_{nd}\rangle$ with excitation state transition probability $\overline{P}_{ex}(t)$ obtained as

$$\overline{P}_{ex}(t) = |\,\langle\,\overline{\phi}_{nd}|\overline{\Psi}_{nd}(t)\rangle\,|^2 \qquad \Rightarrow \qquad \overline{P}_{ex}(t) = \sin^2(\overline{R}_{n+1}t) \qquad (24f)$$

Since an excitation is characterized by absorption of a positive frequency photon by the atom, we interpret the excitation probability $P_{ex}(t)$ or $\overline{P}_{ex}(t)$ as the probability of absorption of a photon by the atom in polariton or anti-polariton qubit dynamics.

Substituting α , $\delta = \omega_0 - \omega$, $\overline{\alpha}$, $\overline{\delta} = \omega_0 + \omega$ from equations $(2a)$, $(2b)$ into R_n , $\overline{R}_{n+1}$ in equations $(21b)$, $(21c)$), respectively and rewriting

$$(\omega_0+\omega)^2 = (\omega_0-\omega)^2 + 4\omega_0\omega \quad ; \qquad 16g^2(n+1) = 16g^2 n + 16g^2$$

we obtain the relation between the anti-polariton qubit Rabi oscillation frequency $\overline{R}_n$ and the polariton qubit Rabi oscillation frequency R_n in the form

$$\overline{R}_{n+1} = R_n\sqrt{1+\frac{\omega_0\omega+4g^2}{R_n^2}} \;=\; R_n\sqrt{1+\frac{\omega_0\omega}{R_n^2}\left(1+\frac{g^2}{g_c^2}\right)} \qquad ; \qquad \omega_0\omega + 4g^2 = \omega_0\omega\left(1+\frac{g^2}{g_c^2}\right) \qquad (24g)$$

which we substitute into equation $(24f)$ to express the excitation state transition probability in the anti-polariton qubit in the form

$$\overline{P}_{ex}(t) = \sin^2\left(R_n\sqrt{1 + \frac{\omega_0\omega}{R_n^2}\left(1 + \frac{g^2}{g_c^2}\right)}\ t\right) \tag{24h}$$

where we have introduced the critical coupling constant g_c defined earlier in equation $(19e)$. It follows from equations $(24d)$, $(24h)$ that in the lower Rabi subspace, the excitation probability $\overline{P}_{ex}(t)$ governing blue-sideband transitions in an anti-polariton qubit always oscillates faster than the excitation probability $P_{ex}(t)$ governing red-sideband transitions in a polariton qubit. The critical behavior of red-sideband transitions relative to blue-sideband transitions characterizing dynamics in the coupling regimes $g < g_c$, $g = g_c$, $g > g_c$ in the upper Rabi subspace are not manifestly evident in the lower Rabi subspace.

Applying the interpretation that in the lower Rabi subspace, an excitation of the atom to the spin-up state $|u\rangle$, characterized by absorption of a positive frequency photon, is triggered by emission of a positive energy photon by the rotating positive frequency field mode in a transition from the n-photon spin-down state $|nd\rangle$ to the $(n-1)$-photon spin-up state $|n-1u\rangle$ in a polariton qubit or by emission of a negative energy photon (anti-emission) by the anti-rotating negative frequency field mode in a transition from the n-photon spin-down state $|nd\rangle$ to the $(n+1)$-photon spin-up state $|n+1u\rangle$ in an anti-polariton qubit, we substitute the stationary qubit state vectors $|\psi_{nd}\rangle$, $|\phi_{nd}\rangle$, $|\ \overline{\phi}_{nd}\rangle$ from equations $(21a)$, $(21d)$, $(21c)$ into equations $(24c)$, $(24e)$, respectively, to express the general time evolving polariton qubit state vectors in the form

$$|\Psi_{nd}(t)\rangle = e^{-i\omega(\ n-\frac{1}{2}\ )t}\ (\ (\cos(R_nt) + ic_n\sin(R_nt)\ )|nd\rangle - is_n\sin(R_nt)|n-1u\rangle\ )$$

$$|\Phi_{nd}(t)\rangle = e^{-i\omega(\ n-\frac{1}{2}\ )t}\ (\ (-c_n\cos(R_nt) - i\sin(R_nt)\ )|nd\rangle + s_n\cos(R_nt)|n-1u\rangle\ ) \tag{25a}$$

which explicitly describe Rabi oscillations at frequency R_n between the photon emission-absorption states $|nd\rangle$, $|n-1u\rangle$ with the probability $P_{em}(t)$ of emission of a positive energy photon by the rotating positive frequency field mode in a transition from $|nd\rangle$ to $|n-1u\rangle$ obtained as

$$P_{em}(t) = |\ \langle n-1u|\Psi_{nd}(t)\rangle\ |^2 \qquad \Rightarrow \qquad P_{em}(t) = s_n^2\sin^2(R_nt) \tag{25b}$$

which on substituting s_n , R_n using equations $(2a)$, $(8c)$, $(21b)$ takes the explicit form

$$P_{em}(t) = \frac{16g^2n}{16g^2n + (\omega_0-\omega)^2}\sin^2(\frac{1}{2}\sqrt{16g^2n + (\omega_0-\omega)^2}\ t) \tag{25c}$$

accounting for red-sideband transitions generated by the polariton qubit Hamiltonian H, while the general time evolving anti-polariton qubit state vectors are expressed in the form

$$|\ \overline{\Psi}_{nd}(t)\rangle = e^{-i\omega(\ n+\frac{3}{2}\ )t}\left(\ (\cos(\overline{R}_{n+1}t) + i\bar{c}_{n+1}\sin(\overline{R}_{n+1}t)\ )|nd\rangle - i\bar{s}_{n+1}\sin(\overline{R}_{n+1}t)|n+1u\rangle\ \right)$$

$$|\ \overline{\Phi}_{nd}(t)\rangle = e^{-i\omega(\ n+\frac{3}{2}\ )t}\left(\ (-\bar{c}_{n+1}\cos(\overline{R}_{n+1}t) - i\sin(\overline{R}_{n+1}t)\ )|nd\rangle + \bar{s}_{n+1}\cos(\overline{R}_{n+1}t)|n+1u\rangle\ \right) \tag{25d}$$

which explicitly describe Rabi oscillations at frequency $\overline{R}_{n+1}$ between the photon anti-emission-absorption states $|nd\rangle$, $|n+1u\rangle$ with the probability of emission of a negative energy photon (anti-emission) by the anti-rotating negative frequency field mode in a transition from $|nd\rangle$ to $|n+1u\rangle$, which we call the *anti-emission probability* $\overline{P}_{aem}(t)$, obtained as

$$\overline{P}_{aem}(t) = |\ \langle n+1u|\ \overline{\Psi}_{nd}(t)\rangle\ |^2 \qquad \Rightarrow \qquad \overline{P}_{aem}(t) = \bar{s}_{n+1}^2\sin^2(\overline{R}_{n+1}t) \tag{25e}$$

which on substituting $\bar{s}_{n+1}$, $\bar{R}_{n+1}$ using equations $(2b)$, $(8c)$, $(21c)$ takes the explicit form

$$\overline{P}_{aem}(t) = \frac{16g^2(n+1)}{16g^2(n+1) + (\omega_0+\omega)^2}\sin^2(\frac{1}{2}\sqrt{16g^2(n+1) + (\omega_0+\omega)^2}\ t) \tag{25f}$$

accounting for blue-sideband transitions generated by the anti-polariton Hamiltonian $\overline{H}$, where we recall that *anti-emission* means emission of a negative energy photon by the anti-rotating negative frequency field mode.

Here again, we observe that the general time evolving polariton and anti-polariton qubit state vectors $|\Psi_{nd}(t)|\Psi_{nd}(t)\rangle$, $|\Phi_{nd}(t)\rangle$, $|\ \overline{\Psi}_{nd}(t)\rangle$, $|\ \overline{\Phi}_{nd}(t)\rangle$ obtained in equations $(24c)$, $(24e)$, $(25a)$, $(25d)$ are entangled atom-field state vectors which preserve the normalization, non-orthogonality and state transition algebraic relations of the qubit state vectors in equations (22c) , (22d) , (22e) , (22f), respectively in the form

$$\langle\Psi_{nd}(t)|\Psi_{nd}(t)\rangle = 1 \; ; \; \langle\Phi_{nd}(t)|\Phi_{nd}(t)\rangle = 1 \; ; \; \langle\Psi_{nd}(t)|\Phi_{nd}(t)\rangle = -c_n \; ; \; \langle\Phi_{nd}(t)|\Psi_{nd}(t)\rangle = -c_n$$
$$\hat{\mathcal{E}}\;|\Psi_{nd}(t)\rangle = |\Phi_{nd}(t)\rangle \; ; \quad \hat{\mathcal{E}}\;|\Phi_{nd}(t)\rangle = |\Psi_{nd}(t)\rangle \; ; \quad \hat{I}\;|\Psi_{nd}(t)\rangle = |\Psi_{nd}(t)\rangle \; ; \quad \hat{I}\;|\Phi_{nd}(t)\rangle = |\Phi_{nd}(t)\rangle \qquad (26a)$$

$$\langle\,\overline{\Psi}_{nd}(t)|\,\overline{\Psi}_{nd}(t)\rangle = 1 \; ; \; \langle\,\overline{\Phi}_{nd}(t)|\,\overline{\Phi}_{nd}(t)\rangle = 1 \; ; \; \langle\,\overline{\Psi}_{nd}(t)|\,\overline{\Phi}_{nd}(t)\rangle = -\bar{c}_{n+1} \; ; \; \langle\,\overline{\Phi}_{nd}(t)|\,\overline{\Psi}_{nd}(t)\rangle = -\bar{c}_{n+1}$$
$$\hat{\overline{\mathcal{E}}}\;|\Psi_{nd}(t)\rangle = |\,\overline{\Phi}_{nd}(t)\rangle \; ; \quad \hat{\overline{\mathcal{E}}}\;|\,\overline{\Phi}_{nd}(t)\rangle = |\Psi_{nd}(t)\rangle \; ; \quad \hat{\overline{I}}\;|\Psi_{nd}(t)\rangle = |\Psi_{nd}(t)\rangle \; ; \quad \hat{\overline{I}}\;|\,\overline{\Phi}_{nd}(t)\rangle = |\,\overline{\Phi}_{nd}(t)\rangle \quad (26b)$$

If the interaction begins with the atom in the spin-down (ground) state $|d\rangle$ and the field mode in the vacuum state $|0\rangle$, then polariton or anti-polariton qubit formation starts with *spontaneous excitation process* from the initial 0-photon spin-down state $|\psi_{0d}\rangle$ where the anti-rotating negative frequency component of the field mode *spontaneously emits* a negative energy photon (spontaneous anti-emission), which causes the atom to spontaneously absorb a positive energy photon in an anti-polariton qubit state transition from $|\psi_{0d}\rangle$ to $|\,\overline{\phi}_{0d}\rangle$ or in an event of vacuum fluctuations, the rotating positive frequency component of the field mode spontaneously emits a positive energy photon in a polariton qubit state transition from $|\psi_{0d}\rangle$ to $|\phi_{0d}\rangle$, where the initial excitation polariton or anti-polariton qubit state vectors are specified by setting $n = 0$ in equations (21a) , (21b) , (21c) (noting $|-1u\rangle = 0$) in the form

$$|\psi_{0d}\rangle = |0d\rangle \qquad ; \qquad |\phi_{0d}\rangle = -c_0|0d\rangle \qquad ; \qquad |\,\overline{\phi}_{0d}\rangle = -\bar{c}_1|0d\rangle + \bar{s}_1|1u\rangle \qquad (27a)$$

with

$$R_0 = \frac{1}{2}(\omega_0 - \omega) \; ; \quad c_0 = 1 \; ; \quad s_0 = 0 \; ; \quad \overline{R}_1 = \frac{1}{2}\sqrt{16g^2 + (\omega_0 + \omega)^2} \; ; \quad \bar{c}_1 = \frac{\overline{\omega_0 + \omega}}{2\overline{R}_1} \quad ; \quad \bar{s}_1 = \frac{g}{\overline{R}_1} \qquad (27b)$$

Setting $n = 0$ in equations (24e) , (24f) , (25e) provides the general time evolving spontaneous excitation anti-polariton qubit state vectors $|\,\overline{\Psi}_{0d}(t)\rangle$, $|\,\overline{\Phi}_{0d}(t)\rangle$, with corresponding spontaneous excitation and anti-emission probabilities $P_{ex}^{spt}(t)$, $P_{aem}^{spt}(t)$ in the form

$$|\,\overline{\Psi}_{0d}(t)\rangle = e^{-\frac{3}{2}i\omega t}\left(\;\cos(\overline{R}_1 t)|\psi_{0d}\rangle - i\sin(\overline{R}_1 t)|\,\overline{\phi}_{0d}\rangle\;\right) \; ; \quad \overline{P}_{ex}^{spt}(t) = \sin^2(\overline{R}_1 t) \; ; \quad \overline{P}_{aem}^{spt}(t) = \bar{s}_1^2\overline{P}_{ex}^{spt}(t)$$
$$|\,\overline{\Phi}_{0d}(t)\rangle = e^{-\frac{3}{2}i\omega t}\left(\;\cos(\overline{R}_1 t)|\phi_{0d}\rangle - i\sin(\overline{R}_1 t)|\,\overline{\psi}_{0d}\rangle\;\right) \qquad (27c)$$

which describe spontaneously generated Rabi oscillations at frequency $\overline{R}_1$ between initial and excitation states $|\psi_{0d}\rangle$, $|\,\overline{\phi}_{0d}\rangle$ in the anti-polariton qubit governed by spontaneous excitation probability $\overline{P}_{ex}^{spt}(t)$, while setting $n = 0$ in equations (24c) , (24d) , (25b) provides the general time evolving spontaneous excitation polariton qubit state vectors $|\Psi_{nd}(t)\rangle$, $|\Phi_{nd}(t)\rangle$, with corresponding spontaneous excitation and emission probabilities $P_{ex}^{spt}(t)$, $P_{em}^{spt}(t)$ in the form

$$|\phi_{0d}\rangle = -|\psi_{0d}\rangle \qquad ; \qquad |\Psi_{0d}(t)\rangle = -|\Phi_{0d}(t)\rangle = e^{\frac{i}{2}(\omega_0-\omega)t}|0d\rangle$$
$$P_{ex}^{spt}(t) = \sin^2\frac{1}{2}(\omega_0 - \omega)t \qquad ; \qquad P_{em}^{spt}(t) = 0 \qquad (27d)$$

where we have used equations (27a) , (27b) to obtain the final results in equation (27d).

It emerges from equations (27c) , (27d) that in a spontaneous excitation process starting with the atom in spin-down state $|d\rangle$ and the field mode in vacuum state $|0\rangle$, only the anti-polariton qubit Hamiltonian $\overline{H}$ generates observable dynamical effects characterized by spontaneous anti-emission and blue-sideband transitions described by the general time evolving spontaneous excitation anti-polariton qubit state vectors $|\,\overline{\Psi}_{0d}(t)\rangle$, $|\,\overline{\Phi}_{0d}(t)\rangle$ in equation (27c), while the polariton qubit Hamiltonian H generates only *virtual transitions* between degenerate polariton qubit states $|\psi_{0d}\rangle = -|\,\overline{\phi}_{0d}\rangle$, signified by a non-zero spontaneous excitation probability $P_{ex}^{spt}(t) = \sin^2\frac{1}{2}(\omega_0 - \omega)t$ and vanishing spontaneous emission probability $P_{em}^{spt}(t) = 0$, the process being described by the degenerate time evolving plane wave spontaneous excitation polariton qubit state vectors $|\Psi_{0d}(t)\rangle = -|\Phi_{0d}(t)\rangle$ in equation (27d).

In summary, polariton or anti-polariton qubit dynamics in the lower Rabi subspace, starting with initial photon absorption by the atom, is characterized by photon emission or anti-emission by the field mode. If the field mode is initially in the vacuum state, then the dynamics starting with spontaneous absorption by the atom, triggered by spontaneous anti-emission by the field mode, is dominated by blued-sideband transitions generated by the anti-polariton qubit Hamiltonian, while the polariton qubit dynamics is suppressed into virtual transitions between degenerate qubit states described by plane waves.

3.3 Photospins

We now provide a general physical interpretation by noting that within the two-dimensional subspaces spanned by the respective qubit state vectors, $|\psi_{nu}\rangle$, $|\phi_{nu}\rangle$ and $|\psi_{nu}\rangle$, $|\overline{\phi}_{nu}\rangle$ in the upper Rabi subspace or $|\psi_{nd}\rangle$, $|\phi_{nd}\rangle$ and $|\psi_{nd}\rangle$, $|\overline{\phi}_{nd}\rangle$ in the lower Rabi subspace, the standard form of the algebraic operations and properties of the polariton and anti-polariton qubit state transition operators $\hat{\varepsilon}$, $\hat{\overline{\varepsilon}}$ given in equations $(12b)$, $(12c)$, $(15b)$, $(15c)$ are similar to the algebraic operations and properties of an atomic spin state transition operator (Pauli matrix) σ_x in a two-dimensional subspace spanned by atomic spin-up and spin-down qubit state vectors $|u\rangle$, $|d\rangle$ according to

$$\sigma_x|u\rangle = |d\rangle \quad ; \quad \sigma_x|d\rangle = |u\rangle \quad ; \quad I|u\rangle = |u\rangle \quad ; \quad I|d\rangle = |d\rangle \tag{28a}$$

$$\sigma_x^\dagger = \sigma_x \quad ; \quad \sigma_x^2 = I \quad ; \quad \sigma_x^{2k} = I \quad ; \quad \sigma_x^{2k+1} = \sigma_x \quad ; \quad k = 0, 1, 2, 3, ...$$
$$e^{\pm i\theta I} = e^{\pm i\theta} I \quad ; \quad e^{\pm i\theta\sigma_x} = \cos\theta \; I \pm i \sin\theta \; \sigma_x \tag{28b}$$

The equivalence of the algebraic operations and properties to that of an atomic spin in equations $(28a)$, $(28b)$, combined with the forms of the respective polariton and anti-polariton qubit Hamiltonians in equations $(12f)$, $(15f)$, $(22g)$, $(22h)$ means that we can interpret polariton and anti-polariton qubits as *photospins*. We define a photospin as a *photon-carrying spin-$\frac{1}{2}$ particle* specified by a state transition operator $\hat{\varepsilon}$ or $\hat{\overline{\varepsilon}}$ in a two-dimensional subspace spanned by two normalized, but non-orthogonal photonic-spin qubit state vectors, where a photonic state vector is just a quantized field mode number state vector with the appropriate number of photons. In this respect, we interpret a polariton qubit as a *rotating photospin* specified by a corresponding state transition operator $\hat{\varepsilon}$ and an anti-polariton qubit as an *anti-rotating photospin* specified by a corresponding state transition operator $\hat{\overline{\varepsilon}}$. The action of the state transition operator $\hat{\varepsilon}$ on the rotating photspin state vectors generates reversible state transitions by *alternately* raising (or lowering) and lowering (or raising) the photonic and spin states, while the action of the state transition operator $\hat{\overline{\varepsilon}}$ on the anti-rotating photspin state vectors generates reversible state transitions by *simultaneously* raising or lowering both photonic and spin states. The dynamical picture of a polariton or anti-polariton qubit as a photospin with algebraic operations and properties of a spin-$\frac{1}{2}$ particle is useful for developing geometrical frameworks for models of interacting polariton and anti-polariton qubit systems as presented in the next section.

4 Polariton and anti-polariton qubit interactions : the quantum Rabi optical lattice

We have now determined the basic algebraic and dynamical properties of polariton and anti-polariton qubits generated by the Jaynes-Cummings and anti-Jaynes-Cummings interactions specified within the quantum Rabi model. The next important challenge is to build models of interacting polariton and anti-polariton qubit systems to provide foundations for studying general dynamical properties and devising practical applications in the design and implementation of quantum information processing, quantum computation and all related quantum technologies.

Motivated by the interpretation of a polariton or an anti-polariton qubit as a photsspin specified by two qubit state vectors and a state transition operator, we introduce a *quantum Rabi optical lattice* defined as a regular pattern of coupled arrays of *optical lattice sites*, where an optical lattice site is an *optical cavity* containing a single two-level atom coupled to a single quantized electromagnetic field mode. We therefore interpret an optical lattice site as a quantum Rabi state space, composed of the upper and lower Rabi subspaces. The general theory of polariton and anti-polariton qubits developed above then means that each optical lattice site contains a polariton or anti-polariton qubit specified by a cavity field mode frequency ω_i , atomic state transition frequency ω_{0i} , coupling constant g_i and state transition operator $\hat{\varepsilon}_i$ or $\hat{\overline{\varepsilon}}_i$, respectively, with corresponding Hamiltonian H_i or $\overline{H}_i$.

Applying the algebraic property that the upper and lower Rabi subspaces in each lattice site are disconnected, we identify the quantum Rabi optical lattice as an *upper Rabi optical lattice* or a *lower Rabi optical lattice*. Each site in an upper Rabi optical lattice is defined as an upper Rabi subspace containing a polariton or an anti-polariton qubit formed from the initial n-photon spin-up state $|\psi_{nu}\rangle$, while each site in a lower Rabi optical lattice is defined as a lower Rabi subspace containing a polariton or an anti-polariton qubit formed from the initial n-photon spin-down state $|\psi_{nd}\rangle$. Since the two models are similar, we present only the upper Rabi optical lattice model below.

4.1 Upper Rabi optical lattice

In the i-th site of the upper Rabi optical lattice, the polariton qubit (rotating photospin) is specified by its two state vectors $|\psi_{n_iu_i}\rangle$, $|\phi_{n_iu_i}\rangle$, qubit transition operator $\hat{\mathcal{E}}_i$, identity operator $\hat{\mathcal{I}}_i$ and Hamiltonian H_i defined by

$$|\psi_{n_iu_i}\rangle = |n_iu_i\rangle \quad ; \quad |\phi_{n_iu_i}\rangle = c_{n_i+1}|n_iu_i\rangle + s_{n_i+1}|n_i+1d_i\rangle \quad ; \quad i = 1,2,3,...,S$$

$$c_{n_i+1} = \frac{g_i\alpha_i}{R_{n_i+1}} \quad ; \quad s_{n_i+1} = \frac{2g_i\sqrt{n_i+1}}{R_{n_i+1}} \quad ; \quad R_{n_i+1} = 2g_i\sqrt{n_i+1+\frac{1}{4}\alpha_i^2} \quad ; \quad \alpha_i = \frac{\omega_{0i}-\omega_i}{2g_i} \tag{29a}$$

$$\hat{\mathcal{E}}_i = \frac{\hat{A}_i}{\sqrt{n_i+1+\frac{1}{4}\alpha_i^2}} \; ; \quad \hat{\mathcal{I}}_i = \hat{\mathcal{E}}_i^2 \; ; \quad \hat{\mathcal{E}}_i^{2k} = \hat{\mathcal{I}}_i \; ; \quad \hat{\mathcal{E}}_i^{2k+1} = \hat{\mathcal{E}}_i \quad ; \quad H_i = \hbar\omega_i\left(n_i+\frac{1}{2}\right)\hat{\mathcal{I}}_i + \hbar R_{n_i+1}\hat{\mathcal{E}}_i \tag{29b}$$

$$e^{\pm i\theta_i\hat{\mathcal{I}}_i} = e^{\pm i\theta_i}\hat{\mathcal{I}}_i \quad ; \quad e^{\pm i\theta_i\hat{\mathcal{E}}_i} = \cos\theta_i\ \hat{\mathcal{I}}_i \pm i\sin\theta_i\ \hat{\mathcal{E}}_i \tag{29c}$$

with time evolution operator

$$U_i(t) = e^{-\frac{i}{\hbar}H_it} \quad \Rightarrow \quad U_i(t) = e^{-i\omega_i\left(n_i+\frac{1}{2}\right)t}\left(\cos(R_{n_i+1}t)\ \hat{\mathcal{I}}_i - i\sin(R_{n_i+1}t)\ \hat{\mathcal{E}}_i\right) \tag{29d}$$

while the anti-polariton qubit (anti-rotating photospin) is specified by its two-state vectors $|\psi_{n_iu_i}\rangle$, $|\,\overline{\phi}_{n_iu_i}\rangle$, qubit transition operator $\hat{\overline{\mathcal{E}}}_i$, identity operator $\hat{\overline{\mathcal{I}}}_i$ and Hamiltonian $\overline{\mathcal{H}}_i$ defined by

$$|\psi_{n_iu_i}\rangle = |n_iu_i\rangle \quad ; \quad |\,\overline{\phi}_{n_iu_i}\rangle = \bar{c}_{n_i}|n_iu_i\rangle + \bar{s}_{n_i}|n_i-1d_i\rangle \quad ; \quad i = 1,2,3,...,S$$

$$\bar{c}_{n_i} = \frac{g_i\overline{\alpha}_i}{\overline{R}_{n_i}} \quad ; \quad \bar{s}_{n_i} = \frac{2g_i\sqrt{n_i}}{\overline{R}_{n_i}} \quad ; \quad \overline{R}_{n_i} = 2g_i\sqrt{n_i+\frac{1}{4}\overline{\alpha}_i^2} \quad ; \quad \overline{\alpha}_i = \frac{\omega_{0i}+\omega_i}{2g_i} \tag{30a}$$

$$\hat{\overline{\mathcal{E}}}_i = \frac{\hat{\overline{A}}_i}{\sqrt{n_i+\overline{\lambda}_i^2}} \quad ; \quad \hat{\overline{\mathcal{I}}}_i = \hat{\overline{\mathcal{E}}}_i^2 \quad ; \quad \hat{\overline{\mathcal{E}}}_i^{2k} = \hat{\overline{\mathcal{I}}}_i \quad ; \quad \hat{\overline{\mathcal{E}}}_i^{2k+1} = \hat{\overline{\mathcal{E}}}_i \quad ; \quad \overline{H}_i = \hbar\omega_i\left(n_i+\frac{1}{2}\right)\hat{\overline{\mathcal{I}}}_i + \hbar\overline{R}_{n_i}\hat{\overline{\mathcal{E}}}_i \tag{30b}$$

$$e^{\pm i\theta_i\hat{\overline{\mathcal{I}}}_i} = e^{\pm i\theta_i}\hat{\overline{\mathcal{I}}}_i \quad ; \quad e^{\pm i\theta_i\hat{\overline{\mathcal{E}}}_i} = \cos\theta_i\ \hat{\overline{\mathcal{I}}}_i \pm i\sin\theta_i\ \hat{\overline{\mathcal{E}}}_i \tag{30c}$$

with time evolution operator

$$\overline{U}_i(t) = e^{-\frac{i}{\hbar}\overline{H}_it} \quad \Rightarrow \quad \overline{U}_i(t) = e^{-i\omega_i\left(n_i+\frac{1}{2}\right)t}\left(\cos(\overline{R}_{n_i}t)\ \hat{\overline{\mathcal{I}}}_i - i\sin(\overline{R}_{n_i}t)\ \hat{\overline{\mathcal{E}}}_i\right) \tag{30d}$$

where the integer k in equations $(29b)$, $(30b)$ takes values $k = 0,1,2,3,....$

The polariton and anti-polariton qubit state transition algebraic operations in the i-th site are obtained in the form

$$\hat{\mathcal{E}}_i|\psi_{n_iu_i}\rangle = |\phi_{n_iu_i}\rangle \quad ; \quad \hat{\mathcal{E}}_i|\phi_{n_iu_i}\rangle = |\psi_{n_iu_i}\rangle \quad ; \quad \hat{\mathcal{I}}_i|\psi_{n_iu_i}\rangle = |\psi_{n_iu_i}\rangle \quad ; \quad \hat{\mathcal{I}}_i|\phi_{n_iu_i}\rangle = |\phi_{n_iu_i}\rangle \tag{31a}$$

$$\hat{\overline{\mathcal{E}}}_i|\psi_{n_iu_i}\rangle = |\,\overline{\phi}_{n_iu_i}\rangle \quad ; \quad \hat{\overline{\mathcal{E}}}_i|\,\overline{\phi}_{n_iu_i}\rangle = |\psi_{n_iu_i}\rangle \quad ; \quad \hat{\overline{\mathcal{I}}}_i|\psi_{n_iu_i}\rangle = |\psi_{n_iu_i}\rangle \quad ; \quad \hat{\overline{\mathcal{I}}}_i|\,\overline{\phi}_{n_iu_i}\rangle = |\,\overline{\phi}_{n_iu_i}\rangle \tag{31b}$$

We note that according to the Rabi Hamiltonian $H_{Ri} = \frac{1}{2}(H_i + \overline{H}_i)$, a polariton and an anti-polariton qubit formed in the same lattice site are correlated ($[H_i, \overline{H}_i] \neq 0$), but do not interact. Only polariton and anti-polariton qubits in different lattice sites interact interact through coupling of their state transition operators $\hat{\mathcal{E}}_i$, $\hat{\overline{\mathcal{E}}}_i$ according to

$$H_{ij} = g_{ij}\hat{\mathcal{E}}_i\hat{\mathcal{E}}_j \quad ; \quad \overline{H}_{ij} = g_{ij}\hat{\overline{\mathcal{E}}}_i\hat{\overline{\mathcal{E}}}_j \quad ; \quad \mathcal{H}_{ij} = g_{ij}\hat{\mathcal{E}}_i\hat{\overline{\mathcal{E}}}_j \quad ; \quad i\ ,\ j\ = 1,2,3,...,S \tag{32a}$$

where H_{ij} is a polariton-polariton, $\overline{H}_{ij}$ an anti-polariton-anti-polariton and $\mathcal{H}_{ij}$ a polariton-anti-polariton qubit interaction Hamiltonian. We assume that the state transition operators $\hat{\mathcal{E}}_i$ or $\hat{\overline{\mathcal{E}}}_i$ for different polariton or anti-polariton qubits commute.

The total Hamiltonian of S interacting polariton qubits or S interacting anti-polariton qubits in the upper Rabi optical lattice is easily obtained in the form

$$H = \sum_{i=1}^{S} H_i + \sum_{i \neq j=1}^{S} g_{ij} \hat{\mathcal{E}}_i \hat{\mathcal{E}}_j \quad ; \quad \overline{H} = \sum_{i=1}^{S} \overline{H}_i + \sum_{i \neq j=1}^{S} g_{ij} \hat{\overline{\mathcal{E}}}_i \hat{\overline{\mathcal{E}}}_j \quad ; \quad i \ , \ j \ = 1, 2, 3, ..., S \tag{32b}$$

while the total Hamiltonian of S polariton qubits interacting with S anti-polariton qubits is obtained as

$$\mathcal{H} = \sum_{i=1}^{S} H_i + \sum_{j \neq i}^{S} \overline{H}_j + \sum_{i \neq j}^{S} g_{ij} \hat{\mathcal{E}}_i \hat{\overline{\mathcal{E}}}_j \quad ; \quad i \ , \ j \ = 1, 2, 3, ..., S \tag{32c}$$

where the polariton and anti-polariton qubit Hamiltonians H_i , $\overline{H}_i$ at the i-th site are defined in equations $(29b)$, $(30b)$. The commuting polariton qubit transition operators $\hat{\mathcal{E}}_i$, $\hat{\mathcal{E}}_j$ or anti-polariton qubit transition operators $\hat{\overline{\mathcal{E}}}_i$, $\hat{\overline{\mathcal{E}}}_j$ at different sites generate state transitions according to the algebraic operations in equations $(31a)$, $(31b)$.

Since the qubit transition operators $\hat{\mathcal{E}}_i$, $\hat{\overline{\mathcal{E}}}_i$ commute with the respective Hamiltonians H_i , $\overline{H}_i$, the time evolution operator $U(t)$, $\overline{U}(t)$, $\mathcal{U}(t)$ generated by the total Hamiltonian H , $\overline{H}$, $\mathcal{H}$ of interacting polariton qubits , interacting anti-polariton qubits or interacting polariton-anti-polariton qubits can be factorized in the form

$$U(t) = e^{-\frac{it}{\hbar} H} \qquad \Rightarrow \qquad U(t) = e^{-\frac{it}{\hbar} \sum_{i=1}^{S} H_i} \ e^{-\frac{it}{\hbar} \sum_{i \neq j=1}^{S} g_{ij} \hat{\mathcal{E}}_i \hat{\mathcal{E}}_j} \tag{32d}$$

$$\overline{U}(t) = e^{-\frac{it}{\hbar} \overline{H}} \qquad \Rightarrow \qquad \overline{U}(t) = e^{-\frac{it}{\hbar} \sum_{i=1}^{S} \overline{H}_i} \ e^{-\frac{it}{\hbar} \sum_{i \neq j=1}^{S} g_{ij} \hat{\overline{\mathcal{E}}}_i \hat{\overline{\mathcal{E}}}_j} \tag{32e}$$

$$\mathcal{U}(t) = e^{-\frac{it}{\hbar} \mathcal{H}} \qquad \Rightarrow \qquad \mathcal{U}(t) = e^{-\frac{it}{\hbar} \sum_{i=1}^{S-1} H_i} \ e^{-\frac{it}{\hbar} \sum_{j \neq i=1}^{S} \overline{H}_j} \ e^{-\frac{it}{\hbar} \sum_{i \neq j=1}^{S} g_{ij} \hat{\mathcal{E}}_i \hat{\overline{\mathcal{E}}}_j} \tag{32f}$$

The commutation of the qubit state transition operators at different sites according to $[\hat{\mathcal{E}}_i, \hat{\mathcal{E}}_j] = 0$, $[\hat{\overline{\mathcal{E}}}_i, \hat{\overline{\mathcal{E}}}_j] = 0$, $[\hat{\mathcal{E}}_i, \hat{\overline{\mathcal{E}}}_j] = 0$ means that the time evolution operators in equations $(32d)$, $(32e)$, $(32f)$ can be evaluated exactly. If we consider only nearest-neighbor interactions, then we can set $j = i + 1$ in the interaction Hamiltonians.

In the simplest case of two interacting polariton qubits or two interacting anti-polariton qubits or polariton-anti-polariton qubit interaction, we set $S = 2$ in equations $(32b)$, $(32c)$ to obtain the respective two-qubit Hamiltonians in the form

$$H = H_1 + H_2 + g \hat{\mathcal{E}}_1 \hat{\mathcal{E}}_2 \quad ; \quad \overline{H} = \overline{H}_1 + \overline{H}_2 + g \hat{\overline{\mathcal{E}}}_1 \hat{\overline{\mathcal{E}}}_2 \quad ; \quad \mathcal{H} = H_1 + \overline{H}_2 + g \hat{\mathcal{E}}_1 \hat{\overline{\mathcal{E}}}_2 \quad ; \quad g = g_{12} \tag{33a}$$

where the polariton and anti-polariton qubit Hamiltonians H_i , $\overline{H}_i$, $i = 1, 2$ are defined at two lattice sites $i = 1, 2$ in the respective forms in equations $(29b)$, $(30b)$. The initial state vectors of the two interacting polariton, anti-polariton or polariton-anti-polariton qubits can be expressed as appropriate in any of four entangled forms

$$|\psi_{12}^{\pm}\rangle = \frac{1}{\sqrt{2}} (\ |\psi_{n_1 u_1}\rangle |\psi_{n_2 u_2}\rangle \pm |\phi_{n_1 u_1}\rangle |\phi_{n_2 u_2}\rangle \ ) \quad ; \quad |\phi_{12}^{\pm}\rangle = \frac{1}{\sqrt{2}} (\ |\psi_{n_1 u_1}\rangle |\phi_{n_2 u_2}\rangle \pm |\phi_{n_1 u_1}\rangle |\psi_{n_2 u_2}\rangle \ ) \tag{33b}$$

$$| \ \overline{\psi}_{12}^{\pm}\rangle = \frac{1}{\sqrt{2}} (\ |\psi_{n_1 u_1}\rangle |\psi_{n_2 u_2}\rangle \pm | \ \overline{\phi}_{n_1 u_1}\rangle | \ \overline{\phi}_{n_2 u_2}\rangle \ ) \quad ; \quad | \ \overline{\phi}_{12}^{\pm}\rangle = \frac{1}{\sqrt{2}} (\ |\psi_{n_1 u_1}\rangle | \ \overline{\phi}_{n_2 u_2}\rangle \pm | \ \overline{\phi}_{n_1 u_1}\rangle |\psi_{n_2 u_2}\rangle \ ) \tag{33c}$$

$$| \ \tilde{\psi}_{12}^{\pm}\rangle = \frac{1}{\sqrt{2}} (\ |\psi_{n_1 u_1}\rangle |\psi_{n_2 u_2}\rangle \pm |\phi_{n_1 u_1}\rangle | \ \overline{\phi}_{n_2 u_2}\rangle \ ) \quad ; \quad | \ \tilde{\phi}_{12}^{\pm}\rangle = \frac{1}{\sqrt{2}} (\ |\psi_{n_1 u_1}\rangle | \ \overline{\phi}_{n_2 u_2}\rangle \pm |\phi_{n_1 u_1}\rangle |\psi_{n_2 u_2}\rangle \ ) \tag{33d}$$

where we note that the polariton and anti-polariton qubit state vectors $|\psi_{n_i u_i}\rangle$, $|\phi_{n_i u_i}\rangle$, $| \ \overline{\phi}_{n_i u_i}\rangle$ defined in the respective forms in equations $(29a)$, $(30a)$ at the two lattice sites $i = 1, 2$ are normalized, but non-orthogonal. The Gram-Schmidt orthonormalization procedure may be applied to transform them into the appropriate orthonormal forms. Dynamical evolution of an entangled two-qubit state vector is governed by an appropriate time evolution operator $U(t)$, $\overline{U}(t)$, $\mathcal{U}(t)$ generated by the respective two-qubit Hamiltonians H , $\overline{H}$, $\mathcal{H}$ in equation $(33a)$ according to equations $(32d)$, $(32e)$, $(32f)$. In each case, the time evolution operator can be evaluated explicitly to provide an exact time evolving entangled state vector of the two interacting polariton-polariton, anti-polariton-anti-polariton or polariton-anti-polariton qubits. The determination of exact time evolving entangled state vectors opens greater possibilities for practical applications of systems of interacting polariton, anti-polariton or polariton-anti-polariton qubits in the design and implementation of various aspects of quantum technology, such as teleportation, quantum cryptography, quantum state tomography, quantum information processing, quantum computation, quantum metrology, etc. Here, we have provided only the model and algebraic foundation for studying the dynamics and practical applications of systems of interacting polaritons and anti-polaritons generated within the quantum Rabi model, but details will be presented in other work.

Finally, we observe that the total Hamiltonian H in equation $(32b)$ for interacting polariton qubits is similar in form , but differs significantly from the corresponding Jaynes-Cummings-Hubbard Hamiltonian generally used to describe polariton interactions in coupled arrays of optical cavities [19-24]. While the Hamiltonian H in equation $(32b)$ generates the dynamics of a system of interacting polariton qubits coupled through their state transition operators $\hat{\mathcal{E}}_i$, $\hat{\mathcal{E}}_j$, the Jaynes-Cummings-Hubbard Hamiltonian in [19-24] generates the dynamics of a system of interacting polaritons coupled through the field mode annihilation and creation operators $\hat{a}_i$, $\hat{a}_i^\dagger$, $\hat{a}_j$, $\hat{a}_j^\dagger$ of photons tunneling between the optical lattice sites i , j.

5 Conclusion

We have developed a precise algebraic and physical framework for studying the dynamics and practical applications of polariton or anti-polariton qubits interpreted as two-state quantized particles formed through the coupling of an atomic spin to a rotating positive frequency or anti-rotating negative frequency component of a quantized electromagnetic field mode in a Jaynes-Cummings or anti-Jaynes-Cummings interaction, respectively. Polariton or anti-polariton qubit formation and dynamics is generated in a two-dimensional subspace spanned by the respective qubit state vectors defined within an upper or lower Rabi state space. Conserved excitation number, identity, state transition, $U(1)$-symmetry, parity-symmetry, $SU(2)/U(1)$-symmetry and $SU(1,1)/U(1)$-symmetry operators, as well as eigenvectors and energy eigenvalues of a polariton or an anti-polariton qubit Hamiltonian have been determined explicitly. Since the atom starts in an excited (spin-up) or ground (spin-down) state, dynamics is effectively characterized by absorption or emission of *positive energy* photons by the field mode in *red-sideband* state transitions generated by the polariton qubit Hamiltonian and absorption or emission of *negative energy* photons by the field mode in *blue-sideband* state transitions generated by the anti-polariton qubit Hamiltonian. Exact time evolving state vectors which describe the red-sideband and blue-sideband state transitions at the respective Rabi oscillation frequencies have been determined. In interactions starting with the field mode in an initial vacuum state, dynamics in the upper Rabi state space is dominated by *spontaneous absorption process* in which the field mode absorbs positive energy photons in red-sideband transitions, while dynamics in the lower Rabi state space is dominated by *spontaneous anti-emission process* in which the field mode emits negative energy photons in blue-sideband transitions. Noting the similarity of polariton and anti-polariton qubits to the basic atomic spin qubits, we have introduced a *quantum Rabi optical lattice* as a geometrical framework for studying the dynamics and physical properties of systems of interacting polariton and anti-polariton qubits.

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References

[1] D Braak 2011 On the Integrability of the Rabi Model, Phys.Rev.Lett.**107**, 100401 ; arXiv:1103.2461 v2 [quant-ph]

[2] Q Xie, H Zhong, M T Batchelor and C Lee 2016 The quantum Rabi model: solutions and dynamics, arXiv:1609.00434 v2 [quant-ph]

[3] Q Xie, et al 2014 Anisotropic Rabi Model, Phys.Rev.**X 4**, 021046 ; arXiv:1401.5865v2[quant-ph]

[4] H H Zhong, et al 2013 Analytical eigenstates for the quantum Rabi Model, J.Phys.**A**: Math.Theor. **46**, 415302

[5] Q H Chent, et al 2012 Exact solvability of the quantum Rabi model using the Bogoliubov operators, Phys.Rev.**A 86**, 023822 ; arXiv:1204.3668[quant-ph]

[6] B Hu, et al 2017 Dynamical properties of the Rabi Model, J.Phys.**A**: Math.Theor. **50**, 074004 ; arXiv: 1608.07709 [quant-ph]

[7] F A Wolf, M Kollar and D Braak 2012 Exact real-time dynamics of the quantum Rabi Model, Phys.Rev.**A 85**, 053817

[8] B C da Cunha, M C de Almeida and A R de Queiroz 2016 On the existence of monodromies for the Rabi model, J.Phys.A: Math.Theor.**49**, 194002

[9] C Joshi, J Larson and T P Spiller, 2016 Quantum state engineering in hybrid open quantum systems, Phys.Rev. **A 93**, 043818 ; arXiv:1509.03599v2[quant-ph]

[10] M Born and E Wolf, 1999 *Principles of Optics*, Cambridge University Press, Cambridge, England

[11] E Rubino, et al 2012 Negative-Frequency Resonant Radiation, Phys.Rev.Lett. **108**, 253901

[12] J McLenaghan and F Konig 2014 Few-cycle fiber pulse compression and evolution of negative resonant radiation, New J. Phys.. **16**, 063017

[13] J A Omolo 2017 Conserved excitation number and $U(1)$-symmetry operators for the anti-rotating (anti-Jaynes-Cummings) term of the Rabi Hamiltonian, Preprint-ResearchGate, DOI:10.13140/RG.2.2.30936.80647

[14] P Meystre and M Sargent III 1991 *Elements of Quantum Optics*, Second Edition, Springer-Verlag, Berlin, Heidelberg, New York

[15] A K Rajagopal , K L Jensen and F W Cummings 1999 Quantum entangled supercorrelated states in the Jaynes-Cummings model, arXiv: 9903085 [quant-ph]

[16] X Gu, S Huai, F Nori and Y Liu, 2016 Polariton states in circuit QED for electromagneticall induced transparency, Phys.Rev. **A 93**, 063827

[17] M J Hwang, R Puebla and M B Plenio 2015 Quantum phase transition and universal dynamics in the Rabi model, Phys.Rev.Lett. **115**, 180404 ; arXiv: 1503.03090 [quant-ph]

[18] C Emary and T Brandes 2003 Chaos and the quantum phase transition in the Dicke model, Phys.Rev.E. **67**, 066203 ; arXiv: 0301273 [cond-mat]

[19] M J Hwang and M B Plenio 2016 Quantum phase transitions in the finite Jaynes-Cummings lattice, Phys.Rev.Lett. **117**, 123602 ; arXiv: 1603.03943 [quant-ph]

[20] M J Hwang and M B Plenio 2016 Quantum phase transitions in the finite Jaynes-Cummings lattice, Phys.Rev.Lett. **117**, 123602 ; arXiv: 1603.03943 [quant-ph]

[21] S Schmidt and J Koch 2012 Circuit QED lattices : towards quantum simulation with superconducting circuits, arXiv: 1212.2070 [quant-ph]

[22] A Maggitti, M Radonjic and B M Jelenkovic, 2016 Dark-polariton bound pairs in the modified Jaynes-Cummings-Hubbard model, Phys.Rev. **A 93**, 013835

[23] H Shapourian and D Sadri 2016 Fock space localization of polaritons in the Jaynes-Cummings dimer model, Phys.Rev. **A 93**, 013845 ; arXiv:1510.02118v2[quant-ph]

[24] A D Greentree, C Tahan, J H Cole and L C L Hollenberg 2006 Quantum phase transitions of light, Nat. Phys. **2**, 856 ; arXiv: 0609050 [cond-mat.other]

[25] M J Hartmann, F G S L Brandao and M B Plenio 2006 Strongly interacting polaritons in coupled arrays of cavities, Nat. Phys. **2**, 849 ; arXiv: 0606097 [quant-ph]