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# The Test of Entanglement of Polarization States of a Semi-Classical Optical Parametric Oscillator

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**Abstract:** We study the dynamical continuous variable entanglement in a semi-classical Optical parametric oscillator (OPO). The general time evolving photon polarization state vectors arising from exact analytical solutions of Heisenberg's equations of the system are used to obtain the photon polarization Bell state vectors. The reduced density matrices of photon polarization Bell state vectors of the semi-classical OPO produce a greater violation of CHSH Bell's inequality beyond the Cirel'son's inequality.

**Keywords:** Optical Parametric Oscillator, Bell State Vectors, Reduced Density Matrices, Entanglement

## 1. Introduction

Quantum entanglement is a physical phenomenon that occurs when pairs or groups of particles are generated or interact in such a manner that the quantum state of each particle cannot be described independently; instead, a quantum state is given for the system as a whole. Entanglement has found many applications in the fields of quantum computation and quantum information processing such as quantum cryptography, quantum dense coding, entanglement swapping, quantum lithography and quantum teleportation [1]. Quantum teleportation (transfer of quantum states between distant locations without an intervening medium) has been achieved over long distances [2, 3, 4]. On August 16, 2016, China launched the world's first "quantum satellite", a communication system incapable of being hacked and stretching over a distance of 2000 km [5].

Continuous variable entanglement was demonstrated for the first time in the Optical parametric oscillator (OPO) operating below threshold in 1990. Entanglement in the above-threshold OPO remained an experimental challenge until 2005, when it was first observed by Villar S. and Cassemiro N. (2005) [6]. Entanglement features for a full quantum treatment of OPO has been studied where time evolution equations are solved through writing of the density operator equation in the Wigner representation using equivalent Langevin equations to obtain analytical results [7]. Su and Tan (2006) [8] observed a two-color entanglement measured to the above-threshold OPO only. The multipartite nature of entanglement was verified by evaluating the van Loock-Furusawa criterion for a particular set of entanglement witnesses deduced from physical considerations [9]. Johansson R. (2014) [10] investigated theoretically the conditions under which a multi-mode nano-mechanical resonator, operating as a purely mechanical parametric oscillator, can be driven into highly non-classical states. Quantum entangled states of the system violate Bell inequalities with homodyne quadrature measurements. Chakrabarti R. and Jenisha J. (2015) [11] studied the evolution of a bipartite entangled quasi-bell state in a strongly coupled qubit oscillator system in the presence of a static bias, and extended it to the ultra-strong coupling regime. Adiabatic approximation was used to obtain reduced density matrix of the qubit for the strong coupling domain in closed form involving linear combinations of the Jacobi theta functions. Apart from employing the adiabatic approximation no other simplification has been made for deriving the reduced density matrix elements. A test of Bell inequality using polarization entangled photons from a Spontaneous

Parametric down Conversion (SPDC) has shown a violation of Bell's inequality [12]. In the current paper, we use reduced density matrices of polarization states of semi-classical OPO to test the violation of CHSH Bell's inequality, under all conditions of interaction.

The paper is organized as follows. In section 2, the Hamiltonian of the system is developed and used to obtain the time evolution operator of the system. In section 3, the time evolving photon polarization Bell states are constructed. In section 4, the test of entanglement of polarization states of OPO using reduced density matrices and Bell inequalities is presented. A conclusion is provided in section 5.

## 2. The Time Evolution Operator

The OPO is an alternative tool for non-linear generation of entangled photons where a pump photon is converted into two lower energy beams, known as the signal photon and the idler photon occurring due to excitation of a material medium when struck by an external electromotive force. The signal photon is taken to be initially in the horizontal polarization state and the idler photon is taken to be initially in the vertical polarization state. The photon is therefore in a superposition of horizontal and vertical polarization. The horizontal polarization state vector represents the basic unit vector $|0\rangle$ and the vertical polarization state vector represents the basic unit vector $|1\rangle$ according to the definition

$$|H\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad |V\rangle = |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1}$$

The two-mode Hamiltonian of semi-classical OPO is described by [13]

$$H = \hbar\left(\omega_a \hat{a}^+ \hat{a} + \omega_b \hat{b}^+ \hat{b} + \alpha\, \hat{a}^+ \hat{b} + \alpha^* \hat{b}^+ \hat{a}\ \right) \tag{2a}$$

where $\hat{a}\left(\hat{a}^+\right)$ is the signal annihilation (creation) operators, $\omega_a$ is the signal frequency, $\hat{b}\left(\hat{b}^+\right)$ is the idler annihilation (creation) operators, $\omega_b$ is the idler frequency and $\alpha\left(\alpha^*\right)$ is a time-dependent interaction parameter which varies harmonically with time in the form of a rotating pump field of frequency $\left(\omega_p\right)$, according to

$$\alpha(t) = \alpha e^{-i\omega_p t} \quad ; \quad \alpha^*(t) = \alpha e^{i\omega_p t} \quad ; \quad \alpha = |\alpha(t)| = consta\,nt \tag{2b}$$

The dynamics of the semi-classical OPO is described through Heisenberg's equations

$$i\hbar \frac{d\hat{a}}{dt} = \left[\ \hat{a}, \hat{\mathrm{H}}\right] = \frac{\partial \hat{\mathrm{H}}}{\partial \hat{a}^+} = \hbar\left(\omega_a \hat{a} + \alpha(t)\hat{b}\right) \tag{3a}$$

$$i\hbar \frac{d\hat{b}}{dt} = \left[\ \hat{b}, \hat{\mathrm{H}}\right] = \frac{\partial \hat{\mathrm{H}}}{\partial \hat{b}^+} = \hbar\left(\alpha^*(t)\hat{a} + \omega_b \hat{b}\right) \tag{3b}$$

Equations (3a) and (3b) can be expressed in matrix form as

$$i\hbar \frac{d\hat{A}}{d\,t} = \hbar \begin{pmatrix} \omega_a & \alpha(t) \\ \alpha^*(t) & \omega_b \end{pmatrix} \hat{A} \tag{4a}$$

where $\hat{A}$ is the two-component photon pair polarization vector defined as

$$\hat{A} = \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} = \hat{a}|0\rangle + \hat{b}|1\rangle = \hat{a}|H\rangle + \hat{b}|V\rangle \tag{4b}$$

Equation (4a) can be expressed in alternative form as

$$i\hbar \frac{d\hat{A}}{dt} = \hbar\left(\Omega_{ab}\hat{S}_0 + \omega_{ab}\hat{S}_z + \alpha(t)\hat{S}_+ + \alpha^*(t)\hat{S}_-\right)\hat{A} \tag{5a}$$

where

$$\Omega_{ab} = \omega_a + \omega_b \quad ; \quad \omega_{ab} = \omega_a - \omega_b, \tag{5b}$$

$$\hat{S}_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad ; \quad \hat{S}_z = \frac{1}{2}\sigma_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5c}$$

$$\hat{S}_+ = \frac{1}{2}\left(\sigma_x + i\sigma_y\right) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad \hat{S}_- = \frac{1}{2}\left(\sigma_x - i\sigma_y\right) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{5d}$$

and $\sigma_i;\ i = x, y, z$ are the Pauli spin operators. Equation (5a) is solved to obtain the general form of time evolution photon polarization vector, expressed as

$$\hat{A}(t) = U_A(t)\hat{A}(0) \tag{6a}$$

where $U_A(t)$ is the time evolution operator,

$$U_A(t) = e^{-i\Omega_{ab}\hat{S}_0 t}\, e^{-i\omega_p t\hat{S}_z}\, e^{-it\left(\Delta\hat{S}_z + \alpha(t)e^{i\omega_p t}\hat{S}_+ + \alpha^*(t)e^{-i\omega_p t}\hat{S}_-\right)} \tag{6b}$$

and $\Delta$ is a constant detuning parameter,

$$\Delta = \omega_{ab} - \omega_p \tag{6c}$$

We substitute for spin operators $\hat{S}_0$, $\hat{S}_z$, $\hat{S}_+$ and $\hat{S}_-$ (5c-d) in (6b) to obtain the final matrix as

$$U_A(t) = \begin{pmatrix} \mu(t) e^{-\frac{i}{2}(\Omega_{ab} + \omega_p)t} & \nu(t) e^{-\frac{i}{2}(\Omega_{ab} + \omega_p)t} \\ -\nu^*(t) e^{-\frac{i}{2}(\Omega_{ab} - \omega_p)t} & \mu^*(t) e^{-\frac{i}{2}(\Omega_{ab} - \omega_p)t} \end{pmatrix} \tag{7a}$$

where

$$\mu(t)=\cos(\beta t)-\frac{i\Delta}{2\beta}\sin(\beta t) \tag{7b}$$

$$\nu(t)=-\frac{i\alpha(t)e^{i\omega_p t}}{\beta}\sin(\beta t) \tag{7c}$$

$$\beta=|\alpha|\sqrt{1+k^2} \quad ; \quad k=\frac{\Delta}{2|\alpha|} \tag{7d}$$

Similarly, if we consider the signal photon represented by mode 'a' to be initially in the vertical polarization state $|V\rangle$ and the idler photon represented by mode 'b' to be initially in the horizontal polarization state $|H\rangle$, and follow steps given by equations (2a) to (7d), we obtain the alternative final form of time evolution operator as

$$U_B(t)=\begin{pmatrix} \mu^*(t)e^{-\frac{i}{2}(\Omega_{ab}-\omega_p)t} & -\nu^*(t)e^{-\frac{i}{2}(\Omega_{ab}-\omega_p)t} \\ \nu(t)e^{-\frac{i}{2}(\Omega_{ab}+\omega_p)t} & \mu(t)e^{-\frac{i}{2}(\Omega_{ab}+\omega_p)t} \end{pmatrix} \tag{8}$$

## 3. Bell State Vectors

For a two-particle system where the signal photon is denoted by mode 'a' is taken to be initially in the horizontal polarization state and the idler photon is denoted by mode 'b' is taken to be initially in the vertical polarization state or vice versa. Then the maximum entangled Bell state vector of the photon is the superposition of horizontal and vertical polarization presented in simple form as [14]

$$|\phi_\pm\rangle_{ab}=\frac{1}{\sqrt{2}}\left(|0\rangle_a|0\rangle_b\pm|1\rangle_a|1\rangle_b\right)=\frac{1}{\sqrt{2}}\left(|H\rangle_a|H\rangle_b\pm|V\rangle_a|V\rangle_b\right) \tag{9a}$$

$$|\psi_\pm\rangle_{ab}=\frac{1}{\sqrt{2}}\left(|0\rangle_a|1\rangle_b\pm|1\rangle_a|0\rangle_b\right)=\frac{1}{\sqrt{2}}\left(|H\rangle_a|V\rangle_b\pm|V\rangle_a|H\rangle_b\right) \tag{9b}$$

Using equations (1), (7a) and (8) the four possible eigen states of the two-particle system in (9a) and (9b) takes the form

$$|H\rangle_a=U_A(t)|H\rangle=\left(\mu(t)e^{-\frac{i}{2}(\Omega_{ab}+\omega_p)t}|0\rangle-\nu^*(t)e^{-\frac{i}{2}(\Omega_{ab}-\omega_p)t}|1\rangle\right) \tag{10a}$$

$$|H\rangle_b=U_B(t)|H\rangle=\left(\mu^*(t)e^{-\frac{i}{2}(\Omega_{ab}-\omega_p)t}|0\rangle+\nu(t)e^{-\frac{i}{2}(\Omega_{ab}+\omega_p)t}|1\rangle\right) \tag{10b}$$

$$|V\rangle_a=U_B(t)|V\rangle=\left(-\nu^*(t)e^{-\frac{i}{2}(\Omega_{ab}-\omega_p)t}|0\rangle+\mu(t)e^{-\frac{i}{2}(\Omega_{ab}+\omega_p)t}|1\rangle\right) \tag{10c}$$

$$|V\rangle_b=U_B(t)|V\rangle=\left(\nu(t)e^{-\frac{i}{2}(\Omega_{ab}+\omega_p)t}|0\rangle+\mu^*(t)e^{-\frac{i}{2}(\Omega_{ab}-\omega_p)t}|1\rangle\right) \tag{10d}$$

Using the general time evolving photon polarization state vector equations (10a-d) in (9a) and (9b), the four time evolution photon polarization Bell state vector reduces to

$$|\phi^+\rangle=e^{-i\Omega_{ab}t}\begin{pmatrix} \cos^2(\beta t)-\left(\frac{\Delta^2+4|\alpha|^2}{4\beta^2}\right)\sin^2(\beta t) \\ -\frac{i\alpha}{\beta}\cos(\omega_p t)\sin(2\beta t)+\frac{i\Delta\alpha}{\beta^2}\sin(\omega_p t)\sin^2(\beta t) \\ -\frac{i\alpha}{\beta}\cos(\omega_p t)\sin(2\beta t)+\frac{i\Delta\alpha}{\beta^2}\sin(\omega_p t)\sin^2(\beta t) \\ \cos^2(\beta t)-\left(\frac{\Delta^2+4|\alpha|^2}{4\beta^2}\right)\sin^2(\beta t) \end{pmatrix} \tag{11a}$$

$$|\phi^-\rangle=e^{-i\Omega_{ab}t}\begin{pmatrix} \cos^2(\beta t)-\left(\frac{\Delta^2-4|\alpha|^2}{4\beta^2}\right)\sin^2(\beta t) \\ -\frac{\alpha}{\beta}\sin(\omega_p t)\sin(2\beta t)-\frac{\Delta\alpha}{\beta^2}\cos(\omega_p t)\sin^2(\beta t) \\ \frac{\alpha}{\beta}\sin(\omega_p t)\sin(2\beta t)+\frac{\Delta\alpha}{\beta^2}\cos(\omega_p t)\sin^2(\beta t) \\ -\cos^2(\beta t)+\left(\frac{\Delta^2-4|\alpha(t)|^2}{4\beta^2}\right)\sin^2(\beta t) \end{pmatrix} \tag{11b}$$

$$|\psi^+\rangle=e^{-i\Omega_{ab}t}\begin{pmatrix} -\frac{i\alpha}{\beta}\cos(\omega_p t)\sin(2\beta t)+\frac{i\Delta\alpha}{\beta^2}\sin(\omega_p t)\sin^2(\beta t) \\ \cos^2(\beta t)-\left(\frac{\Delta^2+4|\alpha|^2}{4\beta^2}\right)\sin^2(\beta t) \\ \cos^2(\beta t)-\left(\frac{\Delta^2+4|\alpha|^2}{4\beta^2}\right)\sin^2(\beta t) \\ -\frac{i\alpha}{\beta}\cos(\omega_p t)\sin(2\beta t)+\frac{i\Delta\alpha}{\beta^2}\sin(\omega_p t)\sin^2(\beta t) \end{pmatrix} \tag{11c}$$

$$|\psi^-\rangle=e^{-i\Omega_{ab}t}\begin{pmatrix} -\frac{i\alpha}{\beta}\sin(\omega_p t)\sin(2\beta t)-\frac{\Delta\alpha}{\beta^2}\cos(\omega_p t)\sin^2(\beta t) \\ \cos^2(\beta t)-\left(\frac{\Delta^2-4|\alpha|^2}{4\beta^2}\right)\sin^2(\beta t) \\ -\cos^2(\beta t)+\left(\frac{\Delta^2-4|\alpha(t)|^2}{4\beta^2}\right)\sin^2(\beta t) \\ \frac{\alpha}{\beta}\sin(\omega_p t)\sin(2\beta t)+\frac{\Delta\alpha}{\beta^2}\cos(\omega_p t)\sin^2(\beta t) \end{pmatrix} \tag{11d}$$

## 4. Reduced Density of State

The Clauser–Horne–Shimony-Holt (CHSH) Bell inequality defined as

$$-2\le S\le +2 \tag{12}$$

is commonly used to test the nature of entanglement of photon polarization states[15]. In (12) S represents the reduced density matrix for Bell state vectors [16]. Entanglement is exhibited by violations of the Bell inequality. The larger the violation of the Bell inequality the more the entanglement present in the system.

We determine reduced density matrices by obtaining the trace of density matrices under the following conditions of interaction:

### 4.1. Resonance

At resonance,

$$|\alpha| >> \Delta \tag{13a}$$

and (7d) reduces to

$$\beta = |\alpha| \quad ; \quad k = 0 \tag{13b}$$

We use the condition in (13a) and (13b) in polarization Bell state vectors (11a-d) to obtain the trace of the density matrices under resonance condition as

$$Tr\,\hat{\rho}^{a}_{ab_+}(t) = Tr_b\left(\left|\phi_+\right\rangle\left\langle\phi_+\right|\right) = 2 - 2\sin^2(\omega_p t)\sin^2(2|\alpha| t) \tag{14a}$$

$$Tr\,\hat{\rho}^{a}_{ab_-}(t) = Tr_b\left(\left|\phi_-\right\rangle\left\langle\phi_-\right|\right) = 2 + 2\sin^2(\omega_p t)\sin^2(2|\alpha| t) \tag{14b}$$

$$Tr\,\hat{\rho}^{b}_{ba_+}(t) = Tr_a\left(\left|\psi_+\right\rangle\left\langle\psi_+\right|\right) = 2 - 2\sin^2(\omega_p t)\sin^2(2|\alpha| t) \tag{14c}$$

$$Tr\,\hat{\rho}^{b}_{ba_-}(t) = Tr_a\left(\left|\psi_-\right\rangle\left\langle\psi_-\right|\right) = 2 + 2\sin^2(\omega_p t)\sin^2(2|\alpha| t) \tag{14d}$$

The maximum and least traces in (14a-14d) at time $t_1$ and $t_2$ are tabulated in table 1.

***Table 1.*** *Maximum and least traces of density matrices for Bell states (14a-14d) under resonance at time* $t_1$ *and* $t_2$ *where n=0, 1, 2…… and* $\omega_p = 2|\alpha|$ .

| Time | $Tr_b\left(\left|\phi_+\right\rangle\left\langle\phi_+\right|\right)$ | $Tr_b\left(\left|\phi_-\right\rangle\left\langle\phi_-\right|\right)$ | $Tr_a\left(\left|\psi_+\right\rangle\left\langle\psi_+\right|\right)$ | $Tr_a\left(\left|\psi_-\right\rangle\left\langle\psi_-\right|\right)$ |
|---|---|---|---|---|
| $t_1 = \frac{n\pi}{\omega_p}$ | 2 | 2 | 2 | 2 |
| $t_2 = \left(\frac{1}{2}+n\right)\frac{\pi}{\omega_p}$ | 0 | 4 | 0 | 4 |

According to CHSH Bell inequality (12), the Bell states (14a) and (14c) under resonance produce weak entanglement. The Bell states (14b) and (14d) produce strong entanglement above the Cirel'son's inequality at the time $t_2$ .

### 4.2. Very Weak Interaction

At very weak interaction

$$|\alpha| << \Delta \tag{15a}$$

and (7d) reduces to

$$\beta = |\alpha| k \quad ; \quad k >> 1 \tag{15b}$$

We use the condition in (15a) and (15b) in polarization Bell state vectors (11a-d) to obtain the trace of the density matrices under very weak interaction condition as

$$Tr_b\left(\left|\phi_+\right\rangle\left\langle\phi_+\right|\right) = Tr_b\left(\left|\phi_-\right\rangle\left\langle\phi_-\right|\right) = Tr_a\left(\left|\psi_+\right\rangle\left\langle\psi_+\right|\right) = Tr_a\left(\left|\psi_-\right\rangle\left\langle\psi_-\right|\right) = 2 - 2\sin^2(2|\alpha| kt) \tag{16}$$

The maximum and least traces arising from (16) under very weak interaction are tabulated in table 2.

***Table 2.*** *Maximum and least traces of density matrices for Bell states (16) under very weak interaction at time* $t_1$ *and* $t_2$ *where n=0, 1, 2……..*

| Time | $Tr_b\left(\left|\phi_+\right\rangle\left\langle\phi_+\right|\right)$ | $Tr_b\left(\left|\phi_-\right\rangle\left\langle\phi_-\right|\right)$ | $Tr_a\left(\left|\psi_+\right\rangle\left\langle\psi_+\right|\right)$ | $Tr_a\left(\left|\psi_-\right\rangle\left\langle\psi_-\right|\right)$ |
|---|---|---|---|---|
| $t = \frac{n\pi}{2|\alpha| k}$ | 2 | 2 | 2 | 2 |
| $t = \left(\frac{1}{2}+n\right)\frac{\pi}{2|\alpha| k}$ | 0 | 0 | 0 | 0 |

According to table (2), all the Bell states in (16) under very weak interaction produce weak entanglement.

### 4.3. Weak Interaction

At weak interaction,

$$|\alpha| < \Delta \tag{17a}$$

and (7d) reduces to

$$\beta = |\alpha|\sqrt{1+k^2} \quad ; \quad k > 1 \tag{17b}$$

We use the condition in (17a) and (17b) in polarization Bell state vectors (11a-d) to obtain the trace of the density matrices under weak interaction condition as

$$\begin{aligned} Tr\,\hat{\rho}^{a}_{ab_+}(t) = Tr_b\left(\left|\phi_+\right\rangle\left\langle\phi_+\right|\right) = 2 - \frac{8}{k^2+1}\sin^2(\alpha t\sqrt{1+k^2}) + \frac{8\alpha}{(k^2+1)}\sin^2(\omega_p t)\sin^2(\alpha t\sqrt{1+k^2}) \\ - \frac{\left(8k^2+8\alpha(k^2+1)\right)}{(k^2+1)}\sin^2(\omega_p t)\sin^4(\alpha t\sqrt{1+k^2}) - \frac{(2k^4+4k^2-6)}{(k^2+1)^2}\sin^4(\alpha t\sqrt{1+k^2} \end{aligned} \tag{18a}$$

$$\begin{aligned} Tr\,\hat{\rho}^{a}_{ab_-}(t) = Tr_b\left(\left|\phi_-\right\rangle\left\langle\phi_-\right|\right) = 2 - \frac{2}{(1+k^2)}\sin^2(\omega_p t)\sin^2(2\alpha t\sqrt{1+k^2}) \\ + \frac{8k^2}{(1+k^2)^2}\sin^4(\alpha t\sqrt{1+k^2}) - \frac{8k^2}{(1+k^2)^2}\sin^2(\omega_p t)\sin^4(\alpha t\sqrt{1+k^2}) \end{aligned} \tag{18b}$$

$$Tr\hat{\rho}^{b}_{ba_+}(t)=Tr_a\left(|\psi_+\rangle\langle\psi_+|\right)=2-\frac{8}{k^2+1}\sin^2(\alpha t\sqrt{1+k^2})+\frac{8\alpha}{(k^2+1)}\sin^2(\omega_p t)\sin^2(\alpha t\sqrt{1+k^2}) -\frac{\left(8k^2+8\alpha(k^2+1)\right)}{(k^2+1)}\sin^2(\omega_p t)\sin^4(\alpha t\sqrt{1+k^2})-\frac{(2k^4+4k^2-6)}{(k^2+1)^2}\sin^4(\alpha t\sqrt{1+k^2} \tag{18c}$$

$$Tr\hat{\rho}^{b}_{ba_-}(t)=Tr_a\left(|\psi_-\rangle\langle\psi_-|\right)=2-\frac{2}{(1+k^2)}\sin^2(\omega_p t)\sin^2(2\alpha t\sqrt{1+k^2}) +\frac{8k^2}{(1+k^2)^2}\sin^4(\alpha t\sqrt{1+k^2})-\frac{8k^2}{(1+k^2)^2}\sin^2(\omega_p t)\sin^4(\alpha t\sqrt{1+k^2}) \tag{18d}$$

The maximum and least traces corresponding to (18a-d) under weak interaction are given in table 3.

***Table 3.*** *Maximum and least traces of density matrices for Bell states (18a-18d) under weak interaction at time $t_1$ and $t_2$ where n=0, 1, 2……. and $|\alpha|\sqrt{1+k^2}=\omega_p$.*

| **Time** | $Tr_b\left(|\phi_+\rangle\langle\phi_+|\right)$ | $Tr_b\left(|\phi_-\rangle\langle\phi_-|\right)$ | $Tr_a\left(|\psi_+\rangle\langle\psi_+|\right)$ | $Tr_a\left(|\psi_-\rangle\langle\psi_-|\right)$ |
|---|---|---|---|---|
| $t_1=\frac{n\pi}{|\alpha|\sqrt{1+k^2}}$ | 2 | 4 | 2 | 4 |
| $t_2=\left(\frac{1}{2}+n\right)\frac{\pi}{2|\alpha|k}$ | < 3 | > 1 | < 3 | > 1 |

According to CHSH Bell inequality, the Bell states (18a) and (18c) under weak interaction produce strong entanglement above the Cirel'son's inequality at the time $t_2$. The Bell states (18b) and (18d) produce strong entanglement above the cirel'son's inequality at the time $t_1$. The trace reduces as the value of k increases i.e. k>1.

### *4.4. Medium Strength Interaction*

At medium strength interaction,

$$|\alpha|>\Delta \tag{19}$$

and (7d) reduces to (17b) at k<1. The trace of the density matrices under medium strength interaction condition take the form represented in (18a-d). The maximum and least traces corresponding to (18a-d) under medium strength interaction are given in table 4.

***Table 4.*** *Maximum and least traces of density matrices for Bell states (18a-18d) under medium strength interaction at time $t_1$ and $t_2$ where n=0, 1, 2 …. and $|\alpha|\sqrt{1+k^2}=\omega_p$.*

| **Time** | $Tr_b\left(|\phi_+\rangle\langle\phi_+|\right)$ | $Tr_b\left(|\phi_-\rangle\langle\phi_-|\right)$ | $Tr_a\left(|\psi_+\rangle\langle\psi_+|\right)$ | $Tr_a\left(|\psi_-\rangle\langle\psi_-|\right)$ |
|---|---|---|---|---|
| $t_1=\frac{n\pi}{|\alpha|\sqrt{1+k^2}}$ | 2 | 4 | 2 | 4 |
| $t_2=\left(\frac{1}{2}+n\right)\frac{\pi}{2|\alpha|k}$ | 3 | 0 | 3 | 0 |

According to table (4), the Bell states in (18a) and (18c) under medium strength interaction produce strong entanglement above the Cirel'son's inequality at the time $t_2$. The Bell states (18b) and (18d) produce strong entanglement above the cirel'son's inequality at the time $t_1$. The trace reduces as the value of k decreases i.e. k<1.

### *4.5. Critical (Threshold) Interactions*

At Critical (threshold) interaction,

$$|\alpha|=\Delta \tag{20}$$

and (7d) becomes (17b) at k=1/2. The reduced density matrices take the same format as those presented in equations (18a-d). The maximum and least traces corresponding to (18a-d) under critical strength interaction are given in table 5.

***Table 5.*** *Maximum and least traces of density matrices for Bell states (18a-d) under critical interaction at time $t_1$ and $t_2$ where n=0, 1, 2…… and $|\alpha|\sqrt{1+k^2}=\omega_p$.*

| **Time** | $Tr_b\left(|\phi_+\rangle\langle\phi_+|\right)$ | $Tr_b\left(|\phi_-\rangle\langle\phi_-|\right)$ | $Tr_a\left(|\psi_+\rangle\langle\psi_+|\right)$ | $Tr_a\left(|\psi_-\rangle\langle\psi_-|\right)$ |
|---|---|---|---|---|
| $t_1=\frac{n\pi}{|\alpha|\sqrt{1+k^2}}$ | 2 | 4 | 2 | 4 |
| $t_2=\left(\frac{1}{2}+n\right)\frac{\pi}{2|\alpha|k}$ | 2.64 | 0.72 | 2.64 | 0.72 |

According to CHSH Bell inequality, the Bell states (18a) and (18c) under critical interaction produce strong entanglement at the time $t_2$. The Bell states (18b) and (18d) produce strong entanglement above the cirel'son's inequality at the time $t_1$.

## 5. Conclusion

The semi-classical OPO is a good system for demonstration of dynamical evolution of entanglement of polarization states by use of Bell states whose entanglement is tested by use of reduced density matrices in CHSH Bell inequality.

The CHSH Bell inequality is violated under weak interaction, medium strength and critical interaction for the Bell state vectors (11a) and (11c) hence producing strong entanglement. The traces of density matrices under very weak interaction for all the four Bell state vectors do not violate the CHSH Bell inequality hence producing weak entanglement which is dynamic in nature.

The CHSH Bell inequality is violated to give a higher trace of S=4 under resonance, weak interaction, medium strength and critical interaction for Bell state vectors (11b) and (11d) hence producing a dynamically stronger entanglement beyond the Cirel'son's inequality of $S< 2\sqrt{2}$ stated for quantum theory. This presents the OPO as an important tool in quantum optics for possible implementation of quantum key communication protocols in quantum mechanics such as quantum teleportation, quantum key distribution, and dense coding.

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