Fundamental quantum noise in a fully quantized degenerate parametric amplification process

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Abstract

An exact analytical solution of the time-dependent Schroedinger equation for a fully quantized degenerate parametric amplification process generated by quantized multi-mode pump photons provides a quantized multi-photon squeeze operator which generates dynamics characterized by interaction frequency mixing specified by quantized anti-normal order Rabi frequency operator $\hat{\mathcal{R}} = g\sqrt{\hat{\mathcal{N}} - k^2}$ and normal order Rabi frequency operator $\hat{R} = g\sqrt{\hat{N} - k^2}$, where g is the pump photon coupling constant, $\hat{\mathcal{N}}$ is the anti-normal order and $\hat{N}$ is the normal order pump photon number operator, while k is the frequency de-tuning parameter. Due to the quantum frequency mixing mechanism, the time evolving annihilation and creation operators of the initially degenerate signal-idler photon modes become effectively non-degenerate. The full quantum interaction thus destroys the initial degeneracy of the signal-idler photon modes. The loss of degeneracy means that all time evolving signal-idler photon operators become complex, thus developing purely quantum Hermitian imaginary parts. Fluctuations of these purely quantum Hermitian operators generate *fundamental quantum noise* effects. General fundamental quantum noise arising from cross-correlations of the Hermitian real and imaginary parts of the complex time evolving operators modify the minimum values of the corresponding Heisenberg uncertainty products. The full quantum squeezing phenomenon signified by the smaller, purely quantum fluctuations, of the imaginary parts, compared to the larger fluctuations of the corresponding Hermitian parts, takes the form of a sum and difference of two time evolving interaction variables and does not display extreme behavior such as exponential growth or exponential decay of one or the other quadrature fluctuation usually obtained in the corresponding semiclassical model.

Keywords: quantized parametric amplification, quantum squeeze operator, quantized Rabi frequency operator, loss of degeneracy, operator state amplitudes, fundamental quantum noise, correlations and uncertainty relations.

1 Introduction

Parametric processes in which signal-idler photon pairs are created through the interaction between light or laser generated pump photons and matter provide the physical foundations for the design and implementation of quantum technologies. Semiclassical models of the parametric processes reveal fundamental quantum mechanical, i.e., nonclassical, properties such as quadrature squeezing and entanglement, which have been identified as the basic resources for developing general quantum communication protocols, quantum computation, quantum information processing and other quantum technologies. In [4-6], the present author has established that in fully quantized parametric oscillation or amplification processes, Jaynes-Cummings or anti-Jaynes-Cummings modes of interaction drive signal-idler photon polarization state dynamics in which the mean total intensities or mean positive-negative helicity (mean horizontal-vertical) intensity inversion and the related mean polarizations are characterized by fundamental quantum phenomena of fractional revivals or general collapses and revivals when the pump photons are in Fock or coherent states. It is important to note that, while squeezing and entanglement emerge as nonclassical properties in both semiclassical and fully quantized parametric processes, the more fundamental features of collapses, revivals and fractional revivals occur only in fully quantized models. The full quantum models thus provide more detailed information about the internal dynamics of parametric oscillation or amplification processes. The exact analytical solutions of the equations of dynamics obtained in [4-6] and in the present paper now open possibilities for more comprehensive theoretical analysis of all the detailed features of the dynamics of fully quantized parametric processes, which can lead to improved precision in the design and implementation of related quantum technologies.

In the present paper, we establish that full quantum interaction creates two different quantized Rabi frequency channels, which destroys the degeneracy of the initial signal-idler photon modes in a degenerate parametric amplification process driven by quantized pump photon modes. The resulting non-degenerate time evolving signal-idler photon annihilation and creation operators develop purely quantum quadrature components, which arise as their Hermitian imaginary parts. In general, the loss of degeneracy due to the full quantum interaction means that all time evolving observable operators such as number operators, polarization operators, etc, now become complex, thus developing purely quantum quadrature components arising as their Hermitian imaginary parts. The fluctuations of these purely quantum quadrature components constitute fundamental quantum noise in the dynamics of a fully quantized degenerate parametric amplification process.

In the general framework of non-linear quantum optics, multi-photon interaction modes emerge as progressive terms in the expansion of the electric polarization component of the displacement vector governing the dynamics of an electromagnetic (radiation) field in a material medium, such as a nonlinear crystal. The widely studied optical harmonic generation and four-wave mixing interactions [1] are respectively the second and third order parametric processes of these multi-photon interactions in non-linear quantum optics. The general process is modeled as the dynamics of a signal and idler photon pair coupled by $\bar{N} \geq 1$ quantized pump photon modes. Hence, we take the model Hamiltonian for a fully quantized degenerate parametric amplification process generated by a multi-photon pump in the multi-linear form [2 , 3]

$$H = \hbar\{\omega \hat{a}^{\dagger}\hat{a} + \sum_{j=1}^{\bar{N}} \omega_j \hat{a}_j^{\dagger}\hat{a}_j + \frac{i}{2} g(\hat{f}\ \hat{a}^{\dagger 2} - \hat{f}^{\dagger}\ \hat{a}^2)\} \tag{1a}$$

where $\hat{f}$ and its Hermitian conjugate $\hat{f}^{\dagger}$, is a multi-photon pump intensity amplitude operator defined by

$$\hat{f} = \prod_{j=1}^{\bar{N}} \hat{a}_j = \hat{a}_1\hat{a}_2....\hat{a}_{\bar{N}} \quad ; \quad \hat{f}^{\dagger} = \prod_{j=1}^{\bar{N}} \hat{a}_j^{\dagger} = \hat{a}_1^{\dagger}\hat{a}_2^{\dagger}....\hat{a}_{\bar{N}}^{\dagger} \tag{1b}$$

Here, $(\hat{a}_j,\ \hat{a}_j^{\dagger})$, $j = 1, 2, ..., \bar{N}$ and $(\hat{a},\ \hat{a}^{\dagger})$ denote the annihilation and creation operators, while ω_j

and ω denote the angular frequencies of the pump and the degenerate signal-idler photon modes, respectively. We take the coupling parameter g to be constant and real for simplicity.

The annihilation and creation operators satisfy the usual commutation relations

$$[\hat{a}_p\ ,\ \hat{a}_q^\dagger] = \delta_{pq} \quad ; \quad [\hat{a}_p\ ,\ \hat{a}_q] = 0 \quad ; \quad [\hat{a}_p^\dagger\ ,\ \hat{a}_q^\dagger] = 0 \tag{1c}$$

where we take p, q to label any photon involved in the process.

The parametric process generated by the multi-linear Hamiltonian in equation $(1a)$ is composed of general radiation-matter interactions proceeding through intermediate excited states in which $\bar{N}$ pump photon modes are annihilated, effectively releasing a degenerate signal-idler photon pair in the final stage. It is therefore effectively a fully quantized $(\bar{N}+2)$-wave mixing process.

In recent work [4-6], the present author has obtained exact analytical solutions of Heisenberg's equations for signal-idler photon polarization state vectors in fully quantized parametric processes generated by nonlinear Hamiltonians of the form in equation $(1a)$. The dynamics is then understood to be due to a Jaynes-Cummings or anti-Jaynes-Cummings mode of interaction governing the time evolution of the coupled signal-idler photon pair polarization states.

In the present paper, we provide exact analytical solutions of the time-dependent Schroedinger equation describing the dynamics of fully quantized parametric processes in the alternative Schroedinger picture. The general time evolving state amplitude $|\psi(t)\rangle$ is governed by the time-dependent Schroedinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H\ |\psi(t)\rangle \tag{2}$$

Exact analytical solutions of this time-dependent Schroedinger equation with a full quantum multi-linear Hamiltonian such as H in equation $(1a)$ have never been found, thus posing a major challenge in nonlinear quantum optics and its practical applications within the Schroedinger picture. We address this important challenge and apply the results to determine some uniquely quantum features of the dynamics in the present paper.

Since exact analytical solutions of the time-dependent Schroedinger equation for a fully quantized parametric amplification process has never been obtained in earlier works, we develop fairly elaborate solution procedure and methods of evaluating expectation values of the resulting time evolving operators through corresponding operator state amplitudes in section 2. In section 3 and section 4, we have evaluated fluctuation and cross-correlation functions, as well as the associated Heisenberg uncertainty relations, of the quadrature (or Hermitian) components of the effectively non-degenerate time evolving signal-idler photon annihilation, creation and number operators, which display the expected squeezing properties and the occurrence of fundamental quantum noise due to the full quantum interaction. Comparison of the full quantum model with the semiclassical model of the degenerate parametric amplification process is presented in section 5.

2 General solution of the Schroedinger equation

We put equation (2) in an exactly solvable form by introducing the degenerate signal-idler photon initial polarization operators $\hat{K}_0$, $\hat{K}_+$ and $\hat{K}_-$ defined by

$$\hat{K}_0 = \frac{1}{2}(\hat{a}^\dagger\hat{a} + \frac{1}{2}) \quad ; \quad \hat{K}_+ = \frac{1}{2}\hat{a}^{\dagger 2} \quad ; \quad \hat{K}_- = \frac{1}{2}\hat{a}^2 \tag{3a}$$

satisfying commutation relations

$$[\hat{K}_+\ ,\ \hat{K}_-] = -2\hat{K}_0 \quad ; \quad [\hat{K}_0\ ,\ \hat{K}_+] = \hat{K}_+ \quad ; \quad [\hat{K}_0\ ,\ \hat{K}_+] = -\hat{K}_- \tag{3b}$$

to express the Hamiltonian H in equation $(1a)$ in the form

$$H = \hbar\{\sum_{j=1}^{N} \omega_j \hat{a}_j^\dagger \hat{a}_j + 2\omega \hat{K}_0 + ig(\hat{f}\hat{K}_+ - \hat{f}^\dagger \hat{K}_-)\} \tag{3c}$$

where we have ignored a constant term $-\frac{1}{2}\hbar\omega$ which yields only a global phase factor.

Adding and subtracting $\sum_{j=1}^{N} \hbar\omega_j \hat{K}_0$, we express H in equation $(3c)$ as

$$H = \hbar \sum_{j=1}^{N} (\omega_j \hat{a}_j^\dagger \hat{a}_j + \omega_j \hat{K}_0) + \hbar\{\delta\ \hat{K}_0 + ig(\hat{f}\ \hat{K}_+ - \hat{f}^\dagger\ \hat{K}_-)\} \tag{3d}$$

where we have introduced the frequency detuning δ defined by

$$\delta = 2\omega - \sum_{j=1}^{N} \omega_j \tag{3e}$$

This relation specifies energy transfer due to the multi-wave mixing interactions characterizing the fully quantized degenerate parametric amplification process generated by multi-mode pump photons.

We write H as a sum of two components in the form

$$H = \overline{H}_0 + \overline{H} \tag{3f}$$

after introducing a free evolution Hamiltonian $\overline{H}_0$ and interaction Hamiltonian $\overline{H}$ defined by

$$\overline{H}_0 = \hbar \sum_{j=1}^{N} (\omega_j \hat{a}_j^\dagger \hat{a}_j + \omega_j \hat{K}_0) \quad ; \quad \overline{H} = \hbar\{\delta\ \hat{K}_0 + ig(\hat{f}\ \hat{K}_+ - \hat{f}^\dagger\ \hat{K}_-)\} \tag{3g}$$

Using the algebraic relations for the pump annihilation and creation operators $\hat{a}_j$, $\hat{a}_j^\dagger$ given in equation $(1c)$, noting the definition of the multi-mode pump operator $\hat{f}$, $\hat{f}^\dagger$ in equation $(1b)$ and applying the algebraic relations for the initial polarization operators $\hat{K}_0$, $\hat{K}_+$, $\hat{K}_-$ given in equation $(3b)$ easily gives

$$[\ \overline{H}_0\ ,\ \overline{H}\ ] = 0 \quad \Rightarrow \quad [\ \overline{H}_0\ ,\ H\ ] = 0 \quad ; \quad [\ \overline{H}\ ,\ H\ ] = 0 \tag{3h}$$

Since both components $\overline{H}_0$ and $\overline{H}$ commute with H, they are constants of the motion. The Hamiltonian $H = \overline{H}_0 + \overline{H}$ is therefore time-independent.

With the Hamiltonian H time-independent, we easily solve the Schroedinger equation (2) through simple integration to obtain

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle \tag{4a}$$

where $|\psi(0)\rangle$ is the initial state amplitude and $U(t)$ is the general time evolution operator obtained as

$$U(t) = e^{-\frac{i}{\hbar}\overline{H}_0 t}\hat{S}(\hat{f}) \tag{4b}$$

after applying the commutation of $\overline{H}_0$ and $\overline{H}$ as in equation $(3h)$ to factorize the free evolution component and the interaction component. The interaction time evolution operator $\hat{S}(\hat{f})$ has been obtained as

$$\hat{S}(\hat{f}) = e^{-\frac{i}{\hbar}\overline{H}t} \tag{4c}$$

which on substituting $\overline{H}$ from equation $(3g)$ takes the form

$$\hat{S}(\hat{f}) = e^{-it\{2\delta\ \hat{K}_0 + ig(\hat{f}\ \hat{K}_+ - \hat{f}^\dagger\ \hat{K}_-)\}} \tag{4d}$$

Using the definitions of the initial polarization operators given in equation (3a) and once again ignoring a constant global phase contributing term, we express the interaction time evolution operator in the final form

$$\hat{S}(\hat{f}) = e^{t\{-i\delta\ \hat{a}^\dagger\hat{a} + \frac{1}{2}g(\hat{f}\hat{a}^{\dagger 2} - \hat{f}^\dagger\hat{a}^2)\}} \tag{4e}$$

which we recognize as the quantized squeeze operator for the fully quantized degenerate parametric amplification process. The time evolution operator $U(t)$ in equation (4b) is therefore a general quantized squeeze operator and the corresponding state amplitude $|\psi(t)\rangle$ obtained in equation (4a) is a general quantized squeezed state amplitude.

2.1 Time evolution in the general squeezed state

The time evolution of a general operator $\hat{Q}$ representing some physical quantity of the parametric amplification process described by the general squeezed state amplitude $|\psi(t)\rangle$ in equation (4a) is obtained using the time evolution operator $U(t)$ according to

$$\hat{Q}(t) = U^\dagger(t)\hat{Q}U(t) \tag{5a}$$

In particular, for the annihilation and creation operators $\hat{a}$, $\hat{a}^\dagger$ of the degenerate signal-idler photon, we set $\hat{Q} = \hat{a}$, $\hat{a}^\dagger$ in equation (5a) and use $U(t)$ from equation (4b) with $\overline{H}_0$ from equation (3g) (set $\hat{K}_0 = \hat{a}^\dagger\hat{a}$) to obtain the form

$$\hat{b}_+(t) = e^{-\frac{i}{2}\Omega t}\hat{S}^\dagger(\hat{f})\hat{a}\hat{S}(\hat{f}) \quad ; \quad \hat{b}_-(t) = e^{\frac{i}{2}\Omega t}\hat{S}^\dagger(\hat{f})\hat{a}^\dagger\hat{S}(\hat{f}) \tag{5b}$$

where we have introduced the total pump photon frequency Ω obtained as

$$\Omega = \sum_{j=1}^{\bar{N}} \omega_j \tag{5c}$$

The notation $\hat{b}_+(t)$ and $\hat{b}_-(t)$ in equation (5b) represents time evolution over two different frequency channels, which we clarify below. The main task is to evaluate $\hat{S}^\dagger(\hat{f})\hat{a}\hat{S}(\hat{f})$ and $\hat{S}^\dagger(\hat{f})\hat{a}^\dagger\hat{S}(\hat{f})$ which provide time evolution with respect to the squeeze operator within the Schroedinger picture. We develop the procedure below.

Let us introduce an operator $\hat{A}$ defined by

$$\hat{A} = -\{-i\delta\ \hat{a}^\dagger\hat{a} + \frac{1}{2}g(\hat{f}\hat{a}^{\dagger 2} - \hat{f}^\dagger\hat{a}^2)\} = i\delta\ \hat{a}^\dagger\hat{a} + \frac{1}{2}g(\hat{f}^\dagger\hat{a}^2 - \hat{f}\hat{a}^{\dagger 2}) \tag{6a}$$

to express the squeeze operator and its Hermitian conjugate in the form

$$\hat{S}(\hat{f}) = e^{-t\hat{A}} \quad ; \quad \hat{S}^\dagger(\hat{f}) = e^{t\hat{A}} \tag{6b}$$

The time evolution of the annihilation and creation operators in equation (5b) is then obtained in explicit form through

$$\hat{S}^\dagger(\hat{f})\hat{a}\hat{S}(\hat{f}) = e^{t\hat{A}}\hat{a}e^{-t\hat{A}} \quad ; \quad \hat{S}^\dagger(\hat{f})\hat{a}^\dagger\hat{S}(\hat{f}) = e^{t\hat{A}}\hat{a}^\dagger e^{-t\hat{A}} \tag{6c}$$

These are evaluated through application of the standard operator expansion theorem [1], which in the present case $\hat{Q} = \hat{a}$, $\hat{a}^\dagger$, takes the general form

$$e^{t\hat{A}}\hat{Q}e^{-t\hat{A}} = \hat{Q} + \frac{t}{1!}[\hat{A}\ ,\ \hat{Q}] + \frac{t^2}{2!}[\hat{A}\ ,\ [\hat{A}\ ,\ \hat{Q}]] + \frac{t^3}{3!}[\hat{A}\ ,[\hat{A}\ ,\ [\hat{A}\ ,\ \hat{Q}]]] + ... \tag{7a}$$

We express this as power series expansion in terms of the commutators in the form

$$e^{t\hat{A}}\hat{Q}e^{-t\hat{A}} = \sum_{n=0}^{\infty} \frac{t^n(\hat{A}^n)_{\hat{Q}}}{n!} \tag{7b}$$

where the power series terms are evaluated according to the definition

$$(\hat{A}^0)_{\hat{Q}} = \hat{Q} \quad ; \quad (\hat{A}^1)_{\hat{Q}} = [\hat{A} \ , \ (\hat{A}^0)_{\hat{Q}}] = [\hat{A} \ , \ \hat{Q}] \quad ; \quad (\hat{A}^n)_{\hat{Q}} = [\hat{A} \ , \ (\hat{A}^{n-1})_{\hat{Q}}] \quad , \quad n \geq 1 \tag{7c}$$

It is convenient to express the expansion in equation ($7b$) in even and odd power terms in the form

$$e^{t\hat{A}}\hat{Q}e^{-t\hat{A}} = \sum_{l=0}^{\infty} \frac{t^{2l}(\hat{A}^{2l})_{\hat{Q}}}{(2l)!} + \sum_{l=0}^{\infty} \frac{t^{2l+1}(\hat{A}^{2l+1})_{\hat{Q}}}{(2l+1)!} \tag{7d}$$

The problem then reduces to evaluation of commutation brackets according to equation ($7c$). In the evaluation of the commutation brackets for the time evolution of the degenerate signal-idler photon annihilation and creation operators ($\hat{Q} = \hat{a}$, $\hat{a}^\dagger$) using equation ($7c$), we must take account of the pump photon operators $\hat{f}$, $\hat{f}^\dagger$ in the definition of $\hat{A}$. In particular, these pump photon operators become important in the evaluation of the second-order commutation brackets $[\hat{A} \ , \ [\hat{A} \ , \ \hat{a}]]$ and $[\hat{A} \ , \ [\hat{A} \ , \ \hat{a}^\dagger]]$, which are also involved in the higher-order commutation brackets. Including the commutation brackets $[\hat{A} \ , \ \hat{f}]$, $[\hat{A} \ , \ \hat{f}^\dagger]$ with respect to the pump photon operators contributes seemingly superfluous extra nonlinear terms, which are inconsistent with the exact time evolution obtained within the Heisenberg picture in [4-6].

To obtain results in exact agreement with the time evolution in the Heisenberg picture, we introduce an effective procedure, which we call *partial evaluation of commutation bracket* with respect to signal-idler photon operators $\hat{a}$, $\hat{a}^\dagger$, in which we treat the driving pump photon operators $\hat{f}$, $\hat{f}^\dagger$ as if they were c-numbers, but maintaining their order in the products during the evaluation of $[\hat{A} \ , \ [\hat{A} \ , \ \hat{a}]]$, $[\hat{A} \ , \ [\hat{A} \ , \ \hat{a}^\dagger]]$. This is understood to be equivalent to averaging (tracing) out the pump photon operators with respect to the initial pump photon state amplitude, i.e., we consider $\langle\psi_f|[\hat{A} \ , \ [\hat{A} \ , \ \hat{a}]]|\psi_f\rangle$, $\langle\psi_f|[\hat{A} \ , \ [\hat{A} \ , \ \hat{a}^\dagger]]|\psi_f\rangle$, where $|\psi_f\rangle$ is an arbitrary multi-pump photon state vector to be dropped at the end of the evaluation. The final results obtained through the partial evaluation of the commutation brackets as appropriate are then exactly the same as the results obtained as exact solutions of Heisenberg time evolution equations for signal-idler photon annihilation and creation operators in fully quantized parametric processes in [4-6].

Setting $\hat{Q} = \hat{a}$, $\hat{a}^\dagger$ in equation ($7c$) and carrying out a number of steps of partial evaluation of the commutation brackets $[\hat{A} \ , \ [\hat{A} \ , \ \hat{a}]]$ and $[\hat{A} \ , \ [\hat{A} \ , \ \hat{a}^\dagger]]$ occurring in second and higher-order commutation brackets with respect to $\hat{a}$, $\hat{a}^\dagger$, holding $\hat{f}$, $\hat{f}^\dagger$ as c-numbers, but maintaining their order in the products at each stage, we obtain general results for even and odd power terms in the form

$$\hat{Q} = \hat{a} \ : \qquad (\hat{A}^0)_{\hat{a}} = \hat{a} \quad ; \quad (\hat{A}^{2l})_{\hat{a}} = \left(\sqrt{\frac{\partial}{\partial \hat{a}}[\frac{\partial \hat{A}}{\partial \hat{a}^\dagger} \ , \ \hat{A}]}\right)^{2l} (\hat{A}^0)_{\hat{a}} \quad ; \quad l = 0, 1, 2, ... \tag{8a}$$

$$(\hat{A}^{2l+1})_{\hat{a}} = -\left(\sqrt{\frac{\partial}{\partial \hat{a}}[\frac{\partial \hat{A}}{\partial \hat{a}^\dagger} \ , \ \hat{A}]}\right)^{2l} \frac{\partial \hat{A}}{\partial \hat{a}^\dagger} \tag{8b}$$

$$\hat{Q} = \hat{a}^\dagger \ : \qquad (\hat{A}^0)_{\hat{a}^\dagger} = \hat{a}^\dagger \quad ; \quad (\hat{A}^{2l})_{\hat{a}^\dagger} = \left(\sqrt{\frac{\partial}{\partial \hat{a}^\dagger}[\hat{A} \ , \ \frac{\partial \hat{A}}{\partial \hat{a}}]}\right)^{2l} (\hat{A}^0)_{\hat{a}^\dagger} \quad ; \quad l = 0, 1, 2, ... \tag{8c}$$

$$(\hat{A}^{2l+1})_{\hat{a}^\dagger} = \left(\sqrt{\frac{\partial}{\partial\hat{a}^\dagger}[\hat{A}\ ,\ \frac{\partial\hat{A}}{\partial\hat{a}}]}\right)^{2l}\frac{\partial\hat{A}}{\partial\hat{a}} \tag{8d}$$

where we have used standard procedure to obtain

$$[\hat{a}\ ,\ \hat{A}] = \frac{\partial\hat{A}}{\partial\hat{a}^\dagger} \quad ; \quad [\hat{a}^\dagger\ ,\ \hat{A}] = -\frac{\partial\hat{A}}{\partial\hat{a}} \tag{8e}$$

We use $\hat{A}$ from equation (6a) to obtain

$$\frac{\partial\hat{A}}{\partial\hat{a}} = i\delta\hat{a}^\dagger + g\hat{f}^\dagger\hat{a} \quad ; \quad \frac{\partial\hat{A}}{\partial\hat{a}^\dagger} = i\delta\hat{a} - g\hat{f}\hat{a}^\dagger \tag{8f}$$

$$[\frac{\partial\hat{A}}{\partial\hat{a}^\dagger}\ ,\ \hat{A}] = (g^2\hat{f}\hat{f}^\dagger - \delta^2)\hat{a} \quad ; \quad \frac{\partial}{\partial\hat{a}}[\frac{\partial\hat{A}}{\partial\hat{a}^\dagger}\ ,\ \hat{A}] = g^2\hat{f}\hat{f}^\dagger - \delta^2 \tag{8g}$$

$$[\hat{A}\ ,\ \frac{\partial\hat{A}}{\partial\hat{a}}] = (g^2\hat{f}^\dagger f - \delta^2)\hat{a}^\dagger \quad ; \quad \frac{\partial}{\partial\hat{a}^\dagger}[\hat{A}\ ,\ \frac{\partial\hat{A}}{\partial\hat{a}}] = g^2\hat{f}^\dagger\hat{f} - \delta^2 \tag{8h}$$

where we have applied partial evaluation of commutation brackets in equations (8g)-(8h).

Substituting equations (8f)-(8g) into equations (8a)-(8b) gives

$$(\hat{A}^{2l})_{\hat{a}} = \left(\sqrt{g^2\hat{f}\hat{f}^\dagger - \delta^2}\right)^{2l}\hat{a} \quad ; \quad (\hat{A}^{2l+1})_{\hat{a}} = \left(\sqrt{g^2\hat{f}\hat{f}^\dagger - \delta^2}\right)^{2l}(-i\delta\hat{a} + g\hat{f}\hat{a}^\dagger) \tag{9a}$$

while substituting equations (8f) and (8h) into equations (8c)-(8d) gives

$$(\hat{A}^{2l})_{\hat{a}^\dagger} = \left(\sqrt{g^2\hat{f}^\dagger\hat{f} - \delta^2}\right)^{2l}\hat{a}^\dagger \quad ; \quad (\hat{A}^{2l+1})_{\hat{a}^\dagger} = \left(\sqrt{g^2\hat{f}^\dagger\hat{f} - \delta^2}\right)^{2l}(i\delta\hat{a}^\dagger + g\hat{f}^\dagger\hat{a}) \tag{9b}$$

Substituting equation (6c) into equation (7d) for $\hat{Q} = \hat{a}\ ,\ \hat{a}^\dagger$ and using equations (9a)-(9b) as appropriate gives the time evolution of $\hat{a}$ and $\hat{a}^\dagger$ in the squeezed state in the form

$$\hat{S}^\dagger\hat{a}\hat{S} = \sum_{l=0}^{\infty}\frac{\left(gt\sqrt{\hat{f}\hat{f}^\dagger - k^2}\right)^{2l}}{(2l)!}\hat{a} + \frac{1}{g\sqrt{\hat{f}\hat{f}^\dagger - k^2}}\sum_{l=0}^{\infty}\frac{\left(gt\sqrt{\hat{f}\hat{f}^\dagger - k^2}\right)^{2l+1}}{(2l+1)!}(-i\delta\hat{a} + g\hat{f}\hat{a}^\dagger) \tag{10a}$$

$$\hat{S}^\dagger\hat{a}^\dagger\hat{S} = \sum_{l=0}^{\infty}\frac{\left(gt\sqrt{\hat{f}^\dagger\hat{f} - k^2}\right)^{2l}}{(2l)!}\hat{a}^\dagger + \frac{1}{g\sqrt{\hat{f}^\dagger\hat{f} - k^2}}\sum_{l=0}^{\infty}\frac{\left(gt\sqrt{\hat{f}^\dagger\hat{f} - k^2}\right)^{2l+1}}{(2l+1)!}(i\delta\hat{a}^\dagger + g\hat{f}^\dagger\hat{a}) \tag{10b}$$

where we introduced a detuning interaction parameter k defined by

$$k^2 = \frac{\delta^2}{g^2} \tag{10c}$$

Notice that we have introduced a factor $g\sqrt{\hat{f}\hat{f}^\dagger - k^2}$ or $g\sqrt{\hat{f}^\dagger\hat{f} - k^2}$ in the numerator and denominator to complete the odd power terms as appropriate in equations (10a)-(10b).

We now introduce hyperbolic functions in equations (10a)-(10b) according to the expansions

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \quad ; \quad \sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \tag{10d}$$

together with the antinormal-order and normal-order multi-photon pump number operators $\hat{\mathcal{N}}$ and $\hat{N}$, respectively defined by

$$\hat{\mathcal{N}} = \hat{f}\hat{f}^{\dagger} = \prod_{j=1}^{N}(\hat{n}_j + 1) \quad ; \quad \hat{N} = \hat{f}^{\dagger}\hat{f} = \prod_{j=1}^{N} \hat{n}_j \quad ; \quad \hat{n}_j = \hat{a}_j^{\dagger}\hat{a}_j \tag{10e}$$

to obtain

$$\hat{S}^{\dagger}\hat{a}\hat{S} = \{\cosh\left(\hat{\mathcal{R}}t\right) - i\frac{\delta}{\hat{\mathcal{R}}}\sinh\left(\hat{\mathcal{R}}t\right)\}\ \hat{a} + \frac{g}{\hat{\mathcal{R}}}\sinh\left(\hat{\mathcal{R}}t\right)\ \hat{f}\hat{a}^{\dagger} \tag{10f}$$

$$\hat{S}^{\dagger}\hat{a}^{\dagger}\hat{S} = \{\cosh\left(\hat{R}t\right) + i\frac{\delta}{\hat{R}}\sinh\left(\hat{R}t\right)\}\ \hat{a}^{\dagger} + \frac{g}{\hat{R}}\sinh\left(\hat{R}t\right)\ \hat{f}^{\dagger}\hat{a} \tag{10g}$$

where we have introduced quantized antinormal-order and normal-order Rabi frequency operators $\hat{\mathcal{R}}$ and $\hat{R}$, respectively, defined by

$$\hat{\mathcal{R}} = g\sqrt{\hat{\mathcal{N}} - k^2} \qquad ; \qquad \hat{R} = g\sqrt{\hat{N} - k^2} \tag{10h}$$

Introducing general time evolving pump photon interaction operators $\hat{\mu}_+(t)$, $\hat{\mu}_-(t)$ and $\hat{\nu}_+(t)$, $\hat{\nu}_-(t)$ defined by

$$\hat{\mu}_+(t) = \cosh\left(\hat{\mathcal{R}}t\right) - i\frac{\delta}{\hat{\mathcal{R}}}\sinh\left(\hat{\mathcal{R}}t\right) \qquad ; \qquad \hat{\mu}_-(t) = \cosh\left(\hat{R}t\right) + i\frac{\delta}{\hat{R}}\sinh\left(\hat{R}t\right) \tag{11a}$$

$$\hat{\nu}_+(t) = \frac{g}{\hat{R}}\sinh\left(\hat{R}t\right)\hat{f}^{\dagger} \qquad ; \qquad \hat{\nu}_-(t) = \frac{g}{\hat{\mathcal{R}}}\sinh\left(\hat{\mathcal{R}}t\right)\hat{f} \tag{11b}$$

and denoting the squeezed state time evolving annihilation and creation operators by $\hat{a}_+(t)$ and $\hat{a}_-(t)$, respectively according to

$$\hat{a}_+(t) = \hat{S}^{\dagger}\hat{a}\hat{S} \quad ; \quad \hat{a}_-(t) = \hat{S}^{\dagger}\hat{a}^{\dagger}\hat{S} \tag{11c}$$

we express the results in equations (10f)-(10g) in the final form

$$\hat{a}_+(t) = \hat{\mu}_+(t)\hat{a} + \hat{\nu}_-(t)\hat{a}^{\dagger} \quad ; \quad \hat{a}_-(t) = \hat{\mu}_-(t)\hat{a}^{\dagger} + \hat{\nu}_+(t)\hat{a} \tag{11d}$$

The general solutions in equation (5b) then become

$$\hat{b}_+(t) = e^{-\frac{i}{2}\Omega t}\hat{a}_+(t) \quad ; \quad \hat{b}_-(t) = e^{\frac{i}{2}\Omega t}\hat{a}_-(t) \tag{11e}$$

The results in equations (11a)-(11e) are exactly the same as the results obtained within Heisenberg picture in [4-6], only with an appropriate interchange of notation $\hat{\nu}_+(t) \leftrightarrow \hat{\nu}_-(t)$ to match their definitions as effective pump photon creation and annihilation operators according to equation (11b), noting that $\hat{\mathcal{R}}, \hat{R}$ are defined in terms of pump photon number operators according to equation (10h). The remarkable agreement of solutions (11d)-(11e) in the Schroedinger picture with solutions obtained in the Heisenberg picture in [4-6] is due to the partial evaluation of commutation brackets applied in the present paper.

We observe that according to the definitions of the time evolving pump photon interaction operators $\hat{\mu}_{\pm}(t)$, $\hat{\nu}_{\pm}(t)$ in equations (11a)-(11b), the time evolving annihilation and creation operators $\hat{b}_{+}(t)$ and $\hat{b}_{-}(t)$ obtained in equations (11d)-(11e) are separately specified by the quantized antinormal and normal order Rabi frequencies $\hat{\mathcal{R}}$, $\hat{R}$, respectively. This fact, coupled with the fact that the operators $\hat{\mu}_{\pm}(t)$, $\hat{\nu}_{\pm}(t)$ are not Hermitian conjugates of one another, leads to the important conclusion that the time evolving annihilation and creation operators $\hat{b}_{+}(t)$ and $\hat{b}_{-}(t)$ are not Hermitian conjugates and are effectively non-degenerate. The full quantum interaction has thus *destroyed the initial degeneracy* of the signal-idler photon annihilation and creation operators.

Under the full quantum interaction, the non-degenerate time evolving annihilation and creation operators $\hat{b}_{+}(t)$, $\hat{b}_{-}(t)$ develop *purely quantum quadrature components* in their imaginary parts. Denoting the quadrature components by $(\hat{x}_{+}(t), \hat{y}_{+}(t))$ and $(\hat{x}_{-}(t), \hat{y}_{-}(t))$ for $\hat{b}_{+}(t)$, $\hat{b}_{-}(t)$, respectively, we define

$$\hat{b}_{+}(t) = \hat{x}_{+}(t) + i\ \hat{y}_{+}(t) \quad ; \quad \hat{x}_{+}(t) = \frac{1}{2}(\hat{b}_{+}(t) + \hat{b}_{+}^{\dagger}(t)) \quad ; \quad \hat{y}_{+}(t) = -\frac{i}{2}(\hat{b}_{+}(t) - \hat{b}_{+}^{\dagger}(t)) \quad (11f)$$

$$\hat{b}_{+}(t) = \hat{x}_{-}(t) + i\ \hat{y}_{-}(t) \quad ; \quad \hat{x}_{-}(t) = \frac{1}{2}(\hat{b}_{-}(t) + \hat{b}_{-}^{\dagger}(t)) \quad ; \quad \hat{y}_{-}(t) = -\frac{i}{2}(\hat{b}_{-}(t) - \hat{b}_{-}^{\dagger}(t)) \quad (11g)$$

where the Hermitian conjugates $\hat{b}_{+}^{\dagger}(t)$, $\hat{b}_{-}^{\dagger}(t)$ follow easily from equations (11d)-(11e) as

$$\hat{b}_{+}^{\dagger}(t) = e^{\frac{i}{2}\Omega t}\hat{a}_{+}^{\dagger}(t) \quad ; \quad \hat{a}_{+}^{\dagger}(t) = \hat{\mu}_{+}^{\dagger}(t)\hat{a}^{\dagger} + \hat{\nu}_{-}^{\dagger}(t)\hat{a} \quad (11h)$$

$$\hat{b}_{-}^{\dagger}(t) = e^{-\frac{i}{2}\Omega t}\hat{a}_{-}^{\dagger}(t) \quad ; \quad \hat{a}_{-}^{\dagger}(t) = \hat{\mu}_{-}^{\dagger}(t)\hat{a} + \hat{\nu}_{+}^{\dagger}(t)\hat{a}^{\dagger} \quad (11i)$$

It will become clear in the course of our study that the imaginary parts $\hat{y}_{+}(t)$ and $\hat{y}_{-}(t)$ are purely quantum quadrature components.

2.2 Interaction-induced pump photon operators: an interpretation

To facilitate the evaluation of mean values, correlation functions and fluctuations of the time evolving operators which characterize the dynamics of the signal-idler photon pair in the fully quantized parametric amplification process, we provide an appropriate physical interpretation of the pump photon interaction operators $\hat{\mu}_{\pm}(t)$, $\hat{\nu}_{\pm}(t)$ defined in equations (11a)-(11b) and specify their action on a given initial pump photon state amplitude.

Let us consider the N pump photon modes to be in an initial Fock state described by state amplitude $|N\rangle$ defined as usual in the product form

$$|N\rangle = \prod_{j=1}^{N} |n_j\rangle \quad ; \quad n_j = 0, 1, 2, ... \quad (12a)$$

We interpret the operator $\hat{f}$ defined in equation (1b) as the non-degenerate N-photon pump annihilation operator, while the Hermitian conjugate operator $\hat{f}^{\dagger}$ is the non-degenerate N-photon pump creation operator. The antinormal and normal order pump photon number operators $\hat{\mathcal{N}}$, $\hat{N}$ have been defined in equation (10e). These operators act on the initial pump photon Fock state amplitude $|N\rangle$ in the usual manner, giving

$$\hat{f}|N\rangle = \sqrt{N}\ |\prod_{j=1}^{\bar{N}}(n_j - 1)\rangle \quad ; \quad \hat{f}^{\dagger}|N\rangle = \sqrt{\mathcal{N}}\ |\prod_{j=1}^{\bar{N}}(n_j + 1)\rangle \quad (12b)$$

$$\hat{N}|N\rangle = N|N\rangle \qquad ; \qquad \hat{\mathcal{N}}|N\rangle = \mathcal{N}|N\rangle \tag{12c}$$

where $\mathcal{N}$ is the anti-normal order pump photon number and N is the normal-order pump photon number obtained in the form

$$\mathcal{N} = \prod_{j=1}^{\bar{N}}(n_j + 1) \quad ; \quad N = \prod_{j=1}^{\bar{N}} n_j \tag{12d}$$

We observe that the parametric interaction mechanism effectively transforms the pump photon annihilation, creation and number operators to the corresponding time evolving interaction operators $\hat{\mu}_+(t)$, $\hat{\nu}_+(t)$, $\hat{\mu}_-(t)$, $\hat{\nu}_-(t)$ defined in equations (11a)-(11b), which we interpret as the *interaction-induced pump photon* number, creation and annihilation operators as appropriate.

According to the set of equations (10h)-(11b), the time evolving operator $\hat{\mu}_+(t)$ is essentially defined in terms of the anti-normal order pump photon number operator $\hat{\mathcal{N}}$, which determines $\hat{\mathcal{R}}$, while the time evolving operator $\hat{\mu}_-(t)$ is essentially defined in terms of the normal order pump photon number operator $\hat{N}$, which determines $\hat{R}$. We therefore interpret $\hat{\mu}_+(t)$ defined in equation (11a) as the *interaction-induced anti-normal order pump photon number operator* and $\hat{\mu}_-(t)$ defined in equation (11b) as the *interaction-induced normal order pump photon number operator*.

The Hermitian conjugates of the interaction-induced pump photon number operators are obtained as

$$\hat{\mu}_+^\dagger(t) = \cosh\left(\hat{\mathcal{R}}t\right) + i\frac{\delta}{\hat{\mathcal{R}}}\sinh\left(\hat{\mathcal{R}}t\right) \qquad ; \qquad \hat{\mu}_-^\dagger(t) = \cosh\left(\hat{R}t\right) - i\frac{\delta}{\hat{R}}\sinh\left(\hat{R}t\right) \tag{13a}$$

Using equations (12b)-(12c) and recalling the hyperbolic function expansions in equation (10d), noting the definitions of $\hat{R}_\pm$ given in equation (10h), we determine the action of the interaction-induced pump photon number operators and their Hermitian conjugates on the initial pump photon Fock state amplitude $|N\rangle$ in the final form

$$\hat{\mu}_+(t)|N\rangle = \mu_+(t)|N\rangle \qquad ; \qquad \hat{\mu}_+^\dagger(t)|N\rangle = \mu_+^*(t)|N\rangle \tag{13b}$$

$$\hat{\mu}_-(t)|N\rangle = \mu_-(t)|N\rangle \qquad ; \qquad \hat{\mu}_-^\dagger(t)|N\rangle = \mu_-^*(t)|N\rangle \tag{13c}$$

where the interaction-induced time evolving pump photon number eigenvalues $\mu_+(t)$ and $\mu_-(t)$ have been obtained in the form

$$\mu_+(t) = \cosh(\mathcal{R}t) - i\frac{\delta}{\mathcal{R}}\sinh(\mathcal{R}t) \qquad ; \qquad \mu_-(t) = \cosh(Rt) + i\frac{\delta}{R}\sinh(Rt) \tag{13d}$$

We have introduced quantized Rabi frequencies $\mathcal{R}$, R obtained as

$$\mathcal{R} = g\sqrt{\mathcal{N} - k^2} \qquad ; \qquad R = g\sqrt{N - k^2} \tag{13e}$$

Noting that, in equation (11b), the definition of the time evolving pump photon interaction operator $\hat{\nu}_-(t)$ involves an effective annihilation operator $\hat{f}$ to the right, while the definition of the time evolving pump photon interaction operator $\hat{\nu}_+(t)$ in equation (11b) involves an effective creation operator $\hat{f}^\dagger$ to the right, we interpret $\hat{\nu}_-(t)$ as the *interaction-induced pump photon annihilation operator* specified by $\hat{\mathcal{R}}$ and $\hat{\nu}_+(t)$ as the *interaction-induced pump photon creation operator* in the *normal order frequency channel* specified by $\hat{R}$. The Hermitian conjugates of the interaction-induced pump photon annihilation and creation operators follow from equations (11a) and (11b), respectively, as

$$\hat{\nu}_+^\dagger(t) = \hat{f}^\dagger\frac{g}{\hat{\mathcal{R}}}\sinh\left(\hat{\mathcal{R}}t\right) \qquad ; \qquad \hat{\nu}_-^\dagger(t) = \hat{f}\frac{g}{\hat{R}}\sinh\left(\hat{R}t\right) \tag{14a}$$

Using the definitions given in equations (11a), (11b), (13a) and the hyperbolic function expansions, we determine the action of the interaction-induced pump photon annihilation and creation operators, together with their Hermitian conjugates, on the initial pump photon Fock state amplitude $|N\rangle$ in the final form

$$\hat{\nu}_+(t)|N\rangle = \nu_+(t)|\prod_{j=1}^{\bar{N}}(n_j-1)\rangle \quad ; \quad \hat{\nu}_+^\dagger(t)|N\rangle = \nu_-(t)|\prod_{j=1}^{\bar{N}}(n_j+1)\rangle \tag{14b}$$

$$\hat{\nu}_-(t)|N\rangle = \nu_-(t)|\prod_{j=1}^{\bar{N}}(n_j+1)\rangle \quad ; \quad \hat{\nu}_-^\dagger(t)|N\rangle = \nu_+(t)|\prod_{j=1}^{\bar{N}}(n_j-1)\rangle \tag{14c}$$

where the c-numbers $\nu_+(t)$ and $\nu_-(t)$ have been obtained in the form

$$\nu_-(t) = \frac{g\sqrt{N}}{R}\sinh(Rt) \quad ; \quad \nu_+(t) = \frac{g\sqrt{N}}{\mathcal{R}}\sinh(\mathcal{R}t) \tag{14d}$$

2.3 Time evolving operator state amplitudes

The calculation of the mean value and correlation functions of a general operator $\hat{Q}$ in the general time evolving state amplitude $|\psi(t)\rangle = U(t)|\psi(0)\rangle$ according to

$$\overline{Q}(t) = \langle\psi(t)|\hat{Q}|\psi(t)\rangle \;=\; \langle\psi(0)|\hat{Q}(t)|\psi(0)\rangle \qquad ; \qquad \hat{Q}(t) = U^\dagger(t)\hat{Q}U(t) \tag{15a}$$

which usually requires expansion of the general time evolving operator $\hat{Q}(t) = U^\dagger(t)\hat{Q}U(t)$ before evaluating in the initial state amplitude $|\psi(0)\rangle$ can be simplified by introducing a time evolving operator state amplitude $|\hat{Q}(t)\rangle$ and its dual $|\hat{Q}^\dagger(t)\rangle$ generated through the action of the time evolving operator $\hat{Q}(t)$ on the initial state amplitude $|\psi(0)\rangle$ in the form

$$|\hat{Q}(t)\rangle = \hat{Q}(t)|\psi(0)\rangle \qquad ; \qquad |\hat{Q}(t)\rangle = \hat{Q}(t)|\psi(0)\rangle \tag{15b}$$

where $\hat{Q}^\dagger(t)$ is the Hermitian conjugate of $\hat{Q}(t)$. Taking Hermitian conjugation of the operator state amplitudes gives

$$\langle\hat{Q}(t)| = \langle\psi(0)|\hat{Q}^\dagger(t) \qquad ; \qquad \langle\hat{Q}^\dagger(t)| = \langle\psi(0)|\hat{Q}(t) \tag{15c}$$

The mean values of $\hat{Q}(t)$ and $\hat{Q}^\dagger(t)$ are obtained as the overlap of the initial state amplitude $|\psi(0)\rangle$ and the time evolving operator state amplitudes $|\hat{Q}(t)\rangle$, $|\hat{Q}^\dagger(t)\rangle$ from equation (15b) in the form

$$\overline{Q}(t) = \langle\psi(0)|\hat{Q}(t)\rangle \;=\; \langle\psi(0)|\hat{Q}(t)|\psi(0)\rangle \qquad ; \qquad \overline{Q}^*(t) = \langle\psi(0)|\hat{Q}^\dagger(t)\rangle \;=\; \langle\psi(0)|\hat{Q}^\dagger(t)|\psi(0)\rangle \tag{15d}$$

while the second moments or correlation functions of the operators $\hat{Q}(t)$, $\hat{Q}^\dagger(t)$ are obtained as appropriate overlaps of the operator state amplitudes $|\hat{Q}(t)\rangle$, $|\hat{Q}^\dagger(t)\rangle$ using equations (15b)-(15c) according to

$$\langle\hat{Q}^\dagger(t)|\hat{Q}(t)\rangle = \langle\psi(0)|\hat{Q}^2(t)|\psi(0)\rangle \;=\; \overline{Q^2(t)} \quad ; \quad \langle\hat{Q}(t)|\hat{Q}^\dagger(t)\rangle = \langle\psi(0)|\hat{Q}^{\dagger 2}(t)|\psi(0)\rangle \;=\; \overline{Q^{*2}(t)} \tag{15e}$$

$$\langle\hat{Q}^\dagger(t)|\hat{Q}^\dagger(t)\rangle \;=\; \langle\psi(0)|\hat{Q}(t)\hat{Q}^\dagger(t)|\psi(0)\rangle \;=\; \overline{Q(t)Q^*(t)} \tag{15f}$$

$$\langle\hat{Q}(t)|\hat{Q}(t)\rangle \;=\; \langle\psi(0)|\hat{Q}^\dagger(t)\hat{Q}(t)|\psi(0)\rangle \;=\; \overline{Q^*(t)Q(t)} \tag{15g}$$

These relations apply to any set of operators $\hat{Q}_j(t)$, $j = 1, 2, ...$

The concept of operator state amplitudes which we have introduced here for general calculations forms part of a growing interest in quantum state engineering and orthogolization schemes [2 , 7-11] for application in quantum computation, quantum information processing and related quantum technologies. In the present work, the appropriate operators are generated through the general time evolution operator $U(t)$ of the fully quantized degenerate parametric amplification process, while the Fock state amplitude $|Nn\rangle$ has been chosen as the initial state.

We proceed to apply the operator state amplitude methods and general results obtained in this section to calculate the fluctuations and correlation functions of time evolving quadrature components and number operator to determine the squeezing properties and fundamental quantum noise in the general dynamics of a fully quantized degenerate parametric amplification process based on the Hamiltonian in equation $(1a)$.

3 Time evolving quadrature operators

Quadrature components of annihilation and creation operators have been identified as useful observable operators for theoretical studies and practical applications of the statistical properties of parametric oscillation and amplification processes. To determine some important fundamental features of a degenerate parametric amplification process driven by fully quantized pump photon modes, we consider the time evolution of the initial signal-idler photon quadrature operators $\hat{a}_1$, $\hat{a}_2$ defined as usual by

$$\hat{a}_1 = \frac{1}{2}(\hat{a} + \hat{a}^\dagger) \qquad ; \qquad \hat{a}_2 = -\frac{i}{2}(\hat{a} - \hat{a}^\dagger) \tag{16a}$$

These initial quadrature operators evolve in time under the full quantum squeeze operator $U(t)$ according to

$$\hat{a}_1(t) = U^\dagger(t)\hat{a}_1 U(t) \qquad ; \qquad \hat{a}_2(t) = U^\dagger(t)\hat{a}_2 U(t) \tag{16b}$$

which on introducing the time evolving annihilation and creation operators $\hat{b}_+(t)$, $\hat{b}_-(t)$ take the form

$$\hat{a}_1(t) = \frac{1}{2}(\hat{b}_+(t) + \hat{b}_-(t)) \qquad ; \qquad \hat{a}_2(t) = -\frac{i}{2}(\hat{b}_+(t) - \hat{b}_-(t)) \tag{16c}$$

Since $\hat{b}_+(t)$ and $\hat{b}_-(t)$ are not Hermitian conjugates of each other, the time evolving quadrature operators $\hat{a}_1(t)$, $\hat{a}_2(t)$ obtained in equation $(16c)$ are complex and have Hermitian conjugates

$$\hat{a}_1^\dagger(t) = \frac{1}{2}(\hat{b}_+^\dagger(t) + \hat{b}_-^\dagger(t)) \qquad ; \qquad \hat{a}_2^\dagger(t) = \frac{i}{2}(\hat{b}_+^\dagger(t) - \hat{b}_-^\dagger(t)) \tag{16d}$$

By expanding the commutation bracket of the operators $\hat{b}_+(t)$ and $\hat{b}_-(t)$ using equations $(11d)$-$(11g)$, then averaging out the quantized pump photon operators in the pump photon Fock state $|N\rangle$, we obtain the commutation relation

$$\langle N|\ [\hat{b}_+(t)\ ,\ \hat{b}_-(t)]\ |N\rangle = \{\mu_+(t)\mu_-(t) - \nu_+^2(t)\}\hat{a}\hat{a}^\dagger - \{\mu_+(t)\mu_-(t) - \nu_-^2(t)\}\hat{a}^\dagger\hat{a} \tag{16e}$$

which can be used to determine the commutation relation for the time evolving quadrature operators according to

$$\langle N|\ [\hat{a}_1(t)\ ,\ \hat{a}_2(t)]\ |N\rangle = -\frac{i}{4}\langle N|\ [(\hat{b}_+(t)+\hat{b}_-(t))\ ,\ (\hat{b}_+(t)-\hat{b}_-(t))]\ |N\rangle = \frac{i}{2}\langle N|\ [\hat{b}_+(t)\ ,\ \hat{b}_-(t)]\ |N\rangle \tag{16f}$$

The non-vanishing commutation bracket ($16e$) shows that the time evolving non-degenerate signal-idler photon mode operators $\hat{b}_+(t)$ and $\hat{b}_-(t)$ are not independent, in contrast to the independent signal-idler photon modes in a general non-degenerate parametric process [1 , 4-6]. This is due to the fact that the non-degeneracy of the operators $\hat{b}_+(t)$ and $\hat{b}_-(t)$ is a full quantum interaction effect, which develops through the time evolution of the initial degenerate annihilation and creation operators $\hat{a}$, $\hat{a}^\dagger$ of the signal-idler photon modes. The operators $\hat{b}_+(t)$, $\hat{b}_-(t)$ maintain the initial algebraic relations of $\hat{a}$ and $\hat{a}^\dagger$, only modified by the frequency mixing effects in the full quantum interaction as established in the commutation relation in equation ($16e$). Indeed, at initial time $t = 0$ where $\mu_+(0) = \mu_-^*(0) = 1$, $\nu_+(0) = \nu_-(0) = 0$ and in the semiclassical model where $\mu_+(t) = \mu_-^*(t) = \mu(t)$, $\nu_+(t) = \nu_-(t) = \nu(t)$ gives $\mu_+(t)\mu_-(t) - \nu_+^2 = \mu_+(t)\mu_-(t) - \nu_-^2 = |\mu(t)|^2 - \nu^2(t) = 1$ (see section 5 for details), the commutation relation ($16e$) reduces to the expected form

$$[\hat{b}_+(0)\ ,\ \hat{b}_-(0)] = [\hat{b}_+^{sc}(t)\ ,\ \hat{b}_-^{sc}(t)] = 1 \qquad (16g)$$

where the pump photon Fock state amplitude $|N\rangle$ has now been dropped as appropriate and the superscript sc denotes semiclassical.

3.1 Time evolving quadrature state amplitudes

As explained in the previous section, we simplify the evaluation procedure for mean values, correlation functions and fluctuations of the time evolving quadrature operators by introducing general *time evolving quadrature state amplitudes* $|\hat{a}_1(t)\rangle$, $|\hat{a}_2(t)\rangle$ and the corresponding dual amplitudes $|\hat{a}_1^\dagger(t)\rangle$, $|\hat{a}_2^\dagger(t)\rangle$ obtained through the action of the time evolving quadrature operators $\hat{a}_1(t)$, $\hat{a}_2(t)$ and their Hermitian conjugates $\hat{a}_1^\dagger(t)$, $\hat{a}_2^\dagger(t)$ on the initial state amplitude $|Nn\rangle$ according to

$$|\hat{a}_1(t)\rangle = \hat{a}_1(t)|Nn\rangle = \frac{1}{2}(|\phi_+(t)\rangle + |\phi_-(t)\rangle) \quad ; \quad |\hat{a}_1^\dagger(t)\rangle = \hat{a}_1^\dagger(t)|Nn\rangle = \frac{1}{2}(|\phi_+^*(t)\rangle + |\phi_-^*(t)\rangle) \qquad (17a)$$

$$|\hat{a}_2(t)\rangle = \hat{a}_2(t)|Nn\rangle = -\frac{i}{2}(|\phi_+(t)\rangle - |\phi_-(t)\rangle) \quad ; \quad |\hat{a}_2^\dagger(t)\rangle = \hat{a}_2^\dagger(t)|Nn\rangle = \frac{i}{2}(|\phi_+^*(t)\rangle - |\phi_-^*(t)\rangle) \qquad (17b)$$

where we have introduced the basic time evolving annihilation and creation operator state amplitudes $|\phi_+(t)\rangle$, $|\phi_-(t)\rangle$ and their corresponding dual state amplitudes $|\phi_+^*(t)\rangle$, $|\phi_-^*(t)\rangle$ obtained as

$$|\phi_+(t)\rangle = \hat{b}_+(t)|Nn\rangle \quad ; \quad |\phi_+^*(t)\rangle = \hat{b}_+^\dagger(t)|Nn\rangle \qquad (17c)$$

$$|\phi_-(t)\rangle = \hat{b}_-(t)|Nn\rangle \quad ; \quad |\phi_-^*(t)\rangle = \hat{b}_-^\dagger(t)|Nn\rangle \qquad (17d)$$

Substituting $\hat{b}_+(t)$, $\hat{b}_+^\dagger(t)$, $\hat{b}_-(t)$, $\hat{b}_-^\dagger(t)$ from the set of equations ($11d$)-($11g$) into equations ($17a$)-($17b$) as appropriate, then applying the results from equations ($13b$)-($13c$) and ($14b$)-($14c$), we obtain the basic state amplitudes and their dual counterparts in the explicit forms

$$|\phi_+(t)\rangle \;=\; e^{-\frac{i}{2}\Omega t}(\mu_+(t)\sqrt{n}\ |N(n-1)\rangle + \nu_-(t)\sqrt{n+1}\ |\prod_{j=1}^{N}(n_j-1)(n+1)\rangle) \qquad (17e)$$

$$|\phi_-(t)\rangle \;=\; e^{\frac{i}{2}\Omega t}(\mu_-(t)\sqrt{n+1}\ |N(n+1)\rangle + \nu_+(t)\sqrt{n}\ |\prod_{j=1}^{N}(n_j+1)(n-1)\rangle) \qquad (17f)$$

$$|\phi_+^*(t)\rangle \;=\; e^{\frac{i}{2}\Omega t}(\mu_+^*(t)\sqrt{n+1}\ |N(n+1)\rangle + \nu_+(t)\sqrt{n}\ |\prod_{j=1}^{N}(n_j+1)(n-1)\rangle) \qquad (17g)$$

$$|\phi^*_-(t)\rangle \;=\; e^{-\frac{i}{2}\Omega t}(\mu^*_-(t)\sqrt{n}\;|N(n-1)\rangle + \nu_-(t)\sqrt{n+1}\;|\prod_{j=1}^{N}(n_j-1)(n+1)\rangle) \qquad (17h)$$

We notice that, according to the definitions in equations (13*d*), (13*e*) and (14*d*), each basic state amplitude has two components weighted by interaction variable pairs $(\mu_+(t), \nu_-(t))$ or $(\mu_-(t), \nu_+(t))$ specified by different quantized Rabi frequencies $\mathcal{R}$, R, which demonstrates the frequency-mixing arising from the full quantum interaction. We observe that this quantized Rabi frequency-mixing is responsible for the occurrence of purely quantum features such as fractional revivals, collapse-revival effects [4-6] and fundamental quantum noise in the full quantum model, in addition to squeezing and entanglement properties which arise also in the widely studied semiclassical model of the degenerate parametric amplification process [1 , 4-6].

The basic state amplitudes $|\phi_+(t)\rangle$, $|\phi_-(t)\rangle$ and their dual counterparts $|\phi^*_+(t)\rangle$, $|\phi^*_-(t)\rangle$ satisfy the following orthogonality relations

$$\langle\phi_+(t)|\phi_-(t)\rangle = 0 \quad ; \quad \langle\phi_-(t)|\phi_+(t)\rangle = 0 \quad ; \quad \langle\phi^*_+(t)|\phi^*_-(t)\rangle = 0 \quad ; \quad \langle\phi^*_-(t)|\phi^*_+(t)\rangle = 0 \qquad (18a)$$

$$\langle\phi_+(t)|\phi^*_+(t)\rangle = 0 \quad ; \quad \langle\phi_-(t)|\phi^*_-(t)\rangle = 0 \quad ; \quad \langle\phi^*_+(t)|\phi_+(t)\rangle = 0 \quad ; \quad \langle\phi^*_-(t)|\phi_-(t)\rangle = 0 \qquad (18b)$$

We observe the basic state amplitudes are not all mutually orthogonal, since there are *nearly degenerate* pairs $(|\phi_+(t)\rangle\ ,\ |\phi^*_-(t)\rangle)$ and $(|\phi_-(t)\rangle\ ,\ |\phi^*_+(t)\rangle)$, which have the same Fock state amplitudes, but differ only in the component weighting interaction variables $\mu_+(t)$, $\mu_-(t)$ specified by different quantized Rabi frequencies $\mathcal{R}$, R. These non-degenerate pairs which differ only in their quantized Rabi frequencies, but have coinciding Fock state amplitudes, are non-orthogonal according to

$$\langle\phi_+(t)|\phi^*_-(t)\rangle = \mu^*_+(t)\mu^*_-(t)n+\nu^2_-(t)(n+1) \quad ; \quad \langle\phi^*_-(t)|\phi_+(t)\rangle = \mu_-(t)\mu_+(t)n+\nu^2_-(t)(n+1) \quad (18c)$$

$$\langle\phi_-(t)|\phi^*_+(t)\rangle = \mu^*_+(t)\mu^*_-(t)(n+1)+\nu^2_+(t)n \quad ; \quad \langle\phi^*_+(t)|\phi_-(t)\rangle = \mu_-(t)\mu_+(t)(n+1)+\nu^2_+(t)n \quad (18d)$$

3.2 Mean values and correlations of the time evolving quadrature operators

We take the overlaps of the initial state amplitude $|Nn\rangle$ and the time evolving quadrature state amplitudes in equations (17*a*)-(17*b*) and use the basic state amplitudes in equations (17*e*)-(17*h*) to obtain the mean values of the time evolving quadrature operators as

$$\overline{a_1(t)} = 0 \quad ; \quad \overline{a^*_1(t)} = 0 \qquad ; \qquad \overline{a_2(t)} = 0 \quad ; \quad \overline{a^*_2(t)} = 0 \qquad (19a)$$

Taking overlaps of the state amplitudes in equations (17*a*)-(17*b*) as appropriate according to the general definitions in equations (15*e*)-(15*g*) and using the basic state amplitudes in equations (17*e*)-(17*h*), we obtain all possible correlation functions for the time evolving quadrature operators $\hat{a}_1(t)$, $\hat{a}_2(t)$ and their Hermitian conjugates $\hat{a}^\dagger_1(t)$, $\hat{a}^\dagger_2(t)$ in the final forms

$$\langle\hat{a}_1(t)|\hat{a}_1(t)\rangle \;=\; \overline{a^*_1(t)a_1(t)} = \frac{1}{4}\{(|\mu_+|^2+\nu^2_+)n + (|\mu_-|^2+\nu^2_-)(n+1)\} \qquad (19b)$$

$$\langle\hat{a}^\dagger_1(t)|\hat{a}^\dagger_1(t)\rangle \;=\; \overline{a_1(t)a^*_1(t)} = \frac{1}{4}\{(|\mu_+|^2+\nu^2_-)(n+1) + (|\mu_-|^2+\nu^2_+)n\} \qquad (19c)$$

$$\langle\hat{a}^\dagger_1(t)|\hat{a}_1(t)\rangle \;=\; \overline{a^2_1(t)} = \frac{1}{4}\{\mu_+\mu_-(2n+1) + \nu^2_-(n+1) + \nu^2_+n\}$$

$$\langle\hat{a}_1(t)|\hat{a}_1^\dagger(t)\rangle = \overline{a_1^{*2}(t)} = \left(\overline{a_1^2(t)}\right)^* \quad (19d)$$

$$\langle\hat{a}_2(t)|\hat{a}_2(t)\rangle = \overline{a_2^*(t)a_2(t)} = \overline{a_1^*(t)a_1(t)} \quad ; \quad \langle\hat{a}_2^\dagger(t)|\hat{a}_2^\dagger(t)\rangle = \overline{a_2(t)a_2^*(t)} = \overline{a_1(t)a_1^*(t)} \quad (19e)$$

$$\langle\hat{a}_2^\dagger(t)|\hat{a}_2(t)\rangle = \overline{a_2^2(t)} = \overline{a_1^2(t)} \quad ; \quad \langle\hat{a}_2(t)|\hat{a}_2^\dagger(t)\rangle = \overline{a_2^{*2}(t)} = \left(\overline{a_2^2(t)}\right)^* \quad (19f)$$

$$\langle\hat{a}_1(t)|\hat{a}_2(t)\rangle = \overline{a_1^*(t)a_2(t)} = \frac{i}{4} \quad ; \quad \langle\hat{a}_2(t)|\hat{a}_1(t)\rangle = \overline{a_2^*(t)a_1(t)} = \left(\overline{a_1^*(t)a_2(t)}\right)^* = -\frac{i}{4} \quad (19g)$$

$$\langle\hat{a}_1(t)|\hat{a}_2^\dagger(t)\rangle = \overline{a_1^*(t)a_2^*(t)} = \frac{i}{4}\{\mu_+^*\mu_-^* + \nu_+^2 n - \nu_-^2(n+1)\}$$

$$\langle\hat{a}_2^\dagger(t)|\hat{a}_1(t)\rangle = \overline{a_2(t)a_1(t)} = \left(\overline{a_1^*(t)a_2^*(t)}\right)^* \quad (19h)$$

$$\langle\hat{a}_1^\dagger(t)|\hat{a}_2(t)\rangle = \overline{a_1(t)a_2(t)} = -\overline{a_2(t)a_1(t)} \quad ; \quad \langle\hat{a}_2(t)|\hat{a}_1^\dagger(t)\rangle = \overline{a_2^*(t)a_1^*(t)} = -\overline{a_1^*(t)a_2^*(t)} \quad (19i)$$

$$\langle\hat{a}_1^\dagger(t)|\hat{a}_2^\dagger(t)\rangle = \overline{a_1(t)a_2^*(t)} = \frac{i}{4}\{((|\mu_+|^2 - \nu_-^2)(n+1) - (|\mu_-|^2 - \nu_+^2)n\} \quad (19j)$$

$$\langle\hat{a}_2^\dagger(t)|\hat{a}_1^\dagger(t)\rangle = \overline{a_2(t)a_1^*(t)} = \left(\overline{a_1(t)a_2^*(t)}\right)^* \quad (19k)$$

We notice that, according to equations (19a), (19d) and (19f), the uncertainties in the quadratures, $\Delta\overline{a_1(t)}$, $\Delta\overline{a_2(t)}$ are equal, but complex and would pose problems in experimental measurements. This is in great contrast to the familiar case of the semiclassical model where the uncertainties are real, but the uncertainty in one quadrature component grows exponentially, while the uncertainty in the other quadrature component decays exponentially [1 , 6] as demonstrated in section 5 below.

3.3 Hermitian quadrature components: fluctuations, squeezing and fundamental quantum noise

For purposes of experimental measurements and practical applications, we introduce Hermitian components $\hat{X}_1(t)$, $\hat{Y}_1(t)$ and $\hat{X}_2(t)$, $\hat{Y}_2(t)$ of the time evolving quadrature operators $\hat{a}_1(t)$, $\hat{a}_2(t)$ according to

$$\hat{X}_1(t) = \frac{1}{2}(\hat{a}_1(t) + \hat{a}_1^\dagger(t)) \quad ; \quad \hat{Y}_1(t) = -\frac{i}{2}(\hat{a}_1(t) - \hat{a}_1^\dagger(t)) \quad (20a)$$

$$\hat{X}_2(t) = \frac{1}{2}(\hat{a}_2(t) + \hat{a}_2^\dagger(t)) \quad ; \quad \hat{Y}_2(t) = -\frac{i}{2}(\hat{a}_2(t) - \hat{a}_2^\dagger(t)) \quad (20b)$$

It is easy to confirm that these Hermitian quadrature components are expressible in terms of the Hermitian components of $\hat{b}_+(t)$ and $\hat{b}_-(t)$ defined within the frequency channels in equations (11h)-(11i), e.g., $\hat{X}_1(t) = \frac{1}{2}(\hat{x}_+(t) + \hat{x}_-(t))$.

We obtain the time evolving Hermitian quadrature operator state amplitudes $|\hat{X}_1(t)\rangle$, $|\hat{Y}_1(t)\rangle$, $|\hat{X}_2(t)\rangle$, $|\hat{Y}_2(t)\rangle$ together with their corresponding Hermitian conjugates, evolving from the initial Fock state amplitude $|\psi(0)\rangle = |Nn\rangle$ in the form

$$|\hat{X}_1(t)\rangle = \hat{X}_1(t)|Nn\rangle = \frac{1}{2}(|\hat{a}_1(t)\rangle + |\hat{a}_1^\dagger(t)\rangle) \quad ; \quad \langle\hat{X}_1(t)| = \langle nN|\hat{X}_1(t) = \frac{1}{2}(\langle\hat{a}_1(t)| + \langle\hat{a}_1^\dagger(t)|) \quad (21a)$$

$$|\hat{Y}_1(t)\rangle = \hat{Y}_1(t)|Nn\rangle \;=\; -\frac{i}{2}(|\hat{a}_1(t)\rangle - |\hat{a}_1^\dagger(t)\rangle) \quad ; \quad \langle\hat{Y}_1(t)| = \langle nN|\hat{Y}_1(t) \;=\; \frac{i}{2}(\langle\hat{a}_1(t)| - \langle\hat{a}_1^\dagger(t)|) \quad (21b)$$

$$|\hat{X}_2(t)\rangle = \hat{X}_2(t)|Nn\rangle \;=\; \frac{1}{2}(|\hat{a}_2(t)\rangle + |\hat{a}_2^\dagger(t)\rangle) \quad ; \quad \langle\hat{X}_2(t)| = \langle nN|\hat{X}_2(t) \;=\; \frac{1}{2}(\langle\hat{a}_2(t)| + \langle\hat{a}_2^\dagger(t)|) \quad (21c)$$

$$|\hat{Y}_2(t)\rangle = \hat{Y}_2(t)|Nn\rangle \;=\; -\frac{i}{2}(|\hat{a}_2(t)\rangle - |\hat{a}_2^\dagger(t)\rangle) \quad ; \quad \langle\hat{Y}_2(t)| = \langle nN|\hat{Y}_2(t) \;=\; \frac{i}{2}(\langle\hat{a}_2(t)| - \langle\hat{a}_2^\dagger(t)|) \quad (21d)$$

where the quadrature operator state amplitudes $|\hat{a}_1(t)\rangle$, $|\hat{a}_1^\dagger(t)\rangle$, $|\hat{a}_2(t)\rangle$, $|\hat{a}_2^\dagger(t)\rangle$ are defined in the set of equations (17a)-(17h).

The mean values of the time evolving Hermitian quadrature components are obtained as overlaps of the initial state amplitude $|Nn\rangle$ with the time evolving Hermitian quadrature operator state amplitudes $|\hat{X}_1(t)\rangle$, $|\hat{Y}_1(t)\rangle$, $|\hat{X}_2(t)\rangle$, $|\hat{Y}_2(t)\rangle$ defined in equations (21a)-(21d) according to

$$\overline{X}_1(t) = \langle nN|\hat{X}_1(t)\rangle \quad ; \quad \overline{Y}_1(t) = \langle nN|\hat{Y}_1(t)\rangle \quad ; \quad \overline{X}_1(t) = \langle nN|\hat{X}_1(t)\rangle \quad ; \quad \overline{Y}_1(t) = \langle nN|\hat{Y}_1(t)\rangle \quad (22a)$$

while the second moments (auto-correlations) of $\hat{X}_1(t)$, $\hat{Y}_1(t)$, $\hat{X}_2(t)$, $\hat{Y}_2(t)$ are obtained as overlaps of the respective state amplitudes $|\hat{X}_1(t)\rangle$, $|\hat{Y}_1(t)\rangle$, $|\hat{X}_2(t)\rangle$, $|\hat{Y}_2(t)\rangle$ according to

$$\overline{X_1^2(t)} = \langle\hat{X}_1(t)|\hat{X}_1(t)\rangle = \frac{1}{4}(\langle\hat{a}_1(t)|\hat{a}_1(t)\rangle + \langle\hat{a}_1^\dagger(t)|\hat{a}_1^\dagger(t)\rangle + \langle\hat{a}_1(t)|\hat{a}_1^\dagger(t)\rangle + \langle\hat{a}_1^\dagger(t)|\hat{a}_1(t)\rangle) \quad (22b)$$

$$\overline{Y_1^2(t)} = \langle\hat{Y}_1(t)|\hat{Y}_1(t)\rangle = \frac{1}{4}(\langle\hat{a}_1(t)|\hat{a}_1(t)\rangle + \langle\hat{a}_1^\dagger(t)|\hat{a}_1^\dagger(t)\rangle - \langle\hat{a}_1(t)|\hat{a}_1^\dagger(t)\rangle - \langle\hat{a}_1^\dagger(t)|\hat{a}_1(t)\rangle) \quad (22c)$$

$$\overline{X_2^2(t)} = \langle\hat{X}_2(t)|\hat{X}_2(t)\rangle = \frac{1}{4}(\langle\hat{a}_2(t)|\hat{a}_2(t)\rangle + \langle\hat{a}_2^\dagger(t)|\hat{a}_2^\dagger(t)\rangle + \langle\hat{a}_2(t)|\hat{a}_2^\dagger(t)\rangle + \langle\hat{a}_2^\dagger(t)|\hat{a}_2(t)\rangle) \quad (22d)$$

$$\overline{Y_2^2(t)} = \langle\hat{Y}_2(t)|\hat{Y}_2(t)\rangle = \frac{1}{4}(\langle\hat{a}_2(t)|\hat{a}_2(t)\rangle + \langle\hat{a}_2^\dagger(t)|\hat{a}_2^\dagger(t)\rangle - \langle\hat{a}_2(t)|\hat{a}_2^\dagger(t)\rangle - \langle\hat{a}_2^\dagger(t)|\hat{a}_2(t)\rangle) \quad (22e)$$

Substituting the explicit forms of $|\phi_+(t)\rangle$, $|\phi_-(t)\rangle$, $|\phi_+^*(t)\rangle$, $|\phi_-^*(t)\rangle$ from equations (17e)-(17h) into equation (17b), then using the results in equations (21a)-(21d) and (22a) provides the mean values as

$$\overline{X}_1(t) = 0 \quad ; \quad \overline{Y}_1(t) = 0 \quad ; \quad \overline{X}_2(t) = 0 \quad ; \quad \overline{Y}_2(t) = 0 \quad (23a)$$

and the second moments as

$$\overline{X_1^2(t)} = \overline{X_2^2(t)} \;=\; \frac{1}{16}\{(\left|\mu_+(t) + \mu_-^*(t)\right|^2)(2n+1) + 4\nu_-^2(t)(n+1) + 4\nu_+^2(t)n\} \quad (23b)$$

$$\overline{Y_1^2(t)} = \overline{Y_2^2(t)} \;=\; \frac{1}{16}\left|\mu_+(t) - \mu_-^*(t)\right|^2(2n+1) \quad (23c)$$

The fluctuations in the Hermitian time evolving quadrature components are obtained as overlaps of the respective fluctuation state amplitudes $|\Delta\hat{X}_1(t)\rangle = |\hat{X}_1(t)\rangle$, $|\Delta\hat{Y}_1(t)\rangle = |\hat{Y}_1(t)\rangle$, $|\Delta\hat{X}_1(t)\rangle = |\hat{X}_1(t)\rangle$, $|\Delta\hat{Y}_1(t)\rangle = |\hat{Y}_1(t)\rangle$ in the form

$$(\Delta\overline{Q}_j(t))^2 = \langle\Delta\hat{Q}_j(t)|\Delta\hat{Q}_j(t)\rangle \;=\; \overline{Q_j^2(t)} \quad ; \quad Q = X, Y \quad ; \quad j = 1, 2 \quad (23d)$$

which on using the results obtained in equations (23b)-(23c) take the final form

$$(\Delta\overline{X}_1(t))^2 = (\Delta\overline{X}_2(t))^2 \;=\; \frac{1}{16}\{(\left|\mu_+(t) + \mu_-^*(t)\right|^2)(2n+1) + 4\nu_-^2(t)(n+1) + 4\nu_+^2(t)n\} \quad (23e)$$

$$(\Delta\overline{Y}_1(t))^2 = (\Delta\overline{Y}_2(t))^2 \;=\; \frac{1}{16}\Big|\mu_+(t) - \mu_-^*(t)\Big|^2(2n+1) \tag{23f}$$

Comparing the results gives the inequalities

$$\Delta\overline{Y}_1(t) < \Delta\overline{X}_1(t) \qquad ; \qquad \Delta\overline{Y}_2(t) < \Delta\overline{X}_2(t) \tag{23g}$$

which shows that the fluctuations in the Hermitian quadrature components $\hat{Y}_1(t)$, $\hat{Y}_2(t)$ are generally less than the fluctuations in the corresponding Hermitian quadrature component $\hat{X}_1(t)$, $\hat{X}_2(t)$. This reveals the squeezing property of the fully quantized parametric amplification process.

A very important feature which emerges here is that the fluctuations $\Delta\overline{Y}_1(t)$, $\Delta\overline{Y}_2(t)$ in the imaginary parts $\hat{Y}_1(t)$, $\hat{Y}_2(t)$ of the time evolving quadrature operators obtained in equation $(27f)$ constitute *fundamental quadrature quantum noise* spectrum $q_n(t)$ expressed in the form

$$(\Delta\overline{Y}_1(t))^2 = (\Delta\overline{Y}_2(t))^2 \;=\; \frac{1}{4}q_n(t) \tag{23h}$$

where the *fundamental quadrature quantum noise* spectrum $q_n(t)$ is obtained as

$$q_n(t) = \frac{1}{4}\Big|\mu_+(t) - \mu_-^*(t)\Big|^2(2n+1) \quad ; \quad n = 0,1,2,3,..... \tag{23i}$$

It is easy to confirm that this noise spectrum vanishes at initial time $t = 0$ when $\mu_+(0) = \mu_-^*(0) = 1$. We demonstrate in section 5 that $q_n(t)$ vanishes in the semiclassical model of the degenerate parametric amplification process. It is therefore an effect of the full quantum interaction.

The squeezing behavior displayed in equations $(23e)$-$(23g)$ in the general squeezed Fock state in the full quantum model are specified by two different quantized Rabi frequencies $\mathcal{R}$, R, defining two competing quantized squeezing parameters $r_+(t) = \mathcal{R}t$, $r_-(t) = Rt$. The general form of this full quantum squeezing action is different from the squeezing behavior in the semiclassical model in which the quadrature fluctuations either grow or decay exponentially, specified by a single Rabi frequency R and a single squeezing parameter $r(t)$ [1 , 6]. We compare the full quantum and semiclassical models in section 5 below.

3.4 Heisenberg uncertainty relations and fundamental quantum noise in Hermitian quadrature correlations

The Heisenberg uncertainty principle imposes constraints on how quantum noise or fluctuations affect the accuracy of simultaneous measurements of mean values of quantum operators in a specified quantum state at any time. For Hermitian operators $\hat{A}(t)$, $\hat{B}(t)$ with state amplitudes $|\hat{A}(t)\rangle$, $|\hat{B}(t)\rangle$ representing the time evolving Hermitian quadrature components defined above, the Heisenberg uncertainty principle governing the measurements of the mean values of $\hat{A}(t)$ and $\hat{B}(t)$ in an initial state $|Nn\rangle$ is obtained in the mathematical form

$$\Delta\overline{A}(t)\Delta\overline{B}(t) \geq \frac{1}{2}\Big|\langle nN|\ [\hat{A}(t), \hat{B}(t)]\ |Nn\rangle\Big| \tag{24a}$$

where $\langle nN|\ [\hat{A}(t), \hat{B}(t)]\ |Nn\rangle$ is the mean value of the commutation bracket of the operators $\hat{A}(t)$, $\hat{B}(t)$ in the initial state $|Nn\rangle$.

Since the commutation bracket provides a measure of the correlation of the Hermitian quadrature operators, i.e., whether or not the operators $\hat{A}(t)$, $\hat{B}(t)$ are independent of each other, we consider it best to use the cross-correlation functions obtained as appropriate overlaps of the time evolving operator state amplitudes $|\hat{A}(t)\rangle$, $|\hat{B}(t)\rangle$ to determine the mean value of the corresponding commutation bracket. The overlaps of the Hermitian operator state amplitudes

$$|\hat{A}(t)\rangle = \hat{A}(t)|Nn\rangle \qquad ; \qquad |\hat{B}(t)\rangle = \hat{B}(t)|Nn\rangle \tag{24b}$$

provides the cross-correlation functions in the form

$$\overline{A(t)B(t)} - \overline{B(t)A(t)} = \langle\hat{A}(t)|\hat{B}(t)\rangle - \langle\hat{B}(t)|\hat{A}(t)\rangle = \langle nN|\left(\hat{A}(t)\hat{B}(t) - \hat{B}(t)\hat{A}(t)\right)|Nn\rangle \tag{24c}$$

which provides the general relation between cross-correlation function and the mean commutation bracket in the form

$$\overline{A(t)B(t)} - \overline{B(t)A(t)} = \langle nN|\ [\hat{A}(t)\ ,\ \hat{B}(t)]\ |Nn\rangle \tag{24d}$$

We substitute equation ($24d$) into equation ($24a$) to obtain the Heisenberg uncertainty relation in terms of the cross-correlation functions as

$$\Delta\overline{A}(t)\Delta\overline{B}(t) \geq \frac{1}{2}\left|\overline{A(t)B(t)} - \overline{B(t)A(t)}\right| \tag{24e}$$

For the Hermitian quadrature components $\hat{A}(t), \hat{B}(t) \ = \ \hat{X}_1(t), \hat{Y}_1(t), \hat{X}_2(t), \hat{Y}_2(t)$, we obtain

$$\overline{X_1(t)Y_1(t)} - \overline{Y_1(t)X_1(t)} = \langle\hat{X}_1(t)|\hat{Y}_1(t)\rangle - \langle\hat{Y}_1(t)|\hat{X}_1(t)\rangle \ = \ \langle nN|\ [\hat{X}_1(t), \hat{Y}_1(t)]\ |Nn\rangle \tag{25a}$$

$$\overline{X_2(t)Y_2(t)} - \overline{Y_2(t)X_2(t)} = \langle\hat{X}_2(t)|\hat{Y}_2(t)\rangle - \langle\hat{Y}_2(t)|\hat{X}_2(t)\rangle \ = \ \langle nN|\ [\hat{X}_2(t), \hat{Y}_2(t)]\ |Nn\rangle \tag{25b}$$

$$\overline{X_1(t)Y_2(t)} - \overline{Y_2(t)X_1(t)} = \langle\hat{X}_1(t)|\hat{Y}_2(t)\rangle - \langle\hat{Y}_2(t)|\hat{X}_1(t)\rangle \ = \ \langle nN|\ [\hat{X}_1(t), \hat{Y}_2(t)]\ |Nn\rangle \tag{25c}$$

$$\overline{Y_1(t)X_2(t)} - \overline{X_2(t)Y_1(t)} = \langle\hat{Y}_1(t)|\hat{X}_2(t)\rangle - \langle\hat{X}_2(t)|\hat{Y}_1(t)\rangle \ = \ \langle nN|\ [\hat{Y}_1(t), \hat{X}_2(t)]\ |Nn\rangle \tag{25d}$$

$$\overline{X_1(t)X_2(t)} - \overline{X_2(t)X_1(t)} = \langle\hat{X}_1(t)|\hat{X}_2(t)\rangle - \langle\hat{X}_2(t)|\hat{X}_1(t)\rangle \ = \ \langle nN|\ [\hat{X}_1(t), \hat{X}_2(t)]\ |Nn\rangle \tag{25e}$$

$$\overline{Y_1(t)Y_2(t)} - \overline{Y_2(t)Y_1(t)} = \langle\hat{Y}_1(t)|\hat{Y}_2(t)\rangle - \langle\hat{Y}_2(t)|\hat{Y}_1(t)\rangle \ = \ \langle nN|\ [\hat{Y}_1(t), \hat{Y}_2(t)]\ |Nn\rangle \tag{25f}$$

Taking the overlaps of the time evolving Hermitian quadrature state amplitudes from equations ($21a$)-($21d$) and using the explicit expressions for the quadrature correlation functions obtained in equations ($19b$)-($19k$) provides the explicit forms of the cross-correlation functions, which we substitute into equations ($25a$)-($25f$) to obtain the corresponding mean commutation brackets in the final forms

$$\langle nN|\ [\hat{X}_1(t), \hat{Y}_1(t)]\ |Nn\rangle = \langle nN|\ [\hat{X}_2(t), \hat{Y}_2(t)]\ |Nn\rangle = \frac{i}{2}q(t) \tag{26a}$$

$$\langle nN|\ [\hat{X}_1(t), \hat{Y}_2(t)]\ |Nn\rangle = \langle nN|\ [\hat{Y}_1(t), \hat{X}_2(t)]\ |Nn\rangle = 0 \tag{26b}$$

$$\langle nN|\ [\hat{X}_1(t), \hat{X}_2(t)]\ |Nn\rangle = \langle nN|\ [\hat{Y}_1(t), \hat{Y}_2(t)]\ |Nn\rangle = \frac{i}{2}(1 + q_{12}(t)) \tag{26c}$$

which we substitute into equation ($24a$) for $\hat{A}(t), \hat{B}(t) \ = \ \hat{X}_1(t), \hat{Y}_1(t), \hat{X}_2(t), \hat{Y}_2(t)$ as appropriate to obtain Heisenberg's uncertainty relations as the set of inequalities

$$\Delta\overline{X}_1(t)\Delta\overline{Y}_1(t) = \Delta\overline{X}_2(t)\Delta\overline{Y}_2(t) \geq \frac{1}{4}q(t) \tag{26d}$$

$$\Delta\overline{X}_1(t)\Delta\overline{Y}_2(t) \ = \ \Delta\overline{Y}_1(t)\Delta\overline{X}_2(t) \geq 0 \tag{26e}$$

$$\Delta \overline{X}_1(t)\Delta \overline{X}_2(t) = \Delta \overline{Y}_1(t)\Delta \overline{Y}_2(t) \geq \frac{1}{4}\Big|1 + q_{12}(t)\Big| \tag{26f}$$

where we have identified *fundamental quadrature correlation quantum noise* $q(t)$ and $q_{12}(t)$ obtained as

$$q_{11}(t) = q_{22}(t) = q(t) \ : \quad q(t) = \frac{1}{4}\left(|\mu_+(t)|^2 - |\mu_-(t)|^2\right) \tag{26g}$$

$$q_{12}(t) = \frac{1}{4}\left\{|\mu_+(t)|^2 + \mu_+(t)\mu_-(t) + \mu_+^*(t)\mu_-^*(t) - 3\nu_-^2(t) - 3 + (|\mu_+(t)|^2 - |\mu_-(t)|^2 + 3\nu_+^2(t) - 3\nu_-^2(t))n\right\} \tag{26h}$$

We observe that in equation (30c), we have added and subtracted 3 to obtain the final result with $q_{12}(t)$ in the form in equation (30h).

The results we have obtained in equations (26a)-(26h) are very important for physical interpretation of the dynamics of the fully quantized degenerate parametric amplification process. The full quantum interaction generates fundamental quantum noise $q(t)$, $q_{12}(t)$, which affect the process of quadrature measurements according to the uncertainty relations in equations (26d) and (26f). We observe that the fundamental quantum noise arises as *pure quantum fluctuation effects*.

The vanishing of the mean value of the commutation brackets in equation (26b) means that the Hermitian quadrature pairs $(\hat{X}_1(t), \hat{Y}_2(t))$ and $(\hat{Y}_1(t), \hat{X}_2(t))$ are independent. Measurements on $\hat{X}_1(t)$ and $\hat{Y}_2(t)$ can be performed simultaneously. Similarly, measurements on $\hat{Y}_1(t)$ and $\hat{X}_2(t))$ can be performed simultaneously. The Hermitian quadrature pairs $(\hat{X}_1(t), \hat{Y}_2(t))$ and $(\hat{Y}_1(t), \hat{X}_2(t))$ can be interpreted as the effective EPR pairs arising through the full quantum interaction in a degenerate parametric amplification process. Since the Hermitian operators $\hat{Y}_1$ and $\hat{Y}_2$ emerging as the imaginary parts of the time evolving quadrature components $\hat{a}_1(t)$ and $\hat{a}_2(t)$ according to equations (20a)-(20b) are purely quantum operators, their fluctuations obtained in equation (23h) and their uncertainty relations obtained in equations (26d)-(26f) are purely quantum mechanical, i.e., nonclassical. Indeed, in the semiclassical model where the fundamental quantum noise vanishes as demonstrated in section 5 below, equation (26f) reduces to the expected quadrature uncertainty relation $\Delta \overline{X}_1(t)\Delta \overline{X}_2(t) \rightarrow \Delta \overline{X}_1^{sc}(t)\Delta \overline{X}_2^{sc}(t) \geq \frac{1}{4}$ (see equation (36e) below), where *sc* denotes semiclassical as defined earlier.

We notice that, at initial time $t = 0$ when the interaction just begins, the interaction variables $\mu_+(t)$, $\mu_-(t)$ and $\nu_+(t)$, $\nu_-(t)$ in equations (13d) and (14d) take initial forms

$$t = 0 \ : \quad \mu_+(0) = \mu_-^*(0) = 1 \quad , \quad \nu_+(0) = \nu_-(0) = 0 \tag{27a}$$

which we use in equations (23i) , (26g) , (26h) to obtain

$$t = 0 \ : \quad q_n(0) = 0 \qquad ; \qquad q(0) = 0 \qquad ; \qquad q_{12}(t) = 0 \tag{27b}$$

This shows that the fundamental quadrature and quadrature correlation quantum noise do not exist at the beginning, but arise only in the course of the full quantum interaction.

4 Time evolution of the initial number operator

We now consider the time evolution of the initial signal-idler photon number operator under the quantum squeezing operator $U(t)$. Given the initial photon number operator

$$\hat{n} = \hat{a}^\dagger \hat{a} \tag{28a}$$

we obtain the time evolving number operator $\hat{n}(t)$ according to

$$\hat{n}(t)=U^\dagger(t)\hat{a}^\dagger\hat{a}U(t) \;=\; U^\dagger(t)\hat{a}^\dagger U(t)U^\dagger(t)\hat{a}U(t) \tag{28b}$$

in the final form

$$\hat{n}(t)=\hat{b}_-(t)\hat{b}_+(t) \tag{28c}$$

This is complex and has Hermitian conjugate

$$\hat{n}^\dagger(t)=\hat{b}_+^\dagger(t)\hat{b}_-^\dagger(t) \tag{28d}$$

These are decomposed into eigen and fluctuation components in the form

$$\hat{n}(t)=\hat{\overline{n}}(t)+\Delta\hat{n}(t) \quad ; \quad \hat{\overline{n}}(t)=\hat{\mu}_-\hat{\mu}_+\hat{a}^\dagger\hat{a}+\hat{\nu}_-\hat{\nu}_+\hat{a}\hat{a}^\dagger \quad ; \quad \Delta\hat{n}(t)=\hat{\mu}_-\hat{\nu}_+\hat{a}^{\dagger 2}+\hat{\nu}_-\hat{\mu}_+\hat{a}^2 \tag{28e}$$

$$\hat{n}^\dagger(t)=\hat{\overline{n}}^\dagger(t)+\Delta\hat{n}^\dagger(t) \quad ; \quad \hat{\overline{n}}^\dagger(t)=\hat{\mu}_+^\dagger\hat{\mu}_-^\dagger\hat{a}^\dagger\hat{a}+\hat{\nu}_+^\dagger\hat{\nu}_-^\dagger\hat{a}\hat{a}^\dagger \quad ; \quad \Delta\hat{n}^\dagger(t)=\hat{\mu}_+^\dagger\hat{\nu}_-^\dagger\hat{a}^{\dagger 2}+\hat{\nu}_+^\dagger\hat{\mu}_-^\dagger\hat{a}^2 \tag{28f}$$

The time evolving number state amplitude $|\hat{n}(t)\rangle$ and its dual $|\hat{n}^\dagger(t)\rangle$ are then easily obtained as

$$|\hat{n}(t)\rangle=\overline{n}(t)|Nn\rangle+|\Delta\hat{n}(t)\rangle \quad ; \quad |\hat{n}^\dagger(t)\rangle=\overline{n^*}(t)|Nn\rangle+|\Delta\hat{n}^\dagger(t)\rangle \tag{29a}$$

where the eigen and fluctuation state amplitudes are obtained according to

$$\hat{\overline{n}}(t)|Nn\rangle=\overline{n}(t)|Nn\rangle \qquad ; \qquad \Delta\hat{n}(t)|Nn\rangle=|\Delta\hat{n}(t)\rangle \tag{29b}$$

$$\hat{\overline{n}}^\dagger(t)|Nn\rangle=\overline{n^*}(t)|Nn\rangle \qquad ; \qquad \Delta\hat{n}^\dagger(t)|Nn\rangle=|\Delta\hat{n}^\dagger(t)\rangle \tag{29c}$$

The mean values $\overline{n}(t)$, $\overline{n^*}(t)$ and the number fluctuation state amplitudes $|\Delta\hat{n}(t)\rangle$, $|\Delta\hat{n}^\dagger(t)\rangle$ have been obtained in explicit forms

$$\overline{n}(t)=\mu_+\mu_-n+\nu_-^2(n+1) \qquad ; \qquad \overline{n^*}(t)=\mu_+^*\mu_-^*n+\nu_-^2(n+1) \tag{29d}$$

$$|\Delta\hat{n}(t)\rangle=\mu_+\nu_+\sqrt{n(n-1)}|\prod_{j=1}^{N}(n_j+1)(n-2)\rangle+\tilde{\mu}_-\nu_-\sqrt{(n+1)(n+2)}|\prod_{j=1}^{N}(n_j-1)(n+2)\rangle \tag{29e}$$

$$|\Delta\hat{n}^\dagger(t)\rangle=\mu_-^*\nu_+\sqrt{n(n-1)}|\prod_{j=1}^{N}(n_j+1)(n-2)\rangle+\overline{\mu}_+^*\nu_-\sqrt{(n+1)(n+2)}|\prod_{j=1}^{N}(n_j-1)(n+2)\rangle \tag{29f}$$

where in the evaluation of $\hat{\mu}_+^\dagger(t)|\prod_{j=1}^{N}(n_j+1)\rangle$ and $\hat{\mu}_-^\dagger(t)|\prod_{j=1}^{N}(n_j-1)\rangle$ according to equations (13b)-(13c) with $n_j\rightarrow n_j\pm 1$, we have obtained

$$\bar{\mu}_+(t)=\cosh(\bar{\mathcal{R}}t)+i\frac{\delta}{\bar{\mathcal{R}}}\sinh(\bar{\mathcal{R}}t) \quad ; \quad \bar{\mathcal{R}}=g\sqrt{\bar{\mathcal{N}}-k^2} \quad ; \quad \bar{\mathcal{N}}=\prod_{j=1}^{\bar{N}}(n_j+2) \tag{29g}$$

$$\tilde{\mu}_-(t)=\cosh(\tilde{R}t)-i\frac{\delta}{\tilde{R}}\sinh(\tilde{R}t) \quad ; \quad \tilde{R}=g\sqrt{\tilde{N}-k^2} \quad ; \quad \tilde{N}=\prod_{j=1}^{\bar{N}}(n_j-1) \tag{29h}$$

The Hermitian conjugates of the number state amplitudes are obtained as

$$\langle\hat{n}(t)|=\langle nN|\hat{n}^\dagger(t) \qquad ; \qquad \langle\hat{n}^\dagger(t)|=\langle nN|\hat{n}(t) \tag{29i}$$

The fluctuations in the time evolving signal-idler photon numbers are easily obtained as the overlaps of the number fluctuation state amplitudes $|\Delta\hat{n}(t)\rangle$, $|\Delta\hat{n}^\dagger(t)\rangle$ in the form

$$(\ \Delta\overline{n^*(t)n(t)} \ )^2 = \langle\Delta\hat{n}(t)|\Delta\hat{n}(t)\rangle \quad ; \quad (\ \Delta\overline{n(t)n^*(t)} \ )^2 = \langle\Delta\hat{n}^\dagger(t)|\Delta\hat{n}^\dagger(t)\rangle \tag{30a}$$

$$(\ \Delta\overline{n(t)} \ )^2 = \langle\Delta\hat{n}^\dagger(t)|\Delta\hat{n}(t)\rangle \quad ; \quad (\ \Delta\overline{n^*(t)} \ )^2 = \langle\Delta\hat{n}(t)|\Delta\hat{n}^\dagger(t)\rangle \tag{30b}$$

which on substituting the explicit forms of the squeezed number fluctuation state amplitudes from equations (29c)-(29d), together with their Hermitian conjugates take the final forms

$$(\ \Delta\overline{n^*(t)n(t)} \ )^2 = |\mu_+|^2\nu_+^2 n(n-1) + |\tilde{\mu}_-|^2\nu_-^2(n+1)(n+2) \tag{30c}$$

$$(\ \Delta\overline{n(t)n^*(t)} \ )^2 = |\mu_-|^2\nu_+^2 n(n-1) + |\overline{\mu}_+|^2\nu_-^2(n+1)(n+2) \tag{30d}$$

$$(\ \Delta\overline{n(t)} \ )^2 = \mu_+\mu_-\nu_+^2 n(n-1) + \overline{\mu}_+\tilde{\mu}_-\nu_-^2(n+1)(n+2) \tag{30e}$$

$$(\ \Delta\overline{n^*(t)} \ )^2 = \mu_+^*\mu_-^*\nu_+^2 n(n-1) + \overline{\mu}_+^*\tilde{\mu}_-^*\nu_-^2(n+1)(n+2) \tag{30f}$$

We notice that only the conjugate cross-correlation fluctuations in equations (30c)-(30d) are real, while the number fluctuations in equations (30e)-(30f) are complex, which would pose problems in direct number measurements.

4.1 Hermitian time evolved number components: fluctuations and fundamental number quantum noise

For experimental measurements and practical applications, it is necessary to introduce Hermitian components $\hat{r}(t)$, $\hat{s}(t)$, of the time evolving number operator defined by

$$\hat{r}(t) = \frac{1}{2}(\hat{n}(t) + \hat{n}^\dagger(t)) \quad ; \quad \hat{s}(t) = -\frac{i}{2}(\hat{n}(t) - \hat{n}^\dagger(t)) \tag{31a}$$

which we can decompose into eigen and fluctuation components

$$\hat{r}(t) = \hat{\overline{r}}(t) + \Delta\hat{r}(t) \quad ; \quad \hat{\overline{r}}(t) = \frac{1}{2}(\hat{\overline{n}}(t) + \hat{\overline{n}}^\dagger(t)) \quad ; \quad \Delta\hat{r}(t) = \frac{1}{2}(\Delta\hat{n}(t) + \Delta\hat{n}^\dagger(t)) \tag{31b}$$

$$\hat{s}(t) = \hat{\overline{s}}(t) + \Delta\hat{s}(t) \quad ; \quad \hat{\overline{s}}(t) = -\frac{i}{2}(\hat{\overline{n}}(t) - \hat{\overline{n}}^\dagger(t)) \quad ; \quad \Delta\hat{s}(t) = -\frac{i}{2}(\Delta\hat{n}(t) - \Delta\hat{n}^\dagger(t)) \tag{31c}$$

The corresponding state amplitudes starting from an initial Fock state amplitudes

$$|\hat{r}(t)\rangle = \frac{1}{2}(|\hat{n}(t)\rangle + |\hat{n}^\dagger(t)\rangle) \quad ; \quad |\hat{s}(t)\rangle = -\frac{i}{2}(|\hat{n}(t)\rangle - |\hat{n}^\dagger(t)\rangle) \tag{31d}$$

with Hermitian conjugates

$$|\hat{r}(t)\rangle = \hat{r}(t)|Nn\rangle \quad , \quad \langle\hat{r}(t)| = \langle nN|\hat{r}(t) \quad ; \quad |\hat{s}(t)\rangle = \hat{s}(t)|Nn\rangle \quad , \quad \langle\hat{s}(t)| = \langle nN|\hat{s}(t) \tag{31e}$$

are obtained using equation (29a) in the form

$$|\hat{r}(t)\rangle = \overline{r}(t)|Nn\rangle + |\Delta\hat{r}(t)\rangle \quad ; \quad |\hat{s}(t)\rangle = \overline{s}(t)|Nn\rangle + |\Delta\hat{s}(t)\rangle \tag{31f}$$

where the mean values $\overline{r}(t)$, $\overline{s}(t)$ and fluctuation state amplitudes $|\Delta\hat{r}(t)\rangle$, $|\Delta\hat{s}(t)\rangle$ have been obtained as

$$\overline{r}(t) = \frac{1}{2}(\overline{n}(t) + \overline{n^*}(t)) \quad ; \quad \overline{s}(t) = -\frac{i}{2}(\overline{n}(t) - \overline{n^*}(t)) \tag{31g}$$

$$|\Delta\hat{r}(t)\rangle = \frac{1}{2}(|\Delta\hat{n}(t)\rangle + |\Delta\hat{n}^{\dagger}(t)\rangle) \quad ; \quad |\Delta\hat{s}(t)\rangle = -\frac{i}{2}(|\Delta\hat{n}(t)\rangle - |\Delta\hat{n}^{\dagger}(t)\rangle) \qquad (31h)$$

Fluctuations in the measurements of the mean values of the Hermitian number components are obtained as the overlaps of the corresponding fluctuation state amplitudes according to

$$(\ \Delta\overline{r}(t)\ )^2 = \langle\Delta\hat{r}(t)|\Delta\hat{r}(t)\rangle \quad ; \quad (\ \Delta\overline{s}(t)\ )^2 = \langle\Delta\hat{s}(t)|\Delta\hat{s}(t)\rangle \qquad (32a)$$

which on using equation (31d) together with its Hermitian conjugate take the form

$$(\ \Delta\overline{r}(t)\ )^2 = \frac{1}{4}\{(\langle\Delta\hat{n}(t)|\Delta\hat{n}(t)\rangle + \langle\Delta\hat{n}^{\dagger}(t)|\Delta\hat{n}^{\dagger}(t)\rangle) + (\langle\Delta\hat{n}^{\dagger}(t)|\Delta\hat{n}(t)\rangle + \langle\Delta\hat{n}(t)|\Delta\hat{n}^{\dagger}(t)\rangle)\} \qquad (32b)$$

$$(\ \Delta\overline{s}(t)\ )^2 = \frac{1}{4}\{(\langle\Delta\hat{n}(t)|\Delta\hat{n}(t)\rangle + \langle\Delta\hat{n}^{\dagger}(t)|\Delta\hat{n}^{\dagger}(t)\rangle) - (\langle\Delta\hat{n}^{\dagger}(t)|\Delta\hat{n}(t)\rangle + \langle\Delta\hat{n}(t)|\Delta\hat{n}^{\dagger}(t)\rangle)\} \qquad (32c)$$

Noting the definitions given in equations (30a)-(30b) and substituting the results from equations (30c)-(30f) into equations (32b)-(32c), we obtain the fluctuations in explicit form

$$(\ \Delta\overline{r}(t)\ )^2 = \frac{1}{4}\{|\mu_+ + \mu_-^*|^2\nu_+^2 n(n-1) + |\overline{\mu}_+ + \tilde{\mu}_-^*|^2\nu_-^2(n+1)(n+2)\} \qquad (32d)$$

$$(\ \Delta\overline{s}(t)\ )^2 = \frac{1}{4}\{|\mu_+ - \mu_-^*|^2\nu_+^2 n(n-1) + |\overline{\mu}_+ - \tilde{\mu}_-^*|^2\nu_-^2(n+1)(n+2)\} \qquad (32e)$$

Comparing equations (32d) and (32e) gives the inequality

$$(\ \Delta\overline{s}(t)\ )^2 < (\ \Delta\overline{r}(t)\ )^2 \qquad (32f)$$

which reveals the expected squeezing phenomenon.

We notice that the fluctuations $\Delta\overline{s}(t)$ of the imaginary part obtained in equation (32e) constitute fundamental *number* quantum noise spectrum $\tilde{\mathcal{N}}_n(t)$ expressed in the form

$$(\ \Delta\overline{s}(t)\ )^2 = \tilde{\mathcal{N}}_n(t) \qquad (32g)$$

where the fundamental number quantum noise is obtained as

$$\tilde{\mathcal{N}}_n(t) = \frac{1}{4}\{|\mu_+ - \mu_-^*|^2\nu_+^2 n(n-1) + |\overline{\mu}_+ - \tilde{\mu}_-^*|^2\nu_-^2(n+1)(n+2)\} \quad ; \quad n = 0,1,2,3,.... \qquad (32h)$$

We demonstrate in section 5 below that this fundamental number quantum noise spectrum, which arises as an effect of the full quantum interaction, vanishes both at initial time $t = 0$ and in the semiclassical model of the degenerate parametric amplification process. We observe that $\hat{s}(t)$ is a purely quantum Hermitian component of the complex time evolving number operator, which vanishes both at initial time and in the semiclassical model where the number operator is real, having only the real part and a null imaginary part.

4.2 Heisenberg uncertainty relations and fundamental number correlation quantum noise

The cross-correlation functions of the Hermitian number operator components $\hat{r}(t)$, $\hat{s}(t)$ are obtained as overlaps of the corresponding number operator state amplitudes defined in equations (31d)-(31f) according to

$$\overline{r(t)s(t)} = \langle\hat{r}(t)|\hat{s}(t)\rangle = \overline{r}(t)\overline{s}(t) + \langle\Delta\hat{r}(t)|\Delta\hat{s}(t)\rangle \quad ; \quad \overline{s(t)r(t)} = \langle\hat{s}(t)|\hat{r}(t)\rangle = \overline{s}(t)\overline{r}(t) + \langle\Delta\hat{s}(t)|\Delta\hat{r}(t)\rangle \qquad (33a)$$

with corresponding fluctuations following easily in the form

$$(\Delta\overline{r(t)s(t)})^2 = \langle\Delta\hat{r}(t)|\Delta\hat{s}(t)\rangle \quad ; \quad (\Delta\overline{s(t)r(t)})^2 = \langle\Delta\hat{s}(t)|\Delta\hat{r}(t)\rangle \tag{33b}$$

which on using equation (31e) take the form

$$(\Delta\overline{r(t)s(t)})^2 = -\frac{i}{4}\{(\langle\Delta\hat{n}(t)|\Delta\hat{n}(t)\rangle - \langle\Delta\hat{n}^\dagger(t)|\Delta\hat{n}^\dagger(t)\rangle) + (\langle\Delta\hat{n}^\dagger(t)|\Delta\hat{n}(t)\rangle - \langle\Delta\hat{n}(t)|\Delta\hat{n}^\dagger(t)\rangle)\} \tag{33c}$$

$$(\Delta\overline{s(t)r(t)})^2 = \frac{i}{4}\{(\langle\Delta\hat{n}(t)|\Delta\hat{n}(t)\rangle - \langle\Delta\hat{n}^\dagger(t)|\Delta\hat{n}^\dagger(t)\rangle) - (\langle\Delta\hat{n}^\dagger(t)|\Delta\hat{n}(t)\rangle - \langle\Delta\hat{n}(t)|\Delta\hat{n}^\dagger(t)\rangle)\}) \tag{33d}$$

We notice that

$$| (\Delta\overline{s(t)r(t)})^2 | < | (\Delta\overline{r(t)s(t)})^2 | \tag{33e}$$

which again reveals a squeezing phenomenon. The explicit forms are obtained using results presented in equations (30a)-(30f).

We use equations (33a)-(33b) to obtain the cross-correlation function relation

$$\overline{r(t)s(t)} - \overline{s(t)r(t)} = \langle\Delta\hat{r}(t)|\Delta\hat{s}(t)\rangle - \langle\Delta\hat{s}(t)|\Delta\hat{r}(t)\rangle = (\Delta\overline{r(t)s(t)})^2 - (\Delta\overline{s(t)r(t)})^2 \tag{34a}$$

which we substitute into the general form of the Heisenberg uncertainty relation in equation (24e) for $\hat{A}(t) = \hat{r}(s)$, $\hat{B}(t) = \hat{s}(s)$ to the Hermitian number cross-correlation uncertainty inequality

$$(\Delta\overline{r(t)})^2(\Delta\overline{s(t)})^2 \geq \frac{1}{4}\Big| (\Delta\overline{r(t)s(t)})^2 - (\Delta\overline{s(t)r(t)})^2 \Big|^2 \tag{34b}$$

We use equations (33c)-(33d) to obtain

$$(\Delta\overline{r(t)s(t)})^2 - (\Delta\overline{s(t)r(t)})^2 = -\frac{i}{2}(\langle\Delta\hat{n}(t)|\Delta\hat{n}(t)\rangle - \langle\Delta\hat{n}^\dagger(t)|\Delta\hat{n}^\dagger(t)\rangle) \tag{34c}$$

which on using the definitions and results from equations (30a) , (30c) and (30d) takes the explicit form

$$(\Delta\overline{r(t)s(t)})^2 - (\Delta\overline{s(t)r(t)})^2 = -\frac{i}{2}\{(|\mu_+|^2 - |\mu_-|^2)\nu_+^2 n(n-1) + (|\overline{\mu}_-|^2 - |\overline{\mu}_+|^2)\nu_-^2(n+1)(n+2)\} \tag{34d}$$

which we substitute into equation (34e) to obtain the uncertainty relation in the final form

$$(\Delta\overline{r(t)})^2(\Delta\overline{s(t)})^2 \geq \frac{1}{16}\Big| \{(|\mu_+|^2 - |\mu_-|^2)\nu_+^2 n(n-1) + (|\tilde{\mu}_-|^2 - |\overline{\mu}_+|^2)\nu_-^2(n+1)(n+2)\} \Big|^2 \tag{34e}$$

We express this as

$$(\Delta\overline{r(t)})^2(\Delta\overline{s(t)})^2 \geq \Big|\tilde{\mathcal{N}}_n^{rs}(t)\Big|^2 \tag{34f}$$

after introducing *fundamental number correlation quantum noise* $\tilde{\mathcal{N}}_n^{rs}(t)$ obtained as

$$\tilde{\mathcal{N}}_n^{rs}(t) = \frac{1}{4}\{(|\mu_+|^2 - |\mu_-|^2)\nu_+^2 n(n-1) + (|\tilde{\mu}_-|^2 - |\overline{\mu}_+|^2)\nu_-^2(n+1)(n+2)\} \tag{34g}$$

Since $\hat{s}(t)$ as defined in equation (31a) is a purely quantum Hermitian operator and its fluctuation $\Delta\overline{s(t)}$ is therefore a purely quantum fluctuation, which vanishes in the semiclassical model, we conclude that the signal-idler photon number uncertainty relation in equation (34f) is a fundamental quantum mechanical uncertainty relation (i.e., nonclassical uncertainty relation) characterizing the general dynamics of a degenerate parametric amplification process generated by quantized multi-mode pump photons as specified in the Hamiltonian H in equation (1a).

5 Comparison with the semiclassical model

To establish that the fundamental quantum noise is a purely quantum effect, we consider its value in the most commonly applied semiclassical model of the degenerate parametric amplification process. In the semiclassical model (or parametric approximation), it is assumed that the pump photons are generated by very high intensity laser sources such that the pump annihilation and creation operators in the Hamiltonian H in equation ($1a$) are replaced by c-numbers α_j , α_j^*, according to

$$\hat{a}_j \equiv \alpha_j \quad , \quad \hat{a}_j^\dagger \equiv \alpha_j^* \qquad ; \qquad \hat{f} \equiv \prod_{j=1}^{\bar{N}} \alpha_j = \alpha \quad , \quad \hat{f}^\dagger \equiv \alpha^* \tag{35a}$$

$$N_+ = N_- = |\alpha|^2 \qquad ; \qquad \mathcal{R} = R = \tilde{\mathcal{R}} = \tilde{R} = \bar{R} = g\sqrt{|\alpha|^2 - k^2} \tag{35b}$$

$$\mathcal{R}t = Rt = \tilde{\mathcal{R}}t = \tilde{R}t = \bar{R}t \quad \Rightarrow \quad \mu_+(t) = \mu_-^*(t) = \overline{\mu}_+ = \tilde{\mu}_-^* = \mu(t) \quad ; \quad \nu_+(t) = \nu_-(t) = \nu(t) \tag{35c}$$

which show that in the semiclassical model, there is only one Rabi frequency $\bar{R}$ so that the resulting time evolving quadrature operators remain real and therefore have null imaginary parts.

The general calculations give

$$|\psi^{sc}(t)\rangle = U^{sc}(t)|\psi(0)\rangle \tag{35d}$$

where $|\psi(0)\rangle$ is the initial state amplitude and $U^{sc}(t)$ is the semiclassical time evolution operator obtained as

$$U^{sc}(t) = e^{-\frac{i}{\hbar}\overline{H}_0 t}\hat{S}^{sc}(\alpha) \qquad ; \qquad \hat{S}^{sc}(\alpha) = e^{t\{-i\delta\ \hat{a}^\dagger\hat{a} + \frac{1}{2}g(\alpha\hat{a}^{\dagger 2} - \alpha^*\hat{a}^2)\}} \tag{35e}$$

$$\hat{b}_+^{sc}(t) = \hat{a}(t) \quad ; \quad \hat{b}_-^{sc}(t) = \hat{a}^\dagger(t) \quad ; \quad \hat{a}(t) = e^{-\frac{i}{2}\Omega t}(\mu(t)\hat{a} + \nu(t)\hat{a}^\dagger) \quad ; \quad \hat{a}^\dagger(t) = e^{\frac{i}{2}\Omega t}(\mu^*(t)\hat{a}^\dagger + \nu(t)\hat{a}) \tag{35f}$$

$$\mu(t) = \cosh\left(gt\sqrt{|\alpha|^2 - k^2}\right) - i\frac{k}{\sqrt{|\alpha|^2 - k^2}}\sinh\left(gt\sqrt{|\alpha|^2 - k^2}\right) \tag{35g}$$

$$\nu(t) = \frac{|\alpha|}{\sqrt{|\alpha|^2 - k^2}}\sinh\left(gt\sqrt{|\alpha|^2 - k^2}\right) \tag{35h}$$

Substituting the semiclassical results from equation ($35c$) into the definition of the fundamental quantum noise in Hermitian quadrature and number fluctuations and correlations in equations ($23i$) , ($26g$) , ($26h$) , ($32h$) , ($34g$), respectively, and applying

$$|\mu(t)|^2 - \nu^2(t) = 1 \qquad ; \qquad |\nu(t)|^2 = \nu^2(t) \tag{36a}$$

gives

$$q_n^{sc}(t) = 0 \quad ; \quad q^{sc}(t) = 0 \quad ; \quad q_{12}^{sc}(t) = 0 \quad ; \quad \mathcal{N}_n^{sc}(t) = 0 \quad ; \quad \mathcal{N}_n^{rs:sc}(t) = 0 \tag{36b}$$

where the superscript sc denotes semiclassical model as defined earlier. According to equation ($36b$), the mode of interaction within the semiclassical model does not generate fundamental quantum noise. This confirms that the fundamental quantum noise is a purely quantum effect generated through the full quantum interaction mechanism. In general, semiclassical interaction mechanism, which generates only one Rabi frequency channel, cannot reveal the existence of fundamental quantum noise, just as much as it cannot reveal the occurrence of fundamental quantum phenomena characterized as fractional revivals or collapses and revivals demonstrated in exact calculations in fully quantized parametric processes in [4-6] and computer simulations in earlier work cited in [4 , 5].

The semiclassical quadrature operators are obtained as

$$\hat{a}_1(t) \rightarrow \hat{X}_1^{sc}(t) = \frac{1}{2}(\hat{b}_+^{sc}(t) + \hat{b}_-^{sc}(t)) = \frac{1}{2}(\hat{a}(t) + \hat{a}^\dagger(t)) \quad ; \quad \hat{Y}_1^{sc}(t) = 0 \tag{36c}$$

$$\hat{a}_2(t) \rightarrow \hat{X}_2(t) = -\frac{i}{2}(\hat{b}_+^{sc}(t) - \hat{b}_-^{sc}(t)) = -\frac{i}{2}(\hat{a}(t) - \hat{a}^\dagger(t)) \quad ; \quad \hat{Y}_2^{sc}(t) = 0 \tag{36d}$$

where the quadrature operators $\hat{Y}_1^{sc}(t)$, $\hat{Y}_2^{sc}(t)$ corresponding to the imaginary components $\hat{Y}_1(t)$, $\hat{Y}_2(t)$ in equations (20*a*)-(20*b*) of the full quantum treatment vanish since the semiclassical quadrature operators are real.

The semiclassical quadrature operators satisfy commutation and uncertainty relations in the degenerate signal-idler photon state $|n\rangle$

$$\langle n| \; [\hat{X}_1^{sc}(t) \; , \; \hat{X}_2^{sc}] \; |n\rangle = \frac{i}{2} \qquad ; \qquad \Delta\overline{X}_1^{sc}(t)\Delta\overline{X}_2^{sc}(t) \geq \frac{1}{4} \tag{36e}$$

Comparing the semiclassical results in equation (36*e*) with the fully quantized results obtained in equations (26*c*) and (26*f*) reveals that the full quantum interaction contributes additional fundamental quantum noise terms $\frac{i}{2}q_{12}(t)$ and $\frac{1}{4}q_{12}(t)$, respectively. This has important implications for the growing interest in practical applications of the degenerate parametric amplification (degenerate spontaneous down-conversion) process in quantum information processing, quantum computation and related quantum technologies. The full quantum parametric amplification process involves fundamental quantum noise state amplitudes which do not exist in the widely applied semiclassical model.

In particular, equations (36*c*)-(36*d*) show that the semiclassical quadrature operators are real so that the imaginary components $\hat{Y}_1^{sc}(t)$, $\hat{Y}_2^{sc}(t)$ vanish, while the full quantum interaction generates *purely quantum mechanical* quadrature components $\hat{Y}_1(t)$, $\hat{Y}_2(t)$ obtained as the imaginary parts of the complex time evolving quadrature operators in equations (16*c*)-(16*d*) and (20*a*)-(20*b*). These generate purely quantum mechanical time evolving quadrature state amplitudes $|\hat{Y}_1(t)\rangle$, $|\hat{Y}_2(t)\rangle$ defined in equations (21*b*) and (21*d*) describing purely quantum quadrature fluctuations $\Delta\overline{Y}_1(t)$, $\Delta\overline{Y}_2(t)$ obtained in equation (23*f*). We identify the pure quantum quadrature fluctuations $\Delta\overline{Y}_1(t)$, $\Delta\overline{Y}_2(t)$ as fundamental quadrature quantum noise spectrum according to equations (23*h*)-(23*i*).

The basic semiclassical channel state amplitudes $|\phi_+^{sc}(t)\rangle = \hat{b}_+^{sc}(t)|n\rangle$, $|\phi_-^{sc}(t)\rangle = \hat{b}_-^{sc}(t)|n\rangle$ and their dual counterparts $|\phi_+^{sc*}(t)\rangle = \hat{b}_+^{sc\dagger}(t)|n\rangle$, $|\phi_-^{sc*}(t)\rangle = \hat{b}_-^{sc\dagger}(t)|n\rangle$ are obtained using equation (35*f*) in the final form

$$|\phi_+^{sc}(t)\rangle = e^{-\frac{i}{2}\Omega t}(\mu(t)\sqrt{n}\;|n-1\rangle + \nu(t)\sqrt{n+1}\;|n+1\rangle) \qquad ; \qquad |\phi_-^{sc*}(t) = |\phi_+^{sc}(t)\rangle \tag{37a}$$

$$|\phi_-^{sc}(t)\rangle = e^{\frac{i}{2}\Omega t}(\mu^*(t)\sqrt{n+1}\;|n+1\rangle + \nu(t)\sqrt{n}\;|n-1\rangle) \qquad ; \qquad |\phi_+^{sc*}(t)\rangle = |\phi_-^{sc}(t)\rangle \tag{37b}$$

Notice that these can be obtained by substituting the semiclassical model variables obtained as the *c*-number analogues of the quantized pump mode operators in equation (35*c*) and setting the pump mode state amplitudes $|N\rangle$, $|\prod_{j=1}^{N}(n_j \pm 1)\rangle$ to unity in the corresponding full quantum state amplitudes in equations (17*e*)-(17*h*).

We observe that in the semiclassical model, the state space spanned by the basic state amplitudes $|\phi_+^{sc}(t)\rangle$, $|\phi_-^{sc}(t)\rangle$ coincides with the dual state space spanned by the dual state amplitudes $|\phi_+^{sc*}(t)$, $|\phi_-^{sc*}(t)\rangle$ according to the state amplitude merging relations $|\phi_-^{sc*}(t) = |\phi_+^{sc}(t)\rangle$, $|\phi_+^{sc*}(t)\rangle = |\phi_-^{sc}(t)\rangle$. This means that the basic semiclassical state space can be interpreted as a *degenerate 2-dimensional* Hilbert space effectively spanned by $|\phi_+^{sc}(t)\rangle$, $|\phi_-^{sc}(t)\rangle$. In contrast, corresponding compatible basic state amplitude pairs $(|\phi_+(t)\rangle, |\phi_-^*(t)\rangle)$ in equations (17*e*) , (17*h*) and $(|\phi_-(t)\rangle, |\phi_+^*(t)\rangle)$ in equations

$(17f)$, $(17g)$ of the full quantum model have the same Fock state amplitudes, but different respective interaction variable pairs $(\mu_+(t), \mu_-^*(t))$ or $(\mu_-(t), \mu_+^*(t))$, since $\mu_+(t)$ is defined in terms of the quantized Rabi frequency $\mathcal{R}$, while $\mu_-(t)$ is defined in terms of the quantized Rabi frequency R. This quantized Rabi frequency difference prevents the merging of the compatible full quantum state amplitude pairs, i.e., $\mu_-^*(t) \neq \mu_+(t) \Rightarrow |\phi_-^*(t)\rangle \neq |\phi_+(t)\rangle$, $\mu_+^*(t) \neq \mu_-(t) \Rightarrow |\phi_+^*(t)\rangle \neq |\phi_-(t)\rangle$. Hence, the basic full quantum frequency channel state space and its dual can be interpreted as a *non-degenerate* 4-*dimensional* Hilbert space spanned by $|\phi_+(t)\rangle$, $|\phi_-(t)\rangle$, $|\phi_+^*(t)\rangle$, $|\phi_-^*(t)\rangle$ defined in equations $(17e)$-$(17h)$. The basic full quantum state amplitudes satisfy orthogonality and non-orthogonality relations according to equations $(18a)$-$(18d)$, while the corresponding semiclassical state amplitudes in equations $(37a)$-$(37b)$ are strictly non-orthogonal according to

$$\langle\phi_+^{sc}(t)|\phi_-^{sc}(t)\rangle = \mu^*(t)\nu(t)(2n+1) \quad ; \quad \langle\phi_-^{sc}(t)|\phi_+^{sc}(t)\rangle = \mu(t)\nu(t)(2n+1) \tag{37c}$$

The full quantum dynamical state space is therefore larger and more general, being characterized by both orthogonality and non-orthogonality properties. These properties of orthogonality and non-orthogonality have important theoretical consequences, which can prove useful in practical applications of the full quantum model or semiclassical model of the parametric amplification process, noting that orthogonality eliminates interference effects, while non-orthogonal state amplitudes contribute interference terms in the evaluation of operator correlation functions as shown below in the semiclassical model.

The semiclassical quadrature state amplitudes $|\hat{X}_1^{sc}(t)\rangle$ and $|\hat{X}_2^{sc}(t)\rangle$ are obtained as

$$|\hat{X}_1^{sc}(t)\rangle = \hat{X}_1^{sc}(t)|n\rangle = \frac{1}{2}(|\phi_+^{sc}(t)\rangle + |\phi_-^{sc}(t)\rangle) \quad ; \quad |\hat{X}_2^{sc}(t)\rangle = \hat{X}_2^{sc}(t)|n\rangle = -\frac{i}{2}(|\phi_+^{sc}(t)\rangle - |\phi_-^{sc}(t)\rangle) \tag{38a}$$

The mean values and second moments of the quadrature operators $\hat{X}_1^{sc}(t)$ and $\hat{X}_1^{sc}(t)$ are obtained as overlaps of the corresponding state amplitudes according to

$$\overline{X}_1^{sc}(t) = \langle n|\hat{X}_1^{sc}(t)\rangle = \frac{1}{2}(\langle n|\phi_+^{sc}(t)\rangle + \langle n|\phi_-^{sc}(t)\rangle) \quad ; \quad \overline{X}_2^{sc}(t) = \langle n|\hat{X}_2^{sc}(t)\rangle = -\frac{i}{2}(\langle n|\phi_+^{sc}(t)\rangle - \langle n|\phi_-^{sc}(t)\rangle) \tag{38b}$$

$$\overline{X_1^{sc2}(t)} = \langle\hat{X}_1^{sc}(t)|\hat{X}_1^{sc}(t)\rangle = \frac{1}{4}(\langle\phi_+^{sc}(t)|\phi_+^{sc}(t)\rangle + \langle\phi_-^{sc}(t)|\phi_-^{sc}(t)\rangle + \langle\phi_+^{sc}(t)|\phi_-^{sc}(t)\rangle + \langle\phi_-^{sc}(t)|\phi_+^{sc}(t)\rangle) \tag{38c}$$

$$\overline{X_2^{sc2}(t)} = \langle\hat{X}_2^{sc}(t)|\hat{X}_2^{sc}(t)\rangle = \frac{1}{4}(\langle\phi_+^{sc}(t)|\phi_+^{sc}(t)\rangle + \langle\phi_-^{sc}(t)|\phi_-^{sc}(t)\rangle - \langle\phi_+^{sc}(t)|\phi_-^{sc}(t)\rangle - \langle\phi_-^{sc}(t)|\phi_+^{sc}(t)\rangle) \tag{38d}$$

Substituting equations $(37a)$-$(37b)$ into equations $(38b)$-$(38d)$ gives final results

$$\overline{X}_1^{sc}(t) = 0 \quad ; \quad \overline{X}_2^{sc}(t) = 0 \tag{38e}$$

$$\overline{X_1^{sc2}(t)} = \frac{1}{4}(|\mu|^2 + \nu^2 + \mu^*\nu + \mu\nu)(2n+1) = |\mu+\nu|^2(2n+1) \tag{38f}$$

$$\overline{X_2^{sc2}(t)} = \frac{1}{4}(|\mu|^2 + \nu^2 - \mu^*\nu - \mu\nu)(2n+1) = |\mu-\nu|^2(2n+1) \tag{38g}$$

The quadrature fluctuations follow easily from equations $(38e)$-$(38g)$ as

$$(\Delta\overline{X}_1^{sc}(t))^2 = |\mu+\nu|^2(2n+1) \quad ; \quad (\Delta\overline{X}_2^{sc}(t))^2 = |\mu-\nu|^2(2n+1) \tag{38h}$$

According to equations $(38f)$-$(38h)$, the fluctuations $\Delta\overline{X}_1^{sc}(t)$ in the semiclassical quadrature $\hat{X}_1^{sc}(t)$ are generally larger than the fluctuations $\Delta\overline{X}_2^{sc}(t)$ in the semiclassical quadrature $\hat{X}_2^{sc}(t)$. This is a general squeezing property of the semiclassical degenerate parametric amplification process. We notice that the quadrature fluctuations obtained in equations (38f)-(38h) in the semiclassical model correspond to the Hermitian quadrature fluctuations obtained in equations (23e)-(23f) in the full quantum model. It is evident that the fluctuations in the full quantum model are characterized by the different quantized Rabi frequencies $R_{\pm}$ and fundamental quantum noise spectrum $q_n(t)$ arising as fluctuations $\Delta\overline{Y}_1(t)$, $\Delta\overline{Y}_2(t)$ of the imaginary parts $\hat{Y}_1(t)$, $\hat{Y}_2(t)$, which vanish in the semiclassical model, while the corresponding fluctuations $\Delta\overline{X}_2^{sc}(t)$ in the semiclassical model are general in nature, but do not include any fundamental quantum noise. The last two terms in the general expansions in equations (38c)-(38d) are the interference terms arising from the non-orthogonality relations in equation (37c). Without these interference terms, the fluctuations in the two quadratures would be equal. It means that the squeezing property is an interference effect associated with the non-orthogonality of the basic state amplitudes $|\phi_+^{sc}(t)\rangle, |\phi_-^{sc}(t)\rangle$ which constitute the quadrature state amplitudes defined in equation (38a).

To obtain the semiclassical squeezing property in familiar form, we consider the resonance case

$$\delta = 0\ ,\ k = 0 \quad \Rightarrow \quad \mu(t) = \cosh(Rt) \quad ; \quad \nu(t) = \sinh(Rt) \quad ; \quad R = g|\alpha| \tag{39a}$$

which we substitute into equations (38f)-(38g) to obtain the semiclassical quadrature second moments and fluctuations under resonance in the final form

$$k = 0 \quad \Rightarrow \quad (\Delta\overline{X}_1^{sc}(t))^2 = \overline{X_1^{sc2}(t)} = \frac{1}{4}e^{2Rt}(2n+1) \quad ; \quad (\Delta\overline{X}_2^{sc}(t))^2 = \overline{X_2^{sc2}(t)} = \frac{1}{4}e^{-2Rt}(2n+1) \tag{39b}$$

which show that under resonance, the fluctuations in the semiclassical quadrature $\hat{X}_1^{sc}(t)$ grow exponentially, while the fluctuations in the semiclassical quadrature $\hat{X}_2^{sc}(t)$ decay exponentially with time for $R > 0$. This is the familiar squeezing property of the semiclassical degenerate parametric amplification process [1 , 6].

For the sake of completeness, we obtain the quadrature cross-correlation functions according to

$$\overline{X_1^{sc}(t)X_2^{sc}(t)} = \langle \hat{X}_1^{sc}(t)|\hat{X}_2^{sc}(t)\rangle \quad ; \quad \overline{X_2^{sc}(t)X_1^{sc}(t)} = \langle \hat{X}_2^{sc}(t)|\hat{X}_1^{sc}(t)\rangle \tag{39c}$$

which on substituting the quadrature state amplitudes defined in equation (38b) and subtracting the results give

$$\overline{X_1^{sc}(t)X_2^{sc}(t)} - \overline{X_2^{sc}(t)X_1^{sc}(t)} = -\frac{i}{2}(\ \langle\phi_+^{sc}(t)|\phi_+^{sc}(t)\rangle - \langle\phi_-^{sc}(t)|\phi_-^{sc}(t)\rangle\ ) \tag{39d}$$

Using $|\phi_+^{sc}(t)\rangle, |\phi_-^{sc}(t)\rangle$ from equations (37c)-(37d) and applying $|\mu(t)|^2 - \nu^2(t) = 1$ gives the final result

$$\overline{X_1^{sc}(t)X_2^{sc}(t)} - \overline{X_2^{sc}(t)X_1^{sc}(t)} = \frac{i}{2} \tag{39e}$$

which we substitute into the general uncertainty relation in equation (24e) for $\hat{A}(t) = \hat{X}_1^{sc}(t), \hat{B}(t) = \hat{X}_2^{sc}(t)$ to obtain the expected semiclassical quadrature uncertainty relation given earlier in equation (36e).

In the case of the time evolving number operator, we observed earlier that the imaginary part $\hat{s}(t)$ of the complex time evolving number operator is a purely quantum mechanical quadrature operator obtained in the fully quantized degenerate parametric amplification process. According to the definition in equation (31a), the quadrature $\hat{s}(t)$ has *no initial time analogue*, since it reduces to a *null operator* at initial time $t = 0$ when the initial number operator is real according to

$$t = 0: \quad \hat{n}(0) = \hat{n} = \hat{a}^\dagger\hat{a} \quad , \quad \hat{n}^\dagger = \hat{n} \quad \Rightarrow \quad \hat{s}(0) = \hat{s} = 0 \quad ; \quad \hat{r}(0) = \hat{r} = \hat{n} \tag{40a}$$

Notice that only the real part $\hat{r}(t)$ has an initial time analogue as established in equation (40a).

We also note that in the semiclassical degenerate parametric amplification process, the time evolving number operator $\hat{n}(t)$ obtained using equations (28c) and (35f) as

$$\hat{n}^{sc}(t) = \hat{b}^{sc}_{-}(t)\hat{b}^{sc}_{+}(t) = \hat{a}^{\dagger}(t)\hat{a}(t) \tag{40b}$$

remains real and therefore has a null imaginary part, i.e., $\hat{s}(t) \rightarrow \hat{s}^{sc}(t) = 0$. The time evolving number quadrature $\hat{s}(t)$ thus arises only within the full quantum interaction model of the degenerate parametric amplification process. All the fluctuations, cross-correlation functions and the Heisenberg uncertainty relation involving the purely quantum operator $\hat{s}(t)$ obtained in equations (32e)-(32h) , (33a)-(33e) , (34c)-(34g), are therefore purely quantum mechanical in nature and each can easily be confirmed to vanish under the semiclassical mode of interaction.

6 Conclusion

In this paper, we have developed an elaborate procedure for obtaining exact analytical solutions of the time-dependent Schroedinger equation for a fully quantized degenerate parametric amplification process generated by quantized multi-mode pump photons. Time evolution of the signal-idler photon annihilation and creation operators governed by the general time evolution operator is obtained through operator expansion theorem by applying an effective method of partial evaluation of commutation brackets. The results reveal that the full quantum interaction destroys the degeneracy of the initial degenerate signal-idler photon modes. Every time evolving operator of the effectively non-degenerate signal-idler photon pair then becomes complex and can be decomposed into Hermitian real and imaginary parts. The Hermitian imaginary parts are purely quantum operators which arise through the full quantum interaction and do not have initial time or classical analogues. The fluctuations of these purely quantum imaginary parts constitute related fundamental quantum noise, which are generally smaller than the fluctuations of the corresponding real parts, thus displaying a squeezing phenomenon. General fundamental quantum noise also arise through cross-correlations of the real and imaginary parts. The cross-correlation functions have been used to derive Heisenberg uncertainty relations in which the minimum uncertainty products are modified by corresponding cross-correlation fundamental quantum noise. It has been demonstrated through explicit calculations that the fundamental quantum noise vanishes in the corresponding semiclassical model of the degenerate parametric amplification process.

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References

[1] L Mandel and E Wolf, *Optical Coherence and Quantum Optics*, Cambridge University Press, Cambridge, (1995)

[2] F Dell'Anno, S De Siena and F Illuminati, Phys. Rep. **428** 53 (2006)

[3] Y R Shen, Phys. Rev. **155** 921 (1967)

[4] J Akeyo Omolo, J. Mod. Opt. **60** 5767 (2013); DOI:10.1080/09500340.2013.798039

[5] J Akeyo Omolo, J. Mod. Phys. **5** 706 (2014); DOI:10.4236/jmp.2014.58082

[6] J Akeyo Omolo, *Parametric Processes and Quantum States Of Light*, ISBN:978-3-659-40846-5, Lambert Academic Publishing (LAP International), Berlin, Germany, (2014)

[7] O Morin, et al, J. Vis. Exp. **87** e51224 (2014); arXiv:1407.0183 [quant-ph]

[8] M S Kim, J. Phys. **B41** 133001 (2008); DOI:10.1088/0953.4075/41/133001 ; arXiv:0807.4708 [quant-ph]

[9] A S Coelho, et al, *Universal state orthogonalizer and qubit generator*, arXiv:0807.4708 [quant-ph]

[10] M Jezek, et al, Phys. Rev. **A89** 042316 (2014); arXiv:1404.4308 [quant-ph]

[11] M R Vanner, M Aspelmeyer and M S Kim, Phys. Rev. Lett. **110** 010504 (2013); DOI:10.1103/PhysRevLett.110.010504 ; arXiv:1203.4525 [quant-ph]