On a complex spacetime frame : new foundation for physics and mathematics

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March 31, 2016

Abstract

This paper provides a derivation of a unit vector specifying the temporal axis of four-dimensional space-time frame as an imaginary axis in the direction of light or general wave propagation. The basic elements of the resulting complex four-dimensional spacetime frame are complex four-vectors expressed in standard form as four-component quantities specified by four unit vectors, three along space axes and one along the imaginary temporal axis. Consistent mathematical operations with the complex four-vectors have been developed, which provide extensions of standard vector analysis theorems to complex four-dimensional spacetime. The general orientation of the temporal unit vector relative to all the three mutually perpendicular spatial unit vectors leads to appropriate modifications of fundamental results describing the invariant length of the spacetime event interval, time dilation, mass increase with speed and relativistic energy conservation law. Contravariant and covariant forms have been defined, providing appropriate definition of the invariant length of a complex four-vector and appropriate definitions of complex tensors within the complex four-dimensional spacetime frame, thus setting a new geometrical framework for physics and mathematics.

1 Introduction

Four-dimensional space-time frame is the natural reference frame for describing the general dynamics of physical systems. Its basic elements are

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four-vectors, which are suitable for expressing field equations and associated physical quantities in consistent forms.

An interesting point to note is that, while physicists and mathematicians are quite comfortable using four-vectors to develop field theories and generalizations of geometry, no attempt has ever been made to identify a unit vector to specify the temporal direction in a manner equivalent to the specification of the three space axes by corresponding unit vectors $\hat{x}$, $\hat{y}$, $\hat{z}$. The current representation of four-vectors in contravariant and covariant forms is obviously *incomplete*, since a unit vector specifying the temporal axis has not been defined.

The definition and specification of a reference frame for theoretical description and experimental verification of various features of the dynamics of a physical system should be complete and the relative orientations of its coordinate axes should be as general as possible. In general, each *coordinate axis* is a vector specified by a *unit vector* giving its direction and a *coordinate* giving its magnitude. In the current definition of the dynamical spacetime reference frame, only the three spatial axes, namely x-axis, y-axis and z-axis, have been fully defined by specifying their corresponding unit vectors and coordinates. The temporal axis has not been fully defined since a corresponding unit vector giving its direction has not been identified. The current definition of the dynamical spacetime frame is therefore incomplete. It is important to note that *the general orientation of the temporal axis relative to the three spatial axes cannot be fixed without specifying the temporal unit vector*, contrary to the present assumption implicit in conventional four-vector mathematics that the temporal axis is perpendicular to all the three mutually perpendicular spatial axes. We observe that there is no physically justifiable reason why a dynamical reference frame such as the four-dimensional spacetime frame must be composed of mutually perpendicular coordinate axes. We therefore propose that the relative orientations of all the four spacetime coordinate axes should be general enough to accommodate as much information as possible about the dynamics of a physical system.

The purpose of this paper is to derive and specify the temporal unit vector to complete and generalize the specification of the dynamical four-dimensional spacetime frame. The derivation is presented in the next section and the appropriate definition of four-vectors, together with suitable mathematical operations and physical consequences, are developed in the subsequent sections.

2 The temporal unit vector

The starting point is the realization that dynamics in a general four-dimensional spacetime frame is governed by wave equations, which we express in the form

$$(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2)\psi = -f(\mathbf{r}, \mathbf{t}) \tag{1}$$

where $\psi(\mathbf{r}, \mathbf{t})$ is an arbitrary function which may represent a temporal or spatial component of a four-vector, while $f(\mathbf{r}, \mathbf{t})$ is a function representing external sources.

In the standard Cartesian coordinate system, the spatial component of the derivative operator ∇ is a three-component vector defined as

$$\nabla = \frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}} \tag{2a}$$

where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ are the mutually perpendicular unit vectors along the three space directions.

The Laplacian ∇^2 is then obtained as a vector dot product

$$\nabla^2 = \nabla \cdot \nabla \;=\; (\frac{\partial}{\partial x}\hat{\mathbf{x}}) \cdot (\frac{\partial}{\partial x}\hat{\mathbf{x}}) + (\frac{\partial}{\partial y}\hat{\mathbf{y}}) \cdot (\frac{\partial}{\partial y}\hat{\mathbf{y}}) + (\frac{\partial}{\partial z}\hat{\mathbf{z}}) \cdot (\frac{\partial}{\partial z}\hat{\mathbf{z}}) \tag{2b}$$

The second order partial differential operator on the l.h.s. of of the wave equation (1) is then expressed in the form

$$-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla \cdot \nabla = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + (\frac{\partial}{\partial x}\hat{\mathbf{x}}) \cdot (\frac{\partial}{\partial x}\hat{\mathbf{x}}) + (\frac{\partial}{\partial y}\hat{\mathbf{y}}) \cdot (\frac{\partial}{\partial y}\hat{\mathbf{y}}) + (\frac{\partial}{\partial z}\hat{\mathbf{z}}) \cdot (\frac{\partial}{\partial z}\hat{\mathbf{z}}) \tag{2c}$$

The form in equation $(2c)$ makes it evidently clear that some information is missing, since the spatial component ∇^2 can be expressed in vector dot product form with the vector ∇ defined in equation $(2a)$, while the temporal component $-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ cannot be expressed in a similar vector dot product form without defining and specifying a unit vector along the temporal direction.

We observe that the missing information is hidden in the speed of light occurring in the temporal component in the form $\frac{1}{c^2}$ (or for any general wave with phase velocity, the information is hidden in the speed occurring in the temporal component in the form $\frac{1}{v^2}$). This follows from the fact that the speed c, wave number k and angular frequency ω of light are related in the form

$$c = \frac{\omega}{k} \quad \Rightarrow \quad \frac{1}{c^2} = \frac{k^2}{\omega^2} \tag{3a}$$

Note that for a general wave characterized by a phase velocity $\mathbf{v}$, the same relation applies, i.e., $v = \frac{\omega}{k}$ and we can replace c with v everywhere.

We introduce the wave vector $\mathbf{k}$ to obtain the dot product form

$$k^2 = \mathbf{k} \cdot \mathbf{k} \quad \Rightarrow \quad \frac{1}{c^2} = \frac{\mathbf{k}}{\omega} \cdot \frac{\mathbf{k}}{\omega} \tag{3b}$$

Expressing the wave vector $\mathbf{k}$ in terms of its unit vector $\hat{\mathbf{k}}$ and wave number k through the standard definition

$$\mathbf{k} = k\hat{\mathbf{k}} \tag{3c}$$

we write equation $(3b)$ in the useful form

$$\frac{1}{c^2} = \frac{k}{\omega}\hat{\mathbf{k}} \cdot \frac{k}{\omega}\hat{\mathbf{k}} \tag{3d}$$

Substituting $\frac{1}{c^2}$ from equation $(3d)$ into the temporal component, introducing the imaginary number $i = \sqrt{-1}$ and reintroducing the speed c by substituting $\frac{1}{c} = \frac{k}{\omega}$ from equation $(3a)$, we express the temporal component in the desired one-dimensional dot product form

$$-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} = (\frac{i}{c}\frac{\partial}{\partial t}\hat{\mathbf{k}}) \cdot (\frac{i}{c}\frac{\partial}{\partial t}\hat{\mathbf{k}}) \tag{4b}$$

which is now consistent with the dot product forms of the spatial components in equation $(2d)$. This is the main result in this paper. We have succeeded in deriving and identifying the *unit wave vector* $\hat{\mathbf{k}}$ as the unit vector specifying the temporal direction in four-dimensional spacetime frame. The occurrence of the imaginary number $i = \sqrt{-1}$ leads to the interpretation that the temporal axis is an *imaginary axis* specified by the unit wave vector $\hat{\mathbf{k}}$ defining the direction of light / wave propagation within a complex four-dimensional spacetime frame.

2.1 The derivative four-vector

Substituting equation (4) into equation $(2c)$ now provides the complete dot product form obtained as

$$-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2 = (\frac{i}{c}\frac{\partial}{\partial t}\hat{\mathbf{k}}) \cdot (\frac{i}{c}\frac{\partial}{\partial t}\hat{\mathbf{k}}) + \nabla \cdot \nabla \tag{5a}$$

The r.h.s. of equation $(5a)$ can be expressed in vector dot product form

$$(\frac{i}{c}\frac{\partial}{\partial t}\hat{\mathbf{k}}) \cdot (\frac{i}{c}\frac{\partial}{\partial t}\hat{\mathbf{k}}) + \nabla \cdot \nabla = (\frac{i}{c}\frac{\partial}{\partial t}\hat{\mathbf{k}} + \nabla) \cdot (\frac{i}{c}\frac{\partial}{\partial t}\hat{\mathbf{k}} + \nabla) + d \tag{5b}$$

where d is a derivative operator composed of extra terms which arise if the temporal unit vector, $\hat{\mathbf{k}}$, has general orientation not perpendicular to the spatial unit vectors, $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$, thus satisfying

$$\hat{\mathbf{k}} \cdot \hat{\mathbf{x}} \neq 0 \quad ; \quad \hat{\mathbf{k}} \cdot \hat{\mathbf{y}} \neq 0 \quad ; \quad \hat{\mathbf{k}} \cdot \hat{\mathbf{z}} \neq 0 \quad \Rightarrow \quad d \neq 0 \tag{5c}$$

The derivative operator d vanishes in the special case where the temporal unit vector is perpendicular to all the three spatial unit vectors, thus satisfying

$$\hat{\mathbf{k}} \cdot \hat{\mathbf{x}} = 0 \quad ; \quad \hat{\mathbf{k}} \cdot \hat{\mathbf{y}} = 0 \quad ; \quad \hat{\mathbf{k}} \cdot \hat{\mathbf{z}} = 0 \quad \Rightarrow \quad d = 0 \tag{5d}$$

It is then clear that the first term on the r.h.s. of equation (5b) is the dot product of a four-component derivative vector, which we shall call the *spacetime derivative four-vector* denoted by $\vec{\nabla}$ taking the form

$$\vec{\nabla} = \frac{i}{c}\frac{\partial}{\partial t}\hat{\mathbf{k}} + \nabla = \frac{i}{c}\frac{\partial}{\partial t}\hat{\mathbf{k}} + \frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}} \tag{6a}$$

We then obtain

$$\vec{\nabla} \cdot \vec{\nabla} = (\frac{i}{c}\frac{\partial}{\partial t}\hat{\mathbf{k}} + \nabla) \cdot (\frac{i}{c}\frac{\partial}{\partial t}\hat{\mathbf{k}} + \nabla) \tag{6b}$$

which we substitute into equation (5b) to obtain

$$\vec{\nabla} \cdot \vec{\nabla} = -(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + d) \tag{6c}$$

Full expansion of $\vec{\nabla} \cdot \vec{\nabla}$ as defined in equation (6b), considering general orientation of $\hat{\mathbf{k}}$ specified by equation (5c), provides the general form of d to be used in equation (6c), revealing a modification of the standard d'Alembertian operator $\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$ which is a wave propagation operator in classical and quantum field theories. The usual form of the d'Alembertian operator is obtained in the conventional four-vector mathematics based on the assumption that the temporal and spatial axes in a spacetime frame are all mutually perpendicular.

3 Vectors in four-dimensional spacetime

Having defined the spacetime derivative four-vector $\vec{\nabla}$ in equation (6a), we proceed to the general definition of the other physical or mathematical four-vectors in similar form and then developing the appropriate mathematical operations with the four-vectors in the complex four-dimensional spacetime.

We define a general complex four-vector $\vec{V}$ in the form

$$\vec{V} = V_k\hat{\mathbf{k}} + \mathbf{V} = V_k\hat{\mathbf{k}} + V_x\hat{\mathbf{x}} + V_y\hat{\mathbf{y}} + V_z\hat{\mathbf{z}} \tag{7a}$$

with the imaginary temporal component V_k defined by

$$V_k = -ic\mathcal{V} \quad ; \quad \mathcal{V} = \mathcal{V}(t, \mathbf{r}) \tag{7b}$$

where $\mathcal{V}$ is a scalar quantity specifying the nature of the temporal component of the four-vector. In general, the temporal component of each four-vector occurs along the imaginary temporal axis, $i\hat{\mathbf{k}}$, multiplied by a factor $-c$, where c is the speed of light. We observe that if we base the derivation on a general wave equation characterized by a phase velocity $\mathbf{v}$, then c would be replaced everywhere by the speed $v = |\mathbf{v}|$ of the general wave. The speed $v = c$ then becomes specific to light or electromagnetic wave.

According to the general definition in equations (7a)-(7b), the spacetime derivative, $\vec{\nabla}$, current density, $\vec{J}$, linear momentum, $\vec{P}$ and field potential, $\vec{A}$, four-vectors are obtained as

$$\vec{V} = \vec{\nabla} \quad \Rightarrow \quad V_k = \frac{i}{c}\frac{\partial}{\partial t} \quad ; \quad \mathcal{V} = -\frac{1}{c^2}\frac{\partial}{\partial t} \quad ; \quad \mathbf{V} = \nabla \tag{8a}$$

$$\vec{V} = \vec{J} \quad \Rightarrow \quad V_k = -ic\rho \quad ; \quad \mathcal{V} = \rho \quad ; \quad \mathbf{V} = \mathbf{J} \tag{8b}$$

$$\vec{V} = \vec{A} \quad \Rightarrow \quad V_k = -ic\phi \quad ; \quad \mathcal{V} = \phi \quad ; \quad \mathbf{V} = \mathbf{A} \tag{8c}$$

$$\vec{V} = \vec{P} \quad \Rightarrow \quad V_k = -imc \quad ; \quad \mathcal{V} = m \quad ; \quad \mathbf{V} = \mathbf{p} \tag{8d}$$

where m denotes mass.

The spacetime displacement four-vector $\vec{X}$ takes the form

$$\vec{X} = x_k\hat{\mathbf{k}} + \mathbf{r} = x_k\hat{\mathbf{k}} + x\ \hat{\mathbf{x}} + y\ \hat{\mathbf{y}} + z\ \hat{\mathbf{z}} \quad ; \quad x_k = -ict \tag{8e}$$

while the corresponding *event interval* $d\vec{X}$ between two neighbouring spacetime points, takes the form

$$d\vec{X} = dx_k\hat{\mathbf{k}} + \mathbf{dr} = dx_k\hat{\mathbf{k}} + dx\ \hat{\mathbf{x}} + dy\ \hat{\mathbf{y}} + dz\ \hat{\mathbf{z}} \quad ; \quad dx_k = -icdt \tag{8f}$$

4 Mathematical operations with four-vectors

The general four-component vector form in equation (7a) with all unit vectors specified now allows us to carry out four-vector mathematical operations in the complex spacetime frame in exactly the same manner as the standard mathematical operations with the familiar three-component vectors in three-dimensional space.

In developing the mathematical operations in general form, we shall take the temporal unit vector $\hat{\mathbf{k}}$ to be of general orientation relative to the spatial unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$, satisfying the conditions in equation ($5c$). We denote four-vectors with arrows, while the three-component spatial vectors are written in boldface as usual. We use two general four-vectors $\vec{V}$ and $\vec{U}$ defined by

$$\vec{V} = V_k\ \hat{\mathbf{k}} + \mathbf{V} \quad , \quad V_k = -ic\mathcal{V} \quad ; \quad \vec{U} = U_k\ \hat{\mathbf{k}} + \mathbf{U} \quad , \quad U_k = -ic\,\mathcal{U} \tag{9}$$

to develop the mathematical operations with four-vectors. The basic mathematical operations are essentially addition, subtraction, dot product, cross product, divergence and curl.

4.1 Addition and subtraction

Four-vector addition and subtraction is straightforward, taking the form

$$\vec{W} = \vec{U} \pm \vec{V} = (U_k + V_k)\ \hat{\mathbf{k}} \pm (\mathbf{U} + \mathbf{V}) \tag{10}$$

4.2 The dot product

The dot product of the four-vectors $\vec{U}$ and $\vec{V}$ is obtained as

$$\vec{U} \cdot \vec{V} = (U_k\ \hat{\mathbf{k}} + \mathbf{U}) \cdot (V_k\ \hat{\mathbf{k}} + \mathbf{V}) \tag{11a}$$

which we expand term by term, maintaining the order of components in the products and then substitute $U_k = -ic\,\mathcal{U}$, $V_k = -ic\mathcal{V}$ from equation (9), to obtain the dot product in the final form

$$\vec{U} \cdot \vec{V} = \mathbf{U} \cdot \mathbf{V} - c^2\ \mathcal{U}\ \mathcal{V} - ic\ \hat{\mathbf{k}} \cdot (\mathcal{U}\ \mathbf{V} + \mathbf{U}\ \mathcal{V}) \tag{11b}$$

4.3 The cross product

The cross product of the four-vectors $\vec{U}$ and $\vec{V}$ is obtained as

$$\vec{U} \times \vec{V} = (U_k\ \hat{\mathbf{k}} + \mathbf{U}) \times (V_k\ \hat{\mathbf{k}} + \mathbf{V}) \tag{12a}$$

which we expand term by term, using $\hat{\mathbf{k}} \times \hat{\mathbf{k}} = 0$, $\mathbf{U} \times \hat{\mathbf{k}} = -\hat{\mathbf{k}} \times \mathbf{U}$ and then substitute $U_k = -ic\ \mathcal{U}$, $V_k = -ic\mathcal{V}$ from equation (9) to obtain the cross product in the final form

$$\vec{U} \times \vec{V} = \mathbf{U} \times \mathbf{V} - ic\ \hat{\mathbf{k}} \times (\mathcal{U}\ \mathbf{V} - \mathbf{U}\ \mathcal{V}) \tag{12b}$$

4.4 Divergence of a four-vector

Setting $\vec{U}$ equal to the spacetime derivative four-vector $\vec{\nabla}$ according to

$$\vec{U} = \vec{\nabla} = \frac{i}{c}\frac{\partial}{\partial t}\hat{\mathbf{k}} + \nabla = -ic(-\frac{1}{c^2}\frac{\partial}{\partial t})\hat{\mathbf{k}} + \nabla = -ic\,\mathcal{U}\,\hat{\mathbf{k}} + \mathbf{U} \qquad (13a)$$

with

$$\mathcal{U} = -\frac{1}{c^2}\frac{\partial}{\partial t} \quad ; \quad \mathbf{U} = \nabla \qquad (13b)$$

in the general four-vector dot product obtained in equation (11b), we obtain the divergence of a general four-vector $\vec{V}$ in the final form

$$\vec{\nabla} \cdot \vec{V} = \frac{\partial \mathcal{V}}{\partial t} + \nabla \cdot \mathbf{V} + i\hat{\mathbf{k}} \cdot (\frac{1}{c}\frac{\partial \mathbf{V}}{\partial t} - \nabla(c\,\mathcal{V})) \qquad (13c)$$

4.5 Curl of a four-vector

Setting $\vec{U}$ equal to the spacetime derivative four-vector $\vec{\nabla}$ according to equations (13a)-(13b) in the general four-vector cross product obtained in equation (12b), we obtain the curl of a general four-vector $\vec{V}$ in the final form

$$\vec{\nabla} \times \vec{V} = \nabla \times \mathbf{V} - i(\frac{1}{c}\frac{\partial \mathbf{V}}{\partial t} + \nabla(c\,\mathcal{V})) \times \hat{\mathbf{k}} \qquad (14)$$

4.6 Four-vector theorems

We now derive three theorems for four-vectors in complex four-dimensional spacetime frame, which generalize standard vector theorems in three-dimensional space.

4.6.1 Curl of gradient four-vector

A gradient four-vector $\vec{\nabla}\phi$ generated through application of the spacetime derivative four-vector $\vec{\nabla}$ on a scalar function ϕ is obtained as

$$\vec{\nabla}\phi = (\frac{i}{c}\frac{\partial}{\partial t}\hat{\mathbf{k}} + \nabla)\phi = -ic(-\frac{1}{c^2}\frac{\partial \phi}{\partial t})\hat{\mathbf{k}} + \nabla\phi \qquad (15a)$$

Setting the general four-vector $\vec{V}$ equal to the gradient four-vector according to

$$\vec{V} = \vec{\nabla}\phi = -ic(-\frac{1}{c^2}\frac{\partial \phi}{\partial t})\hat{\mathbf{k}} + \nabla\phi = -ic\,\mathcal{V}\,\hat{\mathbf{k}} + \mathbf{V} \qquad (15b)$$

with

$$\mathcal{V} = -\frac{1}{c^2}\frac{\partial \phi}{\partial t} \quad ; \quad \mathbf{V} = \nabla\phi \qquad (15c)$$

in the general curl of a four-vector obtained in equation (14), we obtain the curl of a gradient four-vector $\vec{\nabla}\phi$ in the form

$$\vec{\nabla} \times \vec{\nabla}\phi = \nabla \times \nabla\phi - i(\frac{1}{c}\frac{\partial \nabla\phi}{\partial t} - \nabla(\frac{1}{c}\frac{\partial \phi}{\partial t})) \times \hat{\mathbf{k}} \tag{15d}$$

which on using the standard results

$$\nabla \times \nabla\phi = 0 \quad ; \quad \frac{1}{c}\frac{\partial \nabla\phi}{\partial t} = \nabla(\frac{1}{c}\frac{\partial \phi}{\partial t}) \tag{15e}$$

gives the final result

$$\vec{\nabla} \times \vec{\nabla}\phi = 0 \tag{16}$$

This shows that the curl of a gradient four-vector vanishes. This generalizes the corresponding vector theorem in standard three-dimensional space.

4.6.2 Divergence of curl of a four-vector

Taking the divergence of the curl of the general four-vector $\vec{V}$ in equation (14), we obtain

$$\vec{\nabla} \cdot \vec{\nabla} \times \vec{V} = (\frac{i}{c}\frac{\partial}{\partial t}\hat{\mathbf{k}} + \nabla) \cdot (\nabla \times \mathbf{V} - i(\frac{1}{c}\frac{\partial \mathbf{V}}{\partial t} + \nabla(c\ \mathcal{V})) \times \hat{\mathbf{k}}) \tag{17a}$$

which on expanding term by term becomes

$$\vec{\nabla} \cdot \vec{\nabla} \times \vec{V} = \frac{i}{c}\frac{\partial}{\partial t}\hat{\mathbf{k}} \cdot \nabla \times \mathbf{V} + \frac{1}{c}\frac{\partial}{\partial t}\hat{\mathbf{k}} \cdot \{(\frac{1}{c}\frac{\partial \mathbf{V}}{\partial t} + \nabla(c\ \mathcal{V})) \times \hat{\mathbf{k}}\} + \nabla \cdot \nabla \times \mathbf{V}$$
$$-i\nabla \cdot \{(\frac{1}{c}\frac{\partial \mathbf{V}}{\partial t} + \nabla(c\ \mathcal{V})) \times \hat{\mathbf{k}}\} \tag{17b}$$

Applying standard vector analysis results gives

$$\hat{\mathbf{k}} \cdot \{(\frac{1}{c}\frac{\partial \mathbf{V}}{\partial t} + \nabla(c\ \mathcal{V})) \times \hat{\mathbf{k}}\} = 0 \quad ; \quad \nabla \cdot \nabla \times \mathbf{V} = 0 \tag{17c}$$

which we use in equation (17b) to obtain

$$\vec{\nabla} \cdot \vec{\nabla} \times \vec{V} = i\{\hat{\mathbf{k}} \cdot (\nabla \times \frac{1}{c}\frac{\partial \mathbf{V}}{\partial t}) - \nabla \cdot \{(\frac{1}{c}\frac{\partial \mathbf{V}}{\partial t} + \nabla(c\ \mathcal{V})) \times \hat{\mathbf{k}}\}\} \tag{17d}$$

Application of standard vector identity

$$\nabla \cdot (\mathbf{Q} \times \mathbf{R}) = \mathbf{R} \cdot (\nabla \times \mathbf{Q}) - \mathbf{Q} \cdot (\nabla \times \mathbf{R}) \tag{17e}$$

gives

$$\nabla\cdot\{(\frac{1}{c}\frac{\partial \mathbf{V}}{\partial t}+\nabla(c\,\mathcal{V}))\times\hat{\mathbf{k}}\} = \hat{\mathbf{k}}\cdot(\nabla\times(\frac{1}{c}\frac{\partial \mathbf{V}}{\partial t}+\nabla(c\,\mathcal{V})))-(\frac{1}{c}\frac{\partial \mathbf{V}}{\partial t}+\nabla(c\,\mathcal{V}))\cdot(\nabla\times\hat{\mathbf{k}}) \tag{17f}$$

which on using

$$\nabla \times \nabla(c\,\mathcal{V}) = 0 \quad ; \quad \nabla \times \hat{\mathbf{k}} = 0 \tag{17g}$$

takes the final form

$$\nabla \cdot \{(\frac{1}{c}\frac{\partial \mathbf{V}}{\partial t} + \nabla(c\,\mathcal{V})) \times \hat{\mathbf{k}}\} = \hat{\mathbf{k}} \cdot (\nabla \times \frac{1}{c}\frac{\partial \mathbf{V}}{\partial t}) \tag{17h}$$

Substituting equation ($17h$) into equation ($17d$) gives the final result

$$\vec{\nabla} \cdot \vec{\nabla} \times \vec{V} = 0 \tag{18}$$

This shows that the divergence of curl of a four-vector vanishes. This generalizes the corresponding vector theorem in standard three-dimensional space.

4.6.3 General vanishing four-vector dot product

The important identity on the vanishing of the divergence of curl of a four-vector in equation (18) can be generalized by taking the dot product of the four-vector $\vec{U}$ and the cross product of the four-vectors $\vec{U}$ and $\vec{V}$ which on using the general result in equation ($12b$) is obtained as

$$\vec{U} \cdot (\vec{U} \times \vec{V}) = (U_k\ \hat{\mathbf{k}} + \mathbf{U}) \cdot (\mathbf{U} \times \mathbf{V} + \hat{\mathbf{k}} \times (\mathbf{U_k}\ \mathbf{V} - \mathbf{U}\ \mathbf{V_k})) \tag{19a}$$

which we expand term by term and use standard results

$$\hat{\mathbf{k}} \cdot \{\hat{\mathbf{k}} \times (U_k\mathbf{V} - \mathbf{U}V_k)\} = 0 \quad ; \quad \mathbf{U} \cdot (\mathbf{U} \times \mathbf{V}) = 0 \tag{19b}$$

to obtain

$$\vec{U} \cdot (\vec{U} \times \vec{V}) = \hat{\mathbf{k}} \cdot (U_k\mathbf{U} \times \mathbf{V}) + \mathbf{U} \cdot \{\hat{\mathbf{k}} \times (U_k\ \mathbf{V} - \mathbf{U}\ V_k)\} \tag{19c}$$

Applying a vector identity

$$\mathbf{U} \cdot \{\hat{\mathbf{k}} \times (U_k\ \mathbf{V} - \mathbf{U}\ V_k)\} = \hat{\mathbf{k}} \cdot \{(U_k\ \mathbf{V} - \mathbf{U}\ V_k) \times \mathbf{U}\} \tag{19d}$$

and using

$$\mathbf{U}V_k \times \mathbf{U} = V_k\mathbf{U} \times \mathbf{U} = 0 \tag{19e}$$

gives

$$\mathbf{U} \cdot \{\hat{\mathbf{k}} \times (U_k\ \mathbf{V} - \mathbf{U}\ V_k)\} = -\hat{\mathbf{k}} \cdot (U_k\ \mathbf{U} \times \mathbf{V}) \tag{19f}$$

which we substitute into equation ($19c$) to obtain the final result

$$\vec{U} \cdot (\vec{U} \times \vec{V}) = 0 \tag{19g}$$

This result generalizes the divergence of curl of a four-vector obtained in equation (18). It is a generalization the corresponding vector theorem in standard three-dimensional space.

The three four-vector theorems derived in equations (16) , (18) and ($19g$) confirm the consistency of the definitions of the complex four-component vectors and corresponding mathematical operations within the complex four-dimensional spacetime frame. This means that complex four-dimensional spacetime frame characterized by complex four-component vectors is a consistent mathematical extension of the standard three-dimensional space characterized by the usual three-component vectors.

5 Physical features of dynamics in the complex spacetime frames

The identification of a temporal unit vector has led to the definition of four-dimensional spacetime frames fully specified by unit vectors along temporal and spatial coordinates. The fact that the temporal axis is imaginary means that the basic elements of the complex four-dimensional spacetime frame are *complex four-vectors* with four components, each component defined along an axis specified by a unit vector as presented above. The mathematical operations with these four-vectors follows the well developed procedures in standard vector analysis using three-component vectors defined within three-dimensional space frames. The basic mathematical operations with fully specified spacetime four-vectors yields consistent results with additional information in the imaginary parts arising as features associated with the imaginary temporal axis. The real parts of the four-vector dot and cross products agree with results usually obtained through conventional four-vector mathematical operations. The divergence and curl of a four-vector take interesting forms with important physical consequences as explained below. The vanishing of the curl of a gradient four-vector and the vanishing of the divergence of curl of a general four-vector in complex spacetime frame is a useful extension of well established theorems derived in standard vector analysis in three-dimensional space frame. In this section, we study some of the physical consequences of the dynamics based on the mathematical operations with complex four-vectors defined within the four-dimensional spacetime frame with an imaginary temporal axis, noting that the four-vectors represent physical quantities which characterize the dynamics of a system.

5.1 General field intensities in a spacetime frame

Rewriting the curl of the general four-vector $\vec{V}$ in equation (14) in the form

$$\vec{\nabla} \times \vec{V} = \nabla \times \mathbf{V} + i(-\frac{1}{c}\frac{\partial \mathbf{V}}{\partial t} - \nabla(c\ \mathcal{V})) \times \hat{\mathbf{k}} \tag{20a}$$

we introduce general field intensities $\mathbf{R}$ and $\mathbf{Q}$ characterizing spacetime dynamics defined by

$$\mathbf{R} = \nabla \times \mathbf{V} \quad ; \quad \mathbf{Q} = -\nabla(c\mathcal{V}) - \frac{1}{c}\frac{\partial \mathbf{V}}{\partial t} \tag{20b}$$

to express the curl of the general four-vector in the form

$$\vec{\nabla} \times \vec{V} = \mathbf{R} + i\mathbf{Q} \times \hat{\mathbf{k}} \tag{20c}$$

This is an important result which shows that the curl of the general complex four-vector $\vec{V}$ generates a general complex field intensity $\mathbf{F}$ taking the form

$$\mathbf{F} = \mathbf{R} + i\mathbf{Q} \times \hat{\mathbf{k}} \tag{20d}$$

The general nature of the curl of a general four-vector as presented in equations (20a)-(20d) presents important physical implications for dynamics characterized by quantities which are generally expressed as complex four-vectors within the complex four-dimensional spacetime frame. We illustrate some of these consequences using electromagnetic and gravitational fields as examples.

For dynamics in an electromagnetic field characterized by the field potential four-vector, we set $\vec{V} = \vec{A}$ in equations (20a)-(20d) then means that the curl of the electromagnetic field potential four-vector generates the familiar magnetic and electric field intensities $\mathbf{B}$ and $\mathbf{E}$ obtained by setting $\mathbf{R} = \mathbf{B}$, $\mathbf{Q} = \mathbf{E}$, $\mathbf{V} = \mathbf{A}$, $\mathcal{V} = \phi$ in equation (20b) giving the usual definitions

$$\mathbf{B} = \nabla \times \mathbf{A} \quad ; \quad \mathbf{E} = -\nabla(c\phi) - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \tag{21a}$$

The total electromagnetic field intensity follows from equation (20d) as

$$\mathbf{F} = \mathbf{B} + i\mathbf{E} \times \hat{\mathbf{k}} \tag{21b}$$

In a gravitational field, equation (20c) shows that the curl of the field potential four-vector provides two components of the gravitational field intensity, one component being the familiar Newtonian gravitational field intensity $\mathbf{g}$ specified by $\mathbf{Q} = \mathbf{g}$ and the other component may be interpreted as a *gravitational field induction* $\mathbf{d}$ responsible for deflection of masses within the gravitational field specified by $\mathbf{R} = \mathbf{d}$ in equation (20b), with $\mathcal{V}$ and $\mathbf{V}$ then representing appropriately defined gravitational scalar and vector potentials, respectively, giving

$$\mathbf{d} = \nabla \times \mathbf{V} \quad ; \quad \mathbf{g} = -\nabla(c\mathcal{V}) - \frac{1}{c}\frac{\partial \mathbf{V}}{\partial t} \tag{21c}$$

The total gravitational field intensity follows from equation (20d) as

$$\mathbf{F} = \mathbf{d} + i\mathbf{g} \times \hat{\mathbf{k}} \tag{21d}$$

In this respect, $\mathbf{g}$ corresponds to the electric field intensity $\mathbf{E}$, while $\mathbf{d}$ corresponds to the magnetic field induction $\mathbf{B}$ in an electromagnetic field. This general result is consistent with usual results obtained in the linearized form of Einstein's field equations in the weak gravitational field limit of the general theory of relativity.

5.2 General field equations in a spacetime frame

We now proceed to derive general field equations governing dynamics in a complex spacetime frame specified by general intensities $\mathbf{R}$, $\mathbf{Q}$ derived from the general complex four-vector $\vec{V}$ according to equation (20b).

Following usual procedure, we take three-dimensional divergence $(\nabla\cdot)$ or curl $(\nabla\times)$ of $\mathbf{R}$, $\mathbf{Q}$ in equation (20b) and apply standard three-dimensional vector analysis results as appropriate to obtain the following general field equations governing dynamics in a complex four-dimensional spacetime frame with an imaginary temporal axis:

$$\nabla \cdot \mathbf{Q} = \varrho \quad ; \quad \nabla \cdot \mathbf{R} = 0 \quad ; \quad \nabla \times \mathbf{Q} = -\frac{1}{c}\frac{\partial \mathbf{R}}{\partial t} \quad ; \quad \nabla \times \mathbf{R} = \frac{1}{c}\frac{\partial \mathbf{Q}}{\partial t} + \frac{1}{c}\mathbf{q} \tag{22a}$$

$$\frac{1}{c^2}\frac{\partial^2 \mathcal{V}}{\partial t} - \nabla^2 \mathcal{V} = \frac{\varrho}{c} \quad ; \quad \frac{1}{c^2}\frac{\partial^2 \mathbf{V}}{\partial t^2} - \nabla^2 \mathbf{V} = -\nabla \mathcal{F}(\mathbf{r}) + \frac{1}{c}\mathbf{q} \tag{22b}$$

satisfying the conditions

$$\frac{\partial \mathcal{V}}{\partial t} + \nabla \cdot \mathbf{V} = \mathcal{F}(\mathbf{r}) \quad ; \quad \frac{\partial \varrho}{\partial t} + \nabla \cdot \mathbf{q} = 0 \tag{22c}$$

We interpret ϱ as *source charge density* and $\mathbf{q}$ as *source current density*, which generate the complex four-vector $\vec{V}$. The general complex source current density four-vector $\vec{q}$ takes the form

$$\vec{q} = -ic\varrho\,\hat{\mathbf{k}} + \mathbf{q} \tag{22d}$$

The source charge and current densities have been introduced by considering that the divergence of a vector generates a scalar quantity, while the curl or time derivative of a vector generates another vector, to set a scalar function $\varrho(t, \mathbf{r})$ equal to $\nabla \cdot \mathbf{Q}$ and a vector in the form $\frac{1}{c}\mathbf{q}$ equal to $\nabla \times \mathbf{R} - \frac{1}{c}\frac{\partial \mathbf{Q}}{\partial t}$ at appropriate stages of the derivation of equations $(22a)$-$(22b)$. Notice that the first condition in equation $(22c)$ is a generalization of the familiar Lorentz gauge condition in electrodynamics, which applies when $\mathcal{F}(\mathbf{r}) = 0$.

A general four-vector with time and space varying components satisfies a wave equation governed by a general gauge condition, a continuity equation for the generating sources and Maxwell type equations for the intensities generated through the curl of the general four-vector. These equations apply to the electromagnetic field four-vector $\vec{V} = \vec{A}$, with the corresponding electric and magnetic field intensities $\mathbf{Q} = \mathbf{E}$, $\mathbf{R} = \mathbf{B}$ and sources being the electric charge and current densities $\varrho = \rho$, $\mathbf{q} = \mathbf{J}$. Both electric current density and the electromagnetic energy density four-vectors also propagate as waves within the four-dimensional spacetime frame. As explained earlier, setting $\vec{V}$ equal to a gravitational field potential four-vector leads to the conclusion that the gravitational field is also characterized by a two-component complex field intensity satisfying Maxwell type equations, while the gravitational field potential satisfies a wave equation within the complex four-dimensional spacetime frame.

5.3 Some fundamental physical consequences

It is now quite clear that complex spacetime frame with an imaginary temporal axis specified by a unit vector in the direction of light (wave) propagation has more general dynamical features compared to usual dynamics within conventional real spacetime frame where the temporal axis is not fully specified. The mathematical operations with the complex four-vectors in the four-dimensional complex spacetime frame reveal additional information associated with the general orientation of the temporal unit vector relative to the spatial coordinates, as well as additional information hidden behind the imaginary temporal axis as we demonstrate below.

5.3.1 General field force

In the dynamical field characterized by a general potential four-vector $\vec{V}$ and corresponding general field intensity $\mathbf{F}$ derived as the curl of $\vec{V}$ in equations (20*c*)-(20*d*), we obtain the general field force $\vec{F}$ as the cross product of the general source current density four-vector $\vec{q}$ and the general field intensity $\mathbf{F}$ using equations (20*c*)-(20*d*) and (22*d*) in the form

$$\vec{F} = \vec{q} \times (\vec{\nabla} \times \vec{V}) = \vec{q} \times \mathbf{F} = \vec{q} \times (\mathbf{R} + i\mathbf{Q} \times \hat{\mathbf{k}}) \qquad (23a)$$

which we expand and reorganize to obtain

$$\vec{F} = c\varrho\ \hat{\mathbf{k}} \times (\mathbf{Q} \times \hat{\mathbf{k}}) + \mathbf{q} \times \mathbf{R} + i\ \{c\varrho\ \mathbf{R} \times \hat{\mathbf{k}} + \mathbf{q} \times (\mathbf{Q} \times \hat{\mathbf{k}})\} \qquad (23b)$$

Applying a standard vector identity gives

$$\hat{\mathbf{k}} \times (\mathbf{Q} \times \hat{\mathbf{k}}) = \mathbf{Q} - (\hat{\mathbf{k}} \cdot \mathbf{Q})\hat{\mathbf{k}} \qquad (23c)$$

which we substitute into equation (31*b*) to obtain the general field force in the final form

$$\vec{F} = c\varrho\ \mathbf{Q} + \mathbf{q} \times \mathbf{R} - c\varrho\ (\hat{\mathbf{k}} \cdot \mathbf{Q})\hat{\mathbf{k}} + i\ \{c\varrho\ \mathbf{R} \times \hat{\mathbf{k}} + \mathbf{q} \times (\mathbf{Q} \times \hat{\mathbf{k}})\} \qquad (23d)$$

We notice that the real part of the general force in equation (23*d*) is composed of a Lorentz force term $c\varrho\ \mathbf{Q} + \mathbf{q} \times \mathbf{R}$ and an additional term $-c\varrho\ (\hat{\mathbf{k}} \cdot \mathbf{Q})\hat{\mathbf{k}}$ associated with the general orientation of the temporal unit vector $\hat{\mathbf{k}}$ relative to the spatial unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$. The imaginary part contains a force $c\varrho\ \mathbf{R} \times \hat{\mathbf{k}} + \mathbf{q} \times (\mathbf{Q} \times \hat{\mathbf{k}})$ acting in a plane perpendicular to the field intensities $\mathbf{R}$, $\mathbf{Q}$ and the temporal unit vector $\hat{\mathbf{k}}$. In a conventional dynamical spacetime frame such as the standard electromagnetic field, only the Lorentz force term is obtained in a derivation using Maxwell's equations in a material medium.

5.3.2 General interaction energy

The general interaction energy for source charges under the action of the general field force within the complex four-dimensional spacetime frame, which constitutes a material medium in this respect, is obtained as the dot product of the source current density four-vector $\vec{q}$ defined in equation (22*d*) and the general field force $\vec{F}$ obtained in equation (23*d*) according to

$$\mathcal{E}_{int} = \vec{q} \cdot \vec{F} \qquad (24a)$$

It is easy to use the definition of the general field force $\vec{F}$ in equation $(23a)$ and apply the general vanishing four-vector dot product theorem established earlier in equation $(19g)$ to obtain

$$\vec{q} \cdot \vec{F} = \vec{q} \cdot (\vec{q} \times (\vec{\nabla} \times \vec{V})) = 0 \qquad (24b)$$

giving the final result for the general interaction energy to be

$$\mathcal{E}_{int} = 0 \qquad (24c)$$

The vanishing of the general interaction energy as established above means that the total energy is conserved in the dynamics under the action of the general field force $\vec{F}$ within the complex four-dimensional spacetime frame defined in a material medium. This is in great contrast to the familiar case of dynamics under the Lorentz force in a conventional electromagnetic field in a material medium where the dot product of the electric current density vector and the Lorentz force yields a non-vanishing interaction energy in the form $\mathbf{J} \cdot \mathbf{E}$, leading to violation of the total energy conservation principle.

5.3.3 Angular momentum

Having obtained the general form of four-vector cross product in equation $(12b)$, we now set $\vec{U}$ equal to the spacetime displacement four-vector $\vec{X}$ defined in equation $(8e)$ and $\vec{V}$ equal to the linear momentum four-vector $\vec{P}$ defined in equation $(8d)$ to obtain the angular momentum $\vec{L}$ in the final form

$$\vec{L} = \vec{X} \times \vec{P} = \mathbf{r} \times \mathbf{p} - ic\ \hat{\mathbf{k}} \times (t\mathbf{p} - m\mathbf{r}) = \mathbf{L} + i\ \mathbf{N} \times \hat{\mathbf{k}} \qquad (25a)$$

where the real part $\mathbf{L}$ is the usual orbital angular momentum defined in 3-dimensional space, while $\mathbf{N}$ is the imaginary temporal component. These angular momentum components are defined by

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \quad ; \quad \mathbf{N} = (ct\mathbf{p} - mc\ \mathbf{r}) \qquad (25b)$$

5.3.4 Invariant length of the spacetime event interval

Setting $\vec{U} = \vec{V} = d\vec{X}$ in the dot product in equation $(11b)$ and using the definition of $d\vec{X}$ given in equation $(8f)$, we obtain

$$d\vec{X} \cdot d\vec{X} = (d\vec{X})^2 = (d\mathbf{r})^2 - (cdt)^2 - i\ (2cdt\hat{\mathbf{k}} \cdot d\mathbf{r}) \qquad (26a)$$

which is a complex scalar quantity with square modulus obtained as

$$|(d\vec{X})^2|^2 = (\ (d\mathbf{r})^2 - (cdt)^2\ )^2 + (2cdt\hat{\mathbf{k}} \cdot d\mathbf{r})^2 \tag{26b}$$

We reorganize this and introduce velocity $\mathbf{v}$ defined as usual by

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \quad ; \quad v = |\mathbf{v}| \tag{26c}$$

to obtain the invariant length ds of the spacetime event interval defined by

$$(ds)^2 = |(d\vec{X})^2| \tag{26d}$$

in the final form

$$(ds)^2 = \eta\ (\ (d\mathbf{r})^2 - (cdt)^2\ ) \tag{26e}$$

where we have obtained a temporal-spatial axes general orientation modification factor η in the form

$$\eta = \sqrt{1 + \frac{4(\hat{\mathbf{k}} \cdot \mathbf{v})^2}{c^2(1 - \frac{v^2}{c^2})^2}} \tag{26f}$$

Similar forms of the orientation dependent modifications of the invariant length of the spacetime event interval have also been obtained in works investigating space anisotropy within the framework of Finsler geometry [1-2].

5.3.5 Time dilation

We introduce the event interval $d\vec{X}_0$ in the rest frame defined by

$$d\vec{X}_0 = -icd\tau\ \hat{\mathbf{k}} \tag{27a}$$

where $d\tau$ is the time duration measured in the rest frame. The invariant length of the event interval in the rest frame follows easily from equation (27a) in the form

$$|(d\vec{X}_0)^2| = c^2(d\tau)^2 \tag{27b}$$

Applying the invariance principle according to

$$|(d\vec{X})^2| = |(d\vec{X}_0)^2| \tag{27c}$$

and using equations (26d)-(26e) and (27b), we obtain the time dilation relation in the complex four-dimensional spacetime frame in the final form

$$dt = \eta^{-\frac{1}{2}}\gamma\ d\tau \quad ; \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{27d}$$

which modifies the usual result obtained in Einstein's special theory of relativity by a temporal-spatial orientation factor $\eta^{-\frac{1}{2}}$.

5.3.6 Mass increase

Introducing velocity $\mathbf{v}$ to define the linear momentum four-vector $\vec{P}$ in equation $(8d)$ in the form

$$\mathbf{p} = m\mathbf{v} \quad ; \quad \vec{P} = -imc\,\hat{\mathbf{k}} + m\mathbf{v} \tag{28a}$$

we obtain

$$\vec{P}\cdot\vec{P} = P^2 = m^2c^2(1 + i\frac{2\hat{\mathbf{k}}\cdot\mathbf{v}}{c(1-\frac{v^2}{c^2})})(1-\frac{v^2}{c^2}) \tag{28b}$$

from which the invariant magnitude follows easily in the form

$$\sqrt{|P^2|} = mc\left(1 + \frac{4(\hat{\mathbf{k}}\cdot\mathbf{v})^2}{c^2(1-\frac{v^2}{c^2})^2}\right)^{\frac{1}{4}}\sqrt{1-\frac{v^2}{c^2}} \tag{28c}$$

The linear momentum four-vector $\vec{P}_0$ in the rest frame is defined by

$$\vec{P}_0 = -im_0c\,\hat{\mathbf{k}} \tag{28d}$$

where m_0 is the rest mass. The invariant magnitude of linear momentum in the rest frame is easily obtained as

$$\sqrt{|P_0^2|} = m_0c \tag{28e}$$

Applying the invariance principle according to

$$|P^2| = |P_0^2| \quad \Rightarrow \quad \sqrt{|P^2|} = \sqrt{|P_0^2|} \tag{28f}$$

and using equations $(28c)$ and $(28e)$ gives the mass increase relation in the complex four-dimensional spacetime frame in the final form

$$m = \eta^{-\frac{1}{2}}\gamma\; m_0 \tag{28h}$$

which modifies the usual result obtained in Einstein's special theory of relativity by a temporal-spatial orientation factor $\eta^{-\frac{1}{2}}$.

5.3.7 Energy conservation: dispersion relation

Let us now redefine the linear momentum four-vector $\vec{P}$ as an energy-momentum four-vector using the usual mass-energy equivalence relation $E = mc^2$ to obtain

$$\vec{P} = -ic\,\frac{E}{c^2}\,\hat{\mathbf{k}} + \mathbf{p} \quad ; \quad \vec{V} = \vec{P} \quad ; \quad V_k = -i\,\frac{E}{c} \quad ; \quad \mathcal{V} = \frac{E}{c^2} \quad ; \quad \mathbf{V} = \mathbf{p} \tag{29a}$$

Substituting $\vec{U} = \vec{V} = \vec{P}$ in the general dot product in equation (11b) and using equation (29a) as appropriate gives the square of the energy-momentum four-vector in the form

$$P^2 = p^2 - \frac{E^2}{c^2} - 2i\frac{E}{c}\ \hat{\mathbf{k}} \cdot \mathbf{p} \tag{29b}$$

which we reorganize in the form

$$P^2 = -\left(1 + i\ \frac{2\frac{E}{c}\ \hat{\mathbf{k}} \cdot \mathbf{p}}{(\frac{E^2}{c^2} - p^2)}\right)\left(\frac{E^2}{c^2} - p^2\right) \tag{29c}$$

The invariant magnitude of the energy-momentum four-vector then follows easily in the form

$$|P^2| = \sqrt{1 + \frac{(2\frac{E}{c}\ \hat{\mathbf{k}} \cdot \mathbf{p})^2}{(\frac{E^2}{c^2} - p^2)^2}}\left(\frac{E^2}{c^2} - p^2\right) \tag{29d}$$

We now apply the invariance principle according to equation (28f) and use the results from equations (28e) and (29d) to obtain the general energy conservation law (sometimes called *dispersion relation*) governing dynamics within the complex four-dimensional spacetime frame in the final form

$$E^2 = p^2c^2 + \frac{m_0^2c^4}{\sqrt{1 + \frac{4c^2(\ \hat{\mathbf{k}} \cdot \mathbf{p})^2}{E^2(1 - \frac{p^2c^2}{E^2})^2}}} \tag{29e}$$

after rewriting

$$\frac{(2\frac{E}{c}\ \hat{\mathbf{k}} \cdot \mathbf{p})^2}{(\frac{E^2}{c^2} - p^2)^2} = \frac{4(\ \hat{\mathbf{k}} \cdot \mathbf{p})^2}{\frac{E^2}{c^2}(1 - \frac{p^2c^2}{E^2})^2} = \frac{4c^2(\ \hat{\mathbf{k}} \cdot \mathbf{p})^2}{E^2(1 - \frac{p^2c^2}{E^2})^2} \tag{29f}$$

We may now introduce standard definitions to obtain

$$\mathbf{p} = m\mathbf{v} \quad ; \quad E = mc^2 \quad ; \quad \frac{1}{\sqrt{1 + \frac{4c^2(\ \hat{\mathbf{k}} \cdot \mathbf{p})^2}{E^2(1 - \frac{p^2c^2}{E^2})^2}}} = \frac{1}{\eta} \tag{29g}$$

where η is defined in equation (26e). The general energy conservation law in equation (29e) then expressed in terms of the temporal-spatial axes orientation factor η in the form

$$E^2 = p^2c^2 + \frac{m_0^2c^4}{\eta} \tag{29h}$$

It follows from equation ($29e$) that the general energy conservation law is determined by the general orientation of the temporal unit vector $\hat{\mathbf{k}}$ relative to the spatial unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ (which define the linear momentum $\mathbf{p}$ in this case). In the special case where the temporal axis is perpendicular to all the three spatial axes, the energy conservation law in equation ($29e$) takes the familiar form

$$\hat{\mathbf{k}} \cdot \mathbf{p} = 0 \quad \Rightarrow \quad E^2 = p^2c^2 + m_0^2c^4 \tag{29i}$$

which is the standard result normally obtained in Einstein's special theory of relativity. Confirmed discrepancies between experiment and theoretical predictions based on this long standing energy conservation law has been observed, leading to intensive investigations into the possible violations of Lorentz invariance [3-6] and the effects of anisotropy of space [1-2]. The current theoretical challenges, which include proposed generalizations of Lorentz transformation laws [5-6] to provide for suggested additional non-linear terms to the conservation law in equation ($29i$), may well be addressed by the naturally arising temporal-spatial axes orientation dependent conservation law we have obtained here in equation ($29e$) or ($29h$). We observe that equation ($29e$) takes exactly the form of the modified energy conservation law proposed in [3-6]. In this respect, it may be understood that the observed discrepancies between theory and experiment in relativistic particle physics, gravitation, cosmology and quantum field theory is due to an incomplete specification of the dynamical spacetime frame, which has been addressed in the present paper through derivation and identification of the temporal unit vector to define the imaginary temporal axis. The resulting complex four-dimensional spacetime frame with an imaginary temporal axis then provides the natural geometrical framework for describing the dynamics of physical systems.

6 Contravariant and covariant four-vectors

In standard four-vector mathematics within conventional spacetime frames, contravariant and covariant forms are useful in carrying out mathematical operations. A contravariant four-vector is specified by positive spatial components, while a covariant four-vector is specified by negative spatial components. Changing the signs of the spatial components of four-vectors effects transformations between contravariant and covariant forms.

Adopting conventional notation, we represent a contravariant four-vector $\vec{V}$ by V^{μ}, with corresponding covariant form V_{μ}, with $\mu = 0, 1, 2, 3$ where 0 labels the temporal component, while $1, 2, 3$ label the spatial (x, y, z) components, respectively. We define V^μ and V_μ below.

In the general fully specified complex four-dimensional spacetime frame with imaginary temporal axis, each coordinate axis is specified by a unit vector as explained above. This means that the definition of a four-vector in terms of its components in contravariant or covariant form must take into account the corresponding unit vectors, noting that the temporal unit vector has general orientation relative to all the three mutually perpendicular spatial unit vectors.

Denoting the four unit vectors by $\hat{\mathbf{k}}$, $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ as presented in this paper, we define the contravariant coordinates x^μ of the general complex four-dimensional spacetime frame in the form

$$x^\mu \quad , \quad \mu = 0, 1, 2, 3 : \quad x^0 = ict \quad , \quad x^1 = x \quad , \quad x^2 = y \quad , \quad x^3 = z \tag{30a}$$

The corresponding covariant coordinates x_μ are defined in the form

$$x_\mu \quad , \quad \mu = 0, 1, 2, 3 : \quad x_0 = x^0 = ict \quad , \quad x_1 = -x^1 = -x \quad ,$$
$$x_2 = -x^2 = -y \quad , \quad x_3 = -x^3 = -z \tag{30b}$$

Following earlier definitions, we express the complex contravariant spacetime displacement four-vector X^μ in the fully specified form

$$X^\mu = -x^0\hat{\mathbf{k}} + x^1\hat{\mathbf{x}} + x^2\hat{\mathbf{y}} + x^3\hat{\mathbf{z}} = -ict\ \hat{\mathbf{k}} + \mathbf{r} \quad ; \quad \mathbf{r} = x\ \hat{\mathbf{x}} + y\ \hat{\mathbf{y}} + z\ \hat{\mathbf{z}} \tag{30c}$$

with corresponding covariant form X_μ obtained as

$$X_\mu = -x_0\hat{\mathbf{k}} + x_1\hat{\mathbf{x}} + x_2\hat{\mathbf{y}} + x_3\hat{\mathbf{z}} = -ict\ \hat{\mathbf{k}} - \mathbf{r} \tag{30d}$$

which we express in the final forms

$$X^\mu = -(ict\ \hat{\mathbf{k}} - \mathbf{r}) \quad ; \quad X_\mu = -(ict\ \hat{\mathbf{k}} + \mathbf{r}) \tag{30e}$$

The complex spacetime event interval takes the contravariant and covariant forms

$$dX^\mu = -dx^0\hat{\mathbf{k}} + dx^1\hat{\mathbf{x}} + dx^2\hat{\mathbf{y}} + dx^3\hat{\mathbf{z}} = -(icdt\ \hat{\mathbf{k}} - d\mathbf{r}) \quad ;$$
$$dX_\mu = -dx_0\hat{\mathbf{k}} + dx_1\hat{\mathbf{x}} + dx_2\hat{\mathbf{y}} + dx_3\hat{\mathbf{z}} = -(icdt\ \hat{\mathbf{k}} + d\mathbf{r}) \tag{30f}$$

These have complex conjugates taking final form

$$dX^{\mu *} = (icdt\ \hat{\mathbf{k}} + d\mathbf{r}) \quad ; \quad dX^*_\mu = (icdt\ \hat{\mathbf{k}} - d\mathbf{r}) \tag{30g}$$

from which follows the important relation between contravariant and covariant forms in the complex spacetime frame

$$dX^{\mu *} = -dX_\mu \quad ; \quad dX^*_\mu = -dX^\mu \tag{30h}$$

We use this relation to obtain

$$dX^\mu \cdot dX^*_\mu = -(dX^\mu)^2 \quad ; \quad dX^{\mu *} \cdot dX_\mu = -(dX_\mu)^2 \tag{31a}$$

from which follows the definition of the invariant length ds of the complex spacetime event interval in the form

$$(ds)^2 = |-(dX^\mu)^2| = |-(dX_\mu)^2| \tag{31b}$$

This is expressed in the general form

$$(ds)^2 = |dX^\mu \cdot dX^*_\mu| = |dX^{\mu *} \cdot dX_\mu| \tag{31c}$$

Substituting dX^μ, dX_μ from equation $(30f)$ into equation $(31b)$ or using $dX^{\mu *}$, dX^*_μ from equation $(30g)$ into the general form in equation $(31c)$ and reorganizing gives the final result presented earlier in equations $(26e)$-$(26f)$.

A general complex four-vector $\vec{V}$ as defined earlier is expressed in contravariant and covariant forms according to

$$V^0 = ic\ \mathcal{V} \quad , \quad V^1 = V_x \quad , \quad V^2 = V_y \quad , \quad V^3 = V_z \tag{32a}$$

$$V_0 = V^0 = ic\ \mathcal{V} \quad , \quad V_1 = -V^1 \quad , \quad V_2 = -V^2 \quad , \quad V_3 = -V^3 \tag{32b}$$

with

$$V^\mu = -V^0\hat{\mathbf{k}} + V^1\hat{\mathbf{x}} + V^2\hat{\mathbf{y}} + V^3\hat{\mathbf{z}} \quad ; \quad V_\mu = -V_0\hat{\mathbf{k}} + V_1\hat{\mathbf{x}} + V_2\hat{\mathbf{y}} + V_3\hat{\mathbf{z}} \tag{32c}$$

which we express in the final forms

$$V^\mu = -(ic\mathcal{V}\ \hat{\mathbf{k}} - \mathbf{V}) \quad ; \quad V_\mu = -(ic\mathcal{V}\ \hat{\mathbf{k}} + \mathbf{V}) \tag{32d}$$

related through complex conjugation in the form

$$V^{\mu *} = -V_\mu \quad ; \quad V^*_\mu = -V^\mu \tag{32e}$$

We use this contravariant-covariant four-vector conjugation relation to obtain

$$V^\mu \cdot V^*_\mu = -(V^\mu)^2 \quad ; \quad V^{\mu *} \cdot V_\mu = -(V_\mu)^2 \tag{32f}$$

which provides the definition of the invariant length V of the general complex four-vector V^μ or V_μ according to

$$V^2 = |-(V^\mu)^2| = |-(V_\mu)^2| \qquad (32g)$$

We express this in the general form

$$V^2 = |V^\mu \cdot V_\mu^*| \ = |V^{\mu *} \cdot V_\mu| \qquad (32h)$$

Using V^μ, V_μ from equation $(32d)$, noting the relation in equation $(32e)$, we apply equation $(32g)$ or $(38h)$ to obtain the invariant length in the explicit form

$$V^2 = \sqrt{1 + \frac{(2c\mathcal{V}\ \hat{\mathbf{k}} \cdot \mathbf{V})^2}{(c^2\mathcal{V}^2 - \mathbf{V}^2)^2}}\ \left(c^2\mathcal{V}^2 - \mathbf{V}^2\right) \qquad (32i)$$

6.1 Standard four-vector notation and tensors

We now develop the procedure for defining tensors within the general complex four-dimensional spacetime frame. To put the presentation in familiar form, we adopt the standard contravariant and covariant four-vector notation to express V^μ and V_μ from equation $(32d)$ in the form

$$V^\mu = -(ic\mathcal{V}\ ,\ -\mathbf{V}) \quad ; \quad V_\mu = -(ic\mathcal{V}\ ,\ \mathbf{V}) \qquad (33a)$$

with complex conjugates taking the form

$$V^{\mu *} = (ic\mathcal{V}\ ,\ \mathbf{V}) = -V_\mu \quad ; \quad V_\mu^* = (ic\mathcal{V}\ ,\ -\mathbf{V}) = -V^\mu \qquad (33b)$$

where the usual four-vector mathematics is applied, but now taking account of the general orientation of the temporal unit vector $\hat{\mathbf{k}}$ relative to the spatial unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ to obtain the general results presented above.

Using the complex conjugation relation from equation $(33b)$ gives

$$V^\mu \cdot V_\nu^* = -V^\mu \cdot V^\nu \quad ; \quad V^{\mu *} \cdot V_\nu = -V_\mu \cdot V_\nu \qquad (34a)$$

from which a definition of complex contravariant and covariant rank-2 tensors $T^{\mu\nu}$ and $T_{\mu\nu}$ follows according to

$$T^{\mu\nu} = -V^\mu \cdot V^\nu = V^\mu \cdot V_\nu^* \quad ; \quad T_{\mu\nu} = -V_\mu \cdot V_\nu = V^{\mu *} \cdot V_\nu \qquad (34b)$$

In addition,

$$V^\mu \cdot V^{\nu^*} = -V^\mu \cdot V_\nu \quad ; \quad V^{\mu *} \cdot V^\nu = -V_\mu \cdot V^\nu \qquad (34c)$$

provides a definition of complex rank-2 mixed tensors T^{μ}_{ν} and T^{ν}_{μ} in the form

$$T^{\mu}_{\nu} = -V^{\mu} \cdot V_{\nu} = V^{\mu} \cdot V^{\nu^*} \quad ; \quad T^{\nu}_{\mu} = -V_{\mu} \cdot V^{\nu} = V^{\mu *} \cdot V^{\nu} \qquad (34d)$$

The definition of more general tensors of higher rank follows easily. Some mathematical properties of the rank-2 tensors defined above can be obtained by interchanging the indices μ, ν or taking complex conjugation or carrying out both operations simultaneously.

The complete definition of contravariant and covariant complex four-vectors, which can be used to define tensors of general ranks in contravariant, covariant or mixed forms, provides the necessary foundation for more general vector and tensor analysis, leading to reformulation of differential geometry using complex four-component vectors defined within the complex four-dimensional spacetime frame. This is indeed the origin of a new framework for studying physics, mathematics and related disciplines in the 21^{st}-century and beyond.

7 Conclusion

In this paper, we have successfully derived and specified a unit vector in the temporal direction to define a general complex four-dimensional spacetime frame with an imaginary temporal axis. The basic elements of the complex spacetime frame are complex four-vectors with components defined along the temporal and spatial axes specified by the corresponding unit vectors. Basic mathematical operations developed using the complex four-vectors provide results consistent with standard results of vector analysis within the familiar three-dimensional space frame. Taking account of the general orientation of the temporal unit vector $\hat{\mathbf{k}}$ relative to all the three spatial unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ leads to appropriate modifications of well-known fundamental results in relativistic physics (i.e., dynamics in spacetime frames). Such temporal-spatial axes orientation dependent modifications of the fundamental results may account for the observed discrepancies between experiment and theoretical predictions of phenomena based on the energy conservation principle, Lorentz invariance, etc, which constitute the major challenges to be overcome in current models of relativistic mechanics, quantum field theory and general relativity as a theory of gravitation and cosmology.

The definition of contravariant and covariant forms of the general complex four-vectors, together with the contravariant-covariant complex conjugation relation has provided a consistent definition of the invariant length (or invariant magnitude) of the four-vector, which clearly displays the temporal-spatial axes orientation dependence. The general definition of tensors provided here can lead to a generalization of standard vector analysis in three-dimensional space to include four-dimensional complex spacetime frames and a reformulation of differential geometry based on the new procedures of tensor analysis in four-dimensional complex spacetime. These new and more general mathematical operations with complex four-vectors and related tensors will obviously provide the framework and necessary motivation to reformulate relativistic mechanics, quantum field theory and general relativity theory in particular, which are directly based on the mathematical properties defined within four-dimensional spacetime frames.

Acknowledgement I thank Maseno University, Kenya, for supporting this work by providing facilities and working environment.

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