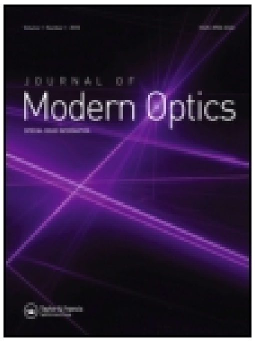

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Exact analytical solutions for fully quantized parametric oscillation dynamics

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In this paper, a simple method for obtaining general analytical solutions of the time evolution equations for a fully quantized parametric oscillation process is developed. Heisenberg's equations for the signal–idler photon annihilation operators are converted into a matrix equation equivalent to a two-state Jaynes–Cummings time evolution equation which has exact analytical solutions. The mean intensity inversion for the coupled signal–idler photon pair is found to undergo fractional revivals for pump photon in a Fock state, provided both signal and idler photons are in occupied Fock states. General collapses and revivals occur for interactions with pump photon in a coherent state, but now with both or either of signal and idler photons in occupied Fock states. An interpretation of the coupled signal–idler photon pair as a circularly polarized two-state system specified by positive and negative helicity states leads to an appropriate description of photon polarization state dynamics governed by the underlying Jaynes–Cummings interaction.

Keywords: quantized parametric oscillation; Jaynes–Cummings interaction; collapses; revivals; fractional revivals; polarization state dynamics

1. Introduction

The fully quantized parametric oscillation/frequency-conversion process treated in this paper is modeled as a quantized three-mode interaction governed by a trilinear Hamiltonian of the form

$$H=\hbar\left(\omega\hat{a}^{\dagger}\hat{a}+\omega_1\hat{a}_1^{\dagger}\hat{a}_1+\omega_2\hat{a}_2^{\dagger}\hat{a}_2+g\left(\hat{a}\hat{a}_1^{\dagger}\hat{a}_2+\hat{a}^{\dagger}\hat{a}_2^{\dagger}\hat{a}_1\right)\right), \tag{1}$$

where g is a constant coupling parameter. In the present work, a-mode, a_1-mode and a_2-mode represent the pump, signal and idler photons, respectively. The annihilation and creation operators for the pump, signal and idler photons are accordingly denoted by $(\hat{a},\hat{a}^{\dagger})$, $(\hat{a}_1,\hat{a}_1^{\dagger})$, $(\hat{a}_2,\hat{a}_2^{\dagger})$, respectively.

Earlier efforts to obtain exact analytical solutions of the equations of dynamics generated by the trilinear Hamiltonian in Equation (1) include the independent works of Carusotto [1] and Jurco [2]. But their approaches yielded expressions which were too complicated and discouraging to work with. As a result, further studies of the fully quantized parametric oscillation process, exemplified by the works of Drobny and Jex [3], Jyotsna and Agarwal [4] and others not cited here, focused attention on the numerical integration methods, which have revealed fundamental quantum mechanical phenomena of collapses, revivals and fractional revivals in the time evolution of the mean signal or idler photon numbers.

The present paper takes the challenge of developing a simple procedure for obtaining exact analytical solutions of the time evolution equations governing the dynamics of the fully quantized parametric oscillation process based on the trilinear Hamiltonian in Equation (1). The main motivation here is that analytical solutions are much easier to work with and very effective in revealing detailed features of the dynamics of a system. The current interest in practical applications of parametric interactions in quantum mechanics based technologies would be highly enhanced by simple exact analytical expressions describing the time evolution of the annihilation and creation operators, as well as the associated state vectors, of a fully quantized parametric oscillation process.

A transformation of H in Equation (1) to the interaction frame through a transformation operator $T_{\mathrm{p}}(t)$ according the transformation law

$$H_{\mathrm{I}}=T_{\mathrm{p}}^{\dagger}\;H\;T_{\mathrm{p}}-\mathrm{i}\hbar T_{\mathrm{p}}^{\dagger}\frac{\mathrm{d}T_{\mathrm{p}}}{\mathrm{d}t} \tag{2a}$$

for the interaction generated by the quantized pump field mode, with

$$T_{\mathrm{p}}(t)=\exp\left(-\mathrm{i}\omega\hat{a}^{\dagger}\hat{a}t\right);\quad T_{\mathrm{p}}^{\dagger}(t)=\exp\left(\mathrm{i}\omega\hat{a}^{\dagger}\hat{a}t\right), \tag{2b}$$

$$T_{\mathrm{p}}^{\dagger}\hat{a}T_{\mathrm{p}}=\exp(-\mathrm{i}\omega t)\hat{a},\quad T_{\mathrm{p}}^{\dagger}\hat{a}^{\dagger}T_{\mathrm{p}}=\exp(\mathrm{i}\omega t)\hat{a}^{\dagger};$$
$$-\mathrm{i}\hbar T^{\dagger}\frac{\mathrm{d}T}{\mathrm{d}t}=-\hbar\omega\hat{a}^{\dagger}\hat{a}, \tag{2c}$$

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yields the interaction Hamiltonian H_{I} in the form

$$H_{\mathrm{I}} = \hbar\left(\omega_1 \hat{a}_1^\dagger \hat{a}_1 + \omega_2 \hat{a}_2^\dagger \hat{a}_2 + g\left(\exp(-\mathrm{i}\omega t)\hat{a}\hat{a}_1^\dagger \hat{a}_2 + \exp(\mathrm{i}\omega t)\hat{a}^\dagger \hat{a}_2^\dagger \hat{a}_1\right)\right). \tag{2d}$$

The dynamics of the interacting signal–idler photon system is described through Heisenberg's equations for the annihilation operator pair ($\hat{a}_1$, $\hat{a}_2$) generated by H_{I} from Equation (2d) in the form

$$\mathrm{i}\hbar\frac{\mathrm{d}\hat{a}_1}{\mathrm{d}t} = \hbar\left(\omega_1 \hat{a}_1 + g\hat{a}\exp(-\mathrm{i}\omega t)\hat{a}_2\right), \tag{3a}$$

$$\mathrm{i}\hbar\frac{\mathrm{d}\hat{a}_2}{\mathrm{d}t} = \hbar\left(\omega_1 \hat{a}_2 + g\hat{a}^\dagger \exp(\mathrm{i}\omega t)\hat{a}_1\right). \tag{3b}$$

These are the time evolution equations governing the dynamics of signal and idler photons driven by a quantized pump photon of angular frequency ω in a fully quantized parametric oscillation process. The task in this paper is to obtain the appropriate analytical solutions. The procedure is developed in the next section.

2. The matrix method: Jaynes–Cummings interaction

The form of Equations (3a)–(3b) suggests the introduction of a two-component annihilation operator column matrix (which may also be called a two-component operator vector) $\hat{B}$ defined by

$$\hat{B} = \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}; \tag{4a}$$

to write Equations (3a)–(3b) in the matrix form

$$\mathrm{i}\hbar\frac{\mathrm{d}\hat{\mathrm{B}}}{\mathrm{d}t} = \mathcal{H}\hat{B}, \tag{4b}$$

where $\mathcal{H}$ is a 2×2 Hamiltonian matrix obtained as

$$\mathcal{H} = \hbar\begin{pmatrix} \omega_1 & g\hat{a}\exp(-\mathrm{i}\omega t) \\ g\hat{a}^\dagger \exp(\mathrm{i}\omega t) & \omega_2 \end{pmatrix}. \tag{4c}$$

Introducing the usual 2×2 identity I and Pauli spin operators $\hat{S}_j = \frac{1}{2}\sigma_j$, $j = x, y, z, 0, +, -$, in the form

$$\hat{S}_0 = \frac{1}{2}I; \quad \hat{S}_z = \frac{1}{2}\sigma_z; \quad \hat{S}_+ = \frac{1}{2}\sigma_+; \quad \hat{S}_- = \frac{1}{2}\sigma_-, \tag{5a}$$

satisfying algebraic relations

$$\hat{S}_+\hat{S}_- = \hat{S}_0 + \hat{S}_z; \; \hat{S}_-\hat{S}_+ = \hat{S}_0 - \hat{S}_z; \; \hat{S}_z\hat{S}_+ + \hat{S}_+\hat{S}_z = 0; \; \hat{S}_z\hat{S}_- + \hat{S}_-\hat{S}_z = 0, \tag{5b}$$

puts the Hamiltonian matrix in Equation (4c) in the form

$$\mathcal{H} = \hbar\left\{\Omega_{12}\hat{S}_0 + \omega_{12}\hat{S}_z + g\left(\hat{a}\exp(-\mathrm{i}\omega t)\hat{S}_+ + \hat{a}^\dagger \exp(\mathrm{i}\omega t)\hat{S}_-\right)\right\}, \tag{6a}$$

after introducing frequency sum Ω_{12} and difference ω_{12} defined by

$$\Omega_{12} = \omega_1 + \omega_2; \quad \omega_{12} = \omega_1 - \omega_2. \tag{6b}$$

The Hamiltonian $\mathcal{H}$ in Equation (6a) (ignoring the constant $\hbar\Omega_{12}\hat{S}_0$ term) takes the form of the Jaynes–Cummings interaction Hamiltonian obtained by Knight and Radmore [5] in their study of the quantum origin of dephasing and revivals in the coherent state Jaynes–Cummings model. The Jaynes–Cummings interaction Hamiltonian has also been obtained in the same form in the excellent textbook of Nielsen and Chuang [6] on quantum computation and quantum information. The matrix approach thus simplifies the problem by converting the equations of dynamics in a fully quantized parametric oscillation process into the form of the standard Jaynes–Cummings interaction, which has exact analytical solutions. This is the main step of the solution procedure developed in the present paper.

Complete understanding of the two-level Jaynes–Cummings mode of interaction in the fully quantized parametric oscillation process may be gained by considering that, according to Equation (4b), the Jaynes–Cummings interaction Hamiltonian $\mathcal{H}$ generates the dynamics of the signal–idler photon system by operating on the two-component operator vector $\hat{B}$ defined in Equation (4a), expressed now in the appropriate form

$$\hat{B} = \hat{a}_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \hat{a}_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hat{a}_1|1\rangle + \hat{a}_2|2\rangle \tag{7a}$$

after introducing the two-dimensional Hilbert space basis vectors $|1\rangle$ and $|2\rangle$ defined as usual by

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{7b}$$

It is clear that $\hat{B}$ takes exactly the form of a two-level atomic state vector as defined within the standard Jaynes–Cummings model in quantum optics [5,8–11] and in general quantum mechanics. The only difference is that the atomic state vector is weighted by c-number probability amplitudes, while $\hat{B}$ as presented in Equation (7a) is weighted by signal–idler photon annihilation operators. In standard photon dynamics, the basis vectors $|1\rangle$ and $|2\rangle$ are interpreted as the basic circular polarization state vectors. In particular, using the Pauli matrix σ_z according to

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \sigma_z|1\rangle = |1\rangle \equiv |+\rangle; \quad \sigma_z|2\rangle = -|2\rangle \equiv |-\rangle \tag{7c}$$

leads to the standard interpretation that $|1\rangle = |+\rangle$ is a positive helicity state vector, while $|2\rangle = |-\rangle$ is a negative helicity state vector for circularly polarized photons [7].

The two-component vector $\hat{B}$ is therefore interpreted as a polarization operator vector for the coupled circularly polarized signal–idler photon pair. The component annihilation operators $\hat{a}_1$ and $\hat{a}_2$ are interpreted as photon intensity (or photon number) operator amplitudes for signal and idler photons in positive and negative helicity states $|1\rangle$ and $|2\rangle$, respectively. This interpretation is clarified by taking the Hermitian conjugate

$$\hat{B}^\dagger = \langle 1|\hat{a}_1^\dagger + \langle 2|\hat{a}_2^\dagger \tag{8a}$$

to be used with Equation (7a) and

$$\langle j|k\rangle = \delta_{jk}, \quad j,k = 1,2, \tag{8b}$$

to obtain the total photon intensity (photon number) operator as

$$\hat{I} = \hat{n} = \hat{B}^\dagger \hat{B} = \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 = \hat{I}_1 + \hat{I}_2, \tag{8c}$$

where $\hat{I}_1$ and $\hat{I}_2$ are interpreted as the intensity operators for circularly polarized signal and idler photons initially in the positive and negative helicity states, respectively. These intensity operators are defined by

$$\hat{I}_1 = \hat{a}_1^\dagger \hat{a}_1 = \hat{n}_1; \quad \hat{I}_2 = \hat{a}_2^\dagger \hat{a}_2 = \hat{n}_2, \tag{8d}$$

where $\hat{n}_1$ and $\hat{n}_2$ are the signal and idler photon number operators, respectively. Additional information on the photon polarization state interpretation will be given in Section 5 below.

The time evolution of $\hat{B}$ according to Equation (4b) is then due to the action of the Jaynes–Cummings interaction Hamiltonian $\mathcal{H}$ in Equation (6a) on the positive and negative helicity state vectors $|1\rangle$ and $|2\rangle$. The underlying dynamics in a fully quantized parametric oscillation process is therefore a two-state dynamics characterized by the time evolution of circularly polarized signal–idler photon pair state vectors generated by a Jaynes–Cummings interaction Hamiltonian. The general interpretation is that the coupled signal–idler photon pair constitutes a composite circularly polarized two-state system specified by the positive and negative helicity states interacting with a single-mode quantized pump field equivalent to a Jaynes–Cummings model for a single two-level atom. In this interpretation, the degenerate signal–idler photon pair is understood as a single two-state circularly polarized photon specified by its positive and negative helicity states.

The Jaynes–Cummings mode of interaction obtained in the present work explicitly accounts for the occurrence of fundamental quantum mechanical phenomena in the form of general collapses, revivals and fractional revivals, revealed in studies of the dynamics of a fully quantized parametric oscillation process using numerical methods [3,4]. The collapse and revival phenomena have been established as general quantum mechanical features of the Jaynes–Cummings mode of interaction in quantum optics [5,8–11].

3. General solution

The general solution of the time evolution Equation (4b) can now be obtained. To realize the full content of the Jaynes–Cummings mode of interaction, Equation (4b) is transformed back to the original frame by applying the inverse operator $T_\mathrm{p}^{-1}(t) = T_\mathrm{p}^\dagger(t) = \exp(\mathrm{i}\omega t\hat{a}^\dagger\hat{a})$ defined earlier in Equation (2b). Using notation $\hat{A}$ for the polarization operator vector in the original frame, the transformation is applied in the form

$$\hat{A}(t) = T_\mathrm{p}^\dagger(t)\hat{B}(t); \quad T_\mathrm{p}^\dagger(0) = 1 \quad \Rightarrow \quad \hat{A}(0) = \hat{B}(0), \tag{9a}$$

which is used in Equation (4b) to obtain the effective time evolution equation in the original frame in the form

$$\mathrm{i}\hbar\frac{\mathrm{d}\hat{A}}{\mathrm{d}t} = H_\mathrm{JC}\hat{A}, \tag{9b}$$

where the Hamiltonian H_JC follows from the transformation in the form

$$H_\mathrm{JC} = T_\mathrm{p}\mathcal{H}T_\mathrm{p}^\dagger - \mathrm{i}\hbar T_\mathrm{p}\frac{\mathrm{d}T_\mathrm{p}^\dagger}{\mathrm{d}t}. \tag{9c}$$

This essentially reverses the general transformation law in Equation (2a) as expected. Substituting $\mathcal{H}$ from Equation (6a) into Equation (9c) and applying standard algebraic relations gives the final form

$$H_\mathrm{JC} = \hbar\left\{\omega\hat{a}^\dagger\hat{a} + \Omega_{12}\hat{S}_0 + \omega_{12}\hat{S}_z + g\left(\hat{a}\hat{S}_+ + \hat{a}^\dagger\hat{S}_-\right)\right\}. \tag{9d}$$

Ignoring the constant $\Omega_{12}\hat{S}_0$ term which yields only a global phase factor, H_JC in Equation (9d) is identified as the standard Jaynes–Cummings Hamiltonian, originally derived by Jaynes and Cummings [10] to describe the interaction between a two-level atom and a quantized single-mode radiation field.

The usual procedure for solving the Jaynes–Cummings problem in quantum optics [12,13] then applies to Equation (9b). Adding and subtracting $\hbar\omega\hat{S}_z$ in Equation (9d) gives

$$H_\mathrm{JC} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \hat{S}_z\right) + \hbar\left\{\Omega_{12}\hat{S}_0 + \delta\hat{S}_z + g\left(\hat{a}\;\hat{S}_+ + \hat{a}^\dagger\;\hat{S}_-\right)\right\} \tag{10a}$$

after introducing the frequency detuning δ defined by

$$\delta = \omega_{12} - \omega = \omega_1 - \omega_2 - \omega. \tag{10b}$$

It is convenient to write H_JC as a sum of two components in the form

$$H_\mathrm{JC} = \hbar\omega\hat{N} + \overline{H} \tag{10c}$$

after introducing an operator $\hat{N}$ and interaction Hamiltonian $\overline{H}$ defined by

$$\hat{N} = \hat{a}^\dagger\hat{a} + \hat{S}_z; \quad \overline{H} = \hbar\left\{\Omega_{12}\hat{S}_0 + \delta\hat{S}_z + g\left(\hat{a}\;\hat{S}_+ + \hat{a}^\dagger\;\hat{S}_-\right)\right\}. \tag{10d}$$

Using standard algebraic relations for $\hat{a}$, $\hat{a}^\dagger$, $\hat{S}_0$, $\hat{S}_z$, $\hat{S}_+$, $\hat{S}_-$ easily gives

$$[\hbar\omega\hat{N}\;,\;\overline{H}] = 0 \quad \Rightarrow \quad [\hbar\omega\hat{N}\;,\;H_\mathrm{JC}] = 0; \quad [\overline{H}\;,\;H_\mathrm{JC}] = 0. \tag{10e}$$

Since both components $\hbar\omega\hat{N}$ and $\overline{H}$ commute with H_JC, they are constants of the motion. The Hamiltonian H_JC is thus time-independent, leading to a solution of the time evolution Equation (9b) through simple integration giving

$$\hat{A}(t) = U_\mathrm{JC}(t)\hat{A}; \quad \hat{A} = \hat{A}(0), \tag{11a}$$

where the general time evolution operator $U_{\mathrm{JC}}(t)$ has been obtained as

$$U_{\mathrm{JC}}(t) = T_N(t)\overline{U}(t) = T_{\mathrm{p}}(t)U(t); \quad U(t) = T(t)\overline{U}(t) \tag{11b}$$

after applying the commutation of $\hbar\omega\hat{N}$ and $\overline{H}$ as in Equation (10e) and using

$$T_N(t) = \exp) - \mathrm{i}\omega t\hat{N}) = T_{\mathrm{p}}(t)T(t);$$
$$T_{\mathrm{p}}(t) = \exp(-\mathrm{i}\omega t\hat{a}^{\dagger}\hat{a}); \quad T(t) = \exp(-\mathrm{i}\omega t\hat{S}_z), \tag{11c}$$

$$\overline{U}(t) = \exp\left(-\frac{\mathrm{i}}{\hbar}\overline{H}t\right). \tag{11d}$$

The initial polarization operator vector $\hat{A} = \hat{A}(0)$ follows from Equations (7a) and (9a) in the form

$$\hat{A} = \hat{a}_1|1\rangle + \hat{a}_2|2\rangle; \quad \hat{a}_1 = \hat{a}_1(0), \quad \hat{a}_2 = \hat{a}_2(0), \tag{12}$$

which is substituted into Equation (11a) to obtain

$$\hat{A}(t) = \hat{a}_1|1;t\rangle + \hat{a}_2^{\dagger}|2;t\rangle, \tag{13a}$$

where the general time evolving positive and negative helicity state vectors $|1;t\rangle$ and $|2;t\rangle$, respectively, are defined by

$$|1;t\rangle = U_{\mathrm{JC}}(t)|1\rangle; \quad |2;t\rangle = U_{\mathrm{JC}}(t)|2\rangle. \tag{13b}$$

Reorganizing $\hat{A}(t)$ in Equation (13a) in the form

$$\hat{A}(t) = \hat{a}_1(t)|1\rangle + \hat{a}_2(t)|2\rangle \tag{13c}$$

gives the desired explicit forms of the time evolving annihilation operators $\hat{a}_1(t)$ and $\hat{a}_2(t)$.

Since the initial operators $\hat{a}_1$ and $\hat{a}_2$ are time-independent, Equation (13a) shows that the time evolution is determined by the general time evolving helicity state vectors $|1;t\rangle$ and $|2;t\rangle$ obtained through the action of the time evolution operator $U_{\mathrm{JC}}(t)$ according to Equation (13b).

3.1. *Evaluating* $U_{\mathrm{JC}}(t)$

To determine the explicit forms of the general time evolving photon helicity state vectors $|1;t\rangle$ and $|2;t\rangle$ in Equation (13b), the time evolution operator $U_{\mathrm{JC}}(t)$ is expressed in an appropriate form. Substituting $\overline{U}(t)$ from Equation (11d) into Equation (11b) and using $\overline{H}$ from Equation (10d) gives $U_{\mathrm{JC}}(t)$ in the form

$$U_{\mathrm{JC}}(t) = \exp\left(-\mathrm{i}\Omega_{12}\hat{S}_0 t\right) T_{\mathrm{p}}(t)T(t)\hat{D}(\delta, g) \tag{14a}$$

after considering that the identity $I = 2\hat{S}_0$ commutes with the rest of the operators to effect a factorization as appropriate. We have introduced a detuning interaction time evolution operator $\hat{D}(\delta, g)$ defined by

$$\hat{D}(\delta, g) = \exp\left(-\mathrm{i}t\left\{\delta\hat{S}_z + g\left(\hat{a}\hat{S}_+ + \hat{a}^{\dagger}(t)\hat{S}_-\right)\right\}\right). \tag{14b}$$

To facilitate evaluation of $|1;t\rangle$, $|2;t\rangle$ in Equation (13b), $\hat{D}(\delta, g)$ is expressed in explicit form to act on $|1\rangle$ and $|2\rangle$ through expansion of the exponential in Equation (14b) and then reorganizing into even and odd power terms in the form

$$\hat{D}(\delta, g) = \sum_{j=0}^{\infty}\frac{(-\mathrm{i}t)^{2j}\left(\delta\hat{S}_z + g\left(\hat{a}\hat{S}_+ + \hat{a}^{\dagger}\hat{S}_-\right)\right)^{2j}}{(2j)!}$$
$$+\sum_{j=0}^{\infty}\frac{(-\mathrm{i}t)^{2j+1}\left(\delta\hat{S}_z + g\left(\hat{a}\hat{S}_+ + \hat{a}^{\dagger}\hat{S}_-\right)\right)^{2j}}{(2j+1)!}$$
$$\times\left(\delta\hat{S}_z + g\left(\hat{a}\hat{S}_+ + \hat{a}^{\dagger}\hat{S}_-\right)\right), \tag{15a}$$

where the odd power term has been expressed in a convenient form for ease of evaluation. Writing

$$\left(\delta\hat{S}_z + g\left(\hat{a}\hat{S}_+ + \hat{a}^{\dagger}\hat{S}_-\right)\right)^{2j} = \left\{\left(\delta\hat{S}_z + g\left(\hat{a}\hat{S}_+ + \hat{a}^{\dagger}\hat{S}_-\right)\right)^2\right\}^j \tag{15b}$$

and carrying out an expansion using Equation (5b) together with

$$\hat{S}_z^2 = \frac{1}{4}I; \quad \hat{S}_+^2 = 0; \quad \hat{S}_-^2 = 0; \quad \hat{a}\hat{a}^{\dagger} = \hat{a}^{\dagger}\hat{a} + 1 \tag{15c}$$

gives ($\hat{S}_z = \hat{S}_z I$)

$$\left(\delta\hat{S}_z + g\left(\hat{a}\hat{S}_+ + \hat{a}^{\dagger}\hat{S}_-\right)\right)^2$$
$$= \left(g\left(\hat{a}^{\dagger}\hat{a} + \hat{S}_z + \frac{1}{2} + k^2\right)^{1/2}\right)^2 I$$
$$= (g\hat{q})^2 I \tag{15d}$$

after introducing a (detuning) parameter k and an operator $\hat{q}$ defined by

$$k^2 = \frac{\delta^2}{4g^2}; \quad \hat{q} = \left(\hat{a}^{\dagger}\hat{a} + \hat{S}_z + \frac{1}{2} + k^2\right)^{1/2}. \tag{15e}$$

Substituting Equation (15d) into Equation (15b) and using the result in Equation (15a), noting

$$(-\mathrm{i}t)^{2j} = (-1)^j t^{2j}; \quad (-\mathrm{i}t)^{2j+1} = -\mathrm{i}(-1)^j t^{2j+1}, \tag{16a}$$

$$\cos x = \sum_{j=0}^{\infty}\frac{(-1)^j x^{2j}}{(2j)!}; \quad \sin x = \sum_{j=0}^{\infty}\frac{(-1)^j x^{2j+1}}{(2j+1)!};$$
$$\sum_{j=0}^{\infty}\frac{(-1)^j x^{2j}}{(2j+1)!} = \frac{1}{x}\sin x \tag{16b}$$

gives

$$\hat{D}(\delta, g) = \cos(gt\hat{q})I - \frac{\mathrm{i}}{g\hat{q}}\sin(gt\hat{q})\left\{\delta\hat{S}_z + g\left(\hat{a}\hat{S}_+ + \hat{a}^{\dagger}\hat{S}_-\right)\right\}. \tag{17}$$

Substituting Equation (17) into Equation (14a) gives the general time evolution operator $U_{\mathrm{JC}}(t)$ in the desired form.

3.2. Evaluating $|1;t\rangle$, $|2;t\rangle$, $\hat{A}(t)$

The time evolving helicity state vectors are evaluated by using Equation (14a) in Equation (13b) and applying

$$\exp\left(-\mathrm{i}\Omega_{12}\hat{S}_0 t\right) = \exp\left(-\frac{\mathrm{i}}{2}\Omega_{12}t\right); \quad \hat{S}_z|1\rangle = \frac{1}{2}|1\rangle;$$
$$\hat{S}_z|2\rangle = -\frac{1}{2}|2\rangle, \tag{18a}$$

$$\exp\left(-\mathrm{i}\omega\hat{S}_z t\right)|1\rangle = \exp\left(-\frac{i}{2}\omega t\right)|1\rangle;$$
$$\exp\left(-\mathrm{i}\omega\hat{S}_z t\right)|2\rangle = \exp\left(\frac{i}{2}\omega t\right)|2\rangle \tag{18b}$$

$$\hat{S}_+|1\rangle = 0, \quad \hat{S}_+|2\rangle = |1\rangle; \quad \hat{S}_-|1\rangle = |2\rangle, \quad \hat{S}_-|2\rangle = 0 \tag{18c}$$

to obtain

$$|1;t\rangle = \exp\left(-\frac{\mathrm{i}}{2}\Omega_{12}t\right)T_{\mathrm{p}}(t)|\delta;g;\omega t\rangle_1;$$
$$|2;t\rangle = \exp\left(-\frac{\mathrm{i}}{2}\Omega_{12}t\right)T_{\mathrm{p}}(t)|\delta;g;\omega t\rangle_2 \tag{19a}$$

after introducing the signal–idler photon pair helicity coherent state vectors $|\delta;g;\omega t\rangle_1$ and $|\delta;g;\omega t\rangle_2$ respectively defined by

$$|\delta;g;\omega t\rangle_1 = T(t)\hat{D}(\delta,g)\,|1\rangle;$$
$$|\delta;g;\omega t\rangle_2 = T(t)\hat{D}(\delta,g)\,|2\rangle \tag{19b}$$

and obtained in the form ($k = \delta/2g$)

$$|k;g;\omega t\rangle_1 = \exp\left(-\frac{\mathrm{i}}{2}\omega t\right)\left\{\left(\cos(gt\hat{q}) - \mathrm{i}\frac{k}{\hat{q}}\sin(gt\hat{q})\right)|1\rangle - \mathrm{i}\frac{\exp(\mathrm{i}\omega t)}{\hat{q}}\sin(gt\hat{q})\,\hat{a}^\dagger\,|2\rangle\right\}, \tag{19c}$$

$$|k;g;\omega t\rangle_2 = \exp\left(\frac{\mathrm{i}}{2}\omega t\right)\left\{\left(\cos(gt\hat{q}) + \mathrm{i}\frac{k}{\hat{q}}\sin(gt\hat{q})\right)|2\rangle - \mathrm{i}\frac{\exp(-\mathrm{i}\omega t)}{\hat{q}}\sin(gt\hat{q})\,\hat{a}\,|1\rangle\right\}, \tag{19d}$$

where the operator ordering from Equation (17) has been maintained as appropriate.

Before taking the final step, the spin operator $\hat{S}_z$ is eliminated from the definition of $\hat{q}$ in Equation (15e) by expanding the trigonometric functions according to Equation (16b) and carrying out repeated applications of $\hat{q}^2$ j-times on $|1\rangle$ and $|2\rangle$ in the form

$$\hat{q}^{2j}|1\rangle = \hat{q}^{2(j-1)}\hat{q}^2|1\rangle; \quad \hat{q}^{2j}|2\rangle = \hat{q}^{2(j-1)}\hat{q}^2|2\rangle \tag{20a}$$

and then using $\hat{S}_z|1\rangle = \frac{1}{2}|1\rangle$, $\hat{S}_z|2\rangle = -\frac{1}{2}|2\rangle$ from Equation (18a) to obtain the final results

$$\hat{q}^{2j}|1\rangle = \left(\hat{a}^\dagger\hat{a}+1+k^2\right)^j|1\rangle = \left(\left(\hat{a}^\dagger\hat{a}+1+k^2\right)^{1/2}\right)^{2j}|1\rangle, \tag{20b}$$

$$\hat{q}^{2j}|2\rangle = \left(\hat{a}^\dagger\hat{a}+k^2\right)^j|2\rangle = \left(\left(\hat{a}^\dagger\hat{a}+k^2\right)^{1/2}\right)^{2j}|2\rangle. \tag{20c}$$

Substituting Equations (20b)–(20c) into the appropriate expanded forms of Equations (19c)–(19d) and introducing operators $\hat{h}_1$, $\hat{h}_0$ defined by

$$\hat{h}_1 = \left(\hat{a}^\dagger\hat{a}+1+k^2\right)^{1/2}; \quad \hat{h}_0 = \left(\hat{a}^\dagger\hat{a}+k^2\right)^{1/2} \tag{21a}$$

gives

$$\cos(gt\hat{q})|1\rangle = \cos(gt\hat{h}_1)|1\rangle; \quad \cos(gt\hat{q})|2\rangle = \cos(gt\hat{h}_0)|2\rangle, \tag{21b}$$

$$\frac{1}{\hat{q}}\sin(gt\hat{q})|1\rangle = \frac{1}{\hat{h}_1}\sin(gt\hat{h}_1)|1\rangle;$$
$$\frac{1}{\hat{q}}\sin(gt\hat{q})|2\rangle = \frac{1}{\hat{h}_0}\sin(gt\hat{h}_0)|2\rangle. \tag{21c}$$

Substituting Equations (21b)–(21c) into Equations (19c)–(19d) gives the final form

$$|k;g;\omega t\rangle_1 = \exp\left(-\frac{\mathrm{i}}{2}\omega t\right)(\hat{\mu}_1\,|1\rangle + \hat{\nu}_0\,|2\rangle);$$
$$|k;g;\omega t\rangle_2 = \exp\left(\frac{\mathrm{i}}{2}\omega t\right)(\hat{\mu}_0\,|2\rangle + \hat{\nu}_1\,|1\rangle) \tag{22}$$

after introducing time evolving operators $\hat{\mu}_1$, $\hat{\nu}_1$, $\hat{\mu}_0$, $\hat{\nu}_0$ defined by

$$\hat{\mu}_1 = \cos(gt\hat{h}_1) - \mathrm{i}\frac{k}{\hat{h}_1}\sin(gt\hat{h}_1);$$
$$\hat{\nu}_1 = -\mathrm{i}\frac{\exp(-\mathrm{i}\omega t)}{\hat{h}_1}\sin(gt\hat{h}_1)\,\hat{a}, \tag{23a}$$

$$\hat{\mu}_0 = \cos(gt\hat{h}_0) + \mathrm{i}\frac{k}{\hat{h}_0}\sin(gt\hat{h}_0);$$
$$\hat{\nu}_0 = -\mathrm{i}\frac{\exp(\mathrm{i}\omega t)}{\hat{h}_0}\sin(gt\hat{h}_0)\,\hat{a}^\dagger \tag{23b}$$

with Hermitian conjugates easily obtained as ($\hat{h}_1^\dagger = \hat{h}_1$, $\hat{h}_0^\dagger = \hat{h}_0$)

$$\hat{\mu}_1^\dagger = \cos(gt\hat{h}_1) + \mathrm{i}\frac{k}{\hat{h}_1}\sin(gt\hat{h}_1);$$
$$\hat{\nu}_1^\dagger = \mathrm{i}\frac{\exp(\mathrm{i}\omega t)\,\hat{a}^\dagger}{\hat{h}_1}\sin(gt\hat{h}_1), \tag{23c}$$

$$\hat{\mu}_0^\dagger = \cos(gt\hat{h}_0) - \mathrm{i}\frac{k}{\hat{h}_0}\sin(gt\hat{h}_0);$$
$$\hat{\nu}_0^\dagger = \mathrm{i}\frac{\exp(-\mathrm{i}\omega t)\,\hat{a}}{\hat{h}_0}\sin(gt\hat{h}_0). \tag{23d}$$

Substituting Equation (22) into Equation (19a) gives

$$|1;t\rangle = \exp\left(-\frac{\mathrm{i}}{2}(\Omega_{12}+\omega)t\right)T_{\mathrm{p}}(t)(\hat{\mu}_1\,|1\rangle + \hat{\nu}_0\,|2\rangle), \tag{24a}$$

$$|2;t\rangle = \exp\left(-\frac{\mathrm{i}}{2}(\Omega_{12}-\omega)t\right)T_{\mathrm{p}}(t)(\hat{\mu}_0\,|2\rangle + \hat{\nu}_1\,|1\rangle), \tag{24b}$$

which are substituted into Equation (13a) to obtain

$$\hat{A}(t) = T_{\rm p}(t)\left\{\exp\left(-\frac{\rm i}{2}(\Omega_{12}+\omega)t\right)(\hat{\mu}_1\,|1\rangle + \hat{\nu}_0\,|2\rangle)\,\hat{a}_1 + \exp\left(-\frac{\rm i}{2}(\Omega_{12}-\omega)t\right)(\hat{\mu}_0\,|2\rangle + \hat{\nu}_1\,|1\rangle)\,\hat{a}_2\right\}, \tag{25a}$$

which is reorganized in the form

$$\hat{A}(t) = T_{\rm p}(t)\left\{\exp\left(-\frac{\rm i}{2}(\Omega_{12}+\omega)t\right) \times (\hat{\mu}_1\hat{a}_1 + \exp({\rm i}\omega t)\hat{\nu}_1\hat{a}_2)\,|1\rangle + \exp\left(-\frac{\rm i}{2}(\Omega_{12}-\omega)t\right) \times (\hat{\mu}_0\hat{a}_2 + \exp(-{\rm i}\omega t)\hat{\nu}_0\hat{a}_1)\,|2\rangle\right\}. \tag{25b}$$

Comparing Equation (25b) with $\hat{A}(t) = \hat{a}_1(t)|1\rangle + \hat{a}_2(t)|2\rangle$ in Equation (13c) gives the general time evolving signal and idler photon annihilation operators $\hat{a}_1(t)$ and $\hat{a}_2(t)$, respectively in the form

$$\hat{a}_1(t) = \exp\left(-\frac{\rm i}{2}(\Omega_{12}+\omega)t\right)T_{\rm p}(t) \times(\hat{\mu}_1\hat{a}_1 + \exp({\rm i}\omega t)\hat{\nu}_1\hat{a}_2), \tag{25c}$$

$$\hat{a}_2(t) = \exp\left(-\frac{\rm i}{2}(\Omega_{12}-\omega)t\right)T_{\rm p}(t) \times(\hat{\mu}_0\hat{a}_2 + \exp(-{\rm i}\omega t)\hat{\nu}_0\hat{a}_1). \tag{25d}$$

These are the desired general time evolution equations for the annihilation operators of the signal and idler photons in a fully quantized parametric oscillation process. They are exact analytical solutions determined within the Heisenberg picture, where they can be used in the calculation of mean values of various physical quantities which characterize the dynamics of a fully quantized parametric oscillation process governed by the trilinear Hamiltonian H in Equation (1).

4. Mean signal–idler photon intensity

The present study shows that the dynamics of the fully quantized parametric oscillation process is characterized by the time evolution of the signal–idler photon pair polarization operator vector $\hat{A}(t)$ related to the total intensity operator $\hat{I}(t)$ according to the definition

$$\hat{I}(t) = \hat{A}^{\dagger}(t)\hat{A}(t), \tag{26a}$$

which on using the form $\hat{A}(t) = \hat{a}_1(t)|1\rangle + \hat{a}_2(t)|2\rangle$ from Equation (13a) gives

$$\hat{I}(t) = \hat{I}_1(t) + \hat{I}_2(t) \tag{26b}$$

with intensity difference operator $\Delta\hat{I}(t)$ following as

$$\Delta\hat{I}(t) = \hat{I}_1(t) - \hat{I}_2(t), \tag{26c}$$

where $\hat{I}_1(t)$ and $\hat{I}_2(t)$ are the time evolving intensity operators for circularly polarized signal–idler photons in positive and negative helicity states, respectively, defined by

$$\hat{I}_1(t) = \hat{a}_1^{\dagger}(t)\hat{a}_1(t); \quad \hat{I}_2(t) = \hat{a}_2^{\dagger}(t)\hat{a}_2(t). \tag{26d}$$

The intensities are the appropriate operators for describing the dynamics of the fully quantized parametric oscillation process. Using $\hat{a}_1(t)$ and $\hat{a}_2(t)$ from Equations (25c)–(25d) in Equation (26 d) gives

$$\hat{I}_1(t) = \left(\hat{\mu}_1^{\dagger}\hat{a}_1^{\dagger} + \exp(-{\rm i}\omega t)\hat{\nu}_1^{\dagger}\hat{a}_2^{\dagger}\right)(\hat{\mu}_1\hat{a}_1 + \exp({\rm i}\omega t)\hat{\nu}_1\hat{a}_2), \tag{26e}$$

$$\hat{I}_2(t) = \left(\hat{\mu}_0^{\dagger}\hat{a}_2^{\dagger} + \exp({\rm i}\omega t)\hat{\nu}_0^{\dagger}\hat{a}_1^{\dagger}\right)(\hat{\mu}_0\hat{a}_2 + \exp(-{\rm i}\omega t)\hat{\nu}_0\hat{a}_1). \tag{26f}$$

To determine the mean signal–idler photon intensity $I(t)$, its positive/negative helicity components $I_1(t)/I_2(t)$ and the mean intensity difference $\Delta I(t)$ in the present work, the initial states of the signal and idler photons are generally taken as the Fock (number) states $|n_1\rangle$ and $|n_2\rangle$, respectively, while the pump photon is generated in the Fock state $|n\rangle$ or the coherent state $|\alpha\rangle$.

4.1. Pump photon in Fock state

Taking the pump photon to be in the Fock state $|n\rangle$, the total initial state vector of the pump, signal and idler photons becomes

$$|nn_1n_2\rangle = |n\rangle|n_1\rangle|n_2\rangle, \quad n, n_1, n_2 = 0, 1, 2, 3, ..., \tag{27a}$$

which satisfies the usual number state algebra. Using $\hat{h}_1$, $\hat{h}_0$ from Equation (21a) gives

$$\hat{h}_1^2|n\rangle = (\hat{a}^{\dagger}\hat{a} + 1 + k^2)|n\rangle = (n + 1 + k^2)|n\rangle, \tag{27b}$$

$$\hat{h}_0^2|n\rangle = (\hat{a}^{\dagger}\hat{a} + k^2)|n\rangle = (n + k^2)|n\rangle. \tag{27c}$$

Taking the expectation values of Equations (26b), (26d), (26 e) and (26 f) with respect to $|nn_1n_2\rangle$ from Equation (27a) gives the mean intensities in the form

$$I(t) = I_1(t) + I_2(t); \quad \Delta I(t) = I_1(t) - I_2(t), \tag{28a}$$

$$I_1(t) = < nn_1n_2|\hat{I}_1(t)|nn_1n_2\rangle = \langle n|\hat{\mu}_1^{\dagger}\hat{\mu}_1|n\rangle\, n_1 + \langle n|\hat{\nu}_1^{\dagger}\hat{\nu}_1|n\rangle\, n_2, \tag{28b}$$

$$I_2(t) = < nn_1n_2|\hat{I}_2(t)|nn_1n_2\rangle = \langle n|\hat{\mu}_0^{\dagger}\hat{\mu}_0|n\rangle\, n_2 + \langle n|\hat{\nu}_0^{\dagger}\hat{\nu}_0|n\rangle\, n_1. \tag{28c}$$

Using $\hat{\mu}_1$, $\hat{\nu}_1$, $\hat{h}_0$, $\hat{\nu}_0$, together with their Hermitian conjugates from Equations (23a)–(23d), expanding the trigonometric functions as appropriate and applying the algebraic results from Equations (27b)–(27c) gives

$$\langle n|\hat{\mu}_1^{\dagger}\hat{\mu}_1|n\rangle = |\mu_1|^2; \quad \langle n|\hat{\nu}_1^{\dagger}\hat{\nu}_1|n\rangle = |\nu_1|^2, \tag{29a}$$

$$\langle n|\hat{\mu}_0^{\dagger}\hat{\mu}_0|n\rangle = |\mu_0|^2; \quad \langle n|\hat{\nu}_0^{\dagger}\hat{\nu}_0|n\rangle = |\nu_0|^2, \tag{29b}$$

where the c-numbers μ_1, ν_1, μ_0 and ν_0 are easily obtained as

$$\hat{\mu}_1|n\rangle = \mu_1|n\rangle;$$
$$\mu_1 = \cos\left(gt(n+1+k^2)^{1/2}\right) - \mathrm{i}\frac{k}{(n+1+k^2)^{1/2}}\sin\left(gt(n+1+k^2)^{1/2}\right), \quad (29c)$$
$$\hat{\nu}_1|n\rangle = \nu_1|n-1\rangle;$$
$$\nu_1 = -\mathrm{i}\frac{\exp(-\mathrm{i}\omega t)n^{1/2}}{(n+k^2)^{1/2}}\sin\left(gt(n+k^2)^{1/2}\right), \quad (29d)$$
$$\hat{\mu}_0|n\rangle = \mu_0|n\rangle;$$
$$\mu_0 = \cos\left(gt(n+k^2)^{1/2}\right) + \mathrm{i}\frac{k}{(n+k^2)^{1/2}}\sin\left(gt(n+k^2)^{1/2}\right), \quad (29e)$$
$$\hat{\nu}_0|n\rangle = \nu_0|n+1\rangle;$$
$$\nu_0 = -\mathrm{i}\frac{\exp(\mathrm{i}\omega t)(n+1)^{1/2}}{(n+1+k^2)^{1/2}}\sin\left(gt(n+1+k^2)^{1/2}\right). \quad (29f)$$

Substituting Equations (29a)–(29b) into Equations (28b)–(28c) gives

$$I_1(t) = |\mu_1|^2\, n_1 + |\nu_1|^2\, n_2; \quad I_2(t) = |\mu_0|^2\, n_2 + |\nu_0|^2\, n_1, \quad (30a)$$

which is used in Equation (28a) to obtain

$$I(t) = (|\mu_1|^2 + |\nu_0|^2)\, n_1 + (|\mu_0|^2 + |\nu_1|^2)\, n_2, \quad (30b)$$
$$\Delta I(t) = (|\mu_1|^2 - |\nu_0|^2)\, n_1 - (|\mu_0|^2 - |\nu_1|^2)\, n_2. \quad (30c)$$

Using explicit expressions from Equations (29c)–(29f) gives

$$|\mu_1|^2 + |\nu_0|^2 = 1; \quad |\mu_0|^2 + |\nu_1|^2 = 1, \quad (31a)$$
$$|\mu_1|^2 - |\nu_0|^2 = \cos^2\left(gt(n+1+k^2)^{1/2}\right) - \frac{n+1-k^2}{n+1+k^2}\sin^2\left(gt(n+1+k^2)^{1/2}\right), \quad (31b)$$
$$|\mu_0|^2 - |\nu_1|^2 = \cos^2\left(gt(n+k^2)^{1/2}\right) - \frac{n-k^2}{n+k^2}\sin^2\left(gt(n+k^2)^{1/2}\right), \quad (31c)$$

which are substituted into Equations (30b)–(30c) to obtain

$$I(t) = n_1 + n_2 = I, \quad (32a)$$
$$\Delta I(t) = n_1\left(\cos^2\left(gt(n+1+k^2)^{1/2}\right) - \frac{n+1-k^2}{n+1+k^2}\sin^2\left(gt(n+1+k^2)^{1/2}\right)\right) - n_2\left(\cos^2\left(gt(n+k^2)^{1/2}\right) - \frac{n-k^2}{n+k^2}\sin^2\left(gt(n+k^2)^{1/2}\right)\right). \quad (32b)$$

As expected, Equation (32a) shows that the mean signal–idler photon pair intensity in the fully quantized parametric oscillation process is conserved. On the other hand, Equation (32b) shows that the mean intensity inversion (or mean intensity difference) for circularly polarized signal–idler photon pair varies with time over different time scales $gt(n+1+k^2)^{1/2}$ and $gt(n+k^2)^{1/2}$, associated with positive and negative helicity channels. Using Equations (29c)–(29f) in Equation (30a) explicitly shows that the individual mean photon intensities $I_1(t)$ and $I_2(t)$ also vary with time over the different time scales according to the positive and negative helicity channels. The beating of oscillations at different Rabi frequencies $g(n+1+k^2)^{1/2}$ and $g(n+k^2)^{1/2}$ over the two different time scales leads to intrinsically quantum mechanical phenomenon of fractional revivals, which is displayed in Figure 1 for detuning parameter and photon number values $k = 3$, $n = 2$, $n_1 = 3$ and $n_2 = 2$.

The occurrence of fractional revivals in the dynamics generated by a pump photon in a Fock state is a remarkable feature of the exact analytical results obtained through the simple solution procedure developed in this paper. Earlier work based on numerical integration of the fully quantized degenerate parametric process [3,4] never yielded fractional revivals when the pump photon is taken in a simple Fock state, leading to the conclusion that this fundamental quantum mechanical phenomenon only occurs when the pump photon is in either a coherent state or a suitable superposition of Fock states.

An interesting physical feature of the dynamics of the fully quantized parametric oscillation process under a Fock state pump photon is that the fractional revivals persist even for very large values of pump photon number n and detuning k as demonstrated in Figure 2 for $n = 10{,}000$, $k = 101$ ($k^2 = 10{,}201$). This persistence of fractional revivals even at very large values of n and k essentially defies naive mathematical expectation that the time scales $gt(n+1+k^2)^{1/2}$ and $gt(n+k^2)^{1/2}$ would coincide for large values $n \gg 1$, $n+1 \approx n$, $k^2 > n+1$, with $gt(n+1+k^2)^{1/2} \approx gt(n+k^2)^{1/2}$, which would yield only one mode of oscillation. The occurrence of the fractional revivals for

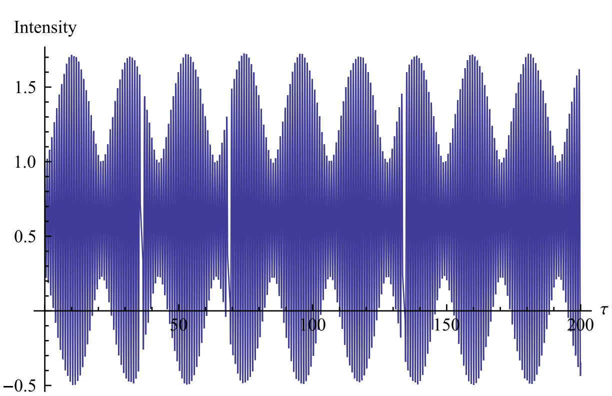

Figure 1. Signal–idler intensity inversion over scaled time $\tau = gt = 0 \to 200$, $n = 2$, $k = 3$, $n_1 = 3$, $n_2 = 1$. (The color version of this figure is included in the online version of the journal.)

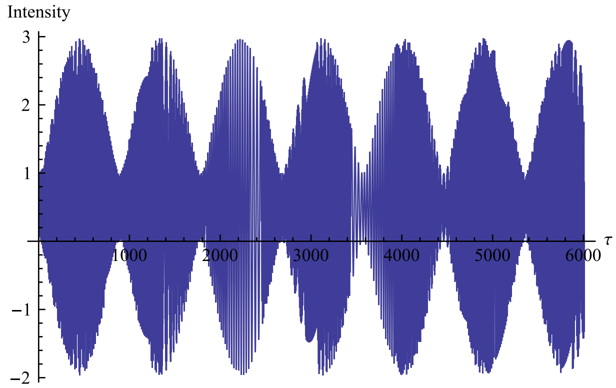

Figure 2. Signal–idler intensity inversion over scaled time $\tau = gt = 0 \to 6010$, $n = 10000$, $k = 101$ ($k^2 = 10201$), $n_1 = 3$, $n_2 = 1$. (The color version of this figure is included in the online version of the journal.)

values as large as $n = 10{,}000$, $k^2 = 10{,}201$ means that the mathematical approximation stated above does not apply here.

The fact that the mean intensity inversion $\Delta I(t)$ in Equation (32b) is composed of two components, one associated with the initial signal photon intensity $I_1 = n_1$ over the time scale $gt(n+1+k^2)^{1/2}$ and the other associated with initial idler photon intensity $I_2 = n_2$ over the time scale $gt(n+k^2)^{1/2}$ means that, taking the signal or idler photon to be in an initial vacuum state $|n_1 = 0\rangle$ ($I_1 = n_1 = 0$) or $|n_2 = 0\rangle$, ($I_2 = n_2 = 0$) removes one component, leaving $\Delta I(t)$ in Equation (32b) with only one mode of oscillation over a single time scale and the fractional revivals disappear in such a case. The occurrence of fractional revivals in the dynamics of a fully quantized parametric oscillator generated through interaction with a pump photon in a simple Fock state is directly governed by the initial intensities (numbers n_1, n_2) of the signal and idler photons; taking either of them equal to zero automatically removes the mechanism responsible for fractional revivals for a Fock state pump photon.

4.1.1. Resonance

The resonance condition is obtained as

$$\delta = \omega_1 - \omega_2 - \omega = 0 \quad \Rightarrow \quad k = 0, \tag{33a}$$

which is applied in Equation (32b) to obtain the mean intensity inversion for circularly polarized signal–idler photon pair under resonance to be

$$\Delta I_r(t) = n_1 \cos\left(2gt(n+1)^{1/2}\right) - n_2 \cos\left(2gtn^{1/2}\right). \tag{33b}$$

4.1.2. Pump field vacuum: natural spontaneous parametric oscillations

An important extreme case to consider under the fully quantized model is the quantized pump field vacuum where $n = 0$. In this case, setting $n = 0$ in Equations (29c)–(29f), (30 c) and (32 b) gives the time varying mean signal–idler photon intensities $I_{10}(t)$, I_{20} and $\Delta I_0(t)$ generated under the pump field vacuum condition in the form

$$n = 0; \quad I_{10}(t) = n_1\left(\cos^2\left(gt(1+k^2)^{1/2}\right) + \frac{k^2}{1+k^2}\sin^2\left(gt(1+k^2)^{1/2}\right)\right), \tag{34a}$$

$$n = 0; \quad I_{20}(t) = n_2 + n_1\left(\frac{1}{1+k^2}\sin^2\left(gt(1+k^2)^{1/2}\right)\right), \tag{34b}$$

$$n = 0; \quad \Delta I_0(t) = n_1\left(\cos^2\left(gt(1+k^2)^{1/2}\right) - \frac{1-k^2}{1+k^2}\sin^2\left(gt(1+k^2)^{1/2}\right)\right) - n_2. \tag{34c}$$

The process under $n = 0$ occurs due to the annihilation of a vacuum fluctuation generated photon of angular frequency ω and an idler photon of angular frequency ω_2 to emit a signal photon of angular frequency ω_1. This is a natural spontaneous parametric oscillation process resulting from fluctuations of the pump field vacuum.

4.2. Pump photon in coherent state

Considering a pump photon generated in a coherent state $|\alpha\rangle$ defined as usual as a superposition of Fock states in the form

$$|\alpha\rangle = \sum_{n=0}^{\infty} \exp\left(-\frac{1}{2}|\alpha|^2\right)\frac{\alpha^n}{(n!)^{1/2}}|n\rangle, \tag{35a}$$

the total initial pump, signal and idler photon state vector $|\alpha n_1 n_2\rangle$ then takes the form

$$|\alpha n_1 n_2 >= \sum_{n=0}^{\infty} \exp\left(-\frac{1}{2}|\alpha|^2\right)\frac{\alpha^n}{(n!)^{1/2}}|n n_1 n_2\rangle, \tag{35b}$$

where the three photon Fock state vector $|n n_1 n_2\rangle$ is already defined in Equation (27a). The mean signal–idler photon pair intensity inversion $\Delta I_\alpha(t)$ is obtained as

$$\Delta I_\alpha(t) = \langle n_2 n_1 \alpha|\Delta\hat{I}(t)|\alpha n_1 n_2\rangle, \tag{36a}$$

where the intensity inversion operator $\Delta\hat{I}(t)$ is given by Equation (26c), with $\hat{I}_1(t)$, $\hat{I}_2(t)$ given in Equations (26e)–(26f). Substituting Equation (35b) into Equation (36a) and then evaluating the expectation values with respect to $|n n_1 n_2\rangle$ as before gives the final results for the mean intensity inversion for the coupled circularly polarized signal–idler photon pair under interaction with a pump photon in a coherent state in the form

$$\Delta I_\alpha(t) = \sum_{n=0}^{\infty} P_n \Delta I(t); \quad P_n = \frac{\exp\left(-|\alpha|^2\right)|\alpha|^{2n}}{n!}, \tag{36b}$$

where P_n is the probability distribution for coherent state pump photons, while $\Delta I(t)$ is the mean intensity inversion obtained earlier with Fock state pump photon in Equation (32b). Substituting $\Delta I(t)$ from Equation (32b) into Equation (36b) gives

$$\Delta I_{\alpha}(t) = \sum_{n=0}^{\infty} \frac{\exp\left(-|\alpha|^2\right)|\alpha|^{2n}}{n!} \Bigg\{ \Big(\cos^2\left(gt(n+1+k^2)^{1/2}\right) - \frac{n+1-k^2}{n+1+k^2} \sin^2\left(gt(n+1+k^2)^{1/2}\right)\Big) n_1 - \Big(\cos^2\left(gt(n+k^2)^{1/2}\right) - \frac{n-k^2}{n+k^2} \sin^2\left(gt(n+k^2)^{1/2}\right)\Big) n_2 \Bigg\}. \tag{36c}$$

Under resonance, set $k = 0$ in Equation (37c) to obtain

$$\Delta I_{\alpha r}(t) = \sum_{n=0}^{\infty} \frac{\exp\left(-|\alpha|^2\right)|\alpha|^{2n}}{n!} \times \left\{ n_1 \cos\left(2gt(n+1)^{1/2}\right) - n_2 \cos\left(2gtn^{1/2}\right)\right\}. \tag{36d}$$

Notice that besides the original different time scales in $\Delta I(t)$ as specified earlier, many more time scales now emerge in $\Delta I_{\alpha}(t)$ in Equation (36c) due to the summation over the pump photon number n. The beating of these oscillations, now enhanced through the summation, causes the mean intensity inversion $\Delta I_{\alpha}(t)$ to undergo general collapses and revivals.

The expected collapses and revivals are demonstrated in Figures 3 and 4, where the mean intensity inversion under resonance in Equation (36d) is plotted against scaled time $\tau = gt$ for two different values $|\alpha|^2 = 5$ and $|\alpha|^2 = 25$ to show how the collapse and revival pattern varies with the pump photon coherent state eigenvalue α.

General collapses and revivals for the off-resonance cases based on Equation (36c) are demonstrated in Figure 5 for $k = 3$, $|\alpha|^2 = 5$. An important feature revealed here is the emergence of fractional revivals in the long time domain. In general, the collapse and revival pattern depends on k and α.

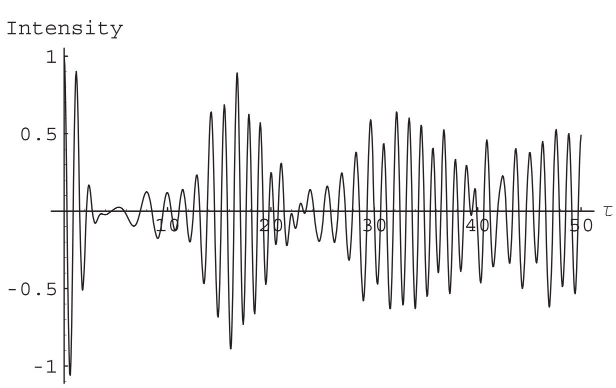

Figure 3. Signal–idler intensity inversion over scaled time $\tau = gt = 0 \to 50$, $k = 0$ (resonance), $|\alpha|^2 = 5$, $n_1 = 3$, $n_2 = 1$: general collapse and revivals.

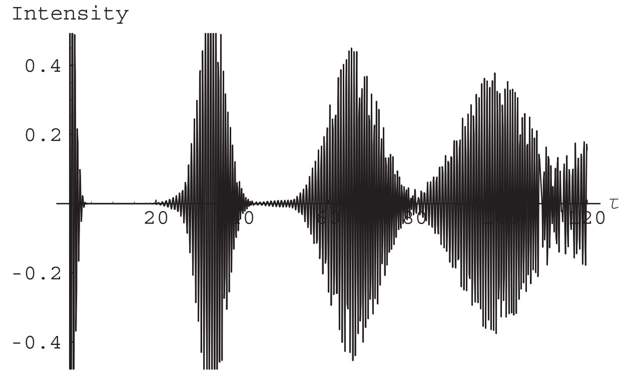

Figure 4. Signal–idler intensity inversion over scaled time $\tau = gt = 0 \to 120$, $k = 0$ (resonance), $|\alpha|^2 = 25$, $n_1 = 3$, $n_2 = 1$: general collapse and revivals.

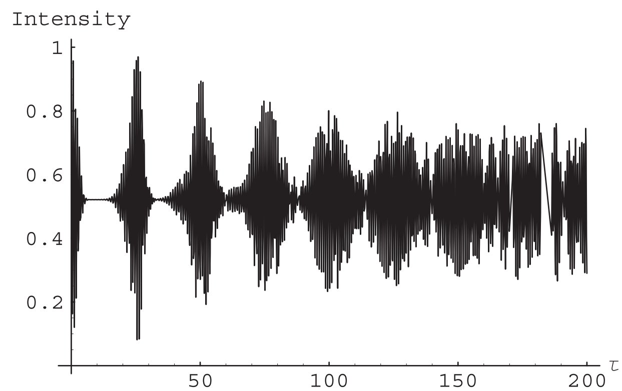

Figure 5. Signal–idler intensity inversion over scaled time $\tau = gt = 0 \to 200$, $k = 3$, $|\alpha|^2 = 5$, $n_1 = 3$, $n_2 = 1$: long time dynamics; fractional revivals emerge.

The above diagrams clearly display the general collapses and revivals deduced from the analytical results in Equations (36c) and (36d). These results agree with the numerical integration result of Jyotsna and Agarwal [4] for a resonant fully quantized degenerate parametric oscillator under interaction with a coherent state pump photon. The emergence of fractional revivals in the long time domain agrees with Averbuck's [9] analysis of fractional revivals in the long time behavior of population inversion in the Jaynes–Cummings model of a two-level atom interacting with a quantized single-mode radiation field.

These results, together with the results of the previous section, lead us to the conclusion that the dynamics of a fully quantized parametric oscillation process generated by pump photons in either Fock state or coherent state, is characterized by fractional revivals or general collapses and revivals. The existence of the general collapses and revivals, as well as the fractional revivals of exactly the same form, in fully quantized parametric interactions, has also been demonstrated in earlier studies based on numerical integration of the time evolution equations [3,4]. The present work has explicitly revealed that the collapse and revival phenomena arise from the Jaynes–Cummings mode of interaction which drives the time evolution of the positive and negative helicity states of the coupled circularly polarized signal–idler photon pair. The matrix based analytical results thus fully account for the phenomena observed in the numerical studies.

5. Polarization state dynamics

It is now clear that the dynamics of the fully quantized parametric oscillation process is determined by the time evolution of the positive and negative helicity state vectors $|1;t\rangle$ and $|2;t\rangle$ which constitute the general time evolving circular polarization state vector of the coupled two-state signal–idler photon pair. In this interpretation, the polarization state dynamics is generated by the Jaynes–Cummings Hamiltonian H_{JC} describing an effective interaction between the two-state circularly polarized signal–idler photon pair and a single-mode quantized pump field.

The general time evolving polarization state vector $|\psi(t)\rangle$ is obtained by applying the general time evolving polarization operator vector $\hat{A}(t)$ on the total initial pump, signal and idler photon state vector $|\psi(0)\rangle$ in the form

$$|\psi(t)\rangle = \hat{A}(t)|\psi(0)\rangle, \tag{37a}$$

where $\hat{A}(t)$ takes the form in Equation (13a) or (13c), with $\hat{a}_1(t)$ and $\hat{a}_2(t)$ obtained in Equations (25c)–(25d). Taking the pump, signal and idler photons to be in initial Fock states $|n\rangle$, $|n_1\rangle$ and $n_2\rangle$, respectively, gives $|\psi(0)\rangle$ in the form

$$|\psi(0)\rangle = |nn_1n_2\rangle, \tag{37b}$$

while taking the pump photon initially in a coherent state $|\alpha\rangle$, with signal–idler photon in initial Fock state $|n_1n_2\rangle$ gives

$$|\psi(0)\rangle = |\alpha n_1 n_2\rangle, \tag{37c}$$

where $|nn_1n_2\rangle$ and $|\alpha n_1n_2\rangle$ are defined in Equations (27a) and (35b), respectively.

5.1. Fock state case

To demonstrate polarization state dynamics, it suffices to consider only the Fock state case specified by Equation (37b), noting that the coherent state case specified by Equation (37c) can be obtained as a simple generalization. Substituting $|\psi(0)\rangle$ from Equation (37b) and $\hat{A}(t)$ from Equation (13c) into Equation (37a) gives

$$|\psi(t)\rangle = (\hat{a}_1(t)|1\rangle + \hat{a}_2(t)|2\rangle)|nn_1n_2\rangle. \tag{38a}$$

Using $\hat{a}_1(t)$, $\hat{a}_2(t)$ from Equations (25c)–(25d) and applying the results presented in Equations (29c)–(29f), together with

$$T_{\mathrm{p}}(t) = \exp(-\mathrm{i}\omega t\hat{a}^{\dagger}\hat{a}); \quad T_{\mathrm{p}}(t)|m\rangle = \exp(-\mathrm{i}m\omega t)|m\rangle; \quad m = n, n-1, n+1 \tag{38b}$$

gives the general time evolving circular polarization state vector for the coupled signal–idler photon pair dynamics under pump photon in the initial Fock state in the form

$$|\psi(t)\rangle = |\psi_1(t)\rangle + |\psi_2(t)\rangle, \tag{38c}$$

where the component $|\psi_1(t)\rangle$ is the general time evolving polarization state vector for (signal) photons initially in the positive helicity state $|1\rangle$, while $|\psi_2(t)\rangle$ is the general time evolving polarization state vector for (idler) photons initially in the negative helicity state $|2\rangle$ obtained as

$$|\psi_1(t)\rangle = \exp\left(-\frac{\mathrm{i}}{2}(\Omega_{12} + (2n+1)\omega)t\right) n_1^{1/2} \times(\mu_1\,|n(n_1-1)n_2\rangle|1\rangle + \nu_0\,|(n+1)(n_1-1)n_2\rangle|2\rangle), \tag{38d}$$

$$|\psi_2(t)\rangle = \exp\left(-\frac{\mathrm{i}}{2}(\Omega_{12} + (2n-1)\omega)t\right) n_2^{1/2} \times(\mu_0\,|nn_1(n_2-1)\rangle|2\rangle + \nu_1\,|(n-1)n_1(n_2-1)\rangle|1\rangle). \tag{38e}$$

Reorganizing Equations (38c)–(38e) in the form

$$|\psi(t)\rangle = \phi_1(t)|1\rangle + \phi_2(t)|2\rangle \tag{38f}$$

yields the time evolving circularly polarized signal–idler photon intensity amplitudes $\phi_1(t)$ and $\phi_2(t)$ in the positive and negative helicity states, respectively, obtained as

$$\phi_1(t) = \exp\left(-\frac{\mathrm{i}}{2}(\Omega_{12} + (2n+1)\omega)t\right) \times\left(n_1^{1/2}\mu_1\,|n(n_1-1)n_2\rangle + \exp(2\mathrm{i}\omega t)n_2^{1/2}\nu_1\,| \times(n-1)n_1(n_2-1)\rangle\right), \tag{38g}$$

$$\phi_2(t) = \exp\left(-\frac{\mathrm{i}}{2}(\Omega_{12} + (2n-1)\omega)t\right) \times\left(n_2^{1/2}\mu_0\,|nn_1(n_2-1)\rangle + \exp(-2\mathrm{i}\omega t)n_1^{1/2}\nu_0\,| \times(n+1)(n_1-1)n_2\rangle\right). \tag{38h}$$

Using Equations (38f)–(38h) gives

$$\langle\psi(t)|\psi(t)\rangle = ||\psi(t)\rangle|^2 = |\phi_1(t)|^2 + |\phi_2(t)|^2, \tag{39a}$$

$$\phi_1^{\dagger}(t)\phi_1(t) = |\phi_1(t)|^2 = |\mu_1|^2 n_1 + |\nu_1|^2 n_2, \tag{39b}$$

$$\phi_2^{\dagger}(t)\phi_2(t) = |\phi_2(t)|^2 = |\mu_0|^2 n_2 + |\nu_0|^2 n_1, \tag{39c}$$

which on comparing with Equations (28a) and (30a) give the expected mean photon intensities for circularly polarized signal–idler photon pair in the form

$$||\psi(t)\rangle|^2 = I(t); \quad |\phi_1(t)|^2 = I_1(t); \quad |\phi_2(t)|^2 = I_2(t). \tag{39d}$$

Within the framework of the photon polarization state dynamics, the mean photon intensity inversion for the circularly polarized signal–idler photon pair is obtained as the expectation value of the discrete operator σ_z in the form

$$\overline{\sigma}_z(t) = <\psi(t)|\sigma_z|\psi(t)\rangle \tag{39e}$$

which on using Equation (38f), together with

$$\langle\psi(t)| = \phi_1^\dagger(t)\langle 1| + \phi_2^\dagger(t)\langle 2|; \quad \sigma_z = |1\rangle\langle 2| - |2\rangle\langle 1| \tag{39f}$$

becomes

$$\overline{\sigma}_z(t) = |\phi_1(t)|^2 - |\phi_2(t)|^2 = I_1(t) - I_2(t) = \Delta I(t). \tag{39g}$$

This establishes that the mean photon intensity inversion $\overline{\sigma}_z(t)$ equals the mean photon intensity difference $\Delta I(t)$ obtained earlier in Equation (30c), giving

$$\overline{\sigma}_z(t) = \Delta I(t) = (|\mu_1|^2 - |\nu_0|^2)\, n_1 - (|\mu_0|^2 - |\nu_1|^2)\, n_2. \tag{39h}$$

The final expression for $\overline{\sigma}_z(t)$ takes the form of $\Delta I(t)$ obtained in Equation (32b), which need not be written here. The usefulness of the photon polarization state vectors of the coupled circularly polarized signal–idler photon pair in the fully quantized parametric oscillation process in determining the density matrix and the probability distribution for studying various physical features, as well as possible applications in quantum communication technology, can follow easily from Glauber's excellent presentation based on the semi-classical model [7]. The mean photon intensity inversion obtained within the photon polarization state dynamics as described in the present work may be useful in the currently developing studies of optical chirality [14].

6. Conclusion

The method developed in this paper is effective in providing general solutions of appropriate time evolution equations for annihilation and creation operators of signal and idler photons in a fully quantized parametric oscillation process governed by a trilinear Hamiltonian. The exact analytical results, which have easily revealed the important quantum mechanical phenomena of collapses and revivals, as well as fractional revivals, of the time evolving mean intensities or intensity inversion for the coupled signal–idler photon pair, will also prove very useful in studying other fundamental quantum mechanical features such as squeezing, photon anti-bunching, super-Poissonian or sub-Poissonian statistics and others generally associated with parametric interactions. The underlying signal–idler photon pair polarization state dynamics governed by a Jaynes–Cummings mode of interaction with pump photon taken in various quantum states, simplifies the determination of photon statistics and can provide deeper insights into fundamental phenomena such as entanglement, decoherence and optical chirality, which currently attract growing research interest in relation to their applications in designs of emerging high precision quantum technologies. The complete understanding of the dynamics under the fully quantized trilinear Hamiltonian achieved through the exact analytical solutions greatly expands the range of possibilities of observable fundamental features, as well as potential applications to quantum information processing, quantum computation and other related quantum mechanics based technologies.

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References

[1] Carusotto, S. *Phys. Rev. A* **1989**, *40*, 1848–1857.
[2] Jurco, B.J. *Math. Phys.* **1989**, *30*, 1739–1743.
[3] Drobny, G.; Jex, I. *Phys. Rev. A* **1992**, *45*, 1816–1821; *ibid.* **1992**, *46*, 499–506.
[4] Jyotsna, I.V.; Agarwal, G.S. *J. Mod. Opt.* **1997**, *44*, 305–312.
[5] Knight, P.L.; Radmore, P.M. *Phys. Rev. A* **1982**, *26*, 676–679.
[6] Nielsen, M.A.; Chuang, I.L. *Quantum Computation and Quantum Information*; Cambridge University Press: Cambridge, UK, 2000.
[7] Glauber, R.J. In *Frontiers in Quantum Optics*; Pike, E.R., Sarkar, S., Eds.; Malvern Physics Series 2; Adam Hilger: Bristol, UK, 1986; p 534.
[8] Shore, B.W.; Knight, P.L. *J. Mod. Opt.* **1993**, *40*, 1195–1238.
[9] Averbukh, I.S.H. *Phys. Rev. A* **1992**, *46*, R2205–R2208.
[10] Jaynes, E.T.; Cummings, F.W. *Proc. IEEE* **1963**, *51*, 89–109.
[11] Meystre, P.; and Sargent III, M. *Elements of Quantum Optics*, 2nd ed. Springer-Verlag: Berlin, 1991.
[12] Azuma, H. *Int. J. Mod. Phys. C* **2010**, *21*, 1021–1049.
[13] El-Orany, F.A.A. *J. Mod. Opt.* **2006**, *53*, 1699–1714.
[14] Coles, M.M.; Andrews, D.L. *Phys. Rev. A* **2012**, *85*, 063810.