NORMS OF GENERALIZED DERIVATIONS ON NORM IDEALS

BY

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Declaration

This thesis is my own work and has not been presented for a degree award in any other institution.

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This thesis has been submitted for examination with our approval as the university supervisors.

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Dedication

To my mother Hilda who encouraged me never to give up in pursuing my dreams. Through your prayer mama, this research has become possible.

Abstract

The norms of inner and generalized derivations on different kinds of algebras have been determined. However, the norms of their restrictions to norm ideals have not been fully explored. For instance, the concept of $S-$universality having been introduced by Fialkow in 1979, has not been fully characterized yet it plays a critical role in the study of norms of derivations. In this study, we have investigated both the algebraic and the norm properties of a generalized derivation. Specifically, we have determined the norm of generalized derivation on a norm ideal, extended the concept of $S-$universality to the setting of a generalized derivation, and established the necessary conditions for the attainment of the optimal value of the circumdiameters of numerical ranges and spectra of two bounded linear operators in a Hilbert space. It turns out that for a pair of $S-$universal, normaloid or spectraloid operators, the circumdiameter of the numerical ranges or the spectra is the sum of the numerical radii or the spectral radii respectively. We have characterized the antidistance from an operator to its similarity orbit in terms of the circumdiameter, norms, numerical and spectral radii. Based on the definition in the context of inner derivations we have extended the concept of $S-$universality to the generalized derivation. Using the relations between the norm of an inner derivation and the diameter of the numerical range as well as spectral inclusion theorem, we have established various relations between the norm of a generalized derivation and the circumdiameters of the numerical ranges and spectra. We hope that the results obtained in this study has greatly contributed to the field of derivations and provided motivations for further research to pure mathematicians in this area of study.

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Chapter 1

Introduction

1.1 Background of the study

The research on norms of elementary operators has been going on for almost six decades. We note that derivations, especially inner and generalized ones, are forms of elementary operators. However, the norm problems of the generalized derivations on norm ideals still remain unresolved. A number of mathematicians such as Stampfli [34], Fialkow [16], Barraa [3], Bonyo and Agure [8], among others have obtained useful results in regard to the norms of derivations. Stampfli's results were used by Barraa and Boumazgour [4] to establish the restriction of generalized derivation implemented by bounded linear operators in a Hilbert space on a norm ideal. The notion of $S-$universality introduced by Fialkow [15, 16, 17] was later used by Barraa and Boumazgour to characterize when the norm of the sum of two bounded operators on a Hilbert space is equal to the sum of their norms. They also provided the necessary and sufficient condition of $S-$universality for arbitrary hyponormal operators. Barraa [3] later compared the norm of a generalized derivations on a Hilbert space with the norm of its restrictions to norm ideals. Barraa's results were later used by Bonyo and Agure [9] to establish the relationship between the norms of inner and generalized derivations. In this study, we have extended the concept of $S-$universality and established its applications as well as determined its properties in the setting of generalized derivations. We have also used the properties of numerical ranges and spectra in the study of the norm of generalized derivation on a norm ideals.

1.2 Basic Concepts

1.2.1 Algebra and Ideals

If S is a subset of an algebra $\mathcal{A}$, then the centre of S is the set $Z(S) = \{x \in \mathcal{A} : xs = sx \text{ for all } s \in S\}$.
Let $\mathcal{A}$ be an algebra, a mapping from $\mathcal{A}$ to $\mathcal{A}$ defined by $a \to a^*$ is called an involution on $\mathcal{A}$ if it satisfies the following four properties: For all $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{K}$, we have:

(i) $(a+b)^* = a^* + b^*$,

(ii) $(\lambda a)^* = \bar{\lambda} a^*$,

(iii) $(ab)^* = b^* a^*$ and

(iv) $a^{**} = a$.

An algebra with an involution is called an involutive algebra or *-algebra.
A norm $\|.\|$ on an algebra $\mathcal{A}$ is said to be sub-multiplicative if it satisfies

the following: $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in \mathcal{A}$.

An algebra $\mathcal{A}$ with a norm which is sub-multiplicative is a normed algebra.

A complete normed algebra is called a Banach algebra. A Banach algebra $\mathcal{A}$ with an involution $a \to a^*$ that satisfies $\|aa^*\| = \|a\|^2$ for all $a \in \mathcal{A}$ is called a $C^* -$ algebra.

A subalgebra of an algebra $\mathcal{A}$ is a vector subspace $\mathcal{J} \subseteq \mathcal{A}$ such that for all $a, b \in \mathcal{J}$, we have $ab \in \mathcal{J}$.

A left (respectively, right) ideal of an algebra $\mathcal{A}$ is a vector subspace $\mathcal{J}$ of $\mathcal{A}$ such that for all $a \in \mathcal{A}$ and $b \in \mathcal{J}$ we have $ab \in \mathcal{J}$ (respectively, $ba \in \mathcal{J}$).

An ideal in $\mathcal{A}$ is a vector subspace that is simultaneously a left and a right ideal in $\mathcal{A}$.

If $\mathcal{J}$ is an ideal of $\mathcal{A}$, then $\mathcal{A}/\mathcal{J}$ is a quotient algebra with the multiplication given by $(a + \mathcal{J})(b + \mathcal{J}) = ab + \mathcal{J}$.

A maximal ideal in $\mathcal{A}$ is a proper ideal that is not contained in any other proper ideal in $\mathcal{A}$.

A subset G of a topological space $\mathcal{T}$ is called dense (in $\mathcal{T}$) if every point t in $\mathcal{T}$ either belongs to G or is a limit point of G, that is closure of G is constituting the whole set $\mathcal{T}$.

An operator T is convexoid if $\overline{W(T)} = conv.\sigma(T)$ (conv. denotes the convex hull). Let H be a complex Hilbert space and let $\mathcal{B}(H)$ be the algebra of all bounded linear operators on H.

A Hilbert space is separable provided it contains a dense countable subset.

A norm ideal $(\mathcal{J}, \|\cdot\|_{\mathcal{J}})$ in $\mathcal{B}(H)$ consists of a proper two-sided ideal $\mathcal{J}$

together with the norm $\|\cdot\|_{\mathcal{J}}$ satisfying the following conditions:

(i) $(\mathcal{J}, \|\cdot\|_{\mathcal{J}})$ is a Banach space,

(ii) $\|AXB\|_{\mathcal{J}} \leq \|A\|\|X\|_{\mathcal{J}}\|B\|$ for all $X \in \mathcal{J}$ and all operators A and B in $\mathcal{B}(H)$, and

(iii) $\|X\|_{\mathcal{J}} = \|X\|$ for X a rank one operator.

For a comprehensive theory of Algebra and ideals refer to [5, 30].

1.2.2 Operators and Functionals

Operators are mappings from one vector space to another vector space while functionals are mappings from a vector space to the space of scalars. Let $\mathcal{X}$ and $\mathcal{Y}$ be vector spaces over a field $\mathbb{K}$. Then a function $T : \mathcal{X} \to \mathcal{Y}$ is called a linear operator if and only if for all $x, y \in \mathcal{X}$ and $\alpha, \beta \in \mathbb{K}$, $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$.
Let $\mathcal{X}$ be a vector space over $\mathbb{K}$. A functional $f : \mathcal{X} \to \mathbb{K}$ is linear if it is a linear operator.
Let $\mathcal{X}$ be a vector space and $\mathcal{X}^*$ the set of all linear functionals on $\mathcal{X}$, then $\mathcal{X}^*$ is called the dual space of $\mathcal{X}$.
A linear operator $T : \mathcal{X} \to \mathcal{Y}$ is bounded if and only if there exists a constant $M > 0$ such that $\|Tx\| \leq M\|x\|$ for all $x \in \mathcal{X}$. For more details, we refer to [18, 30].
An operator $T \in \mathbb{B}(H)$ is said to be;

- Self-adjoint if $T^* = T$.
- Positive if $\langle Tx, x\rangle \geq 0$ for all nonzero $x \in H$

- Normal if $T^*T = TT^*$
- Unitary if $T^*T = TT^* = I$
- Subnormal if there exists a Hilbert space K such that H is a closed linear subspace of K and a normal operator $N \in \mathcal{B}(K)$ such that $Nx = Tx$ for all $x \in H$
- Hyponormal if $T * T \geq TT^*$.

An operator X having trival kernel and dense range is called a quasiaffinity.
Operators A and B are said to be quasisimilar if there exists quasiaffinities X and Y such that $XA = BX$ and $AY = YB$.

1.2.3 Automorphism

An isomorphism is a homomorphism that can be reversed by an inverse morphism. An automorphism on an algebra $\mathcal{A}$ is an isomorphism of $\mathcal{A}$ onto $\mathcal{A}$. Let $Inv(\mathcal{A})$ be inverse of elements in an algebra $\mathcal{A}$. Given an algebra $\mathcal{A}$ with unit, and $A \in Inv(\mathcal{A})$, let α_A be the mapping of $\mathcal{A}$ into $\mathcal{A}$ given by $\alpha_A(X) = A^{-1}XA$, for all $X \in \mathcal{A}$, then we have that α_A is an automorphism on $\mathcal{A}$ and is called an inner automorphism in $\mathcal{A}$. It is clear that $\alpha_A = I$ if A belongs to the centre of $\mathcal{A}$. In particular, if $\mathcal{A}$ is commutative, then I is the only inner automorphism. [7]. Now, for fixed $A, B \in Inv(\mathcal{A})$, we define a mapping $\alpha_{A,B} : \mathcal{A} \to \mathcal{A}$ by $\alpha_{A,B}(X) = A^{-1}XB$ for all $X \in \mathcal{A}$. We shall call $\alpha_{A,B}$ a generalized inner automorphism on $\mathcal{A}$.

1.2.4 Schatten Norm Ideals

A bounded linear operator A over a separable Hilbert space H is said to be in the trace class if for some (and hence all) orthonormal bases $(e_k)_k$ of H, the sum of positive terms

$$\|A\|_1 = tr|A| := \sum_k \langle (A^*A)^{\frac{1}{2}} e_k, e_k \rangle$$

is finite. In this case, the trace of A, which is given by the sum

$$tr(A) := \sum_k \langle Ae_k, e_k \rangle$$

is absolutely convergent and is independent of the choice of the orthonormal basis. When H is finite-dimensional, every operator is trace class and this definition of trace of A coincides with the definition of the trace of a matrix. Let $C_p(H)$ denote the Schatten class p-ideal , $1 \leq p \leq \infty$, see [32]. The class $C_p(H)$ consists of the compact operators X such that $\sum_j S_j^p(X) < \infty$, where $\{S_j(X)\}_j$ denotes the sequence of singular values of X. For $X \in C_p(H)(1 \leq p \leq \infty)$, we set $\|X\|_p = (\sum_j S_j^p(X))^{\frac{1}{p}}$, where, by convention, $\|X\|_\infty = S_1(X)$ is the usual operator norm of X. Then $(C_p(H), \|\cdot\|_p)$ is a norm ideal. Moreover, $(C_2(H), \|\cdot\|_2)$ or simply $C_2(H)$ is a Hilbert-Schimidt class operators with inner product defined by $\langle X, Y \rangle = tr(XY^*), (X, Y \in C_2(H))$, Y^* denotes the adjoint of Y. We refer to [30] for details on norm ideals and algebra.

Following [32], $C_1(H), C_2(H)$ and $C_\infty(H)$ are the trace class, the Hilbert-Schmidt class and the class of compact operators respectively. The Hilbert-Schmidt class is a Hilbert space with respect to the inner product $\langle A, B \rangle =$

$\mathrm{tr}(AB^*)$ where $A, B \in C_2(H)$. We refer to [11, 19, 32] for a comprehensive theory of these classes of compact operators.

1.2.5 Derivations

A derivation δ on an algebra $\mathcal{A}$ is a linear map $\delta : \mathcal{A} \to \mathcal{A}$ such that for all $A, B \in \mathcal{A}$, we have;

$$\delta(AB) = \delta(A)B + A\delta(B).$$

Fix $A, B \in \mathcal{A}$ and define a mapping from $\mathcal{A}$ to $\mathcal{A}$ by $\delta_{A,B}(X) = AX - XB$ for all $X \in \mathcal{A}$. Then $\delta_{A,B}$ is called a generalized derivation in $\mathcal{A}$. In the case that $A = B$, we have an inner derivation $\delta_A = \delta_{A,A}$, that is,

$$\delta_{A,A}(X) = AX - XA \text{ for all } X \in \mathcal{A}$$

or simply,

$$\delta_A(X) = AX - XA, \text{ for all } X \in \mathcal{A}.$$

Now, for a fixed $A \in \mathcal{A}$, the mappings R_A and L_A of $\mathcal{A}$ into $\mathcal{A}$ defined by $L_A(X) = AX$ and $R_A(X) = XA$, for all $X \in \mathcal{A}$, are called the left and the right multiplications by A respectively.

Remark 1.2.1

It can be easily seen that the inner and generalized derivation can also be given in terms of right and left multiplications as $\delta_A = L_A - R_A$ and $\delta_{A,B} = L_A - R_B$, respectively.

Let $\mathcal{A} = \mathcal{B}(H)$ be the algebra of all bounded linear operators on a Hilbert space H and $A = (A_1, A_2, ..., A_n)$ and $B = (B_1, B_2, ..., B_n)$ be $n-$tuples of elements in $\mathcal{B}(H)$. The general elementary operator $\Delta_{A,B}$ associated with A and B is the operator on $\mathcal{B}(H)$ into itself defined by

$$\Delta_{A,B}(X) = \sum_{i=1}^{n} A_i X B_i \text{ for all } X \in \mathcal{B}(H).$$

For details see [7, 11, 30].

Remark 1.2.2

The types of operators defined above, for example, generalized derivations, inner derivations, left multiplication and right multiplication are special types of elementary operators.

The operator norms of an inner derivation and a generalized derivation on $\mathcal{B}(H)$ as computed by Stampfli [34] are given by

$$\|\delta_A\| = 2\inf\{\|A - \lambda I\| : \lambda \in \mathbb{C}\} = 2d(A)$$

where

$$d(A) = \inf\{\|A - \lambda I\| : \lambda \in \mathbb{C}\}$$

is the distance from the operator A to the centre of the algebra $\mathcal{B}(H)$, and

$$\|\delta_{A,B}\| = \inf\{\|A - \lambda I\| + \|B - \lambda I\| : \lambda \in \mathbb{C}\}$$

respectively.

Let $\mathcal{J}$ be a norm ideal in $\mathcal{B}(H)$. An operator A is an $S-$universal operator if $\|\delta_A|\mathcal{J}\| = \|\delta_A\|$.

1.2.6 Spectra and Numerical Ranges

Let $\mathcal{B}(H)$ be the algebra of all bounded linear operators on Hilbert space H and $T \in \mathcal{B}(H)$. The point spectrum, $\sigma_p(T)$ is the set of eigenvalues of T. Approximate point spectrum of T, $\sigma_{ap}(T)$ consists of those complex numbers λ for which there exists a unit sequence $(x_n)_n \subseteq H$ such that $\lim_n \|(T - \lambda I)x_n\| = 0$. The spectrum $\sigma(T)$ of T is a set of complex numbers $\lambda \in \mathbb{C}$ such that the operator $T - \lambda I$ is not invertible. In a finite-dimensional space, $\sigma(T)$ coincides with the point spectrum, $\sigma_p(T)$. The spectral radius of T is given by $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$.
For every $T \in B(H)$, the numerical range of T, $W(T)$ is defined by, $W(T) = \{\langle Tx, x\rangle : x \in H, \|x\| = 1\}$, while the maximal numerical range of T is the set $W_0(T) = \{\lambda : \langle Tx_n, x_n\rangle \to \lambda, \|x_n\| = 1, \|Tx_n\| \to \|T\|\}$. We define the normalized maximal numerical range, $W_N(T)$, of the operator $T \in \mathcal{B}(H)$ to be the set,

$$W_N(T) = \begin{cases} W_0\left(\frac{T}{\|T\|}\right) & \text{if } T \neq 0, \\ 0 & \text{if } T = 0. \end{cases}$$

The numerical radius of T is defined by $\omega(T) = \sup\{|\lambda| : \lambda \in W(T)\}$.
It is known that the numerical range, $W(T)$, is a bounded, compact and closed set. See [28, 18, 30] for a comprehensive theory of spectra and numerical ranges.

1.3 Statement of the Problem

Let $\mathcal{B}(H)$ be the algebra of bounded linear operators on a complex Hilbert space H and $\mathcal{J}$ be a norm ideal in $\mathcal{B}(H)$. For fixed $A, B \in \mathcal{B}(H)$, the mappings $\delta_A(X) = AX - XA$ and $\delta_{A,B}(X) = AX - XB$ for all $X \in \mathcal{B}(H)$ define respectively the inner and generalized derivations on $\mathcal{B}(H)$. The norms of inner derivation, δ_A and generalized derivation, $\delta_{A,B}$ on $\mathcal{B}(H)$ have been fully determined by Stampfli [34] while the norm restrictions of these operators on norm ideals remain unresolved. In this study therefore, we continue to investigate the norm properties of generalized derivations on norm ideals. Moreover, the concept of $S-$universality having been introduced by Fialkow [16] in 1979, has been studied in the context of inner derivation only. We have therefore extended this concept to the setting of a generalized derivation.

1.4 Objective of the study

The main objective of this study was to investigate the norms of generalized derivations on norm ideals.
Specifically, we have;

1. Determined the norm properties of generalized derivation acting on a norm ideal $\mathcal{J}$ in $\mathcal{B}(H)$.
2. Extended the concept of $S-$universality to the generalized derivation and determined its properties in this setting.

3. Established some applications of the concept of $S-$universality as used in generalized derivations.

1.5 Research methodology

In investigating the norm properties of a generalized derivation, we have applied the known results of Stampfli, Barraa and Boumazgour and that of Bonyo and Agure. In extending the concept of S-universality, we have used the definition in the context of inner derivations as provided by Fialkow. Moreover, using the general theories of analysis like spectra, numerical ranges among others as well as the norm properties of the generalized derivations, we have studied the properties of $S-$universal operators in the setting of generalized derivations. Based on the concept of anti-distance and similarity orbits, we have established the applications of $S-$universality as used in the context of generalized derivations.

1.6 Significance of the Study

We hope that the results obtained in this study will greatly contribute to the field of derivations and provide motivation for further research to pure mathematicians in this area of study. The results will also be useful to the theoretical physicists and applied mathematicians alike.

Chapter 2

Literature Review

The norm of generalized derivation was first determined by Stampfli in 1970 [34]. He showed that $\|\delta_A\| = 2\inf\{\|A - \lambda I\| \; \lambda \in \mathbb{C}\} = 2d(A)$ and $\|\delta_{A,B}\| = \inf\{\|A - \lambda I\| + \|B - \lambda I\| : \lambda \in \mathbb{C}\}$. Kyle [20] later established the converse of Stampfli's results. Stampfli also proved that the inner derivation induced by a hyponormal operator has closed range if and only if the operator has finite spectrum. In order to examine the extent to which Stampfli's equality applies, Kittaneh [19] established the orthogonality of the range and the kernel of a normal derivation with respect to the unitary invariant norms associated with norm ideals of operators. Barraa and Pedersen [6] characterized when the product $\delta_{C,D}\delta_{A,B}$ is a generalized derivation in the cases when ring R is the algebra of all bounded operators on a Banach space. In 2001, Timoney [35] made some simple remarks about the norm problem for elementary operators in the contexts of algebras of operators on Banach spaces and C^*- algebra and illustrated a lack of symmetry in the problem for algebras containing the finite rank operators and also for the Calkin algebra. In the same year (2001), Barraa [2] showed that generalized derivation acting on a symmetric norm ideal on a Banach space is convexoid if and only if the operators are convexoid. Later in 2007, he presented a formula for the norm of an elementary operator on a C^*-algebra that seems to be new. The formula involves (matrix) numerical ranges and a kind of geometrical mean for positive matrices, the tracial geometric mean and characterized compactness of elementary operators. He then went ahead to show that a subnormal operator is $S-$universal if and only if the diameter of the spectrum is equal to twice the radius of the smallest disk containing it. In his paper Fialkow [15] proved that if A and B are quasisimilar operators, then the right essential spectrum of A intersects the left essential spectrum of B. Fialkow [17] described the essential spectrum and index function of the operator $X \to AXB$ where A, B and X are Hilbert space operators. Using the same approach, Barraa and Boumazgour later characterized when the norm of the sum of two bounded operators on a Hilbert space is equal to the sum of their norms and provided the necessary and sufficient condition of $S-$universality for hyponormal operators. On the other hand, Barraa and Boumazgour [5] established that if $(\mathcal{J}, \|.\|_{\mathcal{J}})$ is a norm ideal on $\mathcal{B}(H)$, then the restriction of generalized derivation implemented by $A, B \in B(H)$ on a norm ideal is a bounded linear operator on $(\mathcal{J}, \|.\|_{\mathcal{J}})$. Baraa and Boumazgour [28] showed that for every pair (A, B) of operators on H, there exists a positive number α_J satisfying $1 \leq \alpha_J \leq 2$ and $\|\delta_{A,B}\| \leq \alpha_J \|\delta_{A,B}|\mathcal{J}\|$. They also gave some applications to bounds for anti-distance between two boundary operators.

Bonyo and Agure [9] in 2009 established the relationship between the norm of δ_A, δ_B and $\delta_{A,B}$ on $\mathcal{B}(H)$, specifically when the operators A, B are $S-$universal.

Barraa [3] in his paper on generalized derivations and norm equality in normed ideals in 2011 compared the norm of a generalized derivation on a Hilbert space with the norm of its restrictions to Schatten norm ideals. Mecheri [25] in 2009 presented some generalized finite operators and discussed new C^*-algebras generated by generalized finite pairs of operators (A, B). In 2016, Runji, et al [31] examined the relationship between the numerical range of the restriction of a generalized derivation to a norm ideal $\mathcal{J}$ and that of its implementing elements. In the same year, Boumazgour [11] gave some necessary and sufficient conditions for the norm of the restriction of sum of two multiplications to a norm ideal to attain its optimal value. In all the studies done on the norm of generalized derivations, the norm properties of generalized derivation acting on a norm ideal in $\mathcal{B}(H)$ has not been exhaustively researched on. The concept of $S-$universality has been done on inner derivations and not on the generalized derivations. We extended this concept to the setting of generalized derivations and determined its properties. Moreover, we have established some applications of the concept of $S-$universality as used in the setting of generalized derivation.

Chapter 3

Properties of Generalized Derivations

In this chapter, we give some algebraic properties of derivations. We further extend the concept of $S-$universality from the setting of inner derivations to the setting of generalized derivations.

3.1 Algebraic Properties of Generalized Derivations

Recall that a derivation on an algebra $\mathcal{A}$ is a linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ such that for all $A, B \in \mathcal{A}$, $\delta(AB) = \delta(A)B + A\delta(B)$. For fixed $A, B \in \mathcal{A}$, a generalized derivation $\delta_{A,B}$ is defined by $\delta_{A,B}(X) = AX - XB$ for all $X \in \mathcal{A}$. If $A = B$, we obtain an inner derivation $\delta_A(X) = \delta_{A,A}(X) = AX - XA$ for all $X \in \mathcal{A}$. In the next results, we study some properties of the generalized derivations.

Proposition 3.1.1

A generalized derivation $\delta_{A,B}$ is linear but fails to be a derivation on an algebra $\mathcal{A}$ while an inner derivation $\delta_{A,A}$ is a derivation on $\mathcal{A}$.

Proof. First we prove that $\delta_{A,B}$ is linear. Fix $A, B \in \mathcal{A}$ and let $\alpha, \beta \in \mathbb{C}$. Then for all $X, Y \in \mathcal{A}$, we have that $\alpha X + \beta Y \in \mathcal{A}$ and

$$\begin{aligned}\delta_{A,B}(\alpha X + \beta Y) &= A(\alpha X + \beta Y) - (\alpha X + \beta Y)B \\ &= \alpha AX + \beta AY - \alpha XB - \beta YB \\ &= \alpha(AX - XB) + \beta(AY - YB) \\ &= \alpha\delta_{A,B}(X) + \beta\delta_{A,B}(Y).\end{aligned}$$

Hence $\delta_{A,B} : \mathcal{A} \to \mathcal{A}$ is linear.

Next we show that $\delta_{A,B}$ fails to be a derivation on $\mathcal{A}$. Indeed, for all $X, Y \in \mathcal{A}$, we have;

$$\begin{aligned}\delta_{A,B}(XY) &= A(XY) - (XY)B \\ &= AXY - XYB \\ &= AXY - XYB + XBY - XBY \\ &= (AXY - XBY) + (XBY - XYB) \\ &= (AX - XB)Y + X(BY - YB) \\ &= \delta_{A,B}(X)Y + X\delta_{B,B}(Y).\end{aligned}$$

Since $\delta_{A,B}(X)Y + X\delta_{B,B}(Y)$ is not equal to $\delta_{A,B}(X)Y + X\delta_{A,B}(Y)$, it follows that $\delta_{A,B}$ fails to be a derivation on $\mathcal{A}$.

On the other hand, an inner derivation $\delta_{A,A}$ turns out to be a derivation. Indeed, it is clear that δ_A is linear since $\delta_{A,B}$ is linear as proved above.

Now for a fixed $A \in \mathcal{A}$, we have that for all $X, Y \in \mathcal{A}$,

$$\begin{aligned}
\delta_A(XY) &= A(XY) - (XY)A. \\
&= AXY - XAY + XAY - XYA \\
&= (AXY - XAY) + (XAY - XYA) \\
&= (AX - XA)Y + X(AY - YA) \\
&= \delta_A(X)Y + X\delta_A(Y), \text{ as desired.}
\end{aligned}$$

This completes the proof. □

Proposition 3.1.2

If $\mathcal{A}$ is commutative, then 0 is the only inner derivation on $\mathcal{A}$ while a generalized derivation is always nonzero on $\mathcal{A}$.

Proof. For fixed $A \in \mathcal{A}$ and for all $X \in \mathcal{A}$, we have

$$\begin{aligned}
\delta_A(X) &= AX - XA \\
&= AX - AX \text{ since } \mathcal{A} \text{ is commutative} \\
&= 0.
\end{aligned}$$

Now, for fixed $A, B \in \mathcal{A}$, we have for all $X \in \mathcal{A}$, $\delta_{A,B}(X) = AX - XB$ which is nonzero even if $\mathcal{A}$ is commutative. □

In the next proposition, we prove that the sum of two generalized derivations is a generalized derivation.

Proposition 3.1.3

The sum of two generalized derivations is a generalized derivation.

Proof. For fixed $A, B, C, D \in \mathcal{A}$ and for all $X \in \mathcal{A}$, it follows from the linearity of a generalized derivation that,

$$\begin{aligned}(\delta_{A,B}+\delta_{C,D})(X) &= \delta_{A,B}(X)+\delta_{C,D}(X)\\ &= AX-XB+CX-XD\\ &= AX+CX-XB-XD\\ &= (A+C)X-X(B+D)\\ &= \delta_{A+C,B+D}(X),\end{aligned}$$

as desired. □

The following is an immediate consequence of Proposition 3.1.3 above.

Corollary 3.1.4

The sum of two inner derivations is an inner derivation.

PROOF. Let $A = B$ and $C = D$ in the proof of Proposition 3.1.3, then we have; $\delta_{A,A} + \delta_{C,C} = \delta_{A+C,A+C}$. But $\delta_{A,A} = \delta_A, \delta_{C,C} = \delta_C$ and $\delta_{A+C,A+C} = \delta_{A+C}$. The result then follows at once. □

REMARK 3.1.5

The product of two generalized derivations is not necessarily a generalized derivation. The question of when the product of two derivations is a derivation has been considered by a number of authors. For instance Barraa and Pedersen [6] characterized when the product $\delta_{C,D}\delta_{A,B}$ is a generalized derivation in the cases when $\mathcal{A}$ is the algebra of all bounded operators on a Banach space and when $\mathcal{A}$ is a C^*-algebra.

In the next two propositions, we show that a generalized derivation is bounded and further give a generalization of repeated action of $\delta_{A,B}$ on $\mathcal{A}$ by induction.

Proposition 3.1.6

If $\mathcal{A}$ is a normed algebra, then the generalized derivation $\delta_{A,B}$ is a bounded operator on $\mathcal{A}$ with $\|\delta_{A,B}\| \leq \|A\| + \|B\|$. In particular, the inner derivation is bounded with $\|\delta_A\| \leq 2\|A\|$.

Proof. For fixed $A, B \in \mathcal{A}$ and for all $X \in \mathcal{A}$, we have

$$\begin{aligned}\|\delta_{A,B}(X)\| &= \|AX - XB\| \\ &\leq \|AX\| + \|XB\| \\ &\leq \|A\|\|X\| + \|X\|\|B\|.\end{aligned}$$

Now taking supremum over all $X \in \mathcal{A}$ with $\|X\| = 1$, we have, $\|\delta_{A,B}\| \leq \|A\| + \|B\|$. If $A = B$, we immediately conclude that $\|\delta_A\| \leq 2\|A\|$. □

Proposition 3.1.7

Let $\delta_{A,B}$ be a generalized derivation on an algebra $\mathcal{A}$, then for each $n \in \mathbb{N}$,

$$\delta^n_{A,B}(X) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} A^{n-r} X B^r, \tag{3.1}$$

for all $X \in \mathcal{A}$.

Proof. We shall use the principle of mathematical induction to do the proof.

Let $p(n)$ be the statement that for all $n \in \mathbb{N}$,

$$\delta^n_{A,B}(X) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} A^{n-r} X B^r. \tag{3.2}$$

We induct on n.

Base step: $n = 1$

Solving the left hand side (LHS)we have, $\delta_{A,B}(X) = AX - XB$, while for the right hand side (RHS) we get,

$$(-1)^0 \binom{1}{0} AXB^0 + (-1)^1 \binom{1}{1} A^0 XB^1 = AX - XB.$$

Thus $LHS = RHS$ and so $p(1)$ is true. Before proceeding to the induction step, for the sake of clarity, we consider the case $n = 2$. Now, for $n = 2$, we begin by evaluating the RHS;

$$\begin{aligned}\sum_{r=0}^{2} (-1)^r \binom{2}{r} A^{2-r} XB^r &= (-1)^0 \binom{2}{0} A^2 XB^0 + (-1)^1 \binom{2}{1} A^1 XB^1 + \\ &\quad (-1)^2 \binom{2}{2} A^0 XB^2 \\ &= A^2 XB^0 - 2AXB + A^0 XB^2 \\ &= A^2 X - 2AXB + XB^2.\end{aligned}$$

For the LHS;

$$\begin{aligned}\delta^2_{A,B}(X) &= \delta_{A,B}(\delta_{A,B}(X)) \\ &= A\delta_{A,B}(X) - \delta_{A,B}(X)B \\ &= A(AX - XB) - (AX - XB)B \\ &= A^2 X - AXB - AXB + XB^2 \\ &= A^2 X - 2AXB + XB^2.\end{aligned}$$

Since $RHS = LHS$, $p(2)$ is true as well.

Induction step: Suppose that $p(k)$ is true for $k \in \mathbb{N}$. This means that for all $X \in \mathcal{A}$,

$$\delta^k_{A,B}(X) = \sum_{r=0}^{k} (-1)^r \binom{k}{r} A^{k-r} X B^r$$

Then, for $p(k+1)$, we have

$$\begin{aligned}
\delta^{k+1}_{A,B}(X) &= \delta_{A,B}(\delta^k_{A,B}(X)) \\
&= A\delta^k_{A,B}(X) - \delta^k_{A,B}(X)B \\
&= A\left(\sum_{r=0}^{k}(-1)^r\binom{k}{r}A^{k-r}XB^r\right) - \left(\sum_{r=0}^{k}(-1)^r\binom{k}{r}A^{k-r}XB^r\right)B \\
&= \sum_{r=0}^{k}(-1)^r\binom{k}{r}A^{k-r+1}XB^r - \sum_{r=0}^{k}(-1)^r\binom{k}{r}A^{k-r}XB^{r+1} \\
&= \sum_{r=0}^{k+1}(-1)^r\binom{k}{r}A^{k-r+1}XB^r - \sum_{r=1}^{k+1}(-1)^{r-1}\binom{k}{r-1}A^{k-r+1}XB^r \\
&= \sum_{r=0}^{k+1}(-1)^r\binom{k}{r}A^{k-r+1}XB^r + \sum_{r=0}^{k+1}(-1)^r\binom{k}{r-1}A^{k-r+1}XB^r \\
&= \sum_{r=0}^{k+1}(-1)^r\left(\binom{k}{r}+\binom{k}{r-1}\right)A^{k-r+1}XB^r \\
&= \sum_{r=0}^{k+1}(-1)^r\binom{k+1}{r}A^{k-r+1}XB^r.
\end{aligned}$$

Thus $p(k+1)$ is true. Hence $p(k)$ implies $p(k+1)$ and therefore by the principle of mathematical induction, it follows that $p(n)$ is true for all $n \in \mathbb{N}$. □

Remark 3.1.8

Recall that, for a fixed $A \in \mathcal{A}$, the mapping $\alpha_A : \mathcal{A} \to \mathcal{A}$, defined by

$$\alpha_A(X) = A^{-1}XA \quad (X \in \mathcal{A})$$

is an inner automorphism on an algebra $\mathcal{A}$ for all A invertible in the algebra $\mathcal{A}$. Let $Inv(\mathcal{A})$ denote the set of all invertible elements of $\mathcal{A}$, then for $A, B \in \text{Inv}(\mathcal{A})$, we have $\alpha_{A,B}(X) = A^{-1}XB \ \ (X \in \mathcal{A})$ is a generalized inner automorphism. For inner automorphisms, if δ is a continuous derivation on a Banach algebra $\mathcal{A}$, then $\exp(\delta)$ is a continuous automorphism on $\mathcal{A}$ and if A is an element of a Banach algebra $\mathcal{A}$ with unit, then, $exp(\delta_A) = \alpha_{expA}$. See [7] for details.

In the next two propositions, we extend these results to the generalized cases.

Proposition 3.1.9

Let $\mathcal{A}$ be an algebra and $A, B \in Inv\mathcal{A}$, then $\alpha_{A,B}$ is an automorphism on $\mathcal{A}$.

Proof. We first prove that $\alpha_{A,B}$ is a homomorphism. Indeed, for all $X, Y \in \mathcal{A}$, we have;

$$\begin{aligned} \alpha_{A,B}(XY) &= A^{-1}XYB \\ &= A^{-1}((X)(Y))B \\ &= A^{-1}((XB)(YB)) \\ &= (A^{-1}XB)(A^{-1}YB) \\ &= \alpha_{A,B}(X)\alpha_{A,B}(Y). \end{aligned}$$

Thus $\alpha_{A,B}$ is a homomorphism. Now, for all $X, Y \in \mathcal{A}$, $\alpha_{A,B}(X) = \alpha_{A,B}(Y)$ if and only if $A^{-1}XB = A^{-1}YB$. Pre-multiplying both sides by A, then post multiplying by B^{-1}, we obtain $X = Y$. Therefore $\alpha_{A,B}$ is an injective mapping of $\mathcal{A}$ and is hence a bijective mapping of $\mathcal{A}$ onto $\mathcal{A}$. Since $\alpha_{A,B}$ is an invertible homomorphism on $\mathcal{A}$, it follows that $\alpha_{A,B}$ is an automorphism on $\mathcal{A}$. □

Proposition 3.1.10

Let $\delta_{A,B}$ be a generalized derivation on an algebra $\mathcal{A}$. Then, $\exp\ \delta_{A,B}(X) = exp(A)Xexp(-B) = \alpha_{exp(A),exp(B)}(X)$.

Proof. For fixed $A, B \in \mathcal{A}$ and for all $X \in \mathcal{A}$, recall from Proposition 3.1.7 that,

$$\delta^n_{A,B}(X) = \sum_{r=0}^{n}(-1)^r\binom{n}{r}A^{n-r}XB^r. \tag{3.3}$$

Therefore, by definition of exponential,

$$\begin{aligned}
exp\delta_{A,B}(X) &= \sum_{n=0}^{\infty}\frac{1}{n!}\delta^n_{A,B}(X)\\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{r=0}^{n}(-1)^r\binom{n}{r}A^{n-r}XB^r\\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{n}(-1)^r\frac{1}{n!}\frac{n!}{(n-r)!r!}A^{n-r}XB^r\\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{n}(-1)^r\frac{1}{(n-r)!}\frac{1}{r!}A^{n-r}XB^r\\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{n}\left((-1)^r\frac{1}{(n-r)!}A^{n-r}\right)X\left(\frac{1}{r!}B^r\right)\\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{n}\left(\frac{1}{(n-r)!}(A)^{n-r}\right)X(-1)^r\left(\frac{1}{r!}B^r\right)\\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{n}\left(\frac{1}{(n-r)!}(A)^{n-r}\right)X\left((-1)^r\frac{1}{r!}B^r\right)\\
&= exp(A)Xexp(-B)\\
&= \alpha_{exp(A),exp(B)}(X).
\end{aligned}$$

as claimed. □

In the next results, we consider $\mathcal{A} = \mathcal{B}(H)$, the algebra of bounded linear operators on a Hilbert space H, and determine some norm properties of generalized derivations.

Proposition 3.1.11

For $A \in \mathcal{B}(H)$, $\|R_A\| = \|L_A\| = \|A\|$. *Moreover,* $\delta_{A,B} = \delta_A - R_{A-B}$.

Proof. By definition; $\|L_A\| = \sup\{\|L_A(X)\| : X \in \mathcal{B}(H), \|X\| = 1\}$. This implies that $\|L_A\| \geq \|L_A(X)\|$, for all $X \in \mathcal{B}(H)$, with $\|X\| = 1$. So for all $\epsilon > 0$, $\|L_A\| - \epsilon < \|L_A(X)\|$, for all $X \in \mathcal{B}(H)$ with $\|X\| = 1$

But, $\|L_A\| - \epsilon < \|AX\| \leq \|A\|\|X\| = \|A\|$. Since ϵ is arbitrary, this implies that $\|L_A\| \leq \|A\|$

Now; from the definition again, $\|L_A\| \geq \|L_A(X)\| = \|AX\|$ for all $X \in \mathcal{B}(H)$. Taking supremum over all $X \in \mathcal{B}(H)$ with $\|X\| = 1$, we get $\|L_A\| \geq \|A\|$. We now have $\|L_A\| = \|A\|$. Similarly, it can be shown that $\|R_A\| = \|A\|$. It remains to show that $\delta_{A,B} = \delta_A + R_{A-B}$. Now for all $X \in \mathcal{B}(H)$,

$$\begin{aligned} \delta_{A,B}(X) &= AX - XB \\ &= AX - XA + XA - XB \\ &= \delta_A(X) + X(A - B) \\ &= \delta_A(X) + R_{A-B}(X). \end{aligned}$$

which completes the proof. □

The next two propositions give the upper bounds of the numerical and the spectral radii of a bounded operator in H. Even though known in the literature, see for instance [33], we give our proofs.

Proposition 3.1.12

For every $A \in \mathcal{B}(H)$, the following hold;

1. $\omega(A) \leq \|A\|$
2. $r(A) \leq \|A\|$,

where $\omega(A)$ and $r(A)$ are the numerical and the spectral radii of A respectively.

Proof. We recall that $\omega(A) = \sup\{|\lambda| : \lambda \in W(A)\}$, where $W(A) = \{\langle Ax, x\rangle : x \in H, \|x\| = 1\}$.
Now $\lambda \in W(A)$ if and only if $\lambda = \langle Ax, x\rangle$ for some $x \in H$ with $\|x\| = 1$. Then we have $|\lambda| = |\langle Ax, x\rangle| \leq \|Ax\|\|x\| \leq \|A\|\|x\|^2 = \|A\|$. Taking supremum over all $\lambda \in W(A)$, we get

$$\sup\{|\lambda| : \lambda \in W(A)\} \leq \|A\|$$

and therefore it follows that $\omega(A) \leq \|A\|$.
We now use the spectral inclusion, that is, $\sigma(A) \subseteq \overline{W(A)}$ to conclude (2) Then $\lambda \in \sigma(A)$ implies that $\lambda \in \overline{W(A)}$. Since $\omega(A) \leq \|A\|$, it follows that $|\lambda| \leq \|A\|$ for all $\lambda \in \sigma(A)$. Now taking supremum over all $\lambda \in \sigma(A)$, we have, $\sup_{\lambda\in\sigma(A)} |\lambda| \leq \|A\|$, which implies that $r(A) \leq \|A\|$, as claimed. This completes the proof. □

Proposition 3.1.13

For $A \in \mathcal{B}(H)$, $r(A) = \|A\|$ if and only if $\omega(A) = \|A\|$. In particular $\|A\| \in \sigma(A)$ if and only if $\|A\| \in \overline{W(A)}$.

Proof. First, we suppose that $\omega(A) = \|A\|$, and prove that $r(A) = \|A\|$. We may assume, without loss of generality, that $\|A\| = 1$. Otherwise, we multiply by a suitable positive constant. Then $\omega(A) = \|A\|$ implies that there exists a sequence $(x_n)_n$ of unit vectors such that $|\langle Ax_n, x_n\rangle| \to 1$ as $n \to \infty$. Again we may assume without loss of generality that, $\langle Ax_n, x_n\rangle \to 1$ as $n \to \infty$. Since $|\langle Ax_n, x_n\rangle| \leq \|Ax_n\| \leq 1$ and $\langle Ax_n, x_n\rangle \to 1$, it follows that $\|Ax_n - x_n\|^2 = \|A\|^2 - 2Re\langle Ax_n, x_n\rangle + 1 \to 0$ as $n \to \infty$. Consequently, $1 \in \sigma_{ap}(A)$. Since $\sigma_{ap}(A) \subseteq \sigma(A)$, it follows that $1 \in \sigma(A)$ and so $r(A) \geq 1$. But $r(A) \leq \|A\| = 1$. This shows that $r(A) = 1 = \|A\|$. On the other hand, suppose $r(A) = \|A\|$ and prove that $\omega(A) = \|A\|$. If $r(A) = \|A\|$, then there exists $\lambda \in \sigma(A)$ such that $|\lambda| = \|A\|$. Since $\sigma(A) \subseteq \overline{W(A)}$, it follows that $\lambda \in \overline{W(A)}$ and whence $\omega(A) \geq |\lambda| = \|A\|$. Since $\omega(A) \leq \|A\|$ by Proposition 3.1.12, it follows that $\omega(A) = \|A\|$, as desired. □

3.2 Generalized derivation on norm ideals and $S-$universality

In this section, we consider the algebra of bounded linear operators on H, that is $\mathcal{A} = \mathcal{B}(H)$, and we shall simply write $\mathcal{B}(H)$. Recall that an operator $A \in \mathcal{B}(H)$ is $S-$universal if $\|\delta_A|\mathcal{J}\| = 2d(A)$ where $d(A) = \inf\{\|A - \lambda I\| : \lambda \in \mathbb{C}\}$. We then study the norm properties of generalized derivations restricted to norm ideals $\mathcal{J} \subseteq \mathcal{B}(H)$. Most importantly, we extend the concept of $S-$universal operators to the setting of generalized derivations.

Proposition 3.2.1

Let $(\mathcal{J}, \|\cdot\|_{\mathcal{J}})$ be a norm ideal in $\mathcal{B}(H)$ and fix $A \in \mathcal{B}(H)$. If $X \in \mathcal{J}$, then $\delta_A(X) \in \mathcal{J}$ and $\|\delta_A|\mathcal{J}\| \leq 2d(A)$. In particular, the restriction $\delta_A|\mathcal{J}$ of δ_A to $\mathcal{J}$ is a bounded linear operator on $(\mathcal{J}, \|\cdot\|_{\mathcal{J}})$.

PROOF. Since $\delta_A(X) = AX - XA$ and $X \in \mathcal{J}$, it follows that $\delta_A(X) \in \mathcal{J}$ since $\mathcal{J}$ as a 2-sided ideal. Now, for every $X \in \mathcal{J}$,

$$\begin{aligned}\|\delta_A(X)\|_{\mathcal{J}} = \|AX - XA\|_{\mathcal{J}} &= \|(A-\lambda)X - X(A-\lambda)\|_{\mathcal{J}} \\ &\leq \|(A-\lambda)X\|_{\mathcal{J}} + \|X(A-\lambda)\|_{\mathcal{J}} \\ &\leq \|A-\lambda\|\|X\|_{\mathcal{J}} + \|X\|_{\mathcal{J}}\|A-\lambda\|\end{aligned}$$

Taking supremum over all $X \in \mathcal{J}$ with $\|X\|_{\mathcal{J}} = 1$, we get

$$\|\delta_A|\mathcal{J}\| \leq 2\|A-\lambda\|.$$

Now taking infimum over all $\lambda \in \mathbb{C}$, we get;

$$\|\delta_A|\mathcal{J}\| \leq 2\inf_{\lambda\in\mathbb{C}} \|A-\lambda\| = 2d(A).$$

□

Next, we extend Proposition 3.2.1 to the case of a generalized derivation as we give in the following proposition.

Proposition 3.2.2

Let $(\mathcal{J}, \|\cdot\|_{\mathcal{J}})$ be a norm ideal in $\mathcal{B}(H)$ and fix $A, B \in \mathcal{B}(H)$. If $X \in \mathcal{J}$, then, $\delta_{A,B}(X) \in \mathcal{J}$ and $\|\delta_{A,B}|\mathcal{J}\| \leq \|\delta_{A,B}\|$.

In particular, the restriction $\delta_{A,B}|\mathcal{J}$ of $\delta_{A,B}$ to $\mathcal{J}$ is bounded linear oper-

ator on $(\mathcal{J}, \|\cdot\|_{\mathcal{J}})$.

PROOF. Let $X \in \mathcal{J}$, then $\delta_{A,B}(X) = AX - XB \in \mathcal{J}$ since $\mathcal{J}$ is a 2 sided ideal. Now, for fixed $A, B \in \mathcal{B}(H)$, we have that for all $X \in \mathcal{J}$,

$$\begin{aligned}\|\delta_{A,B}(X)\|_{\mathcal{J}} = \|AX - XB\|_{\mathcal{J}} &= \|(A-\lambda)X - X(B-\lambda)\|_{\mathcal{J}} \\ &\leq \|(A-\lambda)X\|_{\mathcal{J}} + \|X(B-\lambda)\|_{\mathcal{J}} \\ &\leq \|A-\lambda\|\|X\|_{\mathcal{J}} + \|X\|_{\mathcal{J}}\|B-\lambda\|.\end{aligned}$$

Taking supremum over all $X \in \mathcal{J}$ with $\|X\|_{\mathcal{J}} = 1$, we get

$$\|\delta_{A,B}|\mathcal{J}\| \leq \|A-\lambda\| + \|B-\lambda\|.$$

Now taking infimum over all $\lambda \in \mathbb{C}$, we get;

$$\|\delta_{A,B}|\mathcal{J}\| \leq \inf_{\lambda\in\mathbb{C}}(\|A-\lambda\| + \|B-\lambda\|) = \|\delta_{A,B}\|.$$

□

Before stating our next result, we state the following two lemmas which are readily available in literature.

Lemma 3.2.3 ([9],Corollary 3.5)
If $A \in \mathcal{B}(H)$ *is* $S-$*universal, then*

$$\|\delta_A\| = 2\|A\|.$$

Lemma 3.2.4 ([9], Theorem 3.5)
Let $A, B \in \mathcal{B}(H)$ *be non-zero S-universal operators. Then,* $\|\delta_{A,B}\| = \frac{1}{2}(\|\delta_A\| + \|\delta_B\|)$ *if and only if* $W_N(A) \cap W_N(-B) \neq \emptyset$.

We now give the following result;

Theorem 3.2.5

Let $A, B \in \mathcal{B}(H)$ be $S-$universal operators and $\mathcal{J}$ a norm ideal in $\mathcal{B}(H)$. Then $\|\delta_{A,B}\| = \|\delta_{A,B}|\mathcal{J}\|$.

Proof. For fixed $A, B \in \mathcal{B}(H)$, we have that $\|\delta_A\| = 2d(A)$ and $\|\delta_B\| = 2d(B)$. Since A, B are $S-$universal, it follows from Lemma 3.2.3 that,

$$\|\delta_A\| = 2\|A\|$$

and

$$\|\delta_B\| = 2\|B\|.$$

Thus, $d(A)+d(B) = \|A\|+\|B\|$. That is; $\inf_{\lambda\in\mathbb{C}} \|A-\lambda\|+\inf_{\lambda\in\mathbb{C}} \|B-\lambda\| = \|A\| + \|B\|$. This implies that $\inf_{\lambda\in\mathbb{C}}(\|A - \lambda\| + \|B - \lambda\|) = \|A\| + \|B\|$. But $\inf_{\lambda\in\mathbb{C}}(\|A - \lambda\| + \|B - \lambda\|) = \|\delta_{A,B}\|$ so that $\|\delta_{A,B}\| = \|A\| + \|B\|$. This is equivalent to $W_N(A) \cap W_N(-B) \neq \emptyset$ which further implies that

$$\begin{aligned}\|\delta_{A,B}\| &= \frac{1}{2}(\|\delta_A\| + \|\delta_B\|)\\ &= \frac{1}{2}(\|\delta_A|\mathcal{J}\| + \|\delta_B|\mathcal{J}\|)\\ &= \|\delta_{A,B}|\mathcal{J}\|.\end{aligned}$$

This completes the proof. □

Remark 3.2.6

Theorem 3.2.5 above extends the notion of $S-$universality from the setting of inner derivation to the setting of a generalized derivation. In particular, we give the following definition:

Definition 3.2.7

Let $A, B \in \mathcal{B}(H)$. We say that the pair (A, B) is $S-$universal if $\|\delta_{A,B}|\mathcal{J}\| = \|\delta_{A,B}\|$ for each norm ideal $\mathcal{J}$ in $\mathcal{B}(H)$

As noted earlier, a special class of norm ideals is the Schatten p-ideal $C_p(H)$. We prove that Theorem 3.2.5 is indeed true for this class of norm ideals.

Theorem 3.2.8

Let $A, B \in B(H)$ be $S-$universal, then

$$\|\delta_{A,B}|C_p(H)\| = \|A\| + \|B\|.$$

PROOF. Since A, B are $S-$universal and $C_p(H)$ is a norm ideal in $\mathcal{B}(H)$, it follows that $\|\delta_{A,B}|C_p\| = \|\delta_{A,B}\| = \inf_{\lambda\in\mathbb{C}}(\|A-\lambda\| + \|B-\lambda\|)$. By a compactness argument, there exists $\mu \in \mathbb{C}$ such that $\inf_{\lambda\in\mathbb{C}}(\|A-\lambda\| + \|B-\lambda\|) = \|A-\mu\| + \|B-\mu\|$. We note that $\delta_{A,B}|C_p = \delta_{A-\mu,B-\mu}|C_p = L_{A-\mu}|C_p - R_{B-\mu}|C_p$. Thus $\|L_{A-\mu}|C_p - R_{B-\mu}|C_p\| = \|A-\mu\| + \|B-\mu\|$. On the other hand, since $\|L_{A-\mu}\| = \|A-\mu\|$ and $\|R_{B-\mu}\| = \|B-\mu\|$, it follows that $\|L_{A-\mu}|C_p - R_{B-\mu}|C_p\| = \|L_{A-\mu}|C_p\| + \|R_{B-\mu}|C_p\|$. Without loss of generality, we may assume that $\mu = 0$. Then $\|L_A|C_p - R_B|C_p\| = \|L_A|C_p\| + \|R_B|C_p\| = \|A\| + \|B\|$. This completes the proof. □

The following are immediate from Theorems 3.2.5 and 3.2.8 above;

Corollary 3.2.9

Let $A, B \in \mathcal{B}(H)$ be $S-$universal operators, then $\|\delta_{A,B}|\mathcal{J}\| = \|A\| + \|B\|$.

PROOF. By Theorem 3.2.5, we have that $\|\delta_{A,B}|\mathcal{J}\| = \|\delta_{A,B}\|$ for A, B $S-$universal operators and that $\|\delta_{A,B}\| = \|A\| + \|B\|$ which gives $\|\delta_{A,B}|\mathcal{J}\| =$

$\|A\| + \|B\|$. □

Corollary 3.2.10

Let $A \in \mathcal{B}(H)$ be $S-$universal, then $\|\delta_A|C_p\| = 2\|A\|$.

PROOF. Let $A = B$ in Theorem 3.2.8 above. Then we immediately get $\|\delta_{A,A}|C_p\| = \|A\| + \|A\| = 2\|A\|$, as desired. □

REMARK 3.2.11

For $A, B \in \mathcal{B}(H)$, the equation

$$\|A - B\| = \|A\| + \|B\|$$

was studied by many authors, see for instance [3, 21].

We state some of the known results in the next two propositions;

Proposition 3.2.12 ([21])

For $A, B \in \mathcal{B}(H)$, $\|A - B\| = \|A\| + \|B\|$ if and only if 0 is in the approximate point spectrum of the operator $\|B\|A + \|A\|B$. The converse holds if either A or B is an isometric operator.

Proposition 3.2.13 ([4])

Let $A, B \in \mathcal{B}(H)$ be nonzero operators then $\|A - B\| = \|A\| + \|B\|$ if and only if $\|A\|\|B\|$ is in the closure of the numerical range of the operator $-A^*B$.

We now give further consequences of Theorem 3.2.8.

Corollary 3.2.14

Let $A, B \in \mathcal{B}(H)$. If the operators A, B are $S-$universal and L_A, R_B are defined on $C_p(H)$, then $0 \in \sigma_{ap}(\|A\|R_B + \|B\|L_A)$. The converse holds if either A or B is isometric.

Proof. By Theorem 3.2.8, we have that, $A, B \in \mathcal{B}(H)$ $S-$universal, $\|L_A|C_p - R_B|C_p\| = \|L_A|C_p\| + \|R_B|C_p\| = \|A\| + \|B\|$. Then by Proposition 3.2.12, $0 \in \sigma_{ap}(\|A\|R_B + \|B\|L_A)$. □

Corollary 3.2.15

Let $A, B \in \mathcal{B}(H)$. If the operators A, B are $S-$universal, then $\|L_A|C_p\|\|R_B|C_p \in \overline{W(-L_{A^*}|C_pR_B|C_p)}$.

Proof. For $A, B \in \mathcal{B}(H)$ $S-$universal, and by Theorem 3.2.8, we have that, $\|L_A|C_p - R_B|C_p\| = \|L_A|C_p\| + \|R_B|C_p\|$. Now by [4, Theorem 2.1], it follows that $\|L_A|C_p\|\|R_B|C_p\| \in \overline{W(-L_{A^*}|C_pR_B|C_p)}$. □

Corollary 3.2.16

Let $A, B \in \mathcal{B}(H)$ be $S-$universal, then

$$\|L_A|C_p\|\|R_B|C_p\| \in \sigma(-L_{A^*}|C_pR_B|C_p).$$

Proof. This follows from using the fact that $\|A\| \in \sigma(A)$ if and only if $\|A\| \in W(A)$, see Proposition 3.1.13 and Corollary 3.2.15 above. □

We now state the following two propositions whose details are readily available in literature which will be used in the next chapter.

Proposition 3.2.17 ([2], Lemma 2.1)

For $A, B \in \mathcal{B}(H)$, we have;

1. $\sigma(\delta_{A,B}) \subseteq \sigma(A) - \sigma(B)$

2. $\overline{W}(\delta_{A,B}) \subseteq \overline{W(A)} - \overline{W(B)}$.

Proposition 3.2.18 ([11], Proposition. 1.1)

If $A, B \in \mathcal{B}(H)$ and $\mathcal{J}$ is a norm ideal in $\mathcal{B}(H)$, then;

1. $\sigma(\delta_{A,B}|\mathcal{J}) = \sigma(A) - \sigma(B)$, .

2. $\overline{W}(\delta_{A,B}|\mathcal{J}) = \overline{W(A)} - \overline{W(B)}$.

In the following example, we have determined the norms of generalized derivation induced by the operators A and B in $\mathcal{B}(H)$.

Example 3.2.19

Consider $M_2(\mathbb{C}) = H$ a set of all 2×2 matrices with complex entries and A, B in $M_2(\mathbb{C}) = H$. Let $\mathbb{C}_2$ be the Hilbert Schmidt class of 2×2 matrix in $M_2(\mathbb{C})$ then we have,

$$\|A\| := \sqrt{\operatorname{tr}(A^T A)}$$

and

$$\|B\| := \sqrt{\operatorname{tr}(B^T B)},$$

where $\operatorname{tr}(A^T A)$ is the trace of $A^T A$ and A^T is the transpose of A. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ be operators in $\mathbb{C}^2$. Then $\|A\|_2 = 1$ and $\|B\|_2 = 1$. Now,

$AB = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, so that, $\|AB\|_2 = 1$.

We have, $A - I\lambda = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix}$.

The eigenvalues of A are given by;

$$|A - \lambda I| = 0 \Rightarrow \ \lambda^2 = 0$$

implying that $\lambda = 0$.

Zero is the only eigenvalue of A and B. Thus $\sigma(A) = \sigma_p(A) = \{0\}$ and $\sigma(B) = \sigma_p(B) = \{0\}$ and $r(A) = r(B) = 0$.

Now with $\|A\| = 1$ and $\|B\| = 1$, we have that,

$$\|A - \lambda I\| = (2\lambda^2 + 1)^{\frac{1}{2}}$$

and

$$\|B - \lambda I\| = (2\lambda^2 + 1)^{\frac{1}{2}}$$

Therefore by the definition of norm of generalized derivation, we get;

$$\begin{aligned} \|\delta_{A,B}\| &= \inf_{\lambda \in \mathbb{C}} \{(2\lambda^2 + 1)^{\frac{1}{2}} + (2\lambda^2 + 1)^{\frac{1}{2}}\} \\ &= \inf_{\lambda \in \mathbb{C}} \{2(2\lambda^2 + 1)^{\frac{1}{2}}\} = 2. \end{aligned}$$

It therefore follows that,

$$\|\delta_{A,B}\| = \|A\| + \|B\|.$$

Now, let $x = (z_1, z_2)$ (i.e $z_1, z_2 \in \mathbb{C}$).

Consequently, $Ax = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_2 \\ 0 \end{pmatrix}$. Now

$\langle Ax, x \rangle = \langle (0, z_1), (z_1, z_2) \rangle = z_1 \overline{z}_2$.

If $\|x\| = 1$, then $|z_1|^2 + |z_2|^2 = 1$. Thus; $W(A) = \{z_1 \overline{z_2} : z_1, z_2 \in \mathbb{C}; |z_1|^2 + |z_2|^2 = 1\}$.

Now let $\lambda = z_1 \overline{z_2}$, we have $|\lambda| = |z_1||z_2| = |z_1|\sqrt{1 - |z_1|^2}$ and hence $W(A) = \{\lambda \in \mathbb{C} : |\lambda|^2 = |z_1|^2(1 - |z_1|^2)$ where $0 \leq |z_1| \leq 1$ and $z_1 \in \mathbb{C}\}$.

We find the maximum value of $|\lambda|$ as $|z_1|$ varies over the closed interval $[0, 1]$. We have,

$|\lambda|^2 = |z_1|^2(1 - |z_1|^2) = (|z_1|^2 - |z_1|^4) = (\frac{1}{4} - (\frac{1}{2} - |z_1|^2)^2)$.

Since $|\lambda| \geq 0$, we note that the maximum value of $|\lambda|$ is $\frac{1}{2}$ and occurs when $|z_1| = \frac{1}{2}$.

Hence $W(A) = \{\lambda \in \mathbb{C} : |\lambda| \leq \frac{1}{2}\}$.

Also; $\omega(A) = \sup\{|\lambda| : \lambda \in W(A)\} = \frac{1}{2}$.

It can be shown that $\omega(B) = \frac{1}{2}$ in the same way.

Chapter 4

$S-$universality on Generalized Derivations

4.1 Introduction

Let S_1 and S_2 be two nonempty sets. We define the circumdiameter of the sets S_1 and S_2 by $\text{diam}_c(S_1, S_2) = \sup\{|\alpha - \beta| : \alpha \in S_1, \beta \in S_2\}$. In the case that $S_1 = S_2$, we define

$$\text{diam}(S_1) := \text{diam}_c(S_1, S_1) = \sup\{|\alpha - \beta| : \alpha, \beta \in S_1\}$$

as the diameter of the set S_1. In this chapter, we shall consider two circumdiameters, namely $\text{diam}_c(\overline{W(A)}, \overline{W(B)})$ and $\text{diam}_c(\sigma(A), \sigma(B))$ where $A, B \in \mathcal{B}(H)$ and $\sigma(A)$ and $\overline{W(A)}$ denote the spectrum and closure of numerical range of A respectively. In particular, we define:

$$\text{diam}_c(\overline{W(A)}, \overline{W(B)}) = \sup\{|\alpha - \beta| : \alpha \in \overline{W(A)}, \beta \in \overline{W(B)}\}$$

and

$$\text{diam}_c(\sigma(A), \sigma(B)) = \sup\{|\alpha - \beta| : \alpha \in \sigma(A), \beta \in \sigma(B)\}.$$

We shall write $\text{diam}_c(\overline{W}(A; B))$ and $\text{diam}_c(\sigma(A; B))$ instead of $\text{diam}_c(\overline{W(A)}, \overline{W(B)})$ and $\text{diam}_c(\sigma(A), \sigma(B))$ respectively. We characterize $S-$universal operators by investigating the relations between the circumdiameters and the norms of these operators as well as that of a generalized derivation on a norm ideal. The equality between these circumdiameters and the sum of spectral radii or numerical radii has also been investigated in the next section. We have also explored the concept of antidistance and similarity orbits by looking at the relationship between similarity orbits and generalized derivations, extending the work of M. Boumazgour [28].

4.2 The circumdiameters $\text{diam}_c(\overline{W}(A; B))$ and $\text{diam}_c(\sigma(A; B))$

We consider a pair of $S-$universal operators $A, B \in \mathcal{B}(H)$ and establish the relationship between the circumdiameter $\text{diam}_c(\overline{W}(A; B))$ and the norm of a generalized derivation implemented by $A, B \in \mathcal{B}(H)$. The following result shows that the concept of $S-$universality ensures the attainment of $\|A\| + \|B\|$ which is the optimal value for $\text{diam}_c(\overline{W}(A; B))$.

Theorem 4.2.1

Let $A, B \in \mathcal{B}(H)$ be $S-$universal, then $\text{diam}_c(\overline{W}(A; B)) = \|A\| + \|B\|$.

PROOF. If the pair (A, B) is $S-$universal, then by Corollary 3.2.16 we have, $\|L_A|C_p\|\|R_B|C_p\| \in \sigma(-L_{A^*}|C_pR_B|C_p)$. But

$$\sigma(-L_{A^*}|C_pR_B|C_p) = -\sigma(A^*)\sigma(B),$$

and

$$\|L_A|C_p\|\|R_B|C_p\| = \|A\|\|B\|.$$

So there exists $\alpha \in \sigma(A)$, $\beta \in \sigma(B)$ such that $\|A\|\|B\| = -\overline{\alpha}\beta$. Since $|\alpha| \leq \|A\|$ and $|\beta| \leq \|B\|$, there exists $\theta \in \mathbb{R}$ such that $\alpha = \|A\|e^{i\theta}$ and $\beta = -\|B\|e^{i\theta}$. Also since $\sigma(\delta_{A,B}|C_p) = \sigma(A) - \sigma(B)$, see Proposition 3.2.18, it follows that

$$r(\delta_{A,B}|C_p) = \text{diam}_c(\sigma(A;B)) \geq |\alpha - \beta| = |\|A\|e^{i\theta} + \|B\|e^{i\theta}| = \|A\| + \|B\|,$$

where $\text{diam}_c(\sigma(A;B)) = \sup\{\|\alpha - \beta| : \alpha \in \sigma(A), \beta \in \sigma(B)\}$. By the spectral inclusion, $\sigma(A) \subseteq \overline{W(A)}, \sigma(B) \subseteq \overline{W(B)}$ and therefore,

$$\text{diam}_c(\overline{W}(A;B)) \geq \text{diam}_c(\sigma(A;B)) \geq \|A\| + \|B\|.$$

For the reverse inequality we have by definition,

$$\begin{aligned}
\text{diam}_c(\overline{W}(A;B)) &= \sup\{|\alpha - \beta| : \alpha \in \overline{W(A)}, \beta \in \overline{W(B)}\} \\
&\leq \sup\{|\alpha| + |\beta| : \alpha \in \overline{W(A)}, \beta \in \overline{W(B)}\} \\
&\leq \sup\{|\alpha| : \alpha \in \overline{W(A)}\} + \sup\{|\beta| : \beta \in \overline{W(B)}\} \\
&= \omega(A) + \omega(B) \\
&\leq \|A\| + \|B\|,
\end{aligned}$$

as desired. □

The following consequences are immediate:

Corollary 4.2.2

Let $A, B \in \mathcal{B}(H)$ be $S-$universal, then

$$diam_c(\overline{W}(A;B)) = \|\delta_{A,B}|\mathcal{J}\|.$$

Proof. By Theorem 4.2.1, where for A, B $S-$universal, $\text{diam}_c(\overline{W}(A;B)) = \|A\| + \|B\|$. But Corollary 3.2.9 gives $\|\delta_{A,B}|\mathcal{J}\| = \|A\| + \|B\|$. The result then follows immediately. □

Corollary 4.2.3

Let $A, B \in \mathcal{B}(H)$ be $S-$universal operators, then $diam_c(\overline{W}(A;B)) = \|\delta_{A,B}\|$.

Proof. Since $A, B \in \mathcal{B}(H)$ are $S-$universal operators, Corollary 4.2.2 asserts that $\text{diam}_c(\overline{W}(A;B)) = \|\delta_{A,B}|\mathcal{J}\|$. But by the extension of $S-$universality to the generalized derivation context, see Theorem 3.2.5, we have that $\|\delta_{A,B}|\mathcal{J}\| = \|\delta_{A,B}\|$ and therefore $\text{diam}_c(\overline{W}(A;B)) = \|\delta_{A,B}\|$. This completes the proof. □

Remark 4.2.4

It is important to note that when $A = B$, then the circumdiameter $\text{diam}_c(W(A;A))$ turns out to be the diameter of the numerical range of A, $\text{diam}(W(A)) := \sup\{|\alpha - \beta| : \alpha, \beta \in W(A)\}$, whose relationship with the norms of derivations was well studied by Bonyo and Agure [9, 8]. Another consequence of Theorem 4.2.1 which interestingly coincides with and is a summary of the results obtained in [8] is the following;

Corollary 4.2.5

Let $A \in \mathcal{B}(H)$ be an $S-$universal operator, $\mathcal{J}$ be a norm ideal in $\mathcal{B}(H)$ and $C_p(H)$ be the Schatten norm ideal. Then,
$diam(\overline{W(A)}) = \|\delta_A\| = \|\delta_A|\mathcal{J}\| = \|\delta_A|C_p\| = 2\|A\|$.

Proof. From Theorem 4.2.1, Corollaries 4.2.2 and 4.2.3 above, we have,

$$\text{diam}_c(\overline{W}(A;B)) = \|\delta_{A,B}|\mathcal{J}\| = \|\delta_{A,B}\| = \|A\| + \|B\|. \tag{4.1}$$

If $A = B$, equation (4.1) reduces to,
$\text{diam}(\overline{W(A)}) = \|\delta_A|\mathcal{J}\| = \|\delta_A|\mathcal{B}(H)\| = 2\|A\|$, as desired. □

Another consequence is the following:

Corollary 4.2.6

Let $A, B \in \mathcal{B}(H)$ be $S-$universal, then the following hold;

1. *$diam_c(\sigma(A;B)) = \|A\|+\|B\| = \|\delta_{A,B}|\mathcal{J}\| = \|\delta_{A,B}\| = diam_c(\overline{W}(A;B))$,*
2. *$diam(\sigma(A)) = 2\|A\| = \|\delta_A|\mathcal{J}\| = \|\delta_A\| = diam(\overline{W(A)})$.*

Proof. Suppose that the pair $A, B \in \mathcal{B}(H)$ is $S-$universal, then as noted in the proof of Theorem 4.2.1,

$$\text{diam}_c(\sigma(A;B)) \geq \|A\| + \|B\|.$$

For the reverse inequality, we have:

$$\begin{aligned}\operatorname{diam}_c(\sigma(A;B)) &= \sup\{|\alpha-\beta| : \alpha\in\sigma(A), \beta\in\sigma(B)\}\\ &\le \sup\{|\alpha|+|\beta| : \alpha\in\sigma(A), \beta\in\sigma(B)\}\\ &\le \sup\{|\alpha| : \alpha\in\sigma(A)\} + \sup\{|\beta| : \beta\in\sigma(B)\}\\ &= r(A)+r(B)\\ &\le \|A\|+\|B\|.\end{aligned}$$

Therefore, $\operatorname{diam}_c(\sigma(A;B)) = \|A\|+\|B\|$. Now, by Theorems 3.2.5 and 4.2.1 and Corollary 4.2.3 we have that for A, B $S-$universal,

$$\|\delta_{A,B}|\mathcal{J}\| = \|\delta_{A,B}\| = \|A\|+\|B\| = \operatorname{diam}_c(\overline{W}(A;B)).$$

The proof of assertion 2 follows immediately from assertions 1 by taking $A = B$ and noting that $\delta_{A,A} = \delta_A$, $\operatorname{diam}_c\sigma(A;A) = \operatorname{diam}(\sigma(A))$ and $\operatorname{diam}_c(\overline{W}(A;A)) = \operatorname{diam}(\overline{W(A)})$. □

We conclude this section by giving the following result which relates $\operatorname{diam}_c(\overline{W}(A,B))$, (respectively $\operatorname{diam}_c(\sigma(A;B))$), with the sum of numerical (respectively spectral) radii of A and B, where A, B are arbitrary operators on $\mathcal{B}(H)$.

Theorem 4.2.7

Let $A, B \in \mathcal{B}(H)$, then;

1. *$\operatorname{diam}_c(\overline{W}(A;B)) \le \omega(A)+\omega(B)$, and*

2. *$\operatorname{diam}_c(\sigma(A;B)) \le r(A)+r(B)$*

PROOF. To prove 1, we have by definition that for $A, B \in \mathcal{B}(H)$,

$$\begin{aligned}
\operatorname{diam}_c(\overline{W}(A;B)) &= \sup\{|\alpha-\beta| : \alpha \in \overline{W(A)}, \beta \in \overline{W(B)}\} \\
&\leq \sup\{|\alpha|+|\beta| : \alpha \in \overline{W(A)}, \beta \in \overline{W(B)}\} \\
&\leq \sup_{\alpha \in \overline{W(A)}} |\alpha| + \sup_{\beta \in \overline{W(B)}} |\beta| \\
&= \omega(A) + \omega(B).
\end{aligned}$$

Now for assertion 2, we have;

$$\begin{aligned}
\operatorname{diam}_c(\sigma(A;B)) &= \sup\{|\alpha-\beta| : \alpha \in \sigma(A), \beta \in \sigma(B)\} \\
&\leq \sup\{|\alpha|+|\beta| : \alpha \in \sigma(A), \beta \in \sigma(B)\} \\
&\leq \sup_{\alpha \in \sigma(A)} |\alpha| + \sup_{\beta \in \sigma(B)} |\beta| \\
&= r(A) + r(B).
\end{aligned}$$

□

REMARK 4.2.8

A natural question then arises. When does equality hold in the assertions 1 and 2 of Theorem 4.2.7 above? In the next results, we answer this question in the affirmative in the case that the operators are $S-$universal.

Theorem 4.2.9

Let $A, B \in \mathcal{B}(H)$ be $S-$universal, then; $\operatorname{diam}_c(\overline{W}(A;B)) = \omega(A)+\omega(B)$.

PROOF. It is clear from Theorem 4.2.7 that for arbitrary operators $A, B \in \mathcal{B}(H)$, we have, $\operatorname{diam}_c(\overline{W}(A;B)) \leq \omega(A)+\omega(B)$.

To prove the reverse inequality we have from Theorem 4.2.1 that, for A, B

S-universal;

$$\begin{aligned}\operatorname{diam}_c(\overline{W}(A;B)) &= \|\delta_{A,B}\| \\ &= \|A\| + \|B\| \\ &\geq \omega(A) + \omega(B).\end{aligned}$$

That is, $\operatorname{diam}_c(\overline{W}(A;B)) \geq \omega(A) + \omega(B)$, which completes the proof. □

Theorem 4.2.10

Let $A, B \in \mathcal{B}(H)$ be $S-$universal operators, then $\operatorname{diam}_c(\sigma(A;B)) = r(A) + r(B)$.

Proof. From Theorem 4.2.7, it is clear that for arbitrary $A, B \in \mathcal{B}(H)$, $\operatorname{diam}_c(\sigma(A;B)) \leq r(A) + r(B)$.
We now prove the reverse inequality. For A, B $S-$universal, we have;

$$\begin{aligned}\operatorname{diam}_c(\sigma(A;B)) &= \|\delta_{A,B}\| \\ &= \|A\| + \|B\| \\ &\geq r(A) + r(B).\end{aligned}$$

This completes the proof. □

Remark 4.2.11

The distances $\operatorname{diam}_c(\overline{W}(A;B))$ and $\operatorname{diam}_c(\sigma(A;B))$ are respectively related to the numerical and spectral radii of a generalized derivation. In fact for a generalized derivation on a norm ideal $\mathcal{J}$ in $\mathcal{B}(H)$, $\operatorname{diam}_c(\overline{W}(A,B))$ and $\operatorname{diam}_c(\sigma(A,B))$ turn out to be exactly the numerical and spectral radii of the generalized derivation respectively, as we give in the following theorem:

Theorem 4.2.12

For $A, B \in \mathcal{B}(H)$, we have;

1. $\omega(\delta_{A,B}) \leq diam_c(\overline{W}(A; B))$
2. $r(\delta_{A,B}) \leq diam_c(\sigma(A; B))$. *Moreover, if $\mathcal{J}$ is a norm ideal in $\mathcal{B}(H)$, then we have;*
3. $\omega(\delta_{A,B}|\mathcal{J}) = diam_c(\overline{W}(A; B))$
4. $r(\delta_{A,B}|\mathcal{J}) = diam_c(\sigma(A; B))$.

Proof. By Proposition 3.2.17, we have that for $A, B \in \mathcal{B}(H)$, $\overline{W}(\delta_{A,B}) \subseteq \overline{W(A)} - \overline{W(B)}$. Now let $\lambda \in \overline{W}(\delta_{A,B})$. Then there exists $\alpha \in \overline{W(A)}$ and $\beta \in \overline{W(B)}$ such that $|\lambda| \leq |\alpha - \beta|$. Taking supremum over all $\lambda \in \overline{W}(\delta_{A,B})$, we obtain $\omega(\delta_{A,B}) \leq |\alpha - \beta|$. Now, taking supremum over all $\alpha \in \overline{W(A)}$ and $\beta \in \overline{W(B)}$, we get, $\omega(\delta_{A,B}) \leq \sup\{|\alpha - \beta| : \alpha \in \overline{W(A)}, \beta \in \overline{W(B)}\} = \text{diam}_c(\overline{W}(A, B))$. On the other hand, again by Proposition 3.2.17, we have; $\sigma(\delta_{A;B}) \subseteq \sigma(A) - \sigma(B)$. Now, by letting $\lambda \in \sigma(\delta_{A,B})$, it follows that there exists $\lambda_1 \in \sigma(A), \lambda_2 \in \sigma(B)$ such that $|\lambda| \leq |\lambda_1 - \lambda_2|$. Taking supremum over all $\lambda \in \sigma(\delta_{A,B})$ and then over all $\lambda_1 \in \sigma(A), \lambda_2 \in \sigma(B)$, we obtain; $r(\delta_{A,B}) \leq \text{diam}_c(\sigma(A; B))$. This proves assertions 1 and 2. To prove assertions 3 and 4, recall from Proposition 3.2.18 that the restriction of $\delta_{A,B}$ to a norm ideal $\mathcal{J}$ yields the equalities; $\overline{W}(\delta_{A,B}|\mathcal{J}) = \overline{W(A)} - \overline{W(B)}$ and $\sigma(\delta_{A,B}) = \sigma(A) - \sigma(B)$. Now by similar arguments as in the proof of assertions 1 and 2 above, we obtain the assertions 3 and 4. □

As an immediate consequence, we give the following;

Corollary 4.2.13

For $A, B \in \mathcal{B}(H)$, we have

1. $\omega(\delta_{A,B}) \leq \omega(A) + \omega(B)$, *and*

2. $r(\delta_{A,B}) \leq r(A) + r(B)$.

   Moreover, if the pair $A, B \in \mathcal{B}(H)$ is $S-$universal, then;

3. $\omega(\delta_{A,B}|\mathcal{J}) = \omega(A) + \omega(B)$, *and*

4. $r(\delta_{A,B}|\mathcal{J}) = r(A) + r(B)$.

Proof. Following Theorems 4.2.7 and 4.2.12, we have;

$$\omega(\delta_{A,B}) \leq \text{diam}_c(\overline{W}(A;B)) \leq \omega(A) + \omega(B)$$

and

$$r(\delta_{A,B}) \leq \text{diam}_c(\sigma(A;B)) \leq r(A) + r(B).$$

Now, assume that $A, B \in \mathcal{B}(H)$ are $S-$universal operators. Then, Theorems 4.2.9, 4.2.10 and 4.2.12 yields;

$$\omega(\delta_{A,B}|\mathcal{J}) = \text{diam}_c(\overline{W}(A;B)) = \omega(A) + \omega(B)$$

and

$$r(\delta_{A,B}|\mathcal{J}) = \text{diam}_c(\sigma(A;B)) = r(A) + r(B),$$

as desired. □

4.3 Normaloid and spectraloid Operators

In this section, we explore other special classes of operators for which we obtain the equalities $\text{diam}_c(\overline{W}(A;B)) = \omega(A)+\omega(B)$ and $\text{diam}_c(\sigma(A;B)) = r(A)+r(B)$ without the operators $A, B \in \mathcal{B}(H)$ being necessarily $S-$universal. We begin by giving the following definitions;

Definition 4.3.1

An operator $A \in \mathcal{B}(H)$ is said to be **normaloid** if $\omega(A) = \|A\|$, and it is said to be **spectraloid** if $r(A) = \omega(A)$.

Remark 4.3.2

Let A be a normaloid operator. Then $\omega(A) = \|A\|$. By the definition 4.3.1 we have $r(A) = \|A\| = \omega(A)$, i.e $r(A) = \omega(A)$. Thus A is spectraloid.

Theorem 4.3.3

If $A, B \in \mathcal{B}(H)$ are both normaloid operators, then;

1. *$\text{diam}_c(\sigma(A;B)) = r(A) + r(B)$*
2. *$\text{diam}_c(\overline{W}(A;B)) = \omega(A) + \omega(B)$.*

Proof. By Theorem 4.2.7, we have that for arbitrary $A, B \in \mathcal{B}(H)$,

$$\text{diam}_c(\sigma(A,B)) \leq r(A) + r(B).$$

Now we suppose that both $A, B \in \mathcal{B}(H)$ are normaloid and prove the reverse inequality. By definition;

$$\begin{aligned}\text{diam}_c(\sigma(A,B)) &= \sup\{|\alpha - \beta| : \alpha \in \sigma(A), \beta \in \sigma(B)\} \\ &\geq |\alpha - \beta| \ \text{ for all } \ \alpha \in \sigma(A), \beta \in \sigma(B)\end{aligned}$$

For $\alpha \in \sigma(A)$ and $\beta \in \sigma(B)$ we have that $|\alpha| \leq \|A\|$ and $|\beta| \leq \|B\|$. Let $\theta \in \mathbb{R}$ such that $\alpha = \|A\|e^{i\theta}$ and $\beta = -\|B\|e^{i\theta}$. Then since A, B are normaloid, it follows that,

$$|\alpha - \beta| = |\|A\|e^{i\theta} + \|B\|e^{i\theta}| = \|A\| + \|B\| = \omega(A) + \omega(B) \geq r(A) + r(B).$$

Therefore

$$\text{diam}_c(\sigma(A; B)) \geq r(A) + r(B)$$

and hence

$$\text{diam}_c(\sigma(A; B)) = r(A) + r(B)$$

as desired. This proves 1. To prove assertion 2, recall from Theorem 4.2.7 that

$$\text{diam}_c(\overline{W}(A; B)) \leq \omega(A) + \omega(B)$$

for arbitrary $A, B \in \mathcal{B}(H)$. Now, by the spectral inclusion, the definition of a normaloid operator as well as the proof of assertion 1 above, we have:

$$\text{diam}_c(\overline{W}(A; B)) \geq \text{diam}_c(\sigma(A; B)) = \|A\| + \|B\| = \omega(A) + \omega(B).$$

This completes the proof. □

Theorem 4.3.4

Let $A, B \in \mathcal{B}(H)$. Then the following are equivalent:

1. *Both A and B are normaloid.*

2. *Both A and B are spectraloid.*

3. *$\text{diam}_c(\overline{W}(A; B)) = \omega(A) + \omega(B)$.*

4. *diam*$_c(\sigma(A;B)) = r(A) + r(B)$.

5. *The pair* $A;B$ *is* $S-$*universal.*

Proof. (1) ⇒ (2): By the fact that a normaloid operator is a spectraloid. Now by Corollary **??**, (2) ⇒ (3): From the proof of Theorem 4.3.3 we have that $\text{diam}_c(\sigma(A;B)) \geq \|A\| + \|B\|$. But we know that $\text{diam}_c(\sigma(A;B)) \leq \|A\| + \|B\|$. Hence $\text{diam}_c(\sigma(A;B)) = \|A\| + \|B\|$. This implies that $\text{diam}_c(\overline{W}(A;B)) = \|A\| + \|B\|$ since it is obvious that $\text{diam}_c(\overline{W}(A;B)) \leq \|A\| + \|B\|$ and $\text{diam}_c(\overline{W}(A;B)) \geq \text{diam}_c(\sigma(A;B)) = \|A\| + \|B\|$. Thus $\text{diam}_c(\sigma(A;B)) = \text{diam}_c(\overline{W}(A;B)) = \omega(A) + \omega(B) \geq r(A) + r(B)$. So $\text{diam}_c(\sigma(A;B)) = r(A) + r(B)$. Hence $3 \Rightarrow 4$.
(4) ⇒ (5): Now, $\text{diam}_c(\sigma(A;B)) = r(A) + r(B) = r(\delta_{A,B}|\mathcal{J})$ which implies that A, B are $S-$universal by Corollary 4.2.13.
(5) ⇒ (1): If A, B are $S-$universal, then by Theorem 4.2.12 and Corollary 4.2.13, we have $\text{diam}_c(\overline{W}(A;B)) = \omega(\delta_{A,B}|\mathcal{J}) = \omega(A) + \omega(B)$ which is only true for normaloid operators. □

4.4 Anti-distance and Similarity orbit

Definition 4.4.1

A *unitary operator* in a Hilbert space H is a bounded linear operator $U : H \to H$ on a Hilbert space H that satisfies $U^*U = UU^* = I$, where U^* is the adjoint of U and $I : H \to H$ is the identity operator.

The simplest unitary operator is the identity operator I.

Example 4.4.2

The matrix $U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is a unitary matrix.

The adjoint of U, $U^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ therefore, $UU^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

Definition 4.4.3

Let $B \in \mathcal{B}(H)$. A *unitary similarity orbit* through B is defined as the set $U_S = \{U^*BU : U \text{ unitary}\}$.

The *anti-distance* from A to the orbit U_S with respect to the norm $\|\cdot\|$ is the $\sup\{\|A - U^*BU\| : U \text{ unitary}\}$.

In [1], T. Ando determined the upper and lower bounds for the anti-distance $\sup\{\|A - U^*BU\|_\infty : U \text{ unitary}\}$ where U runs over the set of unitary matrices.

Just like in the case of a generalized derivation, two operators $A, B \in \mathcal{B}(H)$ must be fixed in order to define the antidistance from A to the unitary similarity orbit through B, U_S.

Therefore, the question about the relation between the norms of generalized derivations and the anti-distance is apparent. From the available literature, very little attempt has been made towards addressing questions in this direction. It is clear that $\sup\{\|A - U^*BU\| : U \text{ unitary}\} \leq \|\delta_{A,B}\|$

In [10], Boumazgour established that for any $A, B \in \mathcal{B}(H)$,

$$\|\delta_{A,B}\| = \sup\{\|A - U^*BU\| : U \text{ unitary}\}. \tag{4.2}$$

Moreover, he proved the following;

1. If $A, B \in \mathcal{B}(H)$, then for $1 \leq p \leq \infty$,

$$\sup\{\|A - U^*BU\| : U \text{ unitary}\} \leq 2^{\frac{1}{p}}\|\delta_{A,B}|C_p\|. \tag{4.3}$$

2. If A, B are hyponormal and cohyponormal operators respectively, (If, in particular both of them are normal) then

$$\sup\{\|A - U^*BU\| : U \ \text{ unitary}\} \leq \sqrt{2}\text{diam}_c(\sigma(A; B)). \tag{4.4}$$

In this section, we give some of results in the same direction.

Proposition 4.4.4

For any $A, B \in \mathcal{B}(H)$, we have:

$$\|\delta_{A,B}\| = \|\delta_{A,B}|C_\infty\|.$$

PROOF. From equation (4.3), we have for $p = \infty$,
$\|\delta_{A,B}\| = \sup\{\|A - U^*BU\| : U \ \text{ unitary}\} \leq \|\delta_{A,B}|C_\infty\| \leq \|\delta_{A,B}\|$.
It therefore immediately follows that $\|\delta_{A,B}\| = \|\delta_{A,B}|C_\infty\|$ □

Theorem 4.4.5

For $S-$universal operators $A, B \in \mathcal{B}(H)$,

1. $\sup\{\|A - U^*BU\| : U \ \text{ unitary}\} = \text{diam}_c(\overline{W}(A; B))$,

2. $\sup\{\|A - U^*BU\| : U \ \text{ unitary}\} = r(A) + r(B)$,

3. $\sup\{\|A - U^*BU\| : U \ \text{ unitary}\} = \|A\| + \|B\|$,

4. $\sup\{\|A - U^*BU\| : U \ \text{ unitary}\} = \omega(A) + \omega(B)$,

5. $\sup\{\|A - U^*BU\| : U \text{ unitary}\} = \omega(\delta_{A,B}|\mathcal{J})$, *and*

6. $\sup\{\|A - U^*BU\| : U \text{ unitary}\} = r(\delta_{A,B}|\mathcal{J})$,

where $\mathcal{J}$ *is a norm ideal in* $\mathcal{B}(H)$.

Proof. Let $A, B \in \mathcal{B}(H)$ be $S-$universal, then by equation (4.2) we get; $\sup\{\|A - U^*BU\| : U \text{ unitary}\} = \|\delta_{A,B}|B(H)\| = \|A\| + \|B\|$.
Now it follows from Corollary 4.2.2 that, $diam_c(\overline{W}(A;B)) = \|\delta_{A,B}\|$ for A, B are $S-$universal. This proves assertions 1 and 3. By Theorems 4.2.9 and 4.2.10 we have that $\text{diam}_c(\overline{W}(A;B)) = \omega(A) + \omega(B)$ and $diam_c(\sigma(A;B)) = r(A) + r(B)$ which implies that $\sup\{\|A - U^*BU\| : U \text{ unitary}\} = \omega(A) + \omega(B)$ and that $\sup\{\|A - U^*BU\| : U \text{ unitary}\} = r(A) + r(B)$ proving the assertion 2 and 4. The proofs for assertions 5 and 6 is clear from Corollary 4.2.13. □

Chapter 5

Summary and Recommendation

5.1 Summary

In this thesis, we have given various algebraic properties of generalized derivation defined on an algebra $\mathcal{A}$ and extended the concept of $S-$universality from the setting of inner derivation to the setting of generalized derivation. For instance, in Proposition 3.1.1, we have proved that a generalized derivation is linear but fails to be a derivation on $\mathcal{A}$, while in Proposition 3.1.3, we show that the sum of two generalized derivations is a generalized derivation. The extension of $S-$universality to the setting generalized derivation is given in Theorems 3.2.5 and 3.2.8 and some of its consequences given in Corollaries 3.2.9 and 3.2.10. We have also extended the concept of inner automorphism to a case of generalized inner automorphism as given in Propositions 3.1.9 and 3.1.10.

The question as to when the circumdiameter of the numerical ranges of A and B, $\text{diam}_c(\overline{W}(A;B))$, attains its optimal value $\|A\|+\|B\|$ has been considered in Chapter 4. In particular, we have proved that if the pair (A, B) is $S-$universal, the $\operatorname{diam}_c(\overline{W}(A; B)) = \|A\| + \|B\|$, see Theorem 4.2.1 and for its consequences, see Corollaries 4.2.2, 4.2.3 and 4.2.5. The corresponding results for $\operatorname{diam}_c(\sigma(A; B))$ are given in Theorem 4.2.7. We have also investigated other classes of operators for which one of the above results would still hold with the operators not being necessarily $S-$universal. Specifically, we have proved that if $A, B \in \mathcal{B}(H)$ are normaloid or spectraloid operators, then circumdiameters of $\operatorname{diam}_c(\overline{W}(A; B))$ and $\operatorname{diam}_c(\sigma(A; B))$ equals $\omega(A) + \omega(B)$ and $r(A) + r(B)$, respectively. Finally, we have characterized the antidistance from an operator $A \in \mathcal{B}(H)$ to the similarity orbit through $B \in \mathcal{B}(H)$ in terms of circumdiameters, spectral and numerical radii as well as the norms of A and B.

5.2 Recommendation

From the results obtained from this study, we recommend the following for further research;

1. In this study, we have investigated the properties of generalized derivations and extended the concept $S-$universality to a generalized derivations. The extension of the concept of $S-$universality to the setting of a more general elementary operators can still be investigated.

2. We have established the attainment of the optimal value of the circumdiameter $\operatorname{diam}_c(\overline{W}(A; B))$ as $\|A\| + \|B\|$ for the pair (A, B) of $S-$universal operators. However, only necessary conditions for

the attainment of this optimal value have been established in this study. It would be interesting to investigate the sufficient conditions as well.

3. In this study, we have established the equality between the circumdiameters $\text{diam}_c(\overline{W}(A;B))$ and the numerical radii, $\text{diam}_c(\sigma(A;B))$ and spectral radii of operators A and B when the pair (A, B) is $S-$universal or normaloid operators. Further investigations should be done to determine whether the equality holds for other classes of operators such as convexoid operators.

4. We have also looked at the applications of the concept of $S-$universality to the antidistance and similarity orbits. In particular, we have characterized the antidistance from an operator $A \in \mathcal{B}(H)$ to the similarity orbit through $B \in \mathcal{B}(H)$ in terms of circumdiameters, spectral and numerical radii as well as the norms of A and B. It would be important to investigate other areas where the concept of $S-$universality can be applied.

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