SPECTRAL PROPERTIES OF SEMIGROUPS OF WEIGHTED COMPOSITION OPERATORS ON THE BLOCH SPACE

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ABSTRACT

Semigroups of composition operators on spaces of analytic functions have interested mathematicians for many years. The breakthrough in Hardy spaces was given by Berkson and Porta. On Bergman and Dirichlet spaces, Siskakis did extensive research whereby he determined semigroup as well as spectral properties of these operators on the unit disk. Later, other researchers like Arvanitidis, Bonyo, Blasco, Matache and others extended the work to Hardy and Bergman spaces of the unit disk and upper half plane. However, very little is known about semigroups of weighted composition operators on Bloch spaces. In this study therefore we investigated the properties of semigroups of weighted composition operators on the Bloch space. In particular, we determined a semigroup of weighted composition operators on the Bloch space; investigated its semigroup properties; and determined its spectral properties on the Bloch space of the unit disk. We used the duality properties of the non reflexive Bergman space to identify a semigroup of composition operators. To obtain the semigroup properties, we employed the theory of semigroups of linear operators and functional analysis where we determined infinitesimal generator of the semigroup and established the strong continuity property. We then determined the resolvents of the infinitesimal generator which are obtained as integral operators. Using the spectral mapping theorems as well as the Hille Yosida theorem, we obtained the spectral properties of the resulting integral operators. The results of this study will indeed contribute new knowledge and we hope it will motivate further research in this area of study.

Chapter 1

Introduction

1.1 Background of the study

Semigroups of composition operators on spaces of analytic functions was first studied by Berkson and Porta [8] on the Hardy spaces of the unit disk and upper half plane. In the said paper, the structure of the semigroups was determined and basic properties of semigroups were obtained. Siskakis extended this study of semigroups of composition operators on the unit disk to Bergman spaces [29] and Dirichlet spaces [30] where he proved strong continuity and identified their infinitesimal generators in the spaces. A lot of research has since been done on the semigroups of weighted composition operators on the Hardy and reflexive Bergman spaces of the disk. More details can be found in [3, 5, 6, 12, 17, 32].

It was proved in [34] that the dual and predual spaces of $L_a^1(\mathbb{D}, m_\alpha)$, the non-reflexive Bergman space, are respectively given as the Bloch space, $B_\infty(\mathbb{D})$ and the little Bloch space,$B_{\infty,0}(\mathbb{D})$ of the disk. In [7], all the self analytic maps of the upper half plane $\mathbb{U}$, of the Bergman spaces were classified according to the location of their fixed points into three distinct classes namely; scaling, translation and rotation groups. The corresponding groups of weighted composition operators for each group was defined and their semigroup and spectral properties studied in detail. For the rotation group, the induced group of weighted composition operators $(T_t)_{t\in\mathbb{R}}$ are defined on the analytic spaces of the unit disk and are given by $T_tf(z) = e^{ict}f(e^{ikt}z)$ with c, $k \in \mathbb{R}$, $k \neq 0$. Both the semigroup and spectral properties of the group $(T_t)_{t\in\mathbb{R}}$ were studied in detail in [7] and [11]. But for adjoint properties, Bonyo [11] considered the reflexive case of Bergman spaces, that is, for $1 < p < \infty$. For the non reflexive case, that is, for $p = 1$, the analysis of the adjoint group still remains open and forms the basis of this study.

The study of semigroups of weighted composition operators on the Bloch and little Bloch space has not been exhaustive. Some of the researches that have been done on the Bloch spaces include but not limited to the following: Madigan and Matheson [20] gave sufficient and necessary condition for composition operators to be compact on Bloch space; Shi and Luo [27] studied compactness and boundedness of composition operators on the Bloch spaces of several complex variables. This was followed by the study of Ohno and Zhao [23] who studied compactness and boundedness of weighted composition operators on Bloch space.

In this research therefore, we studied semigroups of weighted composition operators on Bloch space which are obtained as adjoints to strongly continuous groups on $L^1_a(\mathbb{D}, m_\alpha)$. In particular, we determined the semigroup properties and spectral picture of the semigroup on the Bloch space.

1.2 Organization of the study

In Chapter 1, we give the background of the study and highlight basic concepts necessary in the development of other chapters. In Chapter 2, we highlight the literature review on semigroups of composition operators on spaces of analytic functions. In Chapter 3, we obtain a group of weighted composition operators on the Bloch spaces and prove basic semigroup properties. In Chapter 4, we have obtained the spectral properties of the groups of isometries obtained in Chapter 3. Finally in Chapter 5, we give the summary and recommendations for further research.

1.3 Statement of the Problem

Let $\mathbb{D}$ be an open unit disk of the complex plane $\mathbb{C}$, $B_\infty(\mathbb{D})$ the Bloch space and $B_{\infty,0}(\mathbb{D})$ the Little Bloch space of analytic functions on $\mathbb{D}$. The study of semigroups of composition operators on reflexive Bergman spaces $L^p_a(\mathbb{D}, m_\alpha)$, $1 < p < \infty$, has been done extensively in the literature. However, on the non-reflexive Bergman space $L^1_a(\mathbb{D}, m_\alpha)$, the study has not been exhaustive. Specifically, the properties of the adjoints of semigroups of composition operators defined on $L^1_a(\mathbb{D}, m_\alpha)$ has not been done even though the dual and predual are known. In this research therefore, we studied semigroups of weighted composition operators on the Bloch space which are obtained as adjoints of strongly continuous groups on the non-reflexive Bergman space $L^1_a(\mathbb{D}, m_\alpha)$. In particular, we determined the semigroup and spectral properties of the semigroup of composition operators on the Bloch space of the unit disk.

1.4 Objectives of the Study

The main objective of this research was to investigate the properties of semigroups of weighted composition operators on the Bloch space of the unit disk. The specific objectives were;

1. To determine a semigroup of weighted composition operators on the Bloch space of the unit disk.
2. To determine the semigroup properties of the semigroup identified in (1) above.
3. To investigate the spectral properties of the semigroup identified in (1) above.

1.5 Significance of the Study

The study of the semigroups of weighted composition operators on spaces of analytic functions have been investigated by many scholars but has not been fully exhausted. Specifically, on the non-reflexive Bergman space the analysis of the adjoint group still remains open. It is therefore of great importance to complete the analysis the adjoint group on the Bloch space, the dual space of the non-reflexive Bergman space. The results of this study has contributed new knowledge in this area of mathematics and we hope will advance further research in this and other related areas.

1.6 Research Methodology

To determine a semigroup of composition operator on Bloch space $B_{\infty,0}(\mathbb{D})$, we used the duality properties of the non reflexive Bergman space, $L^1_a(\mathbb{D}, m_\alpha)$ as well as the definition of semigroups of weighted composition operators on $L^1_a(\mathbb{D}, m_\alpha)$. Using the duality pairing, we then obtained the adjoint of this semigroup which naturally constituted a semigroup on the little Bloch space. To investigate the semigroup properties of composition operators, we employed the theory of semigroups of linear operators and functional analysis where we determined the infinitesimal generator of the semigroup and established the strong continuity property. Using spectral theory, we obtained the spectrum, point spectrum and resolvent of the infinitesimal generator which are given as an integral operator. Finally we used the Hille-Yosida theorem and the spectral mapping theorem to obtain the spectrum, point spectrum, spectral radius and norm of the resolvent.

1.7 Basic concepts

1.7.1 The unit disk $\mathbb{D}$ and Upper half plane $\mathbb{U}$

Let $\mathbb{C}$ be the complex plane. The set $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is called the (open) unit disk. Let dA denotes the area measure on $\mathbb{D}$, normalized so that the area of $\mathbb{D}$ is 1. For $\alpha \in \mathbb{R}$, $\alpha > -1$, we define a positive Borel measure dm_α on $\mathbb{D}$ by $dm_\alpha(z) = (1 - |z|^2)^\alpha dA(z)$. dm_α is a finite measure on $\mathbb{D}$. In fact if $\alpha = 0$, then dm_0 and dA coincide and we simply denote it by dA. dm_α can thus be considered as a weighted measure on

$\mathbb{D}$ and a generalization of dA.

On the other hand, the set $\mathbb{U} := \{\omega \in \mathbb{C} : \Im(\omega) > 0\}$ denotes the upper half of the complex plane $\mathbb{C}$ with $\Im(\omega)$ being the imaginary part of $\omega \in \mathbb{C}$. We define a weighted measure on $\mathbb{U}$ by $d\mu_\alpha(\omega) = (\Im(\omega))^\alpha dA(\omega)$ where $\omega \in \mathbb{U}$.

The **Cayley transform** $\psi(z) = \frac{i(1+z)}{1-z}$ maps the unit disk $\mathbb{D}$ conformally onto the upper half plane $\mathbb{U}$ with its inverse $\psi^{-1}(\omega) = \frac{\omega - i}{\omega + i}$. For further details, see [18, 25].

1.7.2 Vector spaces and Normed Spaces

Definition 1.7.1

Let E be a vector space over a field $\mathbb{F}$. A function $\|.\| : E \to \mathbb{F}$ is called a **norm** if it satisfies the following conditions:

1. $\|x\| = 0 \Leftrightarrow x = 0$,
2. $\|\lambda x\| = |\lambda|\|x\|$ for every $x \in E$ and $\lambda \in \mathbb{F}$,
3. $\|x + y\| \leq \|x\| + \|y\|$ for every $x, y \in E$.

The pair $(E, \|.\|)$ or simply E is called a **normed space**.

Definition 1.7.2

A sequence $(x_n)_n \subseteq E$ is said to be **convergent** if for every $\epsilon > 0$, there exists a number M such that for every $n \geq M$ we have $\|x_n - x\| < \epsilon$ for all $x_n, x \in E$. A sequence of vectors $(x_n)_n \subseteq E$ in a normed space is called a **Cauchy sequence** if for every $\epsilon > 0$, there exists a number M such that $\|x_m - x_n\| < \epsilon$ for all $m, n > M$.

Definition 1.7.3

A normed space E is said to be **complete** if every Cauchy sequence in E converges to an element of E. A complete normed space is called a **Banach space**.

1.7.3 Hardy and Bergman spaces

Definition 1.7.4

Let Ω denote an arbitrary open subset of $\mathbb{C}$ and $\mathcal{H}(\Omega)$ denote the space of analytic functions $f : \Omega \to \mathbb{C}$. In this thesis, we consider Ω to be either the unit disk $\mathbb{D}$ or the upper half plane $\mathbb{U}$. For $1 \leq p < \infty$, the Hardy spaces of the unit disk are defined by

$$H^p(\mathbb{D}) := \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|^p_{H^p(\mathbb{D})} = \sup_{0<r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta < \infty \right\}. \tag{1.1}$$

For every $f \in H^p(\mathbb{D})$, we state the well known growth condition;

$$|f(z)| \leq \frac{C\|f\|_{H^p(\mathbb{D})}}{(1-|z|)^{\frac{1}{p}}} \tag{1.2}$$

where C is a constant and $z \in \mathbb{D}$ [26].

For $1 \leq p < \infty$, $\alpha > -1$, the weighted Bergman spaces of the unit disk $\mathbb{D}$, $L^p_a(\mathbb{D}, m_\alpha)$, are also defined by

$$L^p_a(\mathbb{D}, m_\alpha) := \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_{L^p_a(\mathbb{D},m_\alpha)} = \left(\int_{\mathbb{D}} |f(z)|^p dm_\alpha(z) \right)^{\frac{1}{p}} < \infty \right\}. \tag{1.3}$$

In particular, $L^p_a(\mathbb{D}, m_\alpha) = L^p(\mathbb{D}, m_\alpha) \cap \mathcal{H}(\mathbb{D})$, where $L^p(\mathbb{D}, m_\alpha)$ denotes the classical Lebesque spaces associated with the weighted measure dm_α.

For every $f \in L_a^p(\mathbb{D}, m_\alpha)$, we state the well known growth condition;

$$|f(z)| \leq \frac{K\|g\|_{L_a^p(\mathbb{D},m_\alpha)}}{(1-|z|^2)^\gamma} \tag{1.4}$$

where K is a constant, $\gamma = \frac{\alpha+2}{p}$ and $z \in \mathbb{D}$.

The Hardy and Bergman spaces together with their norms are Banach spaces. For $p = 2$, $H^2(\mathbb{D})$ and $L_a^2(\mathbb{D}, m_\alpha)$ are Hilbert spaces. For a comprehensive theory of Hardy and Bergman spaces we refer to [14, 18, 34].

1.7.4 Bloch and Little Bloch spaces

The Bloch space of the unit disk, denoted by $B_\infty(\mathbb{D})$ is defined to be the space of analytic functions f on $\mathbb{D}$ such that the semi norm

$$\|f\|_{B_{\infty,1}(\mathbb{D})} = \sup_{z\in\mathbb{D}} \left\{(1-|z|^2)|f'(z)|\right\} < \infty. \tag{1.5}$$

$B_\infty(\mathbb{D})$ is a Banach space with respect to the norm $\|f\|_{B_\infty(\mathbb{D})} := |f(0)| + \|f\|_{B_{\infty,1}(\mathbb{D})}$. If $f \in B_\infty(\mathbb{D})$, then it satisifies the growth condition

$$|f(z)| \leq \left(1 + \frac{1}{2}\log\frac{1+|z|}{1-|z|}\right)\|f\|_{B_\infty(\mathbb{D})} \tag{1.6}$$

$$|f(0)| \leq \|f\|_{B_\infty(\mathbb{D})}$$

for every $z \in \mathbb{D}$ [34, page 82].

The Bloch space of the upper half plane is denoted by $B_\infty(\mathbb{U})$ and is defined by

$$B_\infty(\mathbb{U}) := \{f \in \mathcal{H}(\mathbb{U}) : \sup_{\omega \in \mathbb{U}} \Im(\omega)|f'(\omega)| < \infty\}.$$

The norm is given by $\|f\|_{B_\infty(\mathbb{U})} = |f(i)| + \|f\|_{B_{\infty,1}(\mathbb{U})}$ where $\|f\|_{B_{\infty,1}(\mathbb{U})} = \sup_{\omega\in\mathbb{U}} \Im(\omega)|f'(\omega)|$. On the other hand, the little Bloch space of the unit disk $\mathbb{D}$ is denoted by $B_{\infty,0}(\mathbb{D})$ and is defined to be the closed subspace of $B_\infty(\mathbb{D})$ such that $B_{\infty,0}(\mathbb{D}) := cl_{B_\infty}\mathbb{C}[z]$, where $\mathbb{C}[z]$ denotes the analytic complex polynomials in z. Equivalently

$$B_{\infty,0}(\mathbb{D}) := \left\{ f \in \mathcal{H}(\mathbb{D}) : \lim_{z\to 1^-, z\in\mathbb{D}} (1-|z|^2)|f'(z)| = 0 \right\}. \tag{1.7}$$

$B_{\infty,0}(\mathbb{D})$ is also Banach space with respect to the norm $\|.\|_{B_\infty(\mathbb{D})}$. If X is a Banach space and $Y \subseteq X$ be its subspace, then we say that Y is dense in X if its closure is the whole of X. That is, $\overline{Y} = X$.

Proposition 1.7.5

[34, Theorem 5.2.2] $B_{\infty,0}(\mathbb{D})$ is closed subspace of $B_\infty(\mathbb{D})$. Moreover, the set of polynomials is dense in $B_{\infty,0}(\mathbb{D})$.

See [14, 25, 34] for more details on Bloch spaces.

1.7.5 Duality of Bergman Spaces

The dual space of a vector space E, denoted by E^*, is a set of all linear maps $\phi : E \to \mathbb{F}$. Elements of E^* are called **functionals** on E. On the other hand, the predual space is a set of all linear maps $\phi : E^* \to \mathbb{F}$. For $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and α >-1, the dual space of the Bergman space $L^p_a(\mathbb{D}, m_\alpha)$ is given by

$$(L_a^p(\mathbb{D}, m_\alpha))^* \approx L_a^q(\mathbb{D}, m_\alpha),$$

under the integral pairing

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}dm_\alpha(z),$$

where $f \in L_a^p(\mathbb{D}, m_\alpha)$ and $g \in L_a^q(\mathbb{D}, m_\alpha)$.

For $p = 1$, the dual space of the Bergman space $L_a^1(\mathbb{D}, m_\alpha)$ is given by

$$(L_a^1(\mathbb{D}m_\alpha))^* \approx B_\infty(\mathbb{D}),$$

under the pairing

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}dm_\alpha(z),$$

where $f \in L_a^1(\mathbb{D}, m_\alpha)$ and $g \in B_\infty(\mathbb{D})$. The predual space of $L_a^1(\mathbb{D}, m_\alpha)$ is given by

$$(B_{\infty,0}(\mathbb{D}))^* \approx L_a^1(\mathbb{D}, m_\alpha), \tag{1.8}$$

under the pairing

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}dm_\alpha(z),$$

where $f \in B_{\infty,0}(\mathbb{D})$ and $g \in L_a^1(\mathbb{D}, m_\alpha)$. We refer to [18, 34] for comprehensive account of the theory of duality of Bergman spaces.

1.7.6 Other spaces of analytic functions

A Banach space of all analytic functions in the Hardy space $H^2(\mathbb{D})$ whose boundary values have bounded mean oscillation is denoted by $BMOA$.

More precisely $f \in H^2(\mathbb{D})$ belongs to $BMOA$ if and only if there exists a constant $C > 0$ such that

$$\int_{R(I)} |f'(z)|^2(1-|z|^2)dA(z) \le C(I),$$

for any arc $I \subset \partial\mathbb{D}$, where $R(I)$ is the Carleson rectangle determined by I with $|I|$ denoting the length of I. The corresponding $BMOA$ norm is

$$\|f\|^2_{BMOA} := |f(0)|^2 + \sup_{I\subset\mathbb{D}} \left(\frac{I}{|I|}\int_{R(I)} |f'(z)|^2(1-|z|^2)dA(z)\right).$$

The closure of all polynomials in $BMOA$ is denoted by $VMOA$, Vanishing mean oscillation. Equivalently, $VMOA$ is the subspace of $BMOA$ formed by those $f \in BMOA$ such that

$$\lim_{|I|\to 0} \frac{I}{|I|}\int_{R(I)} |f'(z)|^2(1-|z|^2)dA(z) = 0.$$

1.7.7 Spectra of Linear operators

Let X and Y be Banach spaces over $\mathbb{C}$. A **linear operator** $T : X \to Y$ is a linear mapping of a linear subspace $D(T)$ of X into Y, where $D(T)$ is the domain of T.

$T : X \to Y$ is said to be a **closed operator** if its graph $\{(x, Tx)|x \in D(T)\}$ is closed in $X \times Y$ and is bounded if there exists $C \ge 0$ such that $\|Tx\| \le C\|x\|$, for all $x \in X$. We denote the space of linear and bounded operators from X to Y by $\mathcal{L}(X, Y)$, and $\mathcal{L}(X, X) = \mathcal{L}(X)$

Let T be closed operator on X, the **resolvent set** of T, $\rho(T)$ is given by $\rho(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is invertible}\}$ where I is the identity operator on

X and its **spectrum** $\sigma(T) = \mathbb{C}\backslash\rho(T)$. Note that the operator $\lambda I - T$ is not invertible if it is not bijective.

The **spectral radius** of T is defined by $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$ and it is well known that $r(T) \leq \|T\|$.

The **point spectrum** is given by $\sigma_p(T) = \{\lambda \in \mathbb{C} : Tx = \lambda x \text{ for some } 0 \neq x \in D(T)\}$. For $\lambda \in \rho(T)$, the operator $R(\lambda, T) := (\lambda I - T)^{-1}$ is called the **resolvent** of T or simply the resolvent operator. For comprehensive theory of spectra of linear operators, we refer to [14, 18, 19].

1.7.8 Semigroup of Linear operators

Definition 1.7.6

Let X be a Banach space. A one parameter family $(T_t)_{t\geq 0} \subseteq \mathcal{L}(X)$ is a **semigroup** of bounded linear operator on X if;

1. $T_o = \mathrm{I}$ (Identity operator on X).
2. $T_{t+s} = T_t \circ T_s$ for every $t, s \geq 0$.

$(T_t)_{t\geq 0}$ is strongly continuous if $\lim_{t\to 0^+} \|T_t x - x\| = 0$ for all $x \in X$. A strongly continuous semigroup is also called C_o- semigroup.

Definition 1.7.7

The **infinitesimal generator** Γ of $(T_t)_{t\geq 0}$ is defined by

$$\Gamma x = \lim_{t\to 0^+} \frac{T_t x - x}{t} = \frac{\partial}{\partial x}(T_t x)\Big|_{t=0} \tag{1.9}$$

for each $x \in D(\Gamma)$, where the domain of Γ is given by $D(\Gamma) = \{x \in X : \lim_{t\to 0^+} \frac{T_t x - x}{t} \text{ exists } \}$.

Let X and Y be arbitrary Banach spaces and $U \in L(X, Y)$ be an invertible operator. If $(T_t)_{t\in\mathbb{R}} \subseteq L(X)$ is a strongly continuous group, we can generate a new strongly continuous group $(S_t)_{t\in\mathbb{R}} \subseteq L(Y)$ by the relation $S_t = UT_tU^{-1}$. In this case if $(T_t)_{t\in\mathbb{R}}$ has generator Γ, then $(S_t)_{t\in\mathbb{R}}$ has generator $\Delta = U\Gamma U^{-1}$ with domain $D(\Delta) = UD(\Gamma) := \{y \in Y : Uy \in D(\Gamma)\}$. Moreover, $\sigma_p(\Delta, Y) = \sigma_p(\Gamma, X)$ and $\sigma(\Delta, Y) = \sigma(\Gamma, X)$. For λ in the resolvent set $\rho(\Gamma, X) := \mathbb{C}\backslash\sigma(\Gamma, X)$, we have that $R(\lambda, \Delta) = UR(\lambda, \Delta)U^{-1}$.

More details about semigroups can be found in [14, 16, 24].

1.7.9 Composition operators

A function φ is called **self analytic** map on $\mathbb{D}$ if it is analytic in $\mathbb{D}$ and $\varphi(\mathbb{D}) \subset \mathbb{D}$. For $t \geq 0$, consider the semigroup of self analytic maps $\varphi_t : \mathbb{D} \to \mathbb{D}$. The semigroup of composition operators induced by φ_t is defined on $\mathcal{H}(\mathbb{D})$ by $C_{\varphi_t}(f) = f \circ \varphi_t$ for all $f \in \mathcal{H}(\mathbb{D})$.
The corresponding group of weighted composition operators induced by φ_t will therefore be defined on $\mathcal{H}(\mathbb{D})$ by

$$T_t(f)(z) = (\varphi_t'(z))^{\gamma}(f \circ \varphi_t)(z)$$

where γ is an appropriately chosen weight and $f \in \mathcal{H}(\mathbb{D})$. See [28] for more details.

Chapter 2

Literature Review

The theory of semigroups of bounded linear operators was introduced by Hille Yosida [33]. The study of semigroups of composition operators on spaces of analytic functions was first studied by Berkson and Porta [8] where they considered a composition semigroup on Hardy spaces of the form $T_t(f) = f_\circ\varphi_t$ where φ_t is a semigroup of self analytic functions mapping the unit disk $\mathbb{D}$ into itself. In that paper, the structure of the semigroups of the functions T_t was determined and their basic properties obtained. Siskakis extended this study of semigroups of composition operators on the unit disk $\mathbb{D}$ to Bergman spaces in [29] and Dirichlet spaces [30] where he proved strong continuity and identifed their infinitesimal generators in the said spaces. Compactness and conditions for compactness were also given in [32] by Siskakis for the resolvent operators on Hardy spaces. General information of composition operators on classical spaces of analytic functions can be obtained in the excellent monographs of Cowen and MacCluer [13] and Shapiro [26].

In the recent years, the study of semigroups of composition operators has been extended to other spaces of analytic functions. Oscar Blasco et al [9] studied the maximal subspace in the space of Bounded Mean Oscillations (BMOA) where a general semigroup of analytic functions on the unit disk $\mathbb{D}$ generates a strongly continuous semigroup of composition operators. Later, in [10] the same authors studied the maximal spaces of strong continuity on BMOA and the Bloch spaces for semigroups of composition operators. The concept of strong continuity of semigroups of composition operators has also been studied by Arvalo et al [2] on mixed norm spaces. In their study, the maximal closed linear subspace in which the semigroups are strongly continuous was obtained. Matache [21] studied boundedness and compactness of composition operators on Hardy spaces of the upper half plane. The same author in [22], extended the study to weighted composition operators on the open unit disk $\mathbb{D}$. Further research on compactness and boundedness of weighted composition operators was done by Contreras and Hernandez [12] on the space of analytic functions f on $\mathbb{D}$ such that $f' \in H^p(\mathbb{D})$. Eva and Dmitry [17] showed that if an operator generates a $C_\circ$-semigroup then it is automatically a semigroup of composition operators. This was an extension of the earlier work of Avicous et al [6] where the author and his co-authors proved that an (unbounded) operator on the classical Hardy space generates a $C_\circ$-semigroup of composition operators if and only if it generates a quasicontractive semigroup. A $C_\circ$-semigroup $(T_t)_{t\geq 0}$ is called a quasicontactive semigroup if there exists a constant $\omega \geq 0$ such that $\|T_t\| \leq e^{\omega t}$ for all $t \geq 0$. In [11], Bonyo used the similarity theory of semigroups as well as spectral theory to obtain resolvents of generators of strongly continuous groups of isometries on the Hardy and Bergman spaces. These groups were obtained as weighted composition operators associated with specific automorphisms of the upper-half plane. Other properties of semigroups of composition operators can be obtained in [31], a review by Siskakis on Hardy and Bergman spaces. Arevalo and Oliva [3] studied strong continuity of semigroups of weighted composition operators in several spaces of analytic functions.

Arvanitidis and Siskakis [5] obtained the semigroup and spectral properties of semigroups weighted composition operators on the Hardy space of $\mathbb{U}$ of the form $T_t f(z) = e^{-\frac{t}{p}} f(\phi_t(z))$ where $\phi_t(z) = e^{-t}z$, $z \in \mathbb{U}$, and $f \in H(\mathbb{U})$.Ballamoole et al [7] obtained the spectrum, generator, adjoint and decomposability of cesaro-like operator which was identified as the resolvent for appropriate semigroups of composition operators on Hardy and weighted Bergman spaces.

The study of semigroups of composition operators on the Bloch and little Bloch space has not been exhausted. Madigan and Matheson [20] gave sufficient and necessary condition for composition operators to be compact on Bloch spaces. Shi and Luo [27] studied compactness and boundedness of composition operators on the Bloch spaces of several complex variables. This was followed by the study of Ohno and Zhao [23] who studied compactness and boundedness of weighted composition operators on Bloch space of several complex variables. The spectral picture of invertible weighted composition operators, when φ is an elliptic automorphism in $\mathbb{D}$, has also been studied by Eklund et al in [15]. The duality properties of Bergman spaces are well studied in literature. In [34], for $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha > -1$, the dual space of $L^p_a(\mathbb{D}, m_\alpha)$ is given as $L^q_a(\mathbb{D}, m_\alpha)$. For $p = 1$, it is proved in [34] that the dual and predual spaces of $L^1_a(\mathbb{D}, m_\alpha)$, the non-reflexive Bergman space are respectively given as the Bloch space, $B_\infty(\mathbb{D})$ and the little Bloch space, $B_{\infty,0}(\mathbb{D})$. In this research, we extended the study of semigroups of weighted composition operators to the Bloch space. The following theorems from literature were useful in the study.

Theorem 2.0.1 (Spectral mapping theorem for resolvents)
Let T be a closed operator on a Banach space X and $\lambda \in \rho(T)$. Then the following holds;

1. $\sigma(R(\lambda, T))\backslash\{0\} = (\lambda - \sigma(T))^{-1} = \left\{\frac{1}{\lambda-\mu} : \mu \in \sigma(T)\right\}$,

2. $\sigma_p(R(\lambda, T))\backslash\{0\} = (\lambda - \sigma_p(T))^{-1} = \left\{\frac{1}{\lambda-v} : v \in \sigma_p(T)\right\}$,

Theorem 2.0.2 (Spectral mapping theorem for semigroups)
Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X and Γ be its infinitesimal generator then;

1. $\sigma(T_t) \supset e^{t\sigma(\Gamma)}$,

2. $\sigma_p(T_t)\backslash\{0\} = e^{t\sigma_p(\Gamma)}$.

Theorem 2.0.3 (Hille-Yosida theorem)
Let X be a Banach space. A linear operator Γ is the infinitesimal generator of a strongly continuous semigroup of contractions $(T_t)_{t\geq 0}$ if and only if;

1. *Γ is closed and $\overline{D(\Gamma)} = X$*

2. *The resolvent set $\rho(\Gamma)$ of Γ contains $\mathbb{R}^+$ and for every $\lambda > 0$,*

$$\|R(\lambda, \Gamma)\| \leq \frac{1}{|\lambda|}.$$

In this case, if $h \in X$, then

$$R(\lambda, \Gamma)h = \int_0^\infty e^{-\lambda t} T_t h dt$$

is norm convergent.

Theorem 2.0.4 (Open mapping theorem)

If X, Y are Banach spaces and $T \in L(X, Y)$ be invertible, then the inverse map T^{-1} is bounded, that is $T^{-1} \in L(X, Y)$.

Theorem 2.0.5 (Closed graph theorem)

Let X and Y be Banach spaces. Then every closed linear mapping $T : X \to Y$ is continuous.

Chapter 3

Weighted Composition Operators on the Bloch Space

In this chapter, we determine a semigroup of weighted composition operators on the Bloch space. This semigroup is obtained as the adjoint of a semigroup of weighted composition operators on the nonreflexive Bergman space $L_a^1(\mathbb{D}, m_\alpha)$. We prove that this semigroup is a strongly continuous group of isometries on $B_{\infty,0}(\mathbb{D})$. We also determine its infinitesimal generator Γ on $B_{\infty,0}(\mathbb{D})$.

3.1 Self analytic maps and Automorphism groups

A **linear fractional map** on a Banach space X is a linear transformation $\phi : X \to \mathbb{F}$ of the form

$$\phi(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{F}$ and $ad - bc \neq 0$.

On the complex plane, linear fractional transformations are also called **möbius transformations**. An angle preserving transformation is called a **conformal map** while a structure preserving map between Banach spaces is a **Homomorphism**. A bijective homomorphism of an object into itself is an **automorphism**. The set of all automorphisms of an object forms a group. Automorphism groups of $\mathbb{D}$, denoted by $Aut(\mathbb{D})$ consist of analytic functions of the form

$$\varphi(z) = \frac{a(z-b)}{(1-\bar{b}z)} \tag{3.1}$$

for all $z \in \mathbb{D}$ and where a and b are constants with $|a| = 1$ and $|b| \leq 1$. These functions are called möbius transformations of $\mathbb{D}$.

Example 3.1.1

Consider the family of analytic functions

$$\phi_t(z) = \frac{e^{it}z - \frac{1}{2}}{1 - \frac{1}{2}e^{it}z}$$

where $t \geq 0$ and $z \in \mathbb{D}$.

Comparing with equation 3.1, for each $z \in \mathbb{D}$, we have $a = e^{it}$ and $b = \frac{1}{2}e^{-it}$, and therefore ϕ_t is a möbius transformation of $\mathbb{D}$, since $|a| =$

$|e^{it}| = |\cos t + i\sin t| = 1$ and $|b| = |\frac{1}{2}e^{-it}| = \frac{1}{2}$. Moreover,

$$\begin{aligned} |\phi_t(z)|^2 &= \phi_t(z)\overline{\phi_t(z)} \\ &= \frac{e^{it}(z - \frac{1}{2}e^{-it})}{1 - \frac{1}{2}e^{it}z}\frac{e^{-it}(\overline{z} - \frac{1}{2}e^{it})}{1 - \frac{1}{2}e^{-it}\overline{z}} \\ &= \frac{|z|^2 + \frac{1}{4} - \frac{1}{2}(2\Re(ze^{it}))}{1 + \frac{1}{4}|z|^2 - \frac{1}{2}(2\Re(ze^{it}))} \\ &= \frac{4|z|^2 + 1 - 4\Re(ze^{it})}{|z|^2 + 4 - 4\Re(ze^{it})}. \end{aligned}$$

Since $|z| < 1$, then it follows that $4|z|^2 + 1 \leq |z|^2 + 4$ for all $z \in \mathbb{D}$. Thus $|\phi_t(z)| \leq 1$ and therefore $\phi(\mathbb{D}) \subseteq \mathbb{D}$.

A function

$$\psi(z) := \frac{i(1+z)}{(1-z)} \tag{3.2}$$

maps the unit disk $\mathbb{D}$ conformally onto the upper half-plane $\mathbb{U}$ with its inverse given as

$$\psi^{-1}(\omega) = \frac{\omega - i}{\omega + i}. \tag{3.3}$$

On the other hand, the automorphism groups of the upper half-plane, $Aut(\mathbb{U})$, are sets of möbius transformations of the form

$$\varphi(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{R}$, $ad - bc > 0$.

Example 3.1.2

Consider the function

$$\phi(z) = \frac{z}{1-z}.$$

Thus ϕ is self analytic on $\mathbb{U}$. Indeed, for each $z \in \mathbb{U}$, we have $a = 1$,

$b = 0$, $c = -1$, $d = 1$, therefore $ad - bc = 1 > 0$. Moreover,

$$\begin{aligned}\Im(\phi(z)) &= \frac{\frac{z}{1-z} - \frac{\overline{z}}{1-\overline{z}}}{2i} \\ &= \frac{\frac{z - z\overline{z} - \overline{z} + z\overline{z}}{1 - z\Re(z) + |z|^2}}{2i} \\ &= \frac{\Im(z)}{|1-z|^2} > 0\end{aligned}$$

and therefore $\phi(\mathbb{U}) \subseteq \mathbb{U}$. So ϕ is an automorphism in $\mathbb{U}$.

A point $z \in \mathbb{C}$ is called a **fixed point** if $f(z) = z$. That is, it does not change upon application of a map. In [7], all the self analytic maps of the upper half plane $\mathbb{U}$ were characterized according to the location of their fixed points into three distinct classes namely; scaling, translation and rotation groups. In particular, they gave the following result,

Theorem 3.1.3

Let $\varphi : \mathbb{R} \to Aut(\mathbb{U})$ be a nontrivial continuous group homomorphism. Then exactly one of the following holds:

1. *There exists $k > 0$, $k \neq 1$, and $f \in Aut(\mathbb{U})$ so that $\varphi_t(z) = f^{-1}(k^t f(z))$ for all $z \in \mathbb{U}$ and $t \in \mathbb{R}$.*

2. *There exists $k \in \mathbb{R}$, $k \neq 0$, and $f \in Aut(\mathbb{U})$ so that $\varphi_t(z) = f^{-1}(f(z) + kt)$ for all $z \in \mathbb{U}$ and $t \in \mathbb{R}$.*

3. *There exists $k \in \mathbb{R}$, $k \neq 0$, and a conformal mapping g of $\mathbb{U}$ onto $\mathbb{D}$ such that $\varphi_t(z) = f^{-1}(e^{ikt} f(z))$ for all $z \in \mathbb{U}$ and $t \in \mathbb{R}$. Equivalently, there exist $\theta \in \mathbb{R}\backslash 0$ and $h \in Aut(\mathbb{U})$ so that*

$$\varphi_t(z) = h^{-1}\left(\frac{h(z)\cos(\theta t) - \sin(\theta t)}{h(z)\sin(\theta t) + \cos(\theta t)}\right).$$

PROOF. See [7, Theorem 2.2]. □

Thus from assertion 1, we have the scaling group whose self analytic maps of $\mathbb{U}$ are of the form $\varphi_t(z) = k^t z$ for $k > 0$, $k \neq 1$, and $t \in \mathbb{R}$. A perfect example is the one considered in [7, Section 3] or [5] which is given by $\varphi_t(z) = e^{-t}z$. From assertion 2, we have the translation group whose self analytic maps of $\mathbb{U}$ are of the form $\varphi_t(z) = z + kt$ for $k \in \mathbb{R}$, $k \neq 0$. Again we refer to [7, Section 4] for an example, $\varphi_t(z) = z + t$ that is for $k = 1$. For assertion 3, we have rotation group defined on the unit disk $\mathbb{D}$ which can then be mapped back to the upper half plane $\mathbb{U}$ using the Cayley transform. In this case, the self analytic maps of $\mathbb{D}$ are of the form $\varphi_t(z) = e^{ikt}z$. We refer to [7, Section 5] for an example, $\varphi_t(z) = e^{it}z$, for $k = 1$. We can easily show that φ_t is self analytic on $\mathbb{D}$. Let $z \in \mathbb{D}$, without loss of generality let $k = 1$, then $|e^{it}z| = |z| < 1$.

3.2 Composition operators

Let $\{V_1, V_2\} = \{\mathbb{D}, \mathbb{U}\}$ and let $LF(V_i, V_j)$ denote the collection of conformal mappings from V_i to V_j. In particular, $LF(V_i, V_j) = Aut(V_i)$ and if $h \in LF(V_i, V_j)$, then $g \in Aut(V_j) \mapsto h^{-1} \circ g \circ h \in Aut(V_i)$ is an isomorphism from $Aut(V_i)$ into $Aut(V_j)$. For each $h \in LF(V_i, V_j)$, define a weighted composition operator $S_h : \mathcal{H}(V_j) \to \mathcal{H}(V_i)$ by

$$S_h f(z) = (h'(z))^{\gamma} f(h(z)) \tag{3.4}$$

for all $z \in V_i$.

Note that if $g \in LF(V_i, V_j)$ and $h \in LF(V_j, V_i)$, then $S_h S_g = S_{g \circ h}$ and

$S_h^{-1} = S_{h^{-1}}$ by the Chain Rule. Indeed,

$$\begin{aligned} S_h S_g f(z) &= S_h[(g'(z))^\gamma f(g(z))] \\ &= (h'(z))^\gamma (g'(h(z)))^\gamma f[g(h(z))] \\ &= [(g \circ h)'(z)]^\gamma f[g \circ h(z)] \\ &= S_{g o h} f(z). \end{aligned} \tag{3.5}$$

Clearly $S_I = I$. Therefore from equation (3.5), we have

$$S_h S_h^{-1} = I = S_{h^{-1} o h} = S_h S_{h^{-1}},$$

which implies that $S_h^{-1} = S_{h^{-1}}$.

Following [7], the corresponding groups of weighted composition operators induced by the rotation group φ_t on $L_a^p(\mathbb{D}, m_\alpha)$ are defined by $T_t f(z) = S_{\varphi_t} f(z) = e^{ict} f(e^{ikt} z)$ with $c, k \in \mathbb{R}, \ \ k \neq 0$.
Let $k = 1$ and $c = \gamma = \frac{\alpha+2}{p}$, then the group becomes $T_t f(z) = e^{i\gamma t} f(e^{it} z)$ for all $z \in \mathbb{D}$ and $f \in L_a^p(\mathbb{D}, m_\alpha)$. Both the semigroup and spectral properties of the group $(T_t)_{t\in\mathbb{R}}$ on Hardy and Bergman spaces were studied in detail in [7] and [11]. For the adjoint properties, Bonyo [11] considered the reflexive case of Bergman spaces, that is, for $1 < p < \infty$. For the non reflexive case, that is, for $p = 1$, the analysis of the adjoint group still remains open and forms the basis of this study.
Without loss of generality, we let $k = 1$ and therefore consider automorphsm groups of the form $\varphi_t(z) = e^{it} z$ for $z \in \mathbb{D}$. The corresponding group of weighted composition operators on $L_a^1(\mathbb{D}, m_\alpha)$ is therefore given by $T_t f(z) = (e^{it})^\gamma f(e^{it} z)$, where $\gamma = \alpha + 2$. Following [7], this group

is a strongly continuous group of isometries on $L_a^1(\mathbb{D}, m_\alpha)$ as seen in the following theorem.

Theorem 3.2.1

Let T_t be the group of weighted composition operators given by $T_tf(z) = (e^{ict})^\gamma f(e^{it}z)$ for all $f \in L_a^p(\mathbb{D}, m_\alpha)$ and let Γ be its infinitesimal generator. Then;

1. $(T_t)_{t\geq 0}$ *is a strongly continuous group of isometries on* $L_a^p(\mathbb{D}, m_\alpha)$.
2. $\Gamma f(z) = i(cf(z) + zf'(z))$ *with* $D(\Gamma) = \{f \in L_a^p(\mathbb{D}, m_\alpha)\}$.
3. $\sigma(\Gamma) = \sigma_p(\Gamma) = \{i(n+c) : n \in \mathbb{Z}_+\}$ *and for each* $n \geq 0$, $ker(i(n+c) - \Gamma) = span(z^n)$.
4. *If* $\lambda \in \rho(\Gamma)$, *then* $ran(M_z^m)$ *is* $R(\lambda, \Gamma)$*-invariant for every* $m \in \mathbb{Z}_+$ *such that* $m + c > \Im(\lambda)$. *Infact if* $h \in ran(M_z^m)$ *then,*

$$R(\lambda, \Gamma)h(z) = z^{-i(c+i\lambda)} \int_0^z \omega^{c+i\lambda-1} h(\omega)d\omega = z^m \int_0^m t^{m+ci\lambda-1} Q^m h(tz)dt$$

Proof. The proof follows immediately from [7, Theorem 5.1] by letting $p = 1$. □

Now, the predual of $L_a^1(\mathbb{D}, m_\alpha)$ is given by equation (1.8) as $(B_{\infty,0}(\mathbb{D}))^* \approx L_a^1(\mathbb{D}, m_\alpha)$ under the integral pairing

$$\langle g, f\rangle = \int_{\mathbb{D}} g(z)\overline{f(z)}dm_\alpha$$

where $f \in L_a^1(\mathbb{D}, m_\alpha)$ and $g \in B_{\infty,0}(\mathbb{D})$.

Thus we obtain the adjoint of $(T_t)_{t\geq 0}$ as follows:

Let $g \in B_{\infty,0}(\mathbb{D})$, then

$$\langle g, T_t f\rangle = \int_{\mathbb{D}} g(z)\overline{(e^{it})^{\alpha+2} f(e^{it}z)}dm_\alpha(z).$$

The Borel measure dm_α on $\mathbb{D}$ is given by $dm_\alpha(z) = (1-|z|^2)^\alpha dA(z)$, then it follows that

$$\langle g, T_t f\rangle = \int_{\mathbb{D}} (e^{-it})^{\alpha+2} g(z)\overline{f(e^{it}z)}(1-|z|^2)^\alpha dA(z).$$

By change of variables, let $\omega = e^{it}z$ so that $z = e^{-it}\omega$, and $dA(\omega) = |\varphi_t'|^2 dA(z) = dA(z)$ then we obtain,

$$\begin{aligned}
\langle g, T_t f\rangle &= \int_{\mathbb{D}} (e^{-it})^{\alpha+2} g(e^{-it}\omega)\overline{f(\omega)}(1-|e^{-it}\omega|^2)^\alpha dA(\omega)\\
&= \int_{\mathbb{D}} (e^{-it})^{\alpha+2} g(e^{-it}\omega)\overline{f(\omega)}(1-|\omega|^2)^\alpha dA(\omega)\\
&= \int_{\mathbb{D}} (e^{-it})^{\alpha+2} g(e^{-it}\omega)\overline{f(\omega)}dm_\alpha(\omega)\\
&= \int_{\mathbb{D}} T_{-t}g(\omega)\overline{f(\omega)}dm_\alpha(\omega)\\
&= \langle T_{-t}g, f\rangle.
\end{aligned}$$

Therefore, $S_t g(\omega) = T_{-t}g(\omega) = (e^{-it})^{\alpha+2} g(e^{-it}\omega)$ for all $g \in B_{\infty,0}(\mathbb{D})$. This is the adjoint of the semigroup $(T_t)_{t\geq 0} \subseteq \mathcal{L}(L^1_a(\mathbb{D}, m_\alpha))$ and therefore by definition it constitutes a semigroup on $B_{\infty,0}(\mathbb{D})$.

3.3 Semigroup properties

We now consider the group of weighted composition operators $(S_t)_{t\in\mathbb{R}}$ on $B_{\infty,0}(\mathbb{D})$ given by;

$$\begin{aligned} S_t g(z) &= T_{-t} g(z) \\ &= (e^{-it})^{\alpha+2} g(e^{-it} z) \end{aligned}$$

for all $g \in B_{\infty,0}(\mathbb{D})$, as determined in section 3.2 above.

Then we give the following results that detail the semigroup properties of $(S_t)_{t\geq 0}$ on $B_{\infty,0}(\mathbb{D})$.

Proposition 3.3.1

$(S_t)_{t\in\mathbb{R}}$ is a group on $B_{\infty,0}(\mathbb{D})$.

PROOF. It suffices to prove that $(S_t)_{t\geq 0}$ and $(S_{-t})_{t\geq 0}$ are semigroups on $B_{\infty,0}(\mathbb{D})$. To prove that $(S_t)_{t\geq 0}$ and $(S_{-t})_{t\geq 0}$ are semigroups, we verify for each case that

1. $S_o = \text{I}$ (Identity operator on $B_{\infty,0}(\mathbb{D})$).
2. Semigroup property: $S_{t+s} = S_t \circ S_s$
   and $S_{-t+-s} = S_{-t} \circ S_{-s}$ respectively for every $t, s \geq 0$.

Case 1: $(S_t)_{t\geq 0}$ is a semigroup.

Indeed $S_0 g(z) = (e^0)^{\alpha+2} g(e^0 z) = g(z)$ This implies that $S_0 = I$, the identity on $B_{\infty,0}(\mathbb{D})$.

We next proof the semigroup property as follows;

$$\begin{aligned} S_t(S_s g(z)) &= (e^{-it})^{\alpha+2}(S_s g(e^{-it}z)) \\ &= (e^{-it})^{\alpha+2}((e^{-is})^{\alpha+2}g(e^{-it}.e^{-is}z)) \\ &= (e^{-i(s+t)})^{\alpha+2}g(e^{-i(s+t)}z) \\ &= S_{(s+t)}g(z), \end{aligned}$$

for all $t, s \geq 0$, as desired.

Case 2: $(S_{-t})_{t\geq 0}$ is also a semigroup for $S_{-t}g(z) = (e^{it})^{\alpha+2}g(e^{it}z)$.

$S_{-0}g(z) = S_0 g(z) = g(z)$, since $S_0 = I$ as shown in Case 1. So, $S_{-0} = I$, the identity operator on $B_{\infty,0}(\mathbb{D})$.

For semigroup property, we have;

$$\begin{aligned} S_{-t}(S_{-s}g(z)) &= (e^{it})^{\alpha+2}(S_s g(e^{it}z)) \\ &= (e^{it})^{\alpha+2}((e^{is})^{\alpha+2}g(e^{it}e^{is})z) \\ &= (e^{i(s+t)})^{\alpha+2}g(e^{i(t+s)}z) \\ &= S_{(-t+-s)}g(z), \end{aligned}$$

for all $t, s \geq 0$, as desired.

Therefore $(S_t)_{t\in\mathbb{R}}$ is a group. □

Theorem 3.3.2

$(S_t)_{t\geq 0}$ is strongly continuous on $B_{\infty,0}(\mathbb{D})$.

PROOF. To prove that $(S_t)_{t\geq 0}$ is strongly continuous, we apply density of polynomials in $B_{\infty,0}(\mathbb{D})$, see Proposition 1.7.5. Therefore it suffices to show that for $(z^n)_{n\geq 0}$;

$$\lim_{t\to 0^+} \|S_t z^n - z^n\|_{B_{\infty,0}(\mathbb{D})} = 0.$$

Now, $S_t z^n - z^n = (e^{-i(\alpha+2+n)t} - 1)z^n$ so that;

$$\begin{aligned}\lim_{t\to 0^+} \|S_t z^n - z^n\|_{B_{\infty,0}(\mathbb{D})} &= \lim_{t\to 0^+} (\sup_{z\in\mathbb{D}}(1-|z|^2)|(S_t z^n - z^n)'|) \\ &= \lim_{t\to 0^+} \sup_{z\in\mathbb{D}}(1-|z|^2)|n(e^{-i(\alpha+2+n)t} - 1)z^{n-1}| \\ &= \lim_{t\to 0^+} \sup_{z\in\mathbb{D}}(1-|z|^2)|nz^{n-1}||e^{-i(\alpha+2+n)t} - 1|.\end{aligned}$$

And therefore $\lim_{t\to 0^+} \|S_t z^n - z^n\|_{B_{\infty,0}(\mathbb{D})} = 0$ as $t \to 0^+$, as desired. □

Theorem 3.3.3

The infinitesimal generator Γ of $(S_t)_{t\geq 0}$ is given by $\Gamma g(z) = -i(\gamma g(z) + zg'(z))$ with the domain $D(\Gamma) = \{g \in B_{\infty,0}(\mathbb{D}) : zg'(z) \in B_{\infty,0}(\mathbb{D})\}$.

PROOF. To obtain the infinitesimal generator Γ, we evaluate the limit as given in equation (1.9). For all $g \in B_{\infty,0}(\mathbb{D})$, we have

$$\begin{aligned}\Gamma g(z) &= \lim_{t\to 0^+} \frac{(e^{-it})^{\alpha+2} g(e^{-it}z) - g(z)}{t} \\ &= \frac{\partial}{\partial t}\left((e^{-it})^{\alpha+2} g(e^{-it}z)\right)\Big|_{t=0} \\ &= -i(\alpha+2)(e^{-it})^{\alpha+2} g(e^{-it}z) - iz(e^{-it})^{\alpha+2} g'(e^{-it}z)|_{t=0} \\ &= -i(\alpha+2)g(z) - izg'(z) \\ &= -i(\gamma g(z) + zg'(z)),\end{aligned}$$

which implies that $D(\Gamma) \subset \{g \in B_{\infty,0}(\mathbb{D}) : zg'(z) \in B_{\infty,0}(\mathbb{D})\}$. Conversely, let $g \in B_{\infty,0}(\mathbb{D})$ be such that $zg'(z) \in B_{\infty,0}(\mathbb{D})$, then for $z \in \mathbb{D}$

we have by the Fundamental theorem of Calculus,

$$\begin{aligned}
\frac{S_tg(z)-g(z)}{t} &= \frac{1}{t}\int_0^t \frac{\partial}{\partial s}(S_sg(z))ds \\
&= \frac{1}{t}\int_0^t \frac{\partial}{\partial s}((e^{-is})^\gamma g(e^{-is}z))ds \\
&= \frac{1}{t}\int_0^t (-i\gamma(e^{-is})^\gamma g(e^{-is}z) - iz(e^{-is})^\gamma g'(e^{-is}z))ds \\
&= \frac{1}{t}\int_0^t (e^{-is})^\gamma(-i\gamma g(e^{-is}z) - izg'(e^{-is}z))ds \\
&= \frac{1}{t}\int_0^t S_sG(z)ds,
\end{aligned}$$

where $G(z) = -i(\gamma)g(z) - izg'(z)$ is a function in $B_{\infty,0}(\mathbb{D})$. Thus $\lim_{t\to 0^+}\frac{S_tg-g}{t} = \lim_{t\to 0^+}\frac{1}{t}\int_0^t S_sGds$ and strong continuity of $(S_s)_{s\geq 0}$ implies that $\|\frac{1}{t}\int_0^t S_sGds - G\| \leq \frac{1}{t}\int_0^t \|S_sG - G\|ds \to 0^+$ as $t \to 0^+$. Thus $D(\Gamma) \supseteq \{g \in B_{\infty,0}(\mathbb{D}) : zg'(z) \in B_{\infty,0}(\mathbb{D})\}$, as desired. □

Theorem 3.3.4

$(S_t)_{t\in\mathbb{R}}$ is a strongly continuous group of isometries on $B_{\infty,0}(\mathbb{D})$

PROOF. Following Proposition 3.3.1 and Theorem 3.3.2, it remains to prove that for each $t \in \mathbb{R}$, the group $(S_t)_{t\in\mathbb{R}}$ is an isometry on $B_{\infty,0}(\mathbb{D})$. An isometry is a mapping between Banach spaces that preserves distances. Thus, it suffices to prove that for every $g \in B_{\infty,0}(\mathbb{D})$,

$$\|S_tg\|_{B_{\infty,0}(\mathbb{D})} = \|g\|_{B_{\infty,0}(\mathbb{D})}.$$

It follows from the definition that

$$\begin{aligned}
\|S_t g\|_{B_{\infty,0}(\mathbb{D})} &= |S_t g(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|(S_t g)'(z)| \\
&= |(e^{-it})^{\alpha+2} g(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|(e^{-it})^{\alpha+2} e^{-it} g'(e^{-it}z)| \\
&= |g(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|g'(e^{-it}z)|.
\end{aligned}$$

Now let $\omega = e^{-it}z$ so that $z = e^{it}\omega$ then;

$$\begin{aligned}
\|S_t g\|_{B_{\infty,0}(\mathbb{D})} &= |g(0)| + \sup_{\omega\in\mathbb{D}}\{(1-|e^{it}\omega|^2|g'(\omega)|)\} \\
&= |g(0)| + \sup_{\omega\in\mathbb{D}}\{(1-|\omega|^2)|g'(\omega)|\} \\
&= \|g\|_{B_{\infty,0}(\mathbb{D})}.
\end{aligned}$$

□

Lemma 3.3.5

Every isometry is injective.

PROOF. Let $\phi : S \to T$ be a non-injective isometry then there exists $x, y \in S$ such that $x \neq y$ and $\phi(x) = \phi(y)$. But $x \neq y \Rightarrow d_s(x,y) > 0$ while $\phi(x) = \phi(y) \Rightarrow d_T(\phi(x), \phi(y)) = 0$ which contradicts the notion that ϕ is an isometry. □

From the result of Theorem 3.3.3, we can deduce that $(S_t)_{t\in\mathbb{R}}$ is surjective isometry that is, $\|S_t g\|_{B_{\infty,0}\mathbb{D}} = \|g\|_{B_{\infty,0}(\mathbb{D})}$ for every $g \in B_{\infty,0}\mathbb{D}$. Precisely $(S_t)_{t\in\mathbb{R}}$ is invertible isometry since it is bijective (injective and surjective). Since $(S_t)_{t\in\mathbb{R}}$ is a strongly continuous group of invertible isometries, we can carry out a complete analysis of the spectral picture. This is done in the next chapter. We consequently state the following proposition.

Proposition 3.3.6

Let X be a non-zero Banach space and $T \in L(X)$ be an isometry. If T is invertible, then $\sigma(T) \subseteq \partial\mathbb{D}$ and $\partial\sigma(T) \subseteq \sigma_{ap}(T)$. If T is non-invertible isometry then $\sigma(T) = \overline{\mathbb{D}}$ and $\sigma_{ap}(T)$ lies on the boundary of the closed unit disc $\mathbb{D}$.

PROOF. See [1, Proposition 5.2]. □

Chapter 4

Spectral Properties on the Bloch space

In this chapter, we obtain the spectrum $\sigma(\Gamma)$, the point spectrum $\sigma_p(\Gamma)$ as well as the resolvent $R(\lambda, \Gamma)$ of the infinitesmal generator, Γ at λ. The resolvent is obtained as an integral operator. We also obtain the spectrum of the resolvent $\sigma(R(\lambda, \Gamma))$, point spectrum $\sigma_p(R(\lambda, \Gamma))$ and the spectral radius $r(R(\lambda, \Gamma))$ as well as the norm of the resulting resolvents. We complete this chapter by considering a specific automorphism of $\mathbb{U}$ on $B_\infty(\mathbb{D})$.

4.1 Multiplication operator

Definition 4.1.1

The multiplication operator M_z induced on $\mathcal{H}(\mathbb{D})$ by $z \in \mathbb{D}$ is defined by $M_z f := zf$ for all f in the domain $D(M_z) := \{f \in \mathcal{H}(\mathbb{D}) : zf \in \mathcal{H}(\mathbb{D})\}$

Definition 4.1.2

The left inverse of M_z on $\mathcal{H}(\mathbb{D})$ is defined by $Qf(z) = \frac{f(z)-f(0)}{z}$, $(Qf(0) =$

$f'(0))$.

More generally, if $Q^m f(z) = \sum_{k=m}^{\infty} \frac{f^k(0)}{k!} z^{k-m}$ and $Q^m f(0) = \frac{f^m(0)}{m!}$, then $M_z^m Q^m f = \sum_m^{\infty} \frac{f^k(0)}{k!} z^k$. Indeed from the definition it follows that;

$$\begin{aligned} Q^m f(z) &= \Sigma_{k=m}^{\infty} \frac{f^{(k)}(0)}{k!} z^{k-m} \\ &= \frac{f^{(m)}(0)}{m!} z^{m-m} + \frac{f^{(m+1)}(0)}{(m+1)!} z + \frac{f^{(m+2)}(0)}{(m+2)!} z^2 + \ldots \end{aligned}$$

Hence $Q^m f(0) = \frac{f^{(m)}(0)}{m!}$.

Now

$$\begin{aligned} M_z^m Q^m f(z) &= M_z^m \Sigma_{k=m}^{\infty} \frac{f^k(0)}{k!} z^{k-m} = z^m \Sigma_{k=m}^{\infty} \frac{f^k(0)}{k!} z^{k-m} \\ &= \Sigma_{k=m}^{\infty} \frac{f^k(0)}{k!} z^k. \end{aligned}$$

Thus $M_z^m Q^m f(z) = \Sigma_{k=m}^{\infty} \frac{f^k(0)}{k!} z^k$. Moreover, $Q^m M_z^m f = f$.

Proposition 4.1.3

1. *$M_z : B_\infty(\mathbb{D}) \to B_\infty(\mathbb{D})$ is bounded,*
2. *$M_z B_{\infty,0}(\mathbb{D}) \subseteq B_{\infty,0}(\mathbb{D})$,*
3. *$Q : B_{\infty,0}(\mathbb{D}) \to B_{\infty,0}(\mathbb{D})$ is bounded, and*
4. *For $m \geq 0$, $M_z^m B_{\infty,0}(\mathbb{D}) = \{f \in B_{\infty,0}(\mathbb{D}) \mid f^{(k)}(0) = 0 \, \forall k < m\}$. In particular, $M_z^m B_{\infty,0}(\mathbb{D})$ is closed in $B_{\infty,0}(\mathbb{D})$.*

PROOF. If $f \in B_\infty(\mathbb{D})$, then for all $z \in \mathbb{D}$,

$$\begin{aligned} \|M_z f\|_{B_\infty(\mathbb{D})} = \|zf(z)\|_{B_\infty(\mathbb{D})} &\leq \sup_{z\in\mathbb{D}} (1-|z|^2)|(zf)'(z)| \\ &= \sup_{z\in\mathbb{D}} (1-|z|^2)|zf'(z)| + (1-|z|^2)|f(z)|. \end{aligned}$$

Since $|z| < 1$ and using growth condition (1.6) then we have
$\|M_z f\|_{B_\infty} \leq \sup_{z\in\mathbb{D}}(1-|z|^2)|zf'(z)|+(1-|z|^2)\left(1+\frac{1}{2}\log\left(\frac{1+|z|}{1-|z|}\right)\right)\|f\|_{B_\infty(\mathbb{D})}$.
Hence $M_z f$ is bounded. This proves assertion 1.

Further

$$\lim_{z\to 1^-}(1-|z|^2)|zf'(z)|+(1-|z|^2)\left(1+\frac{1}{2}\log\left(\frac{1+|z|}{1-|z|}\right)\right)\|f\|_{B_\infty(\mathbb{D})} = 0,$$

thus we conclude that $M_z f \in B_{\infty,0}(\mathbb{D})$. Hence part 2 follows.
If $f \in B_{\infty,0}(\mathbb{D})$, then for $|z| < 1$,

$$\begin{aligned}(1-|z|^2)|(Qf(z))'| &= (1-|z|^2)\left|\frac{zf'(z)-f(z)+f(0)}{|z|^2}\right| \\ &\leq \frac{(1-|z|^2)|f'(z)|}{|z|^2}+\frac{(1-|z|^2)\left(1+\frac{1}{2}\log(\frac{1+|z|}{1-|z|})\right)\|f\|_{B_\infty(\mathbb{D})}}{|z|^2} \\ &\quad +\frac{(1-|z|^2)\|f\|_{B_\infty(\mathbb{D})}}{|z|^2},\end{aligned}$$

which tends to zero as $|z| \to 1$. Thus $Qf \in B_{\infty,0}(\mathbb{D})$.
Now, if $f \in B_{\infty,0}(\mathbb{D})$ and $f(0) = 0$, then $f = M_z Qf \in M_z B_{\infty,0}(\mathbb{D})$. The reverse inclusion is obvious.
Therefore the one-to-one and onto mapping M_z from $B_{\infty,0}(\mathbb{D})$ onto $\{f \in B_{\infty,0}(\mathbb{D}) \,|\, f(0) = 0\}$ is bounded. So the open mapping theorem 2.0.4, implies that the inverse is bounded. It therefore follows that $Q : \text{span}(1) \oplus M_z B_{\infty,0} \to B_{\infty,0}$ is bounded. □

4.2 Spectral properties

In this section, we obtain the spectrum, point spectrum and the resolvent of the infinitesimal generator Γ. We further obtain the resolvents as integral operator of which we determine the spectra, point spectra, spectral radius as well as the norm.

Theorem 4.2.1

1. $\sigma_p(\Gamma) = \sigma(\Gamma) = \{-i(\gamma+n) : n \in \mathbb{Z}_+\}$.

2. *If* $\lambda \in \rho(\Gamma)$, *then* $M_z^m B_{\infty,0}(\mathbb{D})$ *is* $R(\lambda,\Gamma)$ *invariant* $\forall m \in \mathbb{Z}_+$, $m > \Im(\lambda)$. *Moreover if* $h \in M_z^m B_{\infty,0}(\mathbb{D})$ *then*

$$R(\lambda,\Gamma)h(z) = -iz^m \int_0^1 t^{m+\gamma-i\lambda-1}(Q^m h)(tz)dt = \frac{-i}{z^{\gamma-i\lambda}} \int_0^z \omega^{\gamma-i\lambda-1}h(\omega)d\omega. \tag{4.1}$$

PROOF. We begin by proving that the point spectrum of Γ is given by $\sigma_p(\Gamma) = \{-i(\gamma+n) : n \in \mathbb{Z}_+\}$. For $\lambda \in \sigma_p(\Gamma)$ is equivalent to $\Gamma g = \lambda g$ for some $0 \neq g \in B_{\infty,0}(\mathbb{D})$. Substituting for $\Gamma g(z)$ we have

$$-i(\gamma g(z) + zg'(z)) = \lambda g(z)$$

which implies that

$$\gamma g(z) + zg'(z) = i\lambda g(z)$$

By simplifying we have

$$zg'(z) = (i\lambda - \gamma)g(z)$$

and therefore

$$\frac{dg}{dz} = \frac{i\lambda - \gamma}{z} g(z).$$

By integrating both sides we have

$$\int \frac{dg}{g} = \int \frac{i\lambda - \gamma}{z} dz.$$

Thus

$$\ln g(z) = (i\lambda - \gamma) \ln z + lnc$$

which implies $g(z) = Cz^{(i\lambda - \gamma)}$ But, $z^n \in \mathcal{H}(\mathbb{D})$ if and only if $n \in \mathbb{Z}_+$. Since polynomials are dense in the little Bloch space, $B_{\infty,0}(\mathbb{D})$, it therefore follows that $z^n \in B_{\infty,0}(\mathbb{D})$ if and only if $n \in \mathbb{Z}_+$. Thus $g \in B_{\infty,0}(\mathbb{D})$ if and only if $i\lambda - \gamma = n$, for $n \in \mathbb{Z}_+$ and therefore $\sigma_p(\Gamma) = \{\lambda \in \mathbb{C} : i\lambda - \gamma = n, n \in \mathbb{Z}_+\}$. By simplifying $i\lambda - \gamma = n$, we have $\lambda = -i(\gamma + n)$. Therefore, $\sigma_p(\Gamma) = \{-i(\gamma + n) : n \in \mathbb{Z}_+\}$, as desired

Next we take note that since S_t is an invertible isometry, its spectrum satisfies $\sigma(S_t) \subseteq \partial\mathbb{D}$. The spectral mapping theorem for strongly continuous groups (Theorem 2.0.2) says $e^{t\sigma(\Gamma)} \subseteq \sigma(S_t)$. Therefore if $\lambda \in \sigma(\Gamma)$, then $e^{t\lambda} \in \partial\mathbb{D}$ implies $|e^{t\lambda}| = 1$. Since $t \geq 0$ then $\lambda \in i\mathbb{R}$. And so, $\sigma(\Gamma) \subseteq i\mathbb{R}$.

We claim that infact $\sigma_p(\Gamma) = \sigma(\Gamma)$.

Fix $\lambda \in \mathbb{C}\backslash\sigma_p(\Gamma)$ and let $n \in \mathbb{Z}_+$. Since polynomials are dense in $B_{\infty,0}(\mathbb{D})$ then for arbitrary $h(z) = z^n$, $h(z) \in X(\mathbb{D})$, and by solving the resolvent equation $(\lambda - \Gamma)g = h(z)$ we have

$$(\lambda + i\gamma)g(z) + izg'(z) = h(z).$$

Multiplying both sides by iz^{-1} and rearranging we have

$$g'(z) - \frac{i(\lambda + i\gamma)}{z}g(z) = -iz^{-1}h(z)$$

which is equivalent to

$$(z^{\gamma - i\lambda}g(z))' = -iz^{\gamma - i\lambda - 1}h(z). \tag{4.2}$$

The equation $(\lambda - \Gamma)g = h(z)$ has unique solution which is obtained by integrating both sides of equation (4.2) and is given by

$$\begin{aligned} z^{-i(\lambda + i\gamma)}g(z) &= -i\int (z^{\gamma + n - i\lambda - 1})dz \\ &= \left(\frac{-i}{\gamma + n - i\lambda}\right) z^{\gamma + n - i\lambda}. \end{aligned}$$

This simplifies to

$$g(z) = \left(\frac{1}{\lambda + i(\gamma + n)}\right) z^n.$$

Note that for $\lambda \notin \sigma_p(\Gamma)$ and $g \in D(\Gamma)$,

$$\begin{aligned} (\lambda - \Gamma)g(0) &= \lambda g(0) - \Gamma g(0) \\ &= \lambda g(0) + i\gamma g(0) + 0g'(0) \\ &= (\lambda + i\gamma)g(0). \end{aligned}$$

More generally, if $g(z) = z^m f(z)$ with $f(0) \neq 0$ then

$$\begin{aligned}(\lambda - \Gamma)g(z) &= \lambda g(z) + i\gamma g(z) + izg'(z) \\ &= \lambda z^m f(z) + i\gamma z^m f(z) + iz(z^m f(z))' \\ &= \lambda z^m f(z) + i\gamma z^m f(z) + imz^m f(z) + iz^{m+1} f'(z) \\ &= z^m(\lambda f(z) + i\gamma f(z) + imf(z) + izf'(z)).\end{aligned}$$

$(\lambda - \Gamma)g$ and g have the same order of zero at 0. Thus $M_z^m B_{\infty,0}(\mathbb{D})$ is invariant under $\lambda - \Gamma$.

With $\lambda \in \mathbb{C}\backslash\sigma_p(\Gamma)$ and let $m < -\Im(\lambda + \gamma)$. If $h = z^m f$ and that $f \in B_{\infty,0}(\mathbb{D})$, then from equation 4.2 we have

$$-i\int_0^z \omega^{\gamma-i\lambda-1}h(\omega)d\omega = -iz^{m+\gamma-i\lambda}\int_0^1 t^{m+\gamma-i\lambda-1}f(tz)dt.$$

Thus $(\lambda - \Gamma)g = h$ has unique solution

$$g(z) = R(\lambda, \Gamma)h(z) = -iz^m \int_0^1 t^{m+\gamma-i\lambda-1}(Q^m h)(tz)dt \qquad (4.3)$$

as desired and can also be simplified to

$$\begin{aligned}R(\lambda, \Gamma)h(z) &= -i\int_0^1 t^{\gamma-i\lambda-1}(tz)^m f(tz)dt \\ &= -i\int_0^1 t^{\gamma-i\lambda-1}h(tz)dt.\end{aligned}$$

Let $\omega = tz$ then $\omega \to 0$ when $t \to 0$ and $\omega \to z$ when $t \to 1$, and so

$$g(z) = \frac{-i}{z^{\gamma-i\lambda}}\int_0^z \omega^{\gamma-i\lambda-1}h(\omega)d\omega.$$

If $u \in B_\infty(\mathbb{D})$ and $0 \leq t < 1$ then

$$\begin{aligned}\|u(tz)\|_{B_\infty(\mathbb{D})} &= \sup_{|z|<1}(1-|z|^2)t|u'(tz)| \\ &\leq \sup_{|z|<1}(1-t^2|z|^2)|u'(tz)| \\ &\leq \|u\|_{B_\infty(\mathbb{D})}.\end{aligned}$$

Thus

$$\|g(z)\| = \|-iz^m \int_0^1 t^{m+\gamma-i\lambda-1}(Q^m h)(tz)dt\| \leq \frac{1}{|m+i\lambda+\gamma|}\|M_z^m\|\|Q^m\|\|h\|. \tag{4.4}$$

Now, for all $m \geq 1$

$$B_{\infty,0}(\mathbb{D}) = \text{span}_{0\leq n<m}(z^n) \oplus M_z^m B_{\infty,0}(\mathbb{D}), \tag{4.5}$$

and

$$R(\lambda,\Gamma)|\text{span}_{0\leq n<m}(z^n) = \begin{pmatrix} \frac{1}{\lambda-i\gamma} & 0 & \cdots & 0 \\ 0 & \frac{1}{\lambda-i(\gamma+1)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\lambda-i(\gamma+m-1)} \end{pmatrix}$$

Therefore, for $\lambda \notin \sigma_p(\Gamma)$, the resolvent operator $R(\lambda,\Gamma)$ is bounded on $B_{\infty,0}(\mathbb{D})$ and so $\lambda \in \rho(\Gamma)$. Thus $\rho(\Gamma) = \mathbb{C}\backslash\sigma_p(\Gamma)$ and hence $\sigma_p(\Gamma) = \sigma(\Gamma)$ as claimed. □

In the next theorem, we obtain the point spectrum, spectrum and the spectral radius of the resolvent operator $R(\lambda,\Gamma)$

Theorem 4.2.2
$\sigma_p(R(\lambda,\Gamma)) = \sigma(R(\lambda,\Gamma)) = \left\{\omega : \left|\omega - \frac{1}{2\Re(\lambda)}\right| = \frac{1}{2\Re(\lambda)}\right\}$. Moreover, $r(R(\lambda,\Gamma)) = \frac{1}{|\Re(\lambda)|}$ and $\|R(\lambda,\Gamma)\| = \frac{1}{|\Re(\lambda)|}$.

PROOF. The spectral mapping theorem for resolvents asserts that $\sigma(R(\lambda,\Gamma))\backslash\{0\} = (\lambda - \sigma(\Gamma))^{-1} = \left\{\frac{1}{\lambda-\mu} : \mu \in \sigma(\Gamma)\right\}$. For $\lambda \in \rho(\Gamma)$. Then,

$$\sigma(R(\lambda,\Gamma)) = \left\{\frac{1}{\lambda + im} : m \in \mathbb{Z}_+\right\} \cup \{0\}$$

where $m = \gamma + n$. Let $\lambda = \Re(\lambda) + i\Im(\lambda)$ and substitute in the equation above to get

$$\sigma(R(\lambda,\Gamma)) = \left\{\frac{1}{\Re(\lambda) + i(m + \Im(\lambda))} : m \in \mathbb{Z}_+\right\}. \tag{4.6}$$

Rationalizing the denominator we get

$$\sigma(R(\lambda,\Gamma)) = \left\{\frac{\Re(\lambda) - i(m + \Im(\lambda))}{(\Re(\lambda))^2 + (m + \Im(\lambda))^2} : m \in \mathbb{Z}_+\right\}.$$

Let

$$\omega = \frac{\Re(\lambda) - i(m + \Im(\lambda))}{(\Re(\lambda))^2 + (m + \Im(\lambda))^2}$$

then subtracting $\frac{1}{2\Re(\lambda)}$ and simplifying we have

$$\begin{aligned}
\omega - \frac{1}{2\Re(\lambda)} &= \frac{\Re(\lambda) - i(m + \Im(\lambda))}{(\Re(\lambda))^2 + (m + \Im(\lambda))^2} - \frac{1}{2\Re(\lambda)} \\
&= \frac{2\Re(\lambda)(\Re(\lambda) - i(m + \Im(\lambda)) - [(\Re(\lambda))^2 + (m + \Im(\lambda))^2]}{2\Re(\lambda)[(\Re(\lambda))^2 + (m + \Im(\lambda))^2]} \\
&= \frac{[(\Re(\lambda))^2 - (m + \Im(\lambda))^2] - 2i\Re(\lambda)(m + \Im(\lambda))}{2\Re(\lambda)[(\Re(\lambda))^2 + (m + \Im(\lambda))^2]}
\end{aligned}$$

Finding the magnitude both sides and simplifying we get

$$\begin{aligned}\left|\omega - \frac{1}{2\Re(\lambda)}\right|^2 &= \left|\frac{[(\Re(\lambda))^2 - (m+\Im(\lambda))^2] - 2i\Re(\lambda)(m+\Im(\lambda)))}{2\Re(\lambda)[(\Re(\lambda))^2 + (m+\Im(\lambda))^2]}\right|^2 \\ &= \frac{[(\Re(\lambda))^2 - (m+\Im(\lambda))^2]^2 + 4(\Re(\lambda))^2(m+\Im(\lambda))^2}{4\Re(\lambda)^2[(\Re(\lambda))^2 + (m+\Im(\lambda))^2]^2}\end{aligned}$$

Since $[(\Re(\lambda))^2+(m+\Im(\lambda))^2]^2 = [(\Re(\lambda))^2-(m+\Im(\lambda))^2]^2+4(\Re(\lambda))^2(m+\Im(\lambda))^2$ then the equation above simplifies to

$$\left|\omega - \frac{1}{2\Re(\lambda)}\right|^2 = \frac{1}{|(2\Re(\lambda))|^2}.$$

That is, $\left|\omega - \frac{1}{2\Re(\lambda)}\right| = \frac{1}{|2\Re(\lambda)|}$.
Therefore, $\sigma(R(\lambda,\Gamma)) = \left\{\omega : \left|\omega - \frac{1}{2\Re(\lambda)}\right| = \frac{1}{|2\Re(\lambda)|}\right\}$.
Similary,

$$\begin{aligned}\sigma_p(R(\lambda,\Gamma)) &= (\lambda - \sigma_p(\Gamma))^{-1} = \left\{\frac{1}{\lambda-\mu} : \mu \in \sigma_p(\Gamma)\right\} \cup \{0\} \\ &= \left\{\omega : \left|\omega - \frac{1}{2\Re(\lambda)}\right| = \frac{1}{|2\Re(\lambda)|}\right\}.\end{aligned}$$

And therefore $\sigma_p(R(\lambda,\Gamma)) = \sigma(R(\lambda,\Gamma))$.

Next we prove that the spectral radius $r(R(\lambda,\Gamma)) = \frac{1}{|\Re(\lambda)|}$ and $\|R(\lambda,\Gamma)\| = \frac{1}{|\Re(\lambda)|}$.
By definition,

$$\begin{aligned}r(R(\lambda,\Gamma)) &= \sup\{|\omega| : \omega \in \sigma(R(\lambda,\Gamma))\} \\ &= \sup\left\{|\omega| : \left|\omega - \frac{1}{2\Re(\lambda)}\right| = \frac{1}{|2\Re(\lambda)|}\right\}\end{aligned}$$

which implies that

$$\begin{aligned} |\omega| &\leq \frac{1}{|2\Re(\lambda)|} + \frac{1}{|2\Re(\lambda)|} \\ &= \frac{1}{|\Re(\lambda)|}. \end{aligned}$$

Therefore, $r(R(\lambda, \Gamma)) = \sup\left\{|\lambda| \,:\, |\lambda| \leq \frac{1}{|\Re(\lambda)|}\right\} = \frac{1}{|\Re(\lambda)|}$
Finally, we prove that $\|R(\lambda, \Gamma)\| = \frac{1}{|\Re(\lambda)|}$.
Since the spectral radius of an operator is always bounded by its norm, we have $r(R(\lambda, \Gamma)) \leq \|R(\lambda, \Gamma)\|$ which implies

$$\frac{1}{|\Re(\lambda)|} = r(R(\lambda, \Gamma)) \leq \|R(\lambda, \Gamma)\|.$$

Hille-Yosida theorem (Theorem 2.0.3) asserts that

$$\|R(\lambda, \Gamma)\| \leq \frac{1}{|\Re(\lambda)|}.$$

Thus

$$\frac{1}{|\Re(\lambda)|} = r(R(\lambda, \Gamma)) \leq \|R(\lambda, \Gamma)\| \leq \frac{1}{|\Re(\lambda)|}$$

So, $\|R(\lambda, \Gamma)\| = \frac{1}{|\Re(\lambda)|}$. □

4.3 Analysis of a Specific Automorphism Group of the half-plane

From Theorem 3.1.3 assertion 3, we consider $f : \mathbb{U} \to \mathbb{D}$. As defined earlier by equations (3.2) and (3.3), let $f(z) = \psi^{-1}(z) = \frac{z-i}{z+i}$ and $f^{-1}(z) =$

$\psi(z) = \frac{i(1+z)}{1-z}$. For $k = 1$, let $u_t(z) = e^{-it}z$ with corresponding group of weighted composition operators defined by $S_{u_t}g(z) = (u_t')^\gamma g(u_t(z))$. Now

$$\begin{aligned}\varphi_t(z) = f^{-1}(e^{-it}f(z)) &= \frac{i(1+e^{-it}(\frac{z-i}{z+i}))}{1-e^{-it}(\frac{z-i}{z+i})} \\ &= \frac{i((z+i)+e^{-it}(z-i))}{(z+i)-e^{-it}(z-i)} \\ &= \frac{iz(1+e^{-it})-(1-e^{-it})}{(1-e^{-it})z+i(1+e^{-it})}.\end{aligned}$$

From Euler's formula we have

$$e^{it} = \cos t + i\sin t \tag{4.7}$$

and

$$e^{-it} = \cos t - i\sin t. \tag{4.8}$$

Adding equations (4.7) and (4.8) we have

$$2\cos t = (e^{it} + e^{-it})$$

which implies that

$$2\cos\frac{t}{2} = (e^{i\frac{t}{2}} + e^{-i\frac{t}{2}}) \tag{4.9}$$

Multiplying both sides of equation (4.9) by $e^{-i\frac{t}{2}}$ we have

$$2\cos\frac{t}{2}e^{-i\frac{t}{2}} = 1 + e^{-it}. \tag{4.10}$$

Subtracting equations (4.7) and (4.8) we have

$$2i\sin t = (e^{it} - e^{-it})$$

which implies that

$$2i\sin\frac{t}{2} = (e^{i\frac{t}{2}} - e^{-i\frac{t}{2}}). \tag{4.11}$$

Multiplying both sides of equation (4.11) by $e^{-i\frac{t}{2}}$ we have

$$2i\sin\frac{t}{2}e^{i\frac{t}{2}} = 1 - e^{-i\frac{t}{2}}. \tag{4.12}$$

Now from equations (4.10) and (4.12) we have

$$\begin{aligned}\varphi_t(z) &= \frac{iz(2\cos\frac{t}{2}e^{-i\frac{t}{2}}) - (2i\sin\frac{t}{2}e^{-i\frac{t}{2}})}{(2i\sin\frac{t}{2}e^{-i\frac{t}{2}})z + 2i\cos\frac{t}{2}e^{-i\frac{t}{2}}}\\ &= \frac{2zi\cos\frac{t}{2} - 2i\sin\frac{t}{2}}{2zi\sin\frac{t}{2} + 2i\cos\frac{t}{2}}\\ &= \frac{z\cos\frac{t}{2} - \sin\frac{t}{2}}{z\sin\frac{t}{2} + \cos\frac{t}{2}}.\end{aligned}$$

Therefore

$$\begin{aligned}\varphi_t(z) &= \frac{z\cos\frac{t}{2} - \sin\frac{t}{2}}{z\sin\frac{t}{2} + \cos\frac{t}{2}} = f^{-1}(e^{-it}f(z))\\ &= f^{-1}\circ u_t \circ f(z).\end{aligned}$$

In this section we obtain the generator, spectrum, point spectrum and the resolvent of the automorphism group given above in the following theorem;

Theorem 4.3.1

Let $\varphi_t \in Aut(\mathbb{U})$ be given by $\varphi_t(z) = \frac{z\cos\frac{t}{2}-\sin\frac{t}{2}}{z\sin\frac{t}{2}+\cos\frac{t}{2}}$ for all $t \in \mathbb{R}$, $z \in \mathbb{U}$, and the corresponding group of isometries in $B_{\infty,0}(\mathbb{D})$ by $S_{\varphi_t}g(z) = (\varphi_t')^{\gamma}g(\varphi_t(z))$. Then

1. *The infinitesimal generator Δ of the group $S_{\varphi_t} \subset B_{\infty,0}(\mathbb{U})$ is given*

by $-\gamma z h(z) - \frac{1}{2}(1+z^2)h'(z))$,
with the domain $D(\Delta) = \{h \in B_{\infty,0}(\mathbb{U}) \,:\, \gamma h(\omega) + \frac{1}{2}(\omega+i)h'(\omega) \in B_{\infty,0}(\mathbb{D})\}$.

2. $\sigma_p(\Delta) = \sigma(\Delta) = \{-i(\gamma+n) \,:\, n \in \mathbb{Z}_+\}$.

3. *If* $\mu \in \rho(\Delta) = \rho(\Gamma)$, *then for* $m \in \mathbb{Z}_+$, $m > \Im(-(\mu+i\gamma))$ *and* $h \in R(M_z^m)$, *we have*

$$R(\mu,\Delta)h(z) \;=\; -i\left(\frac{z-i}{(z+i)^{2\gamma}}\right)^{i\lambda+\gamma}\int_0^z (\omega-i)^{\gamma-i\lambda-1}(\omega+i)^{\gamma+i\lambda+1}h(\omega)d\omega$$

4. $\sigma(R(\mu,\Delta)) = \sigma_p(R(\mu,\Delta)) = \left\{\omega \in \mathbb{C} : \left|\omega - \frac{1}{2\Re(\mu)}\right| = \frac{1}{|2\Re(\mu)|}\right\}$. *Moreover,* $r(R(\mu,\Delta)) = \|R(\mu,\Delta)\| = \frac{1}{\Re(\mu)}$.

PROOF. Since $\varphi_t(z) = f^{-1}\circ u_t \circ f(z)$ it follows that $S_{\varphi_t} = S_f S_{u_t} S_{f^{-1}} = S_f S_{u_t} S_f^{-1}$. If Δ denotes the generator of S_{φ_t} and Γ be generator of S_{u_t}, then $\Delta = S_f \Gamma S_f^{-1}$ with domain $D(\Delta) = S_f D(\Gamma)$.
Let $g' \in B_{\infty,0}(\mathbb{D})$, then $g \in D(\Gamma)$ and define $h := S_f g$ belongs to $D(\Delta)$ with $g = S_f^{-1}(h)$. Then

$$\begin{aligned}\Delta(h(z)) &= S_f\Gamma S_f^{-1}h(z) = S_f\Gamma g(z)\\ &= S_f(-i\gamma g(z) - izg'(z))\\ &= (f'(z))^{\gamma}(-i\gamma g(f(z)) - if(z)g'(f(z))).\end{aligned}$$

As stated earlier $f(z) = \frac{z-i}{z+i}$ implying that $f'(z) = \frac{2i}{(z+i)^2}$ and thus;

$$\Delta(h(z)) = \frac{(2i)^{\gamma}}{(z+i)^{2\gamma}}\left(-i\gamma g(f(z)) - if(z)g'(f(z))\right). \tag{4.13}$$

Since $f^{-1}(z) = \frac{i(1+z)}{1-z}$ and $(f^{-1}(z))' = \frac{2i}{(1-z)^2}$, then we have $g(z) = S_f^{-1}h(z) = S_{f^{-1}}h(z) = \frac{(2i)^\gamma}{(1-z)^{2\gamma}}h(f^{-1}(z))$ implying that

$$g(f(z)) = \frac{(z+i)^{2\gamma}}{(2i)^\gamma}h(z). \tag{4.14}$$

Moreover,

$$\begin{aligned} g'(z) &= 2\gamma(2i)^\gamma(1-z)^{-2\gamma-1}h(f^{-1}(z)) + \frac{(2i)^\gamma}{(1-z)^{2\gamma}}\frac{2i}{(1-z)^2}h'(f^{-1}(z)) \\ &= \frac{(2i)^\gamma}{(1-z)^{2\gamma+2}}(2\gamma(1-z)h(f^{-1}(z)) + 2ih'(f^{-1}(z))) \end{aligned}$$

implying that

$$g'(f(z)) = \frac{(z+i)^{2\gamma+1}}{(2i)^{\gamma+1}}(2\gamma h(z) + (z+i)h'(z)). \tag{4.15}$$

Substituting equations (4.14) and (4.15) in equation (4.13) we have

$$\begin{aligned} \Delta(h(z)) &= \frac{(2i)^\gamma}{(z+i)^{2\gamma}}\left(-i\gamma\frac{(z+i)^{2\gamma}}{(2i)^\gamma}h(z) - i\frac{z-i}{z+i}\frac{(z+i)^{2\gamma+1}}{(2i)^{\gamma+1}}(2\gamma h(z) + (z+i)h'(z))\right) \\ &= -i\gamma h(z) - \gamma(z-i)h(z) - \frac{1}{2}(z-i)(z+i)h'(z) \\ &= -\gamma z h(z) - \frac{1}{2}(1+z^2)h'(z)). \end{aligned}$$

As stated earlier, the domain of Δ, $D(\Delta)$ is given by $D(\Delta) = S_f D(\Gamma) = \{S_f g : g \in D(\Delta)\}$. Now $h \in D(\Delta)$ implies that $S_f^{-1}h \in D(\Gamma)$ which implies that $(S_{f^{-1}}h)' \in B_{\infty,0}(\mathbb{D})$. But

$$\begin{aligned}(S_{f^{-1}}h)' &= \left(\frac{(2i)^\gamma}{(1-z)^{2\gamma}}h(f^{-1}(z))\right)' \\ &= 2\gamma(2i)^\gamma(1-z)^{-2\gamma-1}h(f^{-1}(z)) + \frac{(2i)^\gamma}{(1-z)^{2\gamma}}\frac{2i}{(1-z)^2}h'(f^{-1}(z)) \\ &= \frac{(2i)^\gamma}{(1-z)^{2\gamma}}\left(2\gamma(1-z)^{-1}h(f^{-1}(z)) + \frac{2i}{(1-z)^2}h'(f^{-1}(z))\right).\end{aligned}$$

Also

$$f \circ f^{-1}(z) = \frac{\frac{i(1+z)-i}{1-z} - i}{\frac{i(1+z)}{1-z} + i} = z, \tag{4.16}$$

then

$$(S_{f^{-1}}h)' = \frac{(2i)^\gamma}{(1-z)^{2\gamma}}\left(\frac{2\gamma}{1 - f \circ f^{-1}(z)}h(f^{-1}(z)) + \frac{2i}{(1 - f \circ f^{-1}(z))^2}h'(f^{-1}(z))\right) \tag{4.17}$$

By change of variables, let $\omega = f^{-1}(z) = \frac{i(1+z)}{1-z}$, then from equation (4.16) we have

$$f(\omega) = \frac{i\left(1 + \frac{z-i}{z+i}\right)}{1 - \frac{z-i}{z+i}} = \frac{i(z+i+z-i)}{z+i-z+i} = z = f \circ f^{-1}(z) \tag{4.18}$$

Substituting equation (4.18) in equation (4.17) we have

$$(S_{f^{-1}}h)' = S_{f^{-1}}\left(\frac{2\gamma}{1 - f(\omega)}h(\omega) + \frac{2i}{(1 - f(\omega))^2}h'(\omega)\right). \tag{4.19}$$

Therefore

$$\begin{aligned}h \in D(\Delta) &\Leftrightarrow S_{f^{-1}}\left(\frac{2\gamma}{1-f(\omega)}h(\omega) + \frac{2i}{(1-f(\omega))^2}h'(\omega)\right) \in B_{\infty,0}(\mathbb{D}) \\ &\Leftrightarrow \left(\frac{2\gamma}{1-f(\omega)}h(\omega) + \frac{2i}{(1-f(\omega))^2}h'(\omega)\right) \in B_{\infty,0}(\mathbb{D}).\end{aligned}$$

By change of variables again we have

$$f(\omega) = \frac{\omega - i}{\omega + i} \tag{4.20}$$

$\Rightarrow 1 - f(\omega) = \frac{\omega+i-\omega+i}{\omega+i} = \frac{2i}{\omega+i}$ hence

$$h \in D(\Delta) \Leftrightarrow -(\omega+i)(\gamma h(\omega) + \frac{1}{2}(\omega+i)h'(\omega)) \in B_{\infty,0}(\mathbb{D}),$$

which implies that $D(\Delta) = \left\{h \in B_{\infty,0}(\mathbb{D}) \,:\, \gamma h(\omega) + \frac{1}{2}(\omega+i)h'(\omega) \in B_{\infty,0}(\mathbb{D})\right\}$,

The spectrum and point spectrum of Δ are given as $\sigma_p(\Delta) = \sigma_p(\Gamma) = \sigma(\Gamma) = \sigma(\Delta) = \{-i(\gamma+n) \,:\, n \in \mathbb{Z}_+\}$.

For the resolvents, if $\mu \in \rho(\Delta) = \rho(\Gamma)$, then for $m \in \mathbb{Z}_+$, $m > \Im(-(\mu + i\gamma))$ and if $h \in R(M_z^m)$, we have $R(\mu, \Delta) = S_f R(\mu, \Gamma) S_f^{-1}$ and so

$$\begin{aligned}
R(\mu,\Delta)h(z) &= S_f\left(-iz^{i\lambda+\gamma}\int_0^z \omega^{\gamma-i\lambda-1}S_{f^{-1}}h(\omega)d\omega\right)\\
&= S_f\left(-iz^{i\lambda+\gamma}\int_0^z \omega^{\gamma-i\lambda-1}\frac{(2i)^\gamma}{(1-\omega)^{2\gamma}}h(f^{-1}(\omega))d\omega\right)\\
&= S_f\left(-iz^{i\lambda+\gamma}\int_0^z (f(\omega))^{\gamma-i\lambda-1}\frac{(2i)^\gamma}{(1-f(\omega))^{2\gamma}}h(\omega)\frac{df}{d\omega}d\omega\right)\\
&= -i\frac{(2i)^\gamma}{(z+i)^{2\gamma}}(g(z))^{i\lambda+\gamma}\int_0^z\left(\frac{\omega-i}{\omega+i}\right)^{\gamma-i\lambda-1}(2i)^\gamma\left(\frac{\omega+i}{2i}\right)^{2\gamma}h(\omega)d(\omega)\\
&= -i\left(\frac{z-i}{(z+i)^{2\gamma}}\right)^{i\lambda+\gamma}\int_0^z(\omega-i)^{\gamma-i\lambda-1}(\omega+i)^{\gamma+i\lambda+1}h(\omega)d\omega.
\end{aligned}$$

Finally, from Theorems 2.0.1 and 4.1.2 it follows that for all $\mu \in \rho(\Delta)$, the spectrum of $R(\mu, \Delta)$ is given by

$$\begin{aligned}\sigma(R(\mu,\Delta)) &= \left\{\frac{1}{\mu - z} : z \in \sigma(\Delta)\right\} \cup \{0\} \\ &= \left\{\frac{1}{\mu + i(\gamma + n)} : n \in \mathbb{Z}_+\right\} \cup \{0\} \\ &= \left\{\omega \in \mathbb{C} : \left|\omega - \frac{1}{2\Re(\mu)}\right| = \frac{1}{2\Re(\mu)}\right\}.\end{aligned}$$

Similarly, the point spectrum is given by

$$\begin{aligned}\sigma_p(R(\mu,\Delta)) &= \left\{\frac{1}{\mu - z} : z \in \sigma_p(\Delta)\right\} \cup \{0\} \\ &= \left\{\omega \in \mathbb{C} : \left|\omega - \frac{1}{2\Re(\mu)}\right| = \frac{1}{2\Re(\mu)}\right\}.\end{aligned}$$

Therefore, $\sigma(R(\mu,\Delta)) = \sigma_p(R(\mu,\Delta)) = \left\{\omega \in \mathbb{C} : \left|\omega - \frac{1}{2\Re(\mu)}\right| = \frac{1}{2\Re(\mu)}\right\}$.
Finally, we conclude this section by proving the spectral radius $r(R(\mu,\Delta)) = \frac{1}{|\Re(\mu)|}$ and $\|R(\mu,\Delta)\| = \frac{1}{|\Re(\mu)|}$.
From the spectrum and Theorem 4.2.2, it is clear that the spectrum of the resolvent is $r(R(\mu,\Delta)) = \frac{1}{|\Re(\mu)|}$. Hille-Yosida theorem yields $r(R(\mu,\Delta)) = \frac{1}{|\Re(\mu)|} \leq \|R(\mu,\Delta)\| \leq \frac{1}{|\Re(\mu)|}$.

□

Chapter 5

Summary and Recommendations

5.1 Summary

In this study, we applied the duality properties of the non-reflexive Bergman space, $L_a^1(\mathbb{D}, m_\alpha)$ to obtain a semigroup of weighted composition operators, $(S_t)_{t\geq 0}$ on the little Bloch space, $B_{\infty,0}(\mathbb{D})$. We proved that $(S_t)_{t\geq 0}$ is a strongly continuous group of isometries on $B_{\infty,0}(\mathbb{D})$ with infinitesimal generator given in Theorem 3.3.3.

Using the spectral theory of linear operators, we obtained the spectrum, $\sigma(\Gamma)$, point spectrum, $\sigma_p(\Gamma)$ and the resolvent of the infinitesimal generator in Theorem 4.1.1. The resolvent operator was given as an integral operator. Further, we proved that the spectrum and point spectrum of the infinitesimal generator are equal and they are set of points on the negative imaginary axis of the complex plane. Consequently, we obtained the spectrum and point spectrum of the resolvent operator as well as the spectral radius and the norm of this resolvent in Theorem 4.1.2. This thesis therefore completes our analysis of the adjoint composition semigroup on $L^p_a(\mathbb{D}, m_\alpha)$ for the case $p = 1$. The other case when $1 < p < \infty$ was considered in [11].

5.2 Recommendations

From the results of this study we recommend the following for further research;

1. In this thesis, we considered the groups of weighted composition operators corresponding to the self analytic maps of the rotation group on $B_{\infty,0}(\mathbb{D})$ and studied their semigroup and spectral properties. We therefore recommend the study of groups of weighted composition operators be extended to the scaling and translation groups on $B_{\infty,0}(\mathbb{D})$.

2. In this study, we completed our analysis of the adjoint composition group on the predual of the nonreflexive Bergman space $L^1_a(\mathbb{D}, m_\alpha)$. We recommend the study of groups of weighted composition operators be extended to other spaces of analytic functions like the Dirichlet spaces, Besov spaces, Lipschitz spaces among others.

3. In this study, we considered a specific automorphism of the unit disk $\mathbb{D}$ defined on $B_{\infty,0}(\mathbb{D})$. We recommend an extension of the the study to more general classes of automorphism of the upper half plane on the Bloch space.

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