GROUPS OF COMPOSITION OPERATORS ON DIRICHLET SPACES OF THE UPPER HALF-PLANE

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ABSTRACT

Composition groups has been a topic of interest in the past decades. Most studies have been on spaces of analytic functions of the unit disk. For instance; Matache studied boundedness and compactness of composition operators; Berkson and Porta delve on the structure of semigroups of functions and their basic properties, on Hardy and Bergman spaces of the disk. For the Dirichlet space of the unit disk, Siskakis proved strong continuity of semigroups and compactness of the resolvent operator. Bonyo undertook spectral analysis of certain groups of isometries on Hardy and Bergman spaces of the upper half plane. Little has been done on the Dirichlet space of the upper-half plane and this formed the basis of our study. In this thesis, we determined the composition groups induced by the scaling, translation and rotation groups; investigated both the semigroup as well as the spectral properties of each group on the Dirichlet space of the upper-half plane. To determine the composition groups, known definitions of weighted composition operators as well as the semigroup theory of linear operators on Banach spaces were used. To investigate the semigroup properties, the infinitesimal generators and their domains, the strong continuity property for each group were determined. For the rotation group, we applied the theory of similar semigroups to carry out a complete spectral analysis of the composition group as well as the resulting resolvents which were obtained as integral operators. The results of this study add reasonably to the existing literature and are useful in advancement of research in this area and in optimal control theory where integral equations and integral operators are usually applied.

Chapter 1

INTRODUCTION

1.1 Background of the study

Composition operators have been studied on a number of contexts, primarily on spaces of functions of the unit disk of the complex plane. It has long been understood that all such operators are bounded on the Hardy spaces of the unit disk as well as on a large class of other spaces of functions. Despite all these, there have been complications especially when dealing with composition operators on the half-plane due to the fact that it contains unbounded composition operators. In fact, Matache [17] proved that a composition operator is bounded on the Hardy space of the half- plane if and only if the inducing map fixes the point at infinity and if it has a finite angular derivative there. Later, Stagenga [27], sharpened the result and showed that in cases of bounded composition operators; the norm, the essential norm and spectral radius of the operator are equal to the root of the derivative, thus sharpening the result on non-compactness of composition operators done by Matache [17]. The foregoing being the case, Berkson and Porta [6] went a notch higher and delve on a new development called the semigroups of composition operators or simply composition groups. In their first study, they determined the structure of semigroups of functions and their basic properties in Hardy spaces. Motivated by his works with Porta on semigroups, Berkson [5] went ahead to prove that each composition operator induced by a inner function is isolated, in the operator-norm topology, from every other composition operator. Thereafter, Siskakis [23] extended the study of semigroups of composition operators on the unit disk $\mathbb{D}$ to Bergman spaces. Strong continuity of semigroups and identification of the infinitesimal generator was proved. Besides, Siskakis was able to show compactness and conditions for compactness of the resolvent operator [24]. Recently, a lot of research has been carried out on semigroups of composition operators with Siskakis [23] giving an extensive analysis of semigroups in Hardy and Bergman spaces. Arevalo and Oliva [1] went further to diagnose the strong continuity of semigroups of weighted composition operators in several analytic spaces. In view of all these, quite a magnitude and significant work has been done on composition operators on the Hardy and Bergman spaces of the unit disk $\mathbb{D}$ but little on composition operators on the Dirichlet space. This is the focus of this study. Before giving the statement of the problem, we give the following preliminaries:

1.1.1 Unit disk and Upper half plane

Let $\mathbb{C}$ be the complex plane. The set $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is called the open unit disk. Let dA denote the area measure on $\mathbb{D}$, normalized so that the area of $\mathbb{D}$ is 1. In terms of rectangular and polar co-ordinates, we have

$dA = \frac{1}{\pi} dxdy = \frac{r}{\pi} drd\theta$ where $z = x + iy = re^{i\theta} \in \mathbb{D}$. For $\alpha \in \mathbb{R}$, $\alpha > -1$, we define a weighted measure dm_{α} on $\mathbb{D}$ by $dm_{\alpha}(z) = (1 - |z|^2)^{\alpha} dA(z)$. Moreover, if $\alpha = 0$, then $dm_0(z) = dA(z)$. Thus, we consider dm_{α} as a weighted measure and a generalization of dA.

On the other hand, the set $\mathbb{U} := \{w \in \mathbb{C} : Im(w) > 0\}$ denotes the upper half of the complex plane $\mathbb{C}$, where $Im(w)$ is the imaginary part of a complex number w. For $\alpha > -1$, we define a weighted measure on $\mathbb{U}$ by $d\mu_{\alpha}(w) = (Im(w))^{\alpha} dA(w)$, for each $w \in \mathbb{U}$. Again, it can easily be seen that $\alpha = 0$ coincides with the unweighted measure. The function $\psi(z) = \frac{i(1+z)}{1-z}$ is referred to as the Cayley transform and maps the unit disk $\mathbb{D}$ conformally onto the upper half-plane $\mathbb{U}$ with the inverse $\psi^{-1}(w) = \frac{w-i}{w+i}$ mapping $\mathbb{U}$ onto $\mathbb{D}$. See [12, 19, 30] for details.

1.1.2 Analytic functions

A function $f(z)$ is *analytic* if and only if it is holomorphic (that is, it is complex differentiable within a neighborhood of all points in the domain of f). An analytic self-map is an analytic mapping $f : \mathbb{U} \to \mathbb{U}$, i.e, a self-map on $\mathbb{U}$ is a mapping from $\mathbb{U}$ onto itself.

Two (finite) sets are *isomorphic* if they have the same number of elements or two sets are isomorphic if you can write down a function that assigns each element of one set to another unique element of the other set, such that no elements of either set are missed by the function. More generally, an *isomorphism* is a map that preserves sets and relation among elements. An *automorphism* is an isomorphism from a mathematical object to itself. It is in some sense a symmetry of the object and a way of mapping the object to itself while preserving all its structure. The set of all automorphisms forms a group under composition of morphism called automorphism group and denoted by $Aut(\mathbb{U})$. A linear fractional transformation is a function of the form;

$$f(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{R}$. If $ad - bc \neq 0$, then f is called a *Möbius* transformation. On the other hand, a *homomorphism* is a map between two groups which preserves the group structure, e.g let G_1 and G_2 be two groups and f a map from G_1 to G_2 (for every $g \in G_1$, $f(g) \in G_2$). Then f is a homomorphism if for every $g_1, g_2 \in G_1$ we have $f(g_1 g_2) = f(g_1)f(g_2)$.
Let $D \subset \mathbb{C}$ be open and the function $\phi : D \to \mathbb{C}$ be *holomorphic* and one-to-one. Let $D' := Im(\phi) \in \mathbb{C}$ (hence D' is also open in $\mathbb{C}$). If $\phi^{-1} : D' \to D$ is *holomorphic* then ϕ is said to be *biholomorphic.*
In general if Ω_1 and Ω_2 are open subsets of $\mathbb{C}$, then a *holomorphic* map $f : \Omega_1 \to \Omega_2$ that is one-to-one and onto is called a *biholomorphism* from Ω_1 to Ω_2. A *biholomorphism* from Ω_1 to Ω_2 is called an *automorphism* of Ω_1. The set of all *automorphisms* of Ω_1 is denoted by $Aut(\Omega_1)$.

1.1.3 Banach spaces of Analytic functions

Hardy spaces

For an open subset Ω of $\mathbb{C}$, let $\mathcal{H}(\Omega)$ denotes the space of analytic functions on Ω.
For $1 \leq p < \infty$, the Hardy spaces of the upper half-plane $H^p(\mathbb{U})$ are

defined as;

$$H^p(\mathbb{U}) := \left\{ f \in \mathcal{H}(\mathbb{U}) : \|f\|_{H^p(\mathbb{U})} = \sup_{y>0} \left(\int_{-\infty}^{\infty} |f(x+iy)|^p dx \right)^{\frac{1}{p}} < \infty \right\},$$

and for $p = \infty$, $H^\infty(\mathbb{U})$ is the space of bounded analytic functions on $\mathbb{U}$ with supremum norm. If $f \in H^p(\mathbb{U})$, then f satisfies the growth condition;

$$|f(z)| \leq \frac{C_p\|f\|_p}{(Im(z))^\gamma},\ z \in \mathbb{U},\ C_p \text{ is a constant, } \gamma = \frac{1}{p}.$$

The Hardy spaces of the unit disk are defined by;

$$H^p(\mathbb{D}) := \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|^p_{H^p(\mathbb{D})} = \sup_{0<r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta < \infty \right\}.$$

$H^p(\cdot)$ is a Banach space with respect to the norm $\|\cdot\|_{H^p(\cdot)}$. Infact $H^2(\cdot)$ is a Hilbert space. We refer to [5, 7, 12] for details.

Bergman spaces

For $1 \leq p < \infty$, $\alpha > -1$, the weighted Bergman spaces of the unit disk $\mathbb{D}$, $L^p_a(\mathbb{D}, m_\alpha)$, are defined by;

$$L^p_a(\mathbb{D}, m_\alpha) := \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_{L^p_a(\mathbb{D},m_\alpha)} = \left(\int_{\mathbb{D}} |f(z)|^p dm_\alpha(z) \right)^{\frac{1}{p}} < \infty \right\},$$

while the weighted Bergman space $L^p_a(\mathbb{U}, \mu_\alpha)$ of the upper half plane is given as;

$$L^p_a(\mathbb{U}, \mu_\alpha) := \left\{ f \in \mathcal{H}(\mathbb{U}) : \|f\|_{L^p_a(\mathbb{U},\mu_\alpha)} = \left(\int_{\mathbb{U}} |f(\omega)|^p d\mu_\alpha(\omega) \right)^{\frac{1}{p}} < \infty \right\},$$

where $d\mu_\alpha$ is the weighted measure on $\mathbb{U}$ defined by $d\mu_\alpha(\omega) = (Im(\omega))^\alpha dm(\omega)$, $\omega \in \mathbb{U}$. If $f \in L^p_a(\mathbb{U}, \mu_\alpha)$, then f satisfies the growth condition;

$$|f(z)| \leq \frac{C_p\|f\|_p}{(Im(w))^\gamma},\ \gamma = \frac{\alpha+2}{p},\ w \in \mathbb{U},\ C_p \text{ is a constant.}$$

Take note that $L^p_a(\cdot)$ for $1 \leq p < \infty$ is a Banach space with respect to the norm $\|\cdot\|_{L^p_a(\cdot)}$ and that $L^2_a(\cdot)$ is a Hilbert space. See [5, 19, 30] for details.

Dirichlet spaces

For $\alpha \geq 0$, the weighted Dirichlet space of the unit disk, $\mathcal{D}_\alpha(\mathbb{D})$ consists of those analytic functions f on $\mathbb{D}$, $f \in \mathcal{H}(\mathbb{D})$, such that

$$\mathcal{D}_\alpha(\mathbb{D}) := \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|^2_{\mathcal{D}_{\alpha,1}(\mathbb{D})} = \int_{\mathbb{D}} |f'(z)|^2 dm_\alpha(z) < \infty \right\}$$

with the norm given as:

$$\|f\|_{\mathcal{D}\alpha(\mathbb{D})} = \left(|f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dm_\alpha(z) \right)^{\frac{1}{2}} < \infty.$$

The corresponding weighted Dirichlet space of the upper half-plane $\mathbb{U}$, $\mathcal{D}_\alpha(\mathbb{U})$, consists of those analytic functions f on $\mathbb{U}$, $f \in \mathcal{H}(\mathbb{U})$, such that

$$\mathcal{D}_\alpha(\mathbb{U}) := \left\{ f \in \mathcal{H}(\mathbb{U}) : \|f\|^2_{\mathcal{D}_{\alpha,1}(\mathbb{U})} = \int_{\mathbb{U}} |f'(w)|^2 d\mu_\alpha(w) < \infty \right\}$$

with the norm given as:

$$\|f\|_{\mathcal{D}\alpha(\mathbb{U})} = \left(|f(i)|^2 + \int_{\mathbb{U}} |f'(w)|^2 d\mu_\alpha(w) \right)^{\frac{1}{2}} < \infty.$$

The pair $(\mathcal{D}_\alpha(\cdot), \|\cdot\|_{\mathcal{D}_\alpha(\cdot)})$ or simply $\mathcal{D}_\alpha(\cdot)$ is a Banach space with respect to the norm $\|\cdot\|_{\mathcal{D}_\alpha}$. Moreover, $\mathcal{D}_\alpha(\cdot)$ is a Hilbert space, where the inner product is defined on $\mathbb{D}$ by

$$\langle f, g\rangle = \langle f(0), g(0)\rangle + \int_{\mathbb{D}} f'(z)\overline{g'(z)}dm_\alpha(z).$$

and on the upper half plane as

$$\langle f, g\rangle = \langle f(i), g(i)\rangle + \int_{\mathbb{U}} f'(w)\overline{g'(w)}d\mu_\alpha(w).$$

For $f \in \mathcal{D}_\alpha(\mathbb{D})$, then f satisfies the growth condition:

$$|f(\omega)| \leq c\|f\|\sqrt{\log\frac{1}{1-|\omega|^2}}.$$

Very little is known about the Duirichlet space of the upper half plane $\mathcal{D}_\alpha(\mathbb{U})$ as opposed to the one on the unit disk $\mathcal{D}_\alpha(\mathbb{D})$. For instance, the growth condition for $\mathcal{D}_\alpha(\mathbb{U})$ is not well captured in literature. We refer to [9, 13, 15, 18, 25] for a comprehensive theory of Dirichlet spaces of the unit disk.

1.1.4 Spectra of linear operators

Let X and Y be Banach spaces. The space $\mathcal{L}(X, Y) = \{T : X \to Y$ such that T is linear and continuous$\}$ endowed with the operator norm $\|T\| = \sup_{\|x\|\leq 1}\|Tx\|$, turns out to be a Banach space. We shall write $\mathcal{L}(X)$ instead of $\mathcal{L}(X, X)$.

A linear operator $T : X \to Y$ is a linear map T defined on a linear subspace dom(T), that is, $T : \mathrm{dom}(T) \subseteq X \to Y$.

T is said to be a *closed operator* if its graph $\mathcal{G}(T) = \{(x, Tx) : x \in \mathrm{dom}(T)\}$ is closed in $X \times Y$, that is, for $x_n, x \in X, y \in Y$ such that $x_n \to x$ in X and $Tx_n \to y$ in Y, then we have that $Tx = y$. We refer to [11, 22] for details.

An operator $T : X \to Y$ is continuous at $x \in \mathrm{dom}(T)$ if $(x_n)_n \subset \mathrm{dom}(T)$ and $x_n \to x$ in X. This implies that $Tx_n \to Tx$ in Y. The operator T is said to be *continuous* if it is continuous over its domain dom(T). Strong continuity implies continuity but continuity does not necessarily imply strong continuity. See [11, 22] for details.

Theorem 1.1.1 (Closed graph theorem)

Let X and Y be Banach spaces. Then every closed linear mapping $T : X \to Y$ is continuous.

Let T be a closed operator on X. The *resolvent set* of T, $\rho(T)$ is given by $\rho(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is invertible }\}$ and its *spectrum* $\sigma(T) = \mathbb{C} \setminus \rho(T)$. Therefore, $\sigma(T) \cup \rho(T) = \mathbb{C}$. The *spectral radius* of T is defined by $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$ with the relation $r(T) \leq \|T\|$. The *point spectrum*, $\sigma_p(T) = \{\lambda \in \mathbb{C} : Tx = \lambda x \text{ for some } 0 \neq x \in \mathrm{dom}(T)\}$. For $\lambda \in \rho(T)$, the operator $R(\lambda, T) := (\lambda I - T)^{-1}$ is, by closed graph theorem, a bounded operator on X and is called the resolvent of T at the point λ or simply resolvent operator.

1.1.5 Semigroup theory of linear operators

Let X be a Banach space. A one-parameter family $(T_t)_{t\geq 0} \subset \mathcal{L}(X)$ is a *semigroup of bounded linear operators on* X if it satisfies the following two properties;

(i) $T_0 = I$, the identity operator on X, and

(ii) $T_{t+s} = T_t \circ T_s$ for every $t, s \geq 0$, (Semigroup property).

A semigroup $(T_t)_{t\geq 0}$ of bounded linear operators on X is *strongly continuous* if $\lim_{t\to 0^+}(T_t)x = x$ or $\lim_{t\to 0^+} \|T_t x - x\| = 0$ for all $x \in X$.

The *infinitesimal generator* Γ of a strongly continuous semigroup $(T_t)_{t\geq 0}$ is defined by

$$\Gamma x = \lim_{t\to 0^+} \frac{T_t x - x}{t} = \frac{\partial}{\partial t}(T_t x)\Big|_{t=0} \quad \text{for each } x \in \operatorname{dom}(\Gamma)$$

, where the domain of Γ, $\operatorname{dom}(\Gamma)$ is given by

$$\operatorname{dom}(\Gamma) = \left\{ x \in X : \lim_{t\to 0^+} \frac{T_t x - x}{t} \text{ exists} \right\}.$$

See [12, 19] for details.

1.1.6 Similar semigroups

If X and Y are arbitrary Banach spaces and $U \in \mathcal{L}(X, Y)$ is an invertible operator, then clearly $(A_t)_{t\in\mathbb{R}} \subset \mathcal{L}(X)$ is a strongly continuous group if and only if $B_t := U A_t U^{-1}$, $t \in \mathbb{R}$, is a strongly continuous group in $\mathcal{L}(Y)$. In this case, if $(A_t)_{t\in\mathbb{R}}$ has a generator Γ, then $(B_t)_{t\in\mathbb{R}}$ has generator

$\Delta = U\Gamma U^{-1}$ with domain

$$dom(\Delta) = Udom(\Gamma) := \{Ux \in Y : x \in dom(\Gamma)\}.$$

Moreover,

$$\sigma_p(\Delta, Y) = \sigma_p(\Gamma, X)$$

and

$$\sigma(\Delta, Y) = \sigma(\Gamma, X).$$

Therefore, if λ is in the resolvent set $\rho(\Gamma, X) := \mathbb{C} \setminus \sigma(\Gamma, X)$, we have that $R(\lambda, \Delta) = UR(\lambda, \Gamma)U^{-1}$. For more details we refer to [4] and [7].

1.1.7 Composition groups

For $t \in \mathbb{R}$, consider analytic self-maps $\varphi_t : \mathbb{U} \to \mathbb{U}$. The groups of composition operators induced by $(\varphi_t)_{t\geq 0}$ on $\mathcal{H}(\mathbb{U})$are defined by

$$C_{\varphi_t} f = f \circ \varphi_t \text{ for all } f \in \mathcal{H}(\mathbb{U}).$$

The corresponding groups of weighted composition operators $T_t = S_{\varphi_t}$ associated with φ_t will be defined on $\mathcal{H}(\mathbb{U})$ as

$$T_t f(z) = S_{\varphi_t} f(z) = (\varphi_t'(z))^\gamma (f \circ \varphi_t)(z),$$

where γ is an appropriately chosen weight and $f \in \mathcal{H}(\mathbb{U})$. We refer to [26] for comprehensive study on composition groups.

1.2 Statement of the problem

Groups of composition operators on the spaces of analytic functions have been studied in detail majorly on Hardy and Bergman spaces and on the Dirichlet spaces of the unit disk $\mathbb{D}$. However, the corresponding study on the Dirichlet spaces of the upper half-plane $\mathbb{U}$ has not been done and this formed the basis of our study. In this thesis therefore, we have determined the groups of weighted composition operators induced by the automorphism groups of the upper half plane, and investigated both the semigroup and spectral properties of the obtained composition groups on the Dirichlet space of the upper half plane.

1.3 Objective of the study

The main objective of this study was to determine the groups of composition operators and investigate their properties on the Dirichlet space of the upper half-plane.

The specific objectives of the study were:

1. To determine the groups of composition operators induced by the scaling, translation and the rotation groups of automorphisms of the upper half plane.

2. To investigate the semigroup properties of each of the groups of composition operators determined above.

3. To investigate the spectral properties of each of the groups of composition operators determined above.

1.4 Significance of the study

The study of composition operators have been extensively carried out specifically on the spaces of analytic functions of the unit disk $\mathbb{D}$ but little as far as the upper-half plane $\mathbb{U}$ is concerned. The determination of composition operators corresponding to each automorphism group of the upper half plane together with the investigation of their semigroup as well as their spectral properties will add reasonably to the available literature and therefore will be useful to pure mathematicians in the advancement of research in this area. Besides, the results may also be useful to applied mathematicians especially in optimal control theory where integral equations and integral operators are widely applied.

1.5 Research methodology

To determine the groups of composition operators corresponding to scaling, translation or rotation groups, we used the definition of the weighted composition operator on the Dirichlet space as well as the approaches employed in related works by mathematicians such as Siskakis, Bonyo, Ballamoole, T. Miller and V. Miller. As far as the investigation of the properties of the composition operators for the scaling and translation group is concerned, we first obtained the infinitesimal generators of each of composition operator group together with their domains; then established the strong continuity property of each on the Dirichlet space of the upper- half plane $\mathbb{U}$. All these were based on the known definitions. For the rotation group, we carried out a complete spectral analysis using the theory of similar semigroups of composition operators on the Dirichlet spaces and those on the Bergman spaces which had been earlier considered.

Chapter 2

LITERATURE REVIEW

The study of composition operators on the Dirichet space of the disk was started in ernest by Siskakis [25] where he studied the semigroups of composition operators on the Dirichlet space with the identification of the infinitesimal generator. In the paper, it was proved that such semigroups are strongly continuous. Subsequently, Ross [21] in his survey paper presented a selection of results concerning the class of analytic functions f on the open unit disk $\mathbb{D}$ which have finite Dirichlet integral. In particular, the paper covered the basic structure of these functions - their basic boundary values, and their zeros along with important operators that act on this space of functions.

As the study of Dirichlet spaces gained momentum, Martin and Vukotic in [15], obtained the new upper bounds of the norms of the univalent induced composition operator acting on the Dirichlet space and were able to compute explicitly the norms for univalent symbols whose range is the disk minus a set of measure zero. They were able to show that the spectral radius of induced composition operators on Dirichlet space is equal to one.

Consequently, Charcon, Gerardo and Gimerez in [9] having been impressed by Maria and Vukotic work in [15] went further to investigate the question as to when a bounded analytic function φ on the unit disk $\mathbb{D}$, fixing 0, is such that the family $\varphi^n : n = 0, 1, 2, ...$ is orthogonal in the Dirichlet space $\mathcal{D}$. They also considered the problem of characterizing the self-maps φ on $\mathbb{D}$ in terms of the norm of the induced composition operator $C_\varphi : \mathcal{H}(\mathbb{D}) \to \mathcal{H}(\mathbb{D})$. Besides, Pau and Perez in [18] went further to study composition operators acting on weighted Dirichlet spaces and obtained estimates for the essential norm, described the membership in schatten- von Neumann ideals and characterized the composition operators. Duistemaat and Young [10] in their survey paper delved into the properties of Toeplitz operators acting on Dirichlet spaces. More specifically, they characterised two harmonic symbols of commuting Toeplitz operators. In addition to this paper, Richeter and Sundberg [20] provided a complete characterisation of strongly continuous semigroups on Hardy and Dirichlet spaces with a prescribed generator of the form $\Gamma f = Gf'$ where it was proved that such semigroups are semigroups of composition operators and gives simple sufficient and necessary conditions on G. Moreover, Stylogiannis [28] in his recent works studied strong continuity of semigroups of composition operators on local Dirichlet space.

Recently, Bonyo et al [4] went ahead to classify all the one-parameter groups of automorphisms of $\mathbb{U}$ into three distinct classes namely; scaling, translation and rotation groups according to the location of their fixed points. Besides, Bonyo in [7], obtained the resolvents of the generators of strongly continuous groups of isometries on the Hardy and Bergman spaces which were obtained as weighted composition operators associated with specific groups of automorphisms of the upper-half plane.

From the foregoing, detailed emphasis have been given on composition operators on Dirichlet spaces especially on the unit disk $\mathbb{D}$. However, the corresponding study on the Dirichlet spaces of the upper half-plane $\mathbb{U}$ is much less complete and this will be the focus of our study. Specifically, we are going to determine the composition groups corresponding to the scaling group, translation group and rotation group and further investigate both the semigroup as well as the spectral properties of each composition operator. The following results will be used in this study:

Theorem 2.0.1 (Hille-Yosida theorem)

Let X be a Banach space. A linear operator Γ is the infinitesimal generator of a strongly continuous semigroup of contractions $(T_t)_{t\geq 0} \subseteq \mathcal{L}(X)$ if and only if;

1. *Γ is closed and $\overline{dom(T)} = X$*

2. *The resolvent set $\rho(\Gamma)$ of Γ contains $\mathbb{R}^+$ and for every $\lambda \geq 0$*

$$\|R(\lambda, \Gamma)\| \leq \frac{1}{\lambda}$$

   In this case, if $h \in X$, then

$$R(\lambda, \Gamma)h = \int_0^{\infty} e^{-\lambda t} T_t h dt$$

   is norm convergent.

We refer to [11, 12, 19, 22] for details.

Theorem 2.0.2 (Spectral mapping theorem for resolvents)
Let $(T_t)_{t\geq 0}$ be a closed operator on X and $\lambda \in \rho(\Gamma)$. Then;

$$\begin{aligned}\sigma(R(\lambda,\Gamma)) \setminus \{0\} &= (\lambda - \sigma(\Gamma))^{-1} \\ &= \left\{\frac{1}{\lambda-\mu} : \mu \in \sigma(\Gamma)\right\}.\end{aligned}$$

For details see [12, 19, 22].

Theorem 2.0.3 (Spectral mapping theorem for semigroups)
Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup on Banach space X and Γ be its infinitesimal generator. Then

$$\sigma(T_t) \supset e^{t\sigma(\Gamma)}.$$

For the point spectrum,

$$e^{t\sigma_p(\Gamma)} = \sigma_p(T_t).$$

See [12, 19] for details.

Theorem 2.0.4 (Fatou's lemma)
Let $(X, \mathcal{A}, \mu)$ be a measure space and suppose $f_n : X \to [0, \infty)$ is measurable for all $n \in \mathbb{N}$, then;

$$\int_X \liminf f_n d\mu \leq \liminf \int_X f_n d\mu$$

We refer to [12, 19] for details.

Theorem 2.0.5 (Compact resolvent)
Let X be a Banach space and T be a closed operator on X. If T has a

compact resolvent, then

$$\sigma(T) = \sigma_p(T).$$

See [12, Corollary V.1.15].

Chapter 3

SCALING AND TRANSLATION GROUPS

In this chapter, we consider composition groups induced by the automorphisms corresponding to the scaling and translation groups, and investigate both their semigroup and spectral properties on the Dirichlet space of the upper half-plane.

3.1 Self - Analytic maps on $\mathbb{U}$

A function φ is called a *self-analytic map* on $\mathbb{U}$ if it is analytic on $\mathbb{U}$ and that $\varphi(\mathbb{U}) \subset \mathbb{U}$.These are simply the automorphisms of $\mathbb{U}$. Recall from section 1.1.2 that an automorphism is a biholomorphism from a set onto itself. The set of all automorphisms of an open set $\Omega \subseteq \mathbb{C}$ is denoted by $Aut(\Omega)$. Clearly $Aut(\Omega)$ forms a group under composition as we proof in the following proposition;

Proposition 3.1.1

For every nonempty open set Ω in $\mathbb{C}$, $Aut(\Omega)$ forms a group under the *operation of composition of the maps.*

PROOF. Clearly, $Aut(\Omega)$ contains the identity map and hence is not empty. The composition of homomorphisms is a homomorphism, and the composition of bijections is a bijection.Therefore, the composition of isomorphisms is an isomorphism, and in particular, the composition of automorphisms is an automorphism. Hence, composition is a well defined binary operation on $Aut(\Omega)$.

Composition of functions is always associative. The identity map is an automorphism of Ω. Finally, an isomorphism has an inverse which is an isomorphism; so the inverse of an automorphism of Ω exists and is an automorphism of Ω. □

All self analytic maps of $\mathbb{U}$ ($Aut(\mathbb{U})$) were identified and classified in [4] into three distinct groups according to the location of their fixed points, namely; scaling, translation and rotation groups. Specifically we give the following theorem:

Theorem 3.1.2

Let $\varphi : \mathbb{R} \to Aut(\mathbb{U}); t \mapsto \varphi_t$ be a nontrivial continuous group homomorphism. Then exactly one of the following cases holds:

1. *There exists $k > 0$, $k \neq 1$, and $g \in Aut(\mathbb{U})$ so that $\varphi_t(z) = g^{-1}(k^t g(z))$ for all $z \in \mathbb{U}$ and $t \in \mathbb{R}$.*

2. *There exists $k \in \mathbb{R}$, $k \neq 0$, and $g \in Aut(\mathbb{U})$ so that $\varphi_t(z) = g^{-1}(g(z) + kt)$ for all $z \in \mathbb{U}$ and $t \in \mathbb{R}$.*

3. *There exists $k \in \mathbb{R}$, $k \neq 0$, and a conformal mapping g of $\mathbb{U}$ onto $\mathbb{D}$ such that $\varphi_t(z) = g^{-1}(e^{ikt} g(z))$ for all $z \in \mathbb{U}$ and $t \in \mathbb{R}$. Equiva-*

lently, there exist $\theta \in \mathbb{R} \setminus 0$ and $h \in Aut(\mathbb{U})$ so that:

$$\varphi_t(z) = h^{-1}\left[\frac{h(z)cos(\theta t) - sin(\theta t)}{h(z)sin(\theta t) + cos(\theta t)}\right].$$

See [4, Proposition 2.3] for the proof of Theorem 3.1.2 above.

REMARK 3.1.3

The assertions 1, 2 and 3 of Theorem 3.1.2 above correspond to the automorphism groups: scaling, translation and rotation groups, respectively. In this study and as noted in [4], we shall consider without loss of generality, the following special cases:

1. For the scaling group, we consider $\varphi_t(z) = e^{-t}z, z \in \mathbb{U}$.

2. For the translation group, we consider $\varphi_t(z) = z + t, z \in \mathbb{U}$.

3. For the rotation group, we consider $\varphi_t(z) = e^{ikt}z, z \in \mathbb{D}$.

We begin by highlighting some properties of the maps $(\varphi_t)_{t\geq 0}$.

Proposition 3.1.4

The map $(\varphi_t)_{t\geq 0}$ forms a group on $\mathbb{U}$ for cases 1 and 2. It forms a group on $\mathbb{D}$ for case 3.

PROOF. 1. For the scaling group; by definition, $\varphi_t(z) = e^{-t}z$, for $z \in \mathbb{U}$. Thus, $\varphi_0(z) = e^{-0}z = z$. This implies that $\varphi_0 = I($ identity $)$. Also $\varphi_{t+s}(z) = e^{-(t+s)}z = e^{-t}(e^{-s}z) = \varphi_t \circ \varphi_s(z)$. Thus, $(\varphi_t)_{t\geq 0})$ is a semigroup on $\mathbb{U}$.

Similarly, we show that $(\varphi_{-t})_{t\geq 0}$ is also a semigroup on $\mathbb{U}$. Since

$\varphi_t(z) = e^{-t}z$, for all $z \in \mathbb{U}$, we have $\varphi_{-t}(z) = e^t z$ for all $z \in \mathbb{U}$. Now, $\varphi_{-0}(z) = \varphi_0(z) = I$ as noted earlier and

$$\begin{aligned}\varphi_{-(t+s)}(z) &= e^{t+s}z \\ &= e^t(e^s z) \\ &= (\varphi_{-t} \circ \varphi_{-s})(z).\end{aligned}$$

Therefore, $(\varphi_{-t})_{t\geq 0}$ is also a semigroup, and hence $(\varphi_t)_{t\geq 0}$ is a group, as desired.

2. For the translation group; $\varphi_t(z) = z + t$ for all $z \in \mathbb{U}$, and thus $\varphi_0(z) = z + 0 = z$. This means that $\varphi_0 = I$(identity). Also,

$$\begin{aligned}\varphi_{t+s}(z) &= z + (t+s) \\ &= (z+s) + t \\ &= \varphi_t(\varphi_s(z)) \\ &= (\varphi_t \circ \varphi_s)(z)\end{aligned}$$

implying that $\varphi_{t+s}(z) = (\varphi_t \circ \varphi_s)(z)$ as desired.

On the other hand, since $\varphi_{-t}(z) = z - t$ for all $z \in \mathbb{U}$, we have $\varphi_{-0}(z) = z - 0 = z$. This implies that $\varphi_0 = I$(identity) and also

$$\begin{aligned}\varphi_{-(t+s)}(z) &= z - (t+s) \\ &= z - t - s \\ &= (z-s) - t \\ &= \varphi_{-t}(\varphi_{-s}(z)) \\ &= (\varphi_{-t} \circ \varphi_{-s})(z)\end{aligned}$$

implying that $\varphi_{-(t+s)}(z) = \varphi_{-t} \circ \varphi_{-s}(z))$ as desired. Thus, $(\varphi_{-t})_{t\geq 0}$ is also a semigroup. Hence $(\varphi_t)_{t\geq 0}$ is a group.

3. For the rotation group; by definition $\varphi_t(z) = e^{ikt}z$, for all $z \in \mathbb{D}$. Therefore, $\varphi_0(z) = e^{ik(0)}z = e^0 z = z$. This means that $\varphi_0 = I$(identity) as earlier indicated and

$$\begin{aligned}
\varphi_{t+s}(z) &= e^{ik(t+s)}z \\
&= e^{ikt}(e^{iks}z) \\
&= \varphi_t(\varphi_s(z)) \\
&= (\varphi_t \circ \varphi_s)(z)
\end{aligned}$$

implying that $\varphi_{t+s}(z) = \varphi_t \circ \varphi_s(z)$. Thus, $(\varphi_t)_{t\geq 0}$ is a semigroup on $\mathbb{D}$.

Similarly, we show that $(\varphi_{-t})_{t\geq 0}$ is also a semigroup on $\mathbb{D}$. Since, $\varphi_{-t}(z) = e^{-ikt}z$ for all $z \in \mathbb{D}$, we have $\varphi_{-0}(z) = e^{ik(-0)}z = e^0 z = z$. This shows that $\varphi_0 = I$ (identity) and

$$\begin{aligned}
\varphi_{-(t+s)}(z) &= e^{ik(-(t+s))}z \\
&= e^{-ikt}(e^{-iks}z) \\
&= \varphi_{-t}(\varphi_{-s}(z)) \\
&= \varphi_{-t} \circ \varphi_{-s}(z)
\end{aligned}$$

implying that $\varphi_{-(t+s)} = (\varphi_{-t} \circ \varphi_{-s})$. Thus, $(\varphi_{-t})_{t\geq 0}$ is also a semigroup on $\mathbb{D}$. Hence $(\varphi_t)_{t\geq 0}$ is a group, as desired.

□

Proposition 3.1.5

The maps $(\varphi_t)_{t\geq 0}$ are indeed self-maps on $\mathbb{U}$ for the scaling and translation groups, while a self-map on $\mathbb{D}$ for the rotation group.

PROOF. Recall that for $z \in \mathbb{C}$, $z \in \mathbb{U}$ if $Im(z) > 0$ and $z \in \mathbb{D}$ if $|z| < 1$ and that φ_t is a self map on $\mathbb{U}$ or $\mathbb{D}$ if $\varphi_t(\mathbb{U}) \subset \mathbb{U}$ or $\varphi_t(\mathbb{D}) \subset \mathbb{D}$ respectively. Now for scaling group, $\varphi_t(z) = e^{-t}z, z \in \mathbb{U}$. Then

$$\begin{aligned} Im(\varphi_t(z)) &= \frac{\varphi_t(z) - \overline{\varphi_t(z)}}{2i} \\ &= \frac{e^{-t}z - e^{-t}\overline{z}}{2i} \\ &= \frac{e^{-t}(z - \overline{z})}{2i} \\ &= e^{-t}Im(z) > 0 \end{aligned}$$

since $e^{-t} > 0$ for all $t > 0$ and $Im(z) > 0$ for all $z \in \mathbb{U}$.

For the translation group, $\varphi_t(z) = z + t, z \in \mathbb{U}$. So

$$\begin{aligned} Im(\varphi_t(z)) &= \frac{\varphi_t(z) - \overline{\varphi_t(z)}}{2i} \\ &= \frac{z + t - (\overline{z} + t)}{2i} \\ &= \frac{z - \overline{z}}{2i} = Im(z) > 0, \text{as desired.} \end{aligned}$$

For the rotation group, $\varphi_t(z) = e^{ikt}z$, $z \in \mathbb{D}$. Then, we have for $z \in \mathbb{D}$

$$\begin{aligned} |\varphi_t(z)| &= |e^{ikt}z| \\ &= |z| < 1, \text{as desired.} \end{aligned}$$

□

Proposition 3.1.6

$(\varphi_t)_{t\geq 0}$ *are self-analytic maps on* $\mathbb{U}$ *for the scaling and translation groups while self-analytic map on* $\mathbb{D}$ *for the rotation group.*

PROOF. Following proposition 3.1.5, it remains to show that φ_t are analytic maps.

Analytic maps of the disk $\mathbb{D}$ are of the form

$$\varphi(z) = \frac{a(z-b)}{1-\bar{b}z}$$

where a and b are constants with $|a| = 1$ and $|b| \leq 1$, while analytic maps of the upper half plane are of the form

$$\varphi(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{R}$ with $ad - bc > 0$.

1. For the scaling group, $\varphi_t(z) = e^{-t}z$, for $z \in \mathbb{U}$. Clearly, $a = e^{-t}$, $b = c = 0$ and $d = 1$. So, $ad - bc = e^{-t} > 0$ for all $t \in \mathbb{R}$ and therefore φ_t is analytic on $\mathbb{U}$

2. For the translation group, $\varphi_t(z) = z + t$, for $z \in \mathbb{U}$. Clearly, $a = 1$, $b = t$, $c = 0$ and $d = 1$. So, $ad - bc = 1 > 0$ and hence φ_t is analytic on $\mathbb{U}$.

3. For the rotation group, $\varphi_t(z) = e^{ikt}z$, $k \neq 0$, for all $z \in \mathbb{D}$. Clearly, $a = e^{ikt}$, $b = 0$, and so $|a| = 1$ and $|b| = 0 \leq 1$ and therefore, $\varphi_t(z) = e^{ikt}z$ is analytic on $\mathbb{D}$

This completes the proof. $\square$

3.2 Groups of weighted composition operators

Recall that for $t \in \mathbb{R}$, consider analytic self-maps $\varphi_t : \mathbb{U} \to \mathbb{U}$. The semigroups of composition operators induced by $(\varphi_t)_{t\geq 0}$ are defined by

$$C_{\varphi_t} f = f \circ \varphi_t \text{ for all } f \in \mathcal{H}(\mathbb{U}).$$

The corresponding semigroups of weighted composition operators $T_t = S_{\varphi_t}$ induced by φ_t are defined on $\mathcal{H}(\mathbb{U})$ by

$$T_t f(z) := S_{\varphi_t} f(z) = (\varphi_t'(z))^{\gamma} (f \circ \varphi_t)(z),$$

where γ is an appropriately chosen weight and for all $f \in \mathcal{H}(\mathbb{U})$.

Proposition 3.2.1

$(T_t)_{t\geq 0}$ is a group on $\mathcal{H}(\mathbb{U})$ for the scaling and translation groups, while it is a group on $\mathcal{H}(\mathbb{D})$ for the rotation group.

PROOF. 1. Scaling group: We need to show that a family $(T_t)_{t\geq 0}$ is a semigroup on $\mathcal{H}(\mathbb{U})$. Since for this group, $\varphi_t(z) = e^{-t}(z)$ for all $z \in \mathbb{U}$, the induced composition operators on $\mathcal{H}(\mathbb{U})$ are given by $T_t f(z) = e^{-t\gamma} f(e^{-t} z)$. Thus, $T_0 f(z) = e^0 f(e^0 z) = f(z)$ implying that $T_0 = I($ the identity operator on $\mathcal{H}(\mathbb{U}))$. Also,

$$\begin{aligned}(T_t \circ T_s)f(z) &= T_t(T_s f(z)) \\ &= e^{-t\gamma}(T_s f(e^{-t}z)) \\ &= e^{-t\gamma}.e^{-s\gamma} f(e^{-s}e^{-t}z) \\ &= e^{-(t+s)\gamma} f(e^{-(t+s)}z) \\ &= T_{t+s}f(z).\end{aligned}$$

Therefore, $T_t \circ T_s = T_{t+s}$ and hence $(T_t)_{t\geq 0}$ is a semigroup.

It remains to show that $(T_{-t})_{t\geq 0}$ is also a semigroup on $\mathcal{H}(\mathbb{U})$. Indeed, $T_{-0} = T_0 = I$ as earlier claimed. Now

$$\begin{aligned}T_{-t} \circ T_{-s}f(z) &= T_{-t}(T_{-s}f(z)) \\ &= e^{t\gamma}(T_{-s}f(e^t z)) \\ &= e^{t\gamma}e^{s\gamma} f(e^t e^s z) \\ &= e^{(t+s)\gamma} f(e^{t+s}z) \\ &= T_{-(t+s)}f(z),\end{aligned}$$

implying that $T_{-t} \circ T_{-s}f(z) = T_{-(t+s)}f(z)$. Thus, $(T_{-t})_{t\geq 0}$ is also a semigroup. Therefore $(T_t)_{t\in\mathbb{R}}$ is a group of weighted composition operators on $\mathcal{H}(\mathbb{U})$.

2. Translation group: For this group, $\varphi_t(z) = z + t$ for all $z \in \mathbb{U}$ and the induced composition operators on $\mathcal{H}(\mathbb{U})$ are given by $T_t f(z) = f(z+t))$. Thus, $T_0 f(z) = f(z+0) = f(z)$ implying that $T_0 = I$.

Also,

$$\begin{aligned} T_t \circ T_s f(z) &= T_t(T_s f(z)) = (T_s f)(z+t) \\ &= f(z+t+s) \\ &= f(z+(t+s)) \\ &= T_{s+t} f(z). \end{aligned}$$

Therefore, $T_{t+s}f(z) = (T_t \circ T_s)f(z)$ as desired. Hence $(T_t)_{t\geq 0}$ is a semigroup on $\mathcal{H}(\mathbb{U})$.

Similarly, we show that $(T_{-t})_{t\geq 0}$ is also a semigroup. Since, $T_t f(z) = f(z+t)$ then $T_{-t}f(z) = f(z-t)$. Thus, $T_{-0}f(z) = f(z-0) = f(z)$ implying that $T_{-0} = I($ the identity operator$)$. Also,

$$\begin{aligned} (T_{-t} \circ T_{-s})f(z) &= T_{-t}(T_{-s}f(z)) = T_{-s}f(z-t) \\ &= f(z-t-s) \\ &= f(z-(t+s)) \\ &= T_{-(t+s)}f(z). \end{aligned}$$

Thus, clearly $T_{-(t+s)}f(z) = (T_{-t} \circ T_{-s})f(z)$. Hence, $(T_{-t})_{t\geq 0}$ is also a semigroup as well.

Therefore, $(T_t)_{t\in\mathbb{R}}$ is a group of weighted composition operator on $\mathcal{H}(\mathbb{U})$.

3. Rotation group: For this group, we consider the group of weighted composition operators defined on $\mathcal{H}(\mathbb{D})$ given by

$$T_t f(z) = e^{ict} f(e^{ikt}z), \text{ for } c, k \in \mathbb{R} \text{ and } k \neq 0.$$

Thus, $T_0 f(z) = e^0 f(e^0 z) = f(z)$ implying that $T_0 = I$, and for all $t, s \geq 0$ we have:

$$\begin{aligned}
(T_t \circ T_s) f(z) &= T_t(T_s f(z)) = T_t(e^{ics} f(e^{iks} z)) \\
&= e^{ics} T_t(f(e^{iks} z)) \\
&= e^{ics} e^{ict} f(e^{iks} e^{ikt} z) \\
&= e^{ic(s+t)} f(e^{ik(s+t} z) \\
&= T_{t+s} f(z).
\end{aligned}$$

Thus, $T_t \circ T_s = T_{t+s}$ for all $t, s \geq 0$ as desired. Therefore $(T_t)_{t\geq 0}$ is a semigroup on $\mathcal{H}(\mathbb{D})$.

On the other hand, it remains to show that $(T_{-t})_{t\geq 0}$ is also a semigroup on $\mathcal{H}(\mathbb{D})$. Indeed, $T_{-0} f(z) = T_0 = I$ as above and

$$\begin{aligned}
T_{-t} \circ T_{-s} f(z) &= T_{-t}(T_{-s} f(z)) = T_{-t}(e^{-ics} f(e^{-iks} z)) \\
&= e^{-ics}(T_{-t}(f(e^{-iks} z))) \\
&= e^{-ics} e^{-ict} f(e^{-iks} e^{-ikt} z) \\
&= e^{-ic(t+s)} f(e^{-ik(s+t)} z) \\
&= T_{-(t+s)} f(z), \text{ as desired}.
\end{aligned}$$

Therefore, $T_{-t} \circ T_{-s} = T_{-(t+s)}$ and hence $(T_{-t})_{t\geq 0}$ is a semigroup. Thus, $(T_t)_{t\in\mathbb{R}}$ is a group on $\mathcal{H}(\mathbb{D})$, as desired. □

From now, we restrict our attention to the Dirichlet space of the upper half-plane $\mathcal{D}_\alpha(\mathbb{U})$ and study the properties of the group $T_t : \mathcal{D}_\alpha(\mathbb{U}) \to \mathcal{D}_\alpha(\mathbb{U})$. We begin by investigating the strong continuity of $(T_t)_{t\geq 0}$.

Theorem 3.2.2

Let $\varphi_t \in Aut(\mathbb{U})$ and $T_t = (\varphi_t')^{\gamma}(f \circ \varphi_t)$ be the induced group of weighted composition operators on $\mathcal{D}_\alpha(\mathbb{U})$. Then $(T_t)_{t\geq 0}$ is strongly continuous.

PROOF. Recall that $\|T_t f\|^2_{\mathcal{D}\alpha(\mathbb{U})} = |T_t f(i)|^2 + \int_{\mathbb{U}} |(T_t f - f)'(w)|^2 \, d\mu_\alpha(w)$. To prove strong continuity of $(T_t)_{t\geq 0}$, it suffices to show that:

$$\lim_{t\to 0^+} \|T_t f - f\|_{\mathcal{D}_\alpha(\mathbb{U})} = 0 \text{ for all } f \in \mathcal{D}_\alpha(\mathbb{U}),$$
$$\text{i.e } |(T_t f - f)(i)|^2 + \int_{\mathbb{U}} |(T_t f - f)'(w)|^2 \, d\mu_\alpha(w) \to 0 \text{ as } t \to 0^+.$$

This is equivalent to showing that:

$$|(T_t f - f)(i)|^2 \to 0 \text{ as } t \to 0^+ \text{ and}$$
$$\int_{\mathbb{U}} |(T_t f - f)'(w)|^2 \to 0 \text{ as } t \to 0^+.$$

But it is clear that

$$|(T_t f - f)(i)| \to |(T_0 f - f)(i)| = 0 \text{ as } t \to 0^+.$$

Now, to prove that $\lim_{t\to 0^+} \int_{\mathbb{U}} |(T_t f - f)'(w)|^2 \, d\mu_\alpha(w) = 0$, let $f \in \mathcal{D}_\alpha(\mathbb{U})$ and suppose that $t_n \to 0$ in $\mathbb{R}$. Let $f_n = T_{t_n} f$. Then $f_n(z) \to f(z)$ for each $z \in \mathbb{U}$ as $n \to \infty$ and due to analyticity of each f_n for every n, $f_n' \to f'$ for each n. Let $g_n(z) := 2(|f'|^2 + |f_n'|^2) - |f' - f_n'|^2$, then $g_n \geq 0$ and $g_n(z) \to 2^2|f'(z)|^2$ on $\mathcal{D}_\alpha(\mathbb{U})$ as $n \to \infty$.

By Fatou's lemma, we have

$$\begin{aligned}
\int_{\mathbb{U}} 2^2|f'(z)|^2 d\mu_\alpha &= \int_{\mathbb{U}} \liminf g_n d\mu_\alpha(z) \\
&\leq \liminf \int_{\mathbb{U}} g_n d\mu_\alpha(z) \\
&= \liminf \int_{\mathbb{U}} (2(|f'|^2 + |f_n'|^2) - |f' - f_n'|^2) d\mu_\alpha(z) \\
&= 2\int_{\mathbb{U}} |f'|^2 d\mu_\alpha + 2\int_{\mathbb{U}} |f'|^2 d\mu_\alpha - \limsup_n \int_{\mathbb{U}} |f' - f_n'| d\mu_\alpha(z) \\
&= 2^2 \int_{\mathbb{U}} |f'(z)|^2 d\mu_\alpha - \limsup_n \int_{\mathbb{U}} |f' - f_n'| d\mu_\alpha(z).
\end{aligned}$$

Thus, $0 \leq -\limsup_n \int_{\mathbb{U}} |f' - f_n'|^2 d\mu_\alpha \leq 0$. This implies that $\limsup_n \int_{\mathbb{U}} |f' - f_n'|^2 d\mu_\alpha(z) = 0$. Hence $\lim_n \int_{\mathbb{U}} |f' - f_n'|^2 d\mu_\alpha(z) = 0$, that is, $\lim_n \int_{\mathbb{U}} |(T_{t_n} f - f)'|^2 d\mu_\alpha = 0$.

Therefore,

$$\|T_{t_n} f - f\|_{\mathcal{D}_\alpha(\mathbb{U})} \to 0 \text{ as } t_n \to 0.$$

Thus, $(T_t)_{t \geq 0}$ is strongly continuous, as desired. □

3.3 Scaling group

Here, the self analytic maps on $\mathbb{U}$ are of the form $\varphi_t(z) = e^{-t}z, z \in \mathbb{U}$. By Proposition 3.1.4, $(\varphi_t)_{t \geq 0}$ forms a group on $\mathbb{U}$.

The corresponding groups of weighted composition operators on $\mathcal{D}_\alpha(\mathbb{U})$ are therefore defined by equation (3.1) as;

$$T_t f(z) = e^{-t\gamma} f(e^{-t}z) \tag{3.1}$$

for all $f \in \mathcal{D}_\alpha(\mathbb{U})$, where $\gamma = \frac{\alpha+2}{2}$. By Theorem 3.2.2, this group is strongly continuous and we now find its infinitesimal generator Γ by giving the following theorem,

Theorem 3.3.1

Let $(T_t)_{t\geq 0}$ be the group of weighted composition operators given by equation (3.1). The infinitesimal generator Γ of $(T_t)_{t\geq 0}$ is given by $\Gamma f(z) = -\gamma f(z) - zf'(z)$ with domain $dom(\Gamma) = \{f \in \mathcal{D}_\alpha(\mathbb{U}) : zf'(z) \in \mathcal{D}_\alpha(\mathbb{U})\}$.

PROOF. For $f \in \mathcal{D}_\alpha(\mathbb{U}))$, we have by definition;

$$\begin{aligned}
\Gamma f(z) &= \lim_{t\to 0^+} \frac{e^{-\gamma t} f(e^{-t}z) - f(z)}{t} \\
&= \frac{\partial}{\partial t}\left(e^{-\gamma t} f(e^{-t}z)\right)\Big|_{t=0} \\
&= -\gamma e^{-\gamma t} f(e^{-t}z) + e^{-\gamma t} f'(e^{-t}z)(-e^{-t}z)|_{t=0} \\
&= -\gamma e^{-\gamma t} f(e^{-t}z) - ze^{-\gamma t}e^{-t} f'(e^{-t}z)|_{t=0} \\
&= -\gamma f(z) - zf'(z).
\end{aligned}$$

Therefore, $\mathcal{D}(\Gamma) \subset \{f \in \mathcal{D}_\alpha(\mathbb{U}) : zf'(z) \in \mathcal{D}_\alpha(\mathbb{U})\}$.
Conversely, let $f \in \mathcal{D}_\alpha(\mathbb{U})$ be such that $zf'(z) \in \mathcal{D}_\alpha(\mathbb{U})$. Then for $z \in \mathbb{U}$, by the Fundamental Theorem of Calculus, we have

$$\begin{aligned}
T_t f(z) - f(z) &= \int_0^t \frac{\partial}{\partial s}(e^{-\gamma s} f(\varphi_s(z))ds \\
&= \int_0^t -e^{-\gamma s}\varphi_s(z) f'(\varphi_s(z))) - \gamma e^{-\gamma s} f(\varphi_s(z))ds \\
&= \int_0^t e^{-\gamma s}\left(-\varphi_s(z) f'(\varphi_s(z)) - \gamma f(\varphi_s(z))ds\right) \\
&= \int_0^t T_s F(z)ds, \text{ where }, F(z) = -\gamma f(z) - zf'(z)
\end{aligned}$$

Thus, $\lim_{t\to 0^+} \frac{T_t(f)-f}{t} = \lim_{t\to 0^+} \frac{1}{t}\int_0^t T_s F ds$ and strong continuity of $(T_s)_{s\geq 0}$ implies that $\frac{1}{t}\int_0^t \|T_s F - F\| ds \to 0$ as $t \to 0^+$. Thus

$$\mathcal{D}(\Gamma) \supseteq \{f \in \mathcal{D}_\alpha(\mathbb{U}) : zf'(z) \in \mathcal{D}_\alpha(\mathbb{U})\}.$$

□

In the next theorem, we show that this group fails to be an isometry on $\mathcal{D}_\alpha(\mathbb{U})$.

Theorem 3.3.2

The group $(T_t)_{t\in\mathbb{R}}$ given by equation (3.1) is not an isometry on $\mathcal{D}_\alpha(\mathbb{U})$.

PROOF. By norm definition,

$$\|T_t f\|^2_{\mathcal{D}_\alpha(\mathbb{U})} = |T_t f(i)|^2 + \int_{\mathbb{U}} |(T_t f)'(w)|^2 d\mu_\alpha(w).$$

We have from equation (3.1), $T_t f(w) = e^{-t\gamma} f(e^{-t}w)$ and therefore $(T_t f)'(w) = e^{-t\gamma}e^{-t}f'(e^{-t}w)$. Thus, $|(T_t f)'(w)|^2 = e^{-2t\gamma}e^{-2t}|f'(e^{-t}w)|^2$ and $|T_t f(i)|^2 = e^{-2t\gamma}|f(e^{-t}i)|^2$.

By change of variables: Let $z = e^{-t}w$. Then $w = e^t z$ and $dA(z) = |e^{-t}|^2 dA(w)$ implying that $dA(w) = e^{2t}dA(z)$ and $d\mu_\alpha(w) = (Im(w))^\alpha dA(w) =$

$(e^t Im(z))^\alpha e^{2t} dA(z) = e^{\alpha t} e^{2t} (Im(z))^\alpha dA(z)$. Thus,

$$\begin{aligned}
\|T_t f\|^2_{\mathcal{D}\alpha(\mathbb{U})} &= |T_t f(i)|^2 + \int_{\mathbb{U}} |(T_t f)'(w)|^2_{d\mu_\alpha(w)} \\
&= e^{-2t\gamma}|f(e^{-t}(i))|^2 + \int_{\mathbb{U}} e^{-2t\gamma}.e^{-2t}|f'(e^{-t}w)|^2.e^{\alpha t}.e^{2t} d\mu_{\alpha(z)} \\
&= e^{-2t\gamma}|f(e^{-t}(i))|^2 + \int_{\mathbb{U}} e^{-2t\gamma}.e^{-2t}|f'(e^{-t}w)|^2 e^{\alpha t}.e^{2t} d\mu_\alpha(z) \\
&= e^{-2t\gamma}|f(e^{-t}(i))|^2 + \int_{\mathbb{U}} e^{-2t\gamma} e^{\alpha t}|f'(e^{-t}w)|^2 d\mu_\alpha(z) \\
&= e^{-2t\gamma}\left(|f(e^{-t}(i))|^2 + e^{\alpha t}\int_{\mathbb{U}} |f'(e^{-t}w)|^2 d\mu_\alpha(z)\right)
\end{aligned}$$

But,

$$e^{-2t\gamma}\left(|f(e^{-t}(i))|^2 + e^{\alpha t}\int_{\mathbb{U}} |f'(e^{-t}z)|^2 d\mu_\alpha(z)\right) \neq \|f\|^2_{\mathcal{D}\alpha(\mathbb{U})} \quad (3.2)$$

meaning the Right Hand Side of equation (3.2) is not equal to the norm $\|T_t f\|^2_{\mathcal{D}\alpha(\mathbb{U})}$. This implies that the weighted composition group induced by the scaling group is not an isometry on $\mathcal{D}_\alpha(\mathbb{U})$. □

Remark 3.3.3

The fact that the induced group of composition operators $(T_t)_{t\geq 0}$ fails to be an isometry on $\mathcal{D}_\alpha(\mathbb{U})$ complicates the spectral analysis of the group $(T_t)_{t\geq 0}$. This is because the theory of spectra of semigroups of linear operators are easily applied when we can identify exactly what the spectrum of $(T_t)_{t\geq 0}$ is. For the case when $(T_t)_{t\geq 0}$ is an isometry, then Theorem 2.0.3 readily gives the spectrum of $(T_t)_{t\geq 0}$ and together with the spectral mapping theorems as well as Hille-Yosida theorem, a complete spectral analysis of the infinitesimal generator as well as the resulting resolvents can be easily carried out. For this composition group therefore, we shall only determine the point spectrum of the infinitesimal generator Γ on

$\mathcal{D}_\alpha(\mathbb{U})$, but first we state the following two lemmas;

Lemma 3.3.4

$f \in \mathcal{D}_\alpha(\mathbb{U})$ if and only if $f' \in L^2_a(\mathbb{U}, \mu_\alpha)$.

PROOF. By definition, $f \in \mathcal{D}_\alpha(\mathbb{U})$ if and only if

$$\|f\|_{\mathcal{D}_{\alpha,1}(\mathbb{U})} = \int_{\mathbb{U}} |f'(z)|^2 d\mu_\alpha(z) < \infty.$$

But again by definition of $L^2_a(\mathbb{U}, \mu_\alpha)$, $f' \in L^2_a(\mathbb{U}, \mu_\alpha)$ if and only if

$$\int_{\mathbb{U}} |f'(z)|^2 d\mu_\alpha(z) = \|f'\|^2_{L^2_a(\mathbb{U},\mu_\alpha)} < \infty$$

This means that $f \in \mathcal{D}_\alpha(\mathbb{U})$ if and only if $f \in L^2_a(\mathbb{U}, \mu_\alpha)$, as desired. □

Lemma 3.3.5

Let X denote one of the spaces $H^p(\mathbb{U})$ or $L^p_a(\mathbb{U}, \mu_\alpha)$, $1 \leq p < \infty$ and $\alpha > -1$ ($\alpha = -1$ if $X = H^p(\mathbb{U})$) and let $\gamma = (\alpha + 2)/p$. If $c \in \mathbb{R}$ and $\lambda, \nu \in \mathbb{C}$, then;

1. *$f(\omega) = (\omega - c)^\lambda(\omega + \imath)^\nu \in X$ if and only if $Re(\lambda + \nu) < -\gamma < Re(\lambda)$. In particular: $(\omega - c)^\lambda \notin X$ for any $\lambda \in \mathbb{C}$ and $(\omega + i)^\nu \in X$ if and only if $Re(\nu) < -\gamma$.*

2. *$f(\omega) = e^\omega/\omega^c \in X$ if and only if $1/p < c < \gamma$. In particular, $e^\omega/\omega^c \notin H^p(\mathbb{U})$ for any $c \in \mathbb{R}$.*

For proof see [4, Lemma 3.2].

We now determine the point spectrum of the infinitesimal generator Γ as we give in the following theorem:

Theorem 3.3.6

Let Γ be the infinitesimal generator of the group $(T_t)_{t\geq 0}$ given by equation (3.1). Then the point spectrum of Γ is empty, that is, $\sigma_p(\Gamma) = \emptyset$.

PROOF. Let $\lambda \in \sigma_p(\Gamma)$. Then there exist $0 \neq f \in \mathcal{D}_\alpha(\mathbb{U})$ such that $\Gamma f = \lambda f$. This implies that $-\gamma f(z) - zf'(z) = \lambda f(z)$ which simplifies to $\frac{f'(z)}{f} = -(\gamma + \lambda)\frac{1}{z}$. Integrating both sides with respect to z yields

$$\ln f = -(\gamma + \lambda) \ln z + C_1,$$

which is equivalent to

$$f(z) = Cz^{-(\gamma+\lambda)}, \text{ where } C \text{ is a constant.}$$

It remains to determine for which λ's is $f \in \mathcal{D}_\alpha(\mathbb{U})$ given that $f(z) = Cz^{-(\gamma+\lambda)}$.

Now using Lemma 3.3.4, $f \in \mathcal{D}_\alpha(\mathbb{U})$ if and only if $f' \in L^2_a(\mathbb{U}, \mu_\alpha)$. By differentiation, $f'(z) = -C(\gamma + \lambda)z^{-(\gamma+\lambda+1)}$. Now, by Lemma 3.3.5, $f' \in L^2_a(\mathbb{U}, \mu_\alpha)$ if and only if $Re\,(-(\gamma + \lambda + 1)) < -\gamma < Re\,(-(\gamma + \lambda + 1))$ which is impossible and therefore no such $\lambda \in \mathbb{C}$ exists. Thus, $\sigma_p(\Gamma) = \emptyset$.

$\square$

3.4 Translation group

For this group, we consider the self analytic maps on $\mathbb{U}$ of the form $\varphi_t(z) = z + t$, $z \in \mathbb{U}$ with the corresponding group of weighted composition operators defined on $\mathcal{D}_\alpha(\mathbb{U})$ given by;

$$T_t f(z) = f(z+t), \text{ for all } f \in \mathcal{D}_\alpha(\mathbb{U}). \tag{3.3}$$

This group is strongly continuous by Theorem 3.2.2 and we now determine its infinitesimal generator Γ by giving the following theorem:

Theorem 3.4.1

Let Γ be the infinitesimal generator of the group $(T_t)_{t\geq 0}$ given by equation (3.3) on $\mathcal{D}_\alpha(\mathbb{U})$. Then $\Gamma f(z) = f'(z)$ with domain $dom(\Gamma) = \{f \in \mathcal{D}_\alpha(\mathbb{U}) : f' \in \mathcal{D}_\alpha(\mathbb{U})\}$.

PROOF. If $f \in \mathrm{dom}(\Gamma)$, then by definition

$$\begin{aligned} \Gamma f(z) &= \lim_{t\to 0^+}\left(\frac{(\varphi_t'(z))^\gamma f(\varphi_t(z)) - f(z)}{t}\right) \\ &= \frac{\partial}{\partial t}(f(z+t))\bigg|_{t=0} \\ &= f'(z). \end{aligned}$$

Thus, $\mathrm{dom}(\Gamma) \subset \{f \in \mathcal{D}_\alpha(\mathbb{U}) : f'(z) \in \mathcal{D}_\alpha(\mathbb{U})\}$.

Conversely, if $f \in \mathcal{D}_\alpha(\mathbb{U})$ is such that $f' \in \mathcal{D}_\alpha(\mathbb{U})$, then for $z \in \mathbb{U}$, we have,

$$\begin{aligned} \frac{T_t f - f}{t} &= \frac{1}{t}\int_0^t \frac{\partial}{\partial s}(T_s f)\,ds \\ &= \frac{1}{t}\int_0^t \frac{\partial}{\partial s} f(z+s)ds \\ &= \frac{1}{t}\int_0^t f'(z+s)ds \\ &= \frac{1}{t}\int_0^t T_s F(z)ds, \text{ where } F(z) = f'(z). \end{aligned}$$

Thus, $\|\frac{T_t f - f}{t} - f'\| \leq \frac{1}{t}\int_0^t \|T_s f' - f'\|\,ds \to 0$ as $t \to 0$ by strong continuity. Hence, $\mathrm{dom}(\Gamma) \supseteq \{f \in \mathcal{D}_\alpha(\mathbb{U}) : f' \in \mathcal{D}_\alpha(\mathbb{U})\}$. $\square$

In the next theorem we show that this group fails to be an isometry on $\mathcal{D}_\alpha(\mathbb{U})$.

Theorem 3.4.2

The group $(T_t)_{t\geq 0}$ given by equation (3.3) is not an isometry on $\mathcal{D}_\alpha(\mathbb{U})$.

PROOF. From the norm definition,

$$\|T_t f\|^2_{\mathcal{D}\alpha(\mathbb{U})} = |T_t f(i)|^2 + \int_{\mathbb{U}} |(T_t f)'(w)|^2 d\mu_\alpha(w).$$

But from equation (3.3), $T_t f(w) = f(w+t)$. Therefore, the part $|T_t f(i)|^2$ becomes $T_t f(i) = f(i+t)$ and $|T_t f(i)|^2 = |f(i+t)|^2$. Therefore, $(T_t f)'(w) = f'(w+t)$ and also $|(T_t f)'(w)|^2 = |f'(w+t)|^2$.

Now, for the part $\int_{\mathbb{U}} |(T_t f)'(w)|^2 d\mu_\alpha(w)$, we shall have,

$$\int_{\mathbb{U}} |f'(w+t)|^2 (\mathrm{Im}(w))^\alpha dA(w).$$

By change of variables; let $z = w+t$. Then $w = z-t$ and $dA(z) = dA(w)$. Therefore, $d\mu_\alpha(w) = (Im(z))^\alpha dA(z)$ summing them up in $\|T_t f\|^2_{\mathcal{D}\alpha(\mathbb{U})} = |T_t f(i)|^2 + \int_{\mathbb{U}} |(T_t f)'(w)|^2 d\mu_\alpha(w)$, we have,

$$\begin{aligned}
\|T_t f\|^2_{\mathcal{D}\alpha(\mathbb{U})} &= |f(i+t)|^2 + \int_{\mathbb{U}} |f'(w+t)|^2 (Im(z))^\alpha dA(z) \\
&= |f(i+t)|^2 + \int_{\mathbb{U}} |f'(w+t)|^2 d\mu_\alpha(z) \\
&= |f(i+t)|^2 + \int_{\mathbb{U}} |f'(z)|^2 d\mu_\alpha(z) \\
&= |f(i+t)|^2 + \int_{\mathbb{U}} |f'(z)|^2 d\mu_\alpha(z)
\end{aligned}$$

which is not an isometry since $|f(i+t)|^2 + \int_{\mathbb{U}} |f'(z)|^2 d\mu_\alpha(z) \neq \|f\|^2_{\mathcal{D}_\alpha(\mathbb{U})}$.

□

Next, we determine the point spectrum of the infinitesimal generator Γ by giving the following theorem,

Theorem 3.4.3

Let Γ be the infinitesimal generator of the group $(T_t)_{t\geq 0}$ on $\mathcal{D}_\alpha(\mathbb{U})$ given by equation (3.3). *Then $\sigma_p(\Gamma) = \emptyset$.*

PROOF. Let $\lambda \in \mathbb{C}$ be such that $\lambda \in \sigma_p(\Gamma)$. Then $\Gamma f = \lambda f$ for some $0 \neq f \in \mathcal{D}\alpha(\mathbb{U})$, which is equivalent to the differential equation;

$$f'(z) \;=\; \lambda f(z) \text{ for all } z \in \mathbb{U}. \tag{3.4}$$

By solving equation (3.4), we obtain;

$$\begin{aligned} \frac{df}{f} = \lambda dz \;\; &\Leftrightarrow\;\; \ln f = \lambda z + C_1 \\ &\Leftrightarrow\;\; f(z) = Ce^{\lambda z}. \end{aligned}$$

It remains to check for which λ's is $f \in \mathcal{D}_\alpha(\mathbb{U})$.

Recall that $f \in \mathcal{D}_\alpha(\mathbb{U})$ if and only if $f' \in L^2_a(\mathbb{U}, \mu_\alpha)$. Now, $f'(z) = \lambda Ce^{\lambda z}$. Using Lemma 3.3.5, it follows that $f'(z) = \lambda Ce^{\lambda z} \in L^2_a(\mathbb{U}, \mu_\alpha)$ if and only if $\frac{1}{2} < 0 < \frac{\alpha+2}{2}$ which is impossible and therefore no such $\lambda \in \mathbb{C}$ exists. Hence $\sigma_p(\Gamma) = \emptyset$, as claimed. □

Chapter 4

ROTATION GROUP

For the rotation group, the self-analytic maps are given by $\varphi_t(z) = e^{ikt}z$ for $z \in \mathbb{D}$ and we consider the induced groups of weighted composition operators on $\mathcal{D}_\alpha(\mathbb{D})$ of the form

$$S_{\varphi_t} f(z) \quad = \quad e^{ict} f(e^{ikt} z) \tag{4.1}$$

for all $f \in \mathcal{D}_\alpha(\mathbb{D})$ and $c, k \in \mathbb{R}$ and $k \neq 0$.

Let $\Gamma_{c,k}$ be the generator of the group $(S_{\varphi_t})_{t\in\mathbb{R}}$. Then as remarked in [7], to analyze the group $(S_{\varphi_t})_{t\in\mathbb{R}}$, it is sufficient to consider the case when $c = 0$ and $k = 1$ since the properties are related in the following ways;

Theorem 4.0.1

Let $(S_{\varphi_t})_{t\in\mathbb{R}}$ be a group of weighted composition operators defined on $\mathcal{D}_\alpha(\mathbb{D})$ by $S_{\varphi_t} f(z) = e^{ict} f(e^{ikt} z)$ and let $\Gamma_{c,k}$ be its infinitesimal generator. Then;

1. *$(S_{\varphi_t})_{t\geq 0}$ is an isometry on $\mathcal{D}_\alpha(\mathbb{D})$*

2. *$S_{\varphi_t} = C_{\varphi_t}$ for $c = 0, k = 1$.*

3. $\Gamma_{c,k} = ic + k\Gamma_{0,1}$ *with the domain* $dom(\Gamma_{c,k}) = dom(\Gamma_{0,1}) = \{f \in \mathcal{D}_\alpha(\mathbb{D}) : f' \in \mathcal{D}_\alpha(\mathbb{D})\}$.

4. $\sigma(\Gamma_{c,k}) = \{ic + k\sigma(\Gamma_{0,1})\}$, *and* $\sigma_\rho(\Gamma_{c,k}) = \{ic + k\sigma_\rho(\Gamma_{0,1})\}$

5. $\lambda \in \sigma(\Gamma_{0,1})$ *if and only if* $ic + k\lambda \in \sigma(\Gamma_{c,k})$ *and*

$$R(ic + k\lambda, \Gamma_{c,k}) = \frac{1}{k} R(\lambda, \Gamma_{0,1}).$$

PROOF. 1. A change of variables argument will yield;

$$\begin{aligned}
\|S_{\varphi_t} f\|^2_{\mathcal{D}_\alpha(\mathbb{D})} &= |S_{\varphi_t} f(0)|^2 + \int_{\mathbb{D}} |(S_{\varphi_t} f)'(z)|^2 \, dm_\alpha(z) \\
&= |e^{ict} f(0)|^2 + \int_{\mathbb{D}} |e^{ict} e^{ikt} f'(e^{ikt} z)|^2 \, dm_\alpha(z) \\
&= |f(0)|^2 + \int_{\mathbb{D}} |f'(e^{ikt} z)|^2 \, dm_\alpha(z) \\
&= |f(0)|^2 + \int_{\mathbb{D}} |f'(w)|^2 \, dm_\alpha(w) = \|f\|^2_{\mathcal{D}_\alpha(\mathbb{D})},
\end{aligned}$$

as desired.

2. From definition, $S_{\varphi_t} f(z) = e^{ict} f(e^{ikt} z)$. Taking $c = 0$ and $k = 1$, we get

$$S_{\varphi_t} f(z) = f(e^{ikt} z) = C_{\varphi_t} f(z);$$

where $\varphi_t(z) = e^{ikt} z$ for $z \in \mathbb{D}$.

3. For this case, by the definition of the infinitesimal generator,

$$\begin{aligned}
\Gamma_{c,k} f(z) &= \frac{\partial}{\partial t}(e^{ict} f(e^{ikt} z))\Big|_{t=0} \\
&= ice^{ict} f(e^{ikt} z) + ikze^{ict} f'(e^{ikt} z)|_{t=0} \\
&= icf(z) + ikzf'(z).
\end{aligned}$$

Now if $c = 0, k = 1$, then $\Gamma_{0,1}f(z) = izf'(z)$. This implies that

$$\begin{aligned}\Gamma_{c,k}f(z) &= i(cf(z) + kzf'(z)) \\ &= icf(z) + k\Gamma_{0,1}f(z), \text{ as claimed .}\end{aligned}$$

Now, we obtain the domain of $\Gamma_{c,k}$. First it is clear that $\text{dom}(\Gamma_{c,k}) \subset \{f \in \mathcal{D}_\alpha(\mathbb{D}) : zf' \in \mathcal{D}_\alpha(\mathbb{D})\}$. But $zf' \in \mathcal{D}_\alpha(\mathbb{D})$ implies that $zf' \in ran(M_z)$ and therefore $f' \in \mathcal{D}_\alpha(\mathbb{D})$. Thus $\{f \in \mathcal{D}_\alpha(\mathbb{D}) : zf' \in \mathcal{D}_\alpha(\mathbb{D})\} = \{f \in \mathcal{D}_\alpha(\mathbb{D}) : f' \in \mathcal{D}_\alpha(\mathbb{D})\}$.

Conversely if $f \in \mathcal{D}_\alpha(\mathbb{D})$ is such that $zf' \in X$, then $F(z) = i(cf(z) + zf'(z)) \in \mathcal{D}_\alpha(\mathbb{D})$, and for all $t > 0$

$$\begin{aligned}\frac{T_t f(z) - f(z)}{t} &= \frac{1}{t}\int_0^t \partial s(T_s f(z))ds \\ &= \frac{1}{t}\int_0^t e^{ics}\left(icf(e^{iks}z) + ikzf'(e^{iks}z)\right)ds \\ &= \frac{1}{t}\int_0^t T_sF(z)ds, \text{ where } F(z) = (icf(z) + ikzf'(z))\,.\end{aligned}$$

Again, strong continuity of $(T_s)_{s\geq 0}$ implies that $\|\frac{1}{t}\int_0^t T_sF - F\| \to 0$ as $t \to 0^+$. Thus, $\text{dom}(\Gamma_{c,k}) = \{f \in \mathcal{D}_\alpha(\mathbb{D}) : f' \in \mathcal{D}_\alpha(\mathbb{D})\}$.

4. Now, let $\lambda \in \rho(\Gamma_{0,1})$, then

$$\begin{aligned}(ic + k\lambda - \Gamma_{c,k})\frac{1}{k}R(\lambda, \Gamma_{0,1}) &= (ic + k\lambda - (ic + k\Gamma_{0,1}))\frac{1}{k}R(\lambda, \Gamma_{0,1}) \\ &= \frac{k}{k}(\lambda - \Gamma_{0,1})R(\lambda, \Gamma_{0,1}) = 1\end{aligned}$$

and if $f \in \operatorname{dom}(\Gamma_{c,k})$, then

$$\begin{aligned} R(\lambda, \Gamma_{0,1})(ic + k\lambda - \Gamma_{c,k})f &= \frac{1}{k}R(\lambda, \Gamma_{0,1})(ic + k\lambda - (ic + k\Gamma_{0,1}))f \\ &= \frac{k}{k}(\lambda - \Gamma_{0,1})R(\lambda, \Gamma_{0,1})(\lambda - \Gamma_{0,1})f = f. \end{aligned}$$

Conversely, if $\mu \in \rho(\Gamma_{c,k})$, let $\mu = ic + k\lambda$ so that $\lambda = \frac{\mu - ic}{k}$. Then

$$\begin{aligned} (\lambda - \Gamma_{0,1})kR(\mu, \Gamma_{c,k}) &= k\left(\frac{\mu - ic}{k} - \Gamma_{0,1}\right)R(\mu, \Gamma_{c,k}) = (\mu - ic - k\Gamma_{0,1})R(\mu, \Gamma_{c,k}) \\ &= (\mu - (ic + k\Gamma_{0,1}))R(\mu, \Gamma_{c,k}) = (\lambda - \Gamma_{c,k})kR(\mu, \Gamma_{c,k}) = 1. \end{aligned}$$

5. If $f \in dom(\Gamma_{0,1})$, then by definition,

$$\begin{aligned} kR(\mu, \Gamma_{c,k})(\lambda - \Gamma_{0,1})f &= R(\mu, \Gamma_{c,k})(\mu - ic - k\Gamma_{0,1})f \\ &= R(\mu, \Gamma_{c,k})(\mu - \Gamma_{c,k})f = f \end{aligned}$$

Thus, $\sigma(\Gamma_{c,k}) = \{ic + k\lambda : \lambda \in \sigma(\Gamma_{0,1})\}$, $\sigma\rho(\Gamma_{c,k}) = \{ic + k\lambda : \lambda \in \sigma\rho(\Gamma_{0,1})\}$ and for all $\lambda \in \rho(\Gamma_{0,1})$, $R(ic + k\lambda, \Gamma_{c,k}) = \frac{1}{k}R(\lambda, \Gamma_{0,1})$, as desired.

□

Because of Theorem 4.0.1 above, we shall therefore restrict our attention to the group $C_{\varphi_t}f(z) = f(e^{it}z)$ for all $f \in \mathcal{D}_\alpha(\mathbb{D})$ whose infinitesimal generator is $\Gamma_{0,1}$, and using similarity theory of semigroups, we carry out a complete analysis of both the semigroup and spectral properties of $(C_{\varphi_t})_{t \geq 0}$.

Now, define $C_{\varphi_t} : \mathcal{D}_\alpha(\mathbb{D}) \to \mathcal{D}_\alpha(\mathbb{D})$ by

$$C_{\varphi_t} f(z) = f(e^{it}z)$$

for all $t \geq 0$, $z \in \mathbb{D}$ and $f \in \mathcal{D}_\alpha(\mathbb{D})$, and where $\varphi_t(z) = e^{it}z$. Then we have the following proposition;

Proposition 4.0.2

Let $C_{\varphi_t} : \mathcal{D}_\alpha(\mathbb{D}) \to \mathcal{D}_\alpha(\mathbb{D})$ be given by $C_{\varphi_t} f(z) = f(e^{it}z)$. Then $C_{\tau_t} := \varphi_t' C_{\varphi_t}$ is a group of composition operators on $L^2_a(\mathbb{D}, m_\alpha)$.

PROOF. First, we prove that $f \in \mathcal{D}_\alpha(\mathbb{D})$ if and only if $f' \in L^2_a(\mathbb{D}, m_\alpha)$. Indeed by definition;

$$\begin{aligned} f \in \mathcal{D}_\alpha(\mathbb{D}) &\Leftrightarrow \int_{\mathbb{D}} |f'(z)|^2 dm_\alpha(z) < \infty \\ &\Leftrightarrow f' \in L^2_a(\mathbb{D}, m_\alpha). \end{aligned}$$

Now, let $f \in \mathcal{D}_\alpha(\mathbb{D})$. Then $C_{\varphi_t} f = f \circ \varphi_t$ and so;

$$\begin{aligned} (C_{\varphi_t} f)' &= (f \circ \varphi_t)' \\ &= \varphi_t' f'(\varphi_t) \\ &= \varphi_t' f' \circ \varphi_t \in L^2_a(\mathbb{D}, m_\alpha) \end{aligned}$$

since $f \in \mathcal{D}_\alpha(\mathbb{D})$ if and only if $f' \in L^2_a(\mathbb{D}, m_\alpha)$.

Next, we need to show that the family $(C_{\tau_t})_{t\geq 0}$ defines a group of weighted composition operators on $L^2_a(\mathbb{D}, m_\alpha)$. Since $\varphi_t(z) = e^{it}z$, for $z \in \mathbb{D}$, it follows that for all $f \in L^2_a(\mathbb{D}, m_\alpha)$,

$$\begin{aligned} C_{\mathcal{T}_t} f(z) &= \varphi_t' C_{\varphi_t} f \\ &= \varphi_t' f \circ (\varphi_t) \\ &= e^{it} f(e^{it} z) \end{aligned}$$

Now, $C_{\mathcal{T}_0} f(z) = e^0 f(e^0 z) = f(z)$ and therefore $C_{\mathcal{T}_0} = I$, the identity operator on $L^2_a(\mathbb{D}, m_\alpha)$. For $t, s \geq 0$, we have;

$$\begin{aligned} C_{\mathcal{T}_t} \circ C_{\mathcal{T}_s} f(z) &= C_{\mathcal{T}_t}(C_{\mathcal{T}_s} f(z)) \\ &= \varphi_t' C_{\varphi_t}(C_{\mathcal{T}_s} f) \\ &= \varphi_t' C_{\varphi_t}(\varphi_s' C_{\varphi_s} f) \end{aligned}$$

which on further simplification yields

$$\begin{aligned} C_{\mathcal{T}_t} \circ C_{\mathcal{T}_s} f(z) &= \varphi_t' \varphi_s' C_{\varphi_t}(C_{\varphi_s} f(z)) \\ &= e^{it} e^{is} C_{\varphi_t}(f(\varphi_s(z))) \\ &= e^{i(t+s)} f(\varphi_s(\varphi_t(z))) \\ &= e^{i(t+s)} f(\varphi_s(e^{it} z)) \\ &= e^{i(t+s)} f(e^{is} e^{it} z) = e^{i(t+s)} f(e^{i(t+s)} z) \\ &= C_{\mathcal{T}_{s+t}} f(z). \end{aligned}$$

Thus, $C_{\mathcal{T}_t} \circ C_{\mathcal{T}_s} = C_{\mathcal{T}_{s+t}}$, as desired. Hence, $(C_{\mathcal{T}_t})_{t \geq 0}$ is a semigroup on $L^2_a(\mathbb{D}, m_\alpha)$.

On the other hand, it can be shown similarly that $(C_{\mathcal{T}_{-t}})_{t \geq 0}$ is also a semigroup on $L^2_a(\mathbb{D}, m_\alpha)$. Hence $(C_{\mathcal{T}_t})_{t \geq 0}$ is a group on $L^2_a(\mathbb{D}, m_\alpha)$. □

Now, we define a mapping $U : \mathcal{D}_\alpha(\mathbb{D}) \to L^2_a(\mathbb{D})$ by $Uf = f'$. Then we give the following proposition;

Proposition 4.0.3

Let $U : \mathcal{D}_\alpha(\mathbb{D}) \to L^2_a(\mathbb{D})$ be given by $Uf = f'$. Then U is an isometry upto an additive constant.

PROOF. Using the semi-norm $\|\cdot\|_{\mathcal{D}_\alpha(\mathbb{D})}$ on $\mathcal{D}_\alpha(\mathbb{D})$, we have

$$\begin{aligned} \|Uf\|^2_{L^2_a(\mathbb{D})} &= \|f'\|^2_{L^2_a(\mathbb{D})} \\ &= \int_{\mathbb{D}} |f'(z)|^2\, dm_\alpha(z) \\ &= \|f\|^2_{\mathcal{D}_{\alpha,1}(\mathbb{D})}, \text{ as desired .} \end{aligned}$$

□

Proposition 4.0.4

Let $U : \mathcal{D}_\alpha(\mathbb{D}) \to L^2_a(\mathbb{D})$ be given by $Uf = f'$. Then U is unitary i.e $U^\star U = UU^\star = I$.

PROOF. Note that $U : \mathcal{D}_\alpha(\mathbb{D}) \to L^2_a(\mathbb{D})$ is an isometry by Proposition 4.0.3. Now $U^{-1} : L^2_a(\mathbb{D}) \to \mathcal{D}_\alpha(\mathbb{D})$ is given by $U^{-1}f = \int f dz$ for every $f \in L^2_a(\mathbb{D})$. Let $f \in L^2_a(\mathbb{D})$, then:

$$\begin{aligned} \|U^{-1}f\|^2_{\mathcal{D}_\alpha(\mathbb{D})} &= \int_{\mathbb{D}} |(U^{-1}f)'(z)| dm_\alpha(z) \\ &= \int_{\mathbb{D}} |f(z)|^2 dm_\alpha(z) \\ &= \|f\|^2_{L^2_a(\mathbb{D})}. \end{aligned}$$

Therefore U^{-1} is an isometry as well. Since $\mathcal{D}_\alpha(\mathbb{D})$ and $L^2_a(\mathbb{D})$ are Hilbert spaces with respect to their respective norms, it follows immediately that

U is unitary. □

We can therefore summarize the actions of the mappings U, U^{-1} and C_{τ_t} as we give below;

$$\mathcal{D}_\alpha(\mathbb{D}) \xrightarrow{U} L^2_a(\mathbb{D}) \xrightarrow{C_{\tau_t}} L^2_a(\mathbb{D}) \xrightarrow{U^{-1}} \mathcal{D}_\alpha(\mathbb{D}). \tag{4.2}$$

It is therefore apparent from equation (4.2) that $C_{\varphi_t} = U^{-1}C_{\tau_t}U$ and since U is unitary, we conclude that $C_{\tau_t} = UC_{\varphi_t}U^{-1}$.

Theorem 4.0.5

Let $C_{\varphi_t} : \mathcal{D}_\alpha(\mathbb{D}) \to \mathcal{D}_\alpha(\mathbb{D})$ be given by $C_{\varphi_t}f(z) = f(e^{it}z)$, and $C_{\tau_t} : L^2_a(\mathbb{D}, m_\alpha) \to L^2_a(\mathbb{D}, m_\alpha)$ by $C_{\tau_t} := \varphi' C_{\varphi_t}$. Then C_{φ_t} and C_{τ_t} are similar.

Proof. It is clear from Proposition 4.0.2 that the mapping $U : \mathcal{D}_\alpha(\mathbb{D}) \to L^2_a(m_\alpha)$ given by $Uf = f'$ for all $f \in \mathcal{D}_\alpha(\mathbb{D})$ is an isomorphism. Since $\mathcal{D}_\alpha(\mathbb{D})$ and $L^2_a(\mathbb{D}, m_\alpha)$ are Hilbert spaces, it follows that U is unitary, see Proposition 4.0.4.

Now, let $g \in L^2_a(\mathbb{D}, m_\alpha)$, then $f := U^{-1}g \in \mathcal{D}_\alpha(\mathbb{D})$ and

$$\begin{aligned}
UC_{\varphi_t}U^{-1}g &= UC_{\varphi_t}f \\
&= U(f \circ \varphi_t) \\
&= (f \circ \varphi_t)' \\
&= \varphi_t' f' \circ \varphi_t \\
&= \varphi_t' C_{\varphi_t} f' \\
&= C_{\tau_t} f' \\
&= C_{\tau_t} Uf \\
&= C_{\tau_t} g.
\end{aligned}$$

Therefore, $(C_{\varphi_t})_{t\geq 0}$ and $(C_{\tau_t})_{t\geq 0}$ are similar semigroups. $\square$

Before we state the main result of this chapter, recall that following [4], the multiplication operator M_z given by $M_z f(z) := zf(z)$ is bounded and bounded below on each of the spaces $X(\mathbb{D}) = H^p(\mathbb{D})$ or $L^p_a(\mathbb{D}, m_\alpha)$, $1 \leq p < \infty$ and $\alpha > -1$, with $ran(M_z) = \{f \in X(\mathbb{D}) : f(0) = 0\}$. The left inverse of M_z is the operator $Qf(z) := \frac{f(z)-f(0)}{z}$. For every $m \in \mathbb{N}$, $X(\mathbb{D}) = ran(M_z^m) \oplus \text{span}\{z^n : n \in \mathbb{Z}_+, n < m\}$, and $P_m = M_z^m Q^m$ is the projection of $X(\mathbb{D})$ onto $ran(M_z^m)$ with kernel span $\{z^n : n \in \mathbb{Z}_+, n < m\}$.

Theorem 4.0.6

Let $X(\mathbb{D})$ denote one of the spaces $H^p(\mathbb{D})$ or $L^p_a(\mathbb{D}, m_\alpha)$, $1 \leq p < \infty$ and $(T_t)_{t\geq 0}$ be given by $T_t f(z) = e^{ict} f(e^{ikt} z)$ for all $f \in L^2_a(\mathbb{D}, m_\alpha)$ and Γ be its infinitesimal generator. Then;

1. *$\Gamma f(z) = i(cf(z) + zf'(z))$ with domain $dom(\Gamma) = \{f \in X(\mathbb{D}) : f' \in X(\mathbb{D})\}$.*

2. *$\sigma(\Gamma, X) = \sigma_\rho(\Gamma, X) = \{i(n + c) : n \in \mathbb{Z}_+\}$ and for each $n \geq 0$, $ker\,(i(n + c) - \Gamma) = span(z^n)$.*

3. *If $\lambda \in \rho(\Gamma)$, then $ran(M_z^m)$ is $R(\lambda, \Gamma)$ -invariant for every $m \in \mathbb{Z}_+$ such that $m + c > Im(\lambda)$. Infact, if $h \in ran(M_z^m)$,then*

$$\begin{aligned} R(\lambda, \Gamma)h(z) &= z^{-(c+\lambda i)} \int_0^z w^{c+\lambda i-1} h(w)dw \\ &= z^m \int_0^1 t^{m+c\lambda i-1} Q^m h(tz)dt \end{aligned} \tag{4.3}$$

For proof see [4, Theorem 5.1].

In addition, we also give the following Lemma

Lemma 4.0.7

1. *The infinitesimal generator of* $(C_{\tau_t})_{t\geq 0} \subset L^2_a(\mathbb{D}, m_\alpha)$ *is given by* $\Gamma f = i\,(f(z) + zf'(z))$ *with domain* $\mathrm{dom}(\Gamma) = \{f \in L^2_a(\mathbb{D}) : f' \in L^2_a(\mathbb{D}, m_\alpha)\}$.

2. $\sigma(\Gamma, L^2_a(\mathbb{D}, m_\alpha)) = \sigma_\rho(\Gamma, L^2_a(\mathbb{D}, m_\alpha)) = \{i(n+1) : n \in \mathbb{Z}_+\}$ *and for each* $n \geq 0$, $ker\,(i(n+1) - \Gamma) = span(z^n)$.

3. *If* $\lambda \in \rho(\Gamma)$, *then* $ran(M_z^m)$ *is* $R(\lambda, \Gamma)$ *- invariant for every* $m \in \mathbb{Z}_+$ *such that* $m + 1 > Im(\lambda)$. *Infact, if* $h \in ran(M_z^m)$, *then*

$$R(\lambda, \Gamma)h(z) = z^{-(1+\lambda i)} \int_0^z w^i h(w)dw = z^m \int_0^1 t^{m+\lambda i} Q^m h(tz)dt$$

4. $R(\lambda, \Gamma)$ *is compact on* $L^2_a(\mathbb{D}, m_\alpha)$.

PROOF. Take note that $C_{\tau_t} f(z) = e^{it} f(e^{it}z)$ for all $f \in L^2_a(\mathbb{D}, m_\alpha)$. The result follows at once from Theorem 4.0.6 above by taking $c = k = 1$ and $p = 2$ □

The main result of this chapter which characterizes the properties of the group $(C_{\varphi_t})_{t\geq 0}$ on $\mathcal{D}_\alpha(\mathbb{D})$ is the following.

Theorem 4.0.8

Let $C_{\varphi_t} : \mathcal{D}_\alpha(\mathbb{D}) \to \mathcal{D}_\alpha(\mathbb{D})$ *and* Δ *be its infinitesimal generator. Then the following hold:*

1. $\Delta h(z) = izh'(z)$ *with* $\mathrm{dom}(\Delta) = \{h \in \mathcal{D}_\alpha(\mathbb{D}) : h' \in \mathrm{dom}(\Gamma)\}$

2. $\sigma_p(\Delta) = \sigma(\Delta) = \{i(n+1) : n \in \mathbb{Z}_+\}$, *and for each* $n \geq 0$, $ker\,(i(n+1) - \Gamma) = span(z^n)$

3. *If $\lambda \in \rho(\Delta)$, then $ran(M_z^m)$ is $R(\lambda, \Gamma)$-invariant for every $m \in \mathbb{Z}_+$ such that $m + 1 > Im(\lambda)$. Infact, if $h \in ran(M_z^m)$, then*

$$R(\lambda, \Delta)h(z) = \frac{i}{\lambda}\left(-h(z) + \frac{1}{z^{\lambda i}}\int_0^z w^{\lambda i} h'(w)dw\right)$$

4. *$R(\lambda, \Delta)$ is compact on $\mathcal{D}_\alpha(\mathbb{D})$*

5. $\sigma\left(R(\lambda, \Delta)\right) = \sigma_\rho\left(R(\lambda, \Delta)\right) = \left\{w \in \mathbb{C} : \left|w - \frac{1}{2Re\lambda}\right| = \frac{1}{2Re\lambda}\right\}$

Moreover,

$$r\left(R(\lambda, \Delta)\right) = \|R(\lambda, \Delta)\| = \frac{1}{|Re(\lambda)|}.$$

PROOF. Since $C_{\tau_t} = UC_{\varphi_t}U^{-1}$, it follows that;
$C_{\varphi_t} = U^{-1}C_{\tau_t}U$. Using the similarity theory of semigroups, it follows that if Γ is the generator of the group $(C_{\tau_t})_{t\geq 0}$ and Δ is the generator of the group $(C_{\varphi_t})_{t\geq 0}$, then $\Delta = U^{-1}\Gamma U$ with the domain $\mathrm{dom}(\Delta) = U^{-1}\mathrm{dom}(\Gamma)$.
Now, let $f' \in L_a^2(\mathbb{D})$. Then $f \in \mathrm{dom}(\Gamma)$ and $h := U^{-1}f$ belongs to $\mathrm{dom}(\Delta)$ with $f = Uh$. Then;

$$\begin{aligned}
\Delta\left(h(z)\right) &= U^{-1}\Gamma U h(z) \\
&= U^{-1}\Gamma f(z) \\
&= U^{-1}\left(i(f(z) + zf'(z))\right) \\
&= i\left(U^{-1}f(z) + U^{-1}(zf'(z))\right)
\end{aligned}$$

Since U is the differential operator, it follows that U^{-1} is the integral operator. Now, $U^{-1}f(z) = h(z)$ while $U^{-1}(zf'(z)) = \int zf'(z)dz$ whose solution by the application of integration by parts yields $zh'(z) - h(z)$. Therefore,

$$\begin{aligned}\triangle(h(z)) &= i\left(U^{-1}f(z) + U^{-1}(zf'(z))\right)\\ &= i\left(h(z) + zh'(z) - h(z)\right)\\ &= izh'(z)\end{aligned}$$

as desired. Moreover,

$$\mathrm{dom}(\Delta) = U^{-1}\mathrm{dom}(\Gamma) = \left\{U^{-1}f : f \in \mathrm{dom}(\Gamma)\right\}$$

Now, $h \in \mathrm{dom}(\Delta) \Leftrightarrow Uh \in \mathrm{dom}(\Gamma) \Leftrightarrow h' \in \mathrm{dom}(\Gamma)$, and thus;

$$\mathrm{dom}(\Delta) = \{h \in \mathcal{D}_\alpha(\mathbb{D}) : h' \in \mathrm{dom}(\Gamma)\}$$

This proves assertion (1).

To prove assertion 3, we recall that by the similarity of the corresponding semigroups, we have that if $\lambda \in \rho(\Delta)$, and since $\rho(\Delta) = \rho(\Gamma)$, then for $m \in \mathbb{Z}_+$, $m+1 > Im(\lambda)$, and if $h \in ran(M_z^m)$, then by Lemma 4.0.7

$$\begin{aligned}R(\lambda,\Delta)h(z) &= U^{-1}R(\lambda,\Gamma)Uh(z)\\ &= U^{-1}R(\lambda,\Gamma)h'(z)\\ &= U^{-1}\left(z^{-(1+\lambda i)}\int_0^z w^{\lambda i}h'(w)dw\right)\\ &= \int\left(z^{-(1+\lambda i)}\int_0^z w^{\lambda i}h'(w)dw\right)dz\end{aligned}$$

By integration by parts; Let $u = \int_0^z w^{\lambda i} h'(w)dw$ and $dv = z^{-(1+\lambda i)}$. Then $du = z^{\lambda i} h'(z)dz$ and $v = \frac{z^{-\lambda i}}{-\lambda i}$. Therefore;

$$\begin{aligned}\int \left(z^{-(1+\lambda i)} \int_0^z w^{\lambda i} h'(w)dw \right) dz &= -\frac{1}{\lambda i} z^{-\lambda i} \int_0^z w^{\lambda i} h'(w)dw + \frac{1}{\lambda i} \int z^{-\lambda i} z^{\lambda i} h'(z)dz \\ R(\lambda, \Delta)h(z) &= \frac{i}{\lambda} \left(-h(z) + \frac{1}{z^{\lambda i}} \int_0^z w^{\lambda i} h'(w)dw \right) \text{ as claimed.}\end{aligned}$$

The Compactness of the resolvent $R(\lambda, \Delta)$ follows from the compactness of the resolvent $R(\lambda, \Gamma)$ (See Lemma 4.0.7, assertion 4) since the relation $U^{-1}R(\lambda, \Gamma)U$ preserves compactness which proves (4).

For assertion (5), the spectral mapping theorem and the assertion 2 of this theorem imply that for all $\mu \in \rho(\Delta)$,

$$\begin{aligned}\sigma(R(\lambda, \Delta)) &= \left\{ \frac{1}{\lambda - z} : z \in \sigma(\Delta) \right\} \cup \{0\} \\ &= \left\{ \frac{1}{\lambda + 2(\gamma + n)i} : n \in \mathbb{Z}_+ \right\} \cup \{0\} \\ &= \left\{ \omega \in \mathbb{C} : \left| \omega - \frac{1}{2Re(\lambda)} \right| = \frac{1}{2Re(\lambda)} \right\}\end{aligned}$$

which is a circle centred at $\left(\frac{1}{2Re(\lambda)}, 0 \right)$ with radius $\frac{1}{2Re(\lambda)}$.

The equality $\sigma(R(\lambda, \Delta)) = \sigma_p(R(\lambda, \Delta))$ follows from the compactness of $\mathcal{R}(\lambda, \Delta)$ given by assertion (4) above and Theorem 2.0.5. From the spectrum, it is clear that the spectral radius of the resolvent is $r(R(\lambda, \Delta)) = \frac{1}{Re(\lambda)}$. Finally, the boundedness of the spectral radius $r(R(\lambda, \Delta))$ by the norm $\|R(\lambda, \Delta)\|$ as well as the Hille-Yosida theorem immediately yield $r(R(\lambda, \Delta)) = \frac{1}{Re(\lambda)|} \leq \|R(\lambda, \Delta)\| \leq \frac{1}{Re(\lambda)|}$. This completes the proof. $\square$

Chapter 5

SUMMARY AND RECOMMENDATION

5.1 Summary

We considered groups of composition operators corresponding to the three groups of automorphisms (scaling, translation and rotation), investigated their semigroup as well as their spectral properties on the Dirichlet space of the upper half-plane. More specifically, we considered the self-analytic maps $(\varphi_t)_{t\geq 0}$ of the groups and proved that they form a group and are indeed self-maps on $\mathcal{D}_\alpha(\mathbb{U})$ for scaling and translation and a self-map on $\mathcal{D}_\alpha(\mathbb{D})$ for rotation group. We then proved that the induced groups of composition operators are strongly continuous and their infinitesimal generator Γ and domain exists (Theorem 3.3.1 and Theorem 3.4.1). The fact that the induced group of composition operators $(T_t)_{t\geq 0}$ fails to be an isometry on $\mathcal{D}_\alpha(\mathbb{U})$ complicates the spectral analysis of the group $(T_t)_{t\geq 0}$. This is because the theory of spectra of semigroups of linear operators are easily applied when we can identify exactly what the spectrum of

$(T_t)_{t\geq 0}$ is. Using Lemmas 3.3.4 and 3.3.5, we established that the point spectrum for the groups is empty. For the rotation group, we considered the groups of weighted composition operators on $\mathcal{D}_\alpha(\mathbb{D})$ of the form $S_{\varphi_t}f(t) = e^{ict}f(e^{ikt})$ for all $f \in \mathcal{D}_\alpha(\mathbb{D})$. While analyzing the group, we employed the theory of similarity of semigroups in carrying out complete analysis of both the semigroup and spectral properties of C_{φ_t}. We established that the group is an isometry (Proposition 4.0.3) and by Theorem 4.0.8, we characterized the properties of the group $(C_{\varphi_t})_{t\geq 0}$ which formed our main results of the chapter. For instance, we have proved that the point spectrum of the generator of the group $(S_{\varphi_t})_{t\geq 0}$ coincides with its spectrum (Theorem 4.0.8) and that the resulting resolvent operator is compact.

Moreover, the spectrum and the point spectrum for the resolvents coincide and are circles of radius $\frac{1}{2Re(\lambda)}$ centred at $\left(\frac{1}{2Re(\lambda)}, 0\right)$. The norm as well as the spectral radius of resolvent which is an integral operator also coincide.

5.2 Recommendation

Following the results of the study, we recommend the following open areas for further research:

1. From this thesis, the spectral picture for the groups of weighted composition operators on $\mathcal{D}_\alpha(\mathbb{U})$ is not complete for the scaling and translation groups. This was due to the lack of the property of isometry in the spaces. A different approach can be used to

complete the spectral analysis of these groups on $\mathcal{D}_\alpha(\mathbb{U})$ as well as their corresponding resolvents.

2. In this study, we considered the Dirichlet spaces of the upper half plane for which the value of $p = 2$ which in turn are just but Hilbert spaces. The more general Dirichlet spaces of the upper half plane for which $p \neq 2$ together with composition operators defined on them still remain open for investigation.

3. The properties of the Dirichlet spaces of the upper half-plane are not well established in literature. For instance, the reproducing kernel for $\mathcal{D}_\alpha(\mathbb{U})$ is not known. Even the concept of the Dirichlet projections which map onto the analytic spaces and play a critical role in the theory of reproducing kernels is not well established in literature. These would be very vital in the study of the duality properties of Dirichlet spaces which is an area that has not been explored. It would be interesting to investigate these aspects of the Dirichlet space.

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