REPRODUCING KERNEL FOR THE DIRICHLET SPACE OF THE UPPER HALF - PLANE AND CESÀRO TYPE OPERATOR

BY

OYUGI EFFIE ADHIAMBO

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE IN PURE MATHEMATICS

SCHOOL OF MATHEMATICS, STATISTICS AND ACTUARIAL SCIENCE

MASENO UNIVERSITY

Declaration

This thesis is my own work and has not been presented for a degree award in any other institution.

OYUGI EFFIE ADHIAMBO

MSC/MAT/00177/2017

This thesis has been submitted for examination with our approval as the university supervisors.

Dr. Job O. Bonyo, Supervisor

Dr. David O. Ambogo, Supervisor

Maseno University

2021

Acknowledgments

My profound gratitude goes to my supervisor Dr. Bonyo Job for introducing me to this area of study, for his guidance, patience and availability throughout this thesis research. His thorough supervision, unwavering support and wealth of knowledge made this work possible.

I'm equally very grateful to my second supervisor Dr. Ambogo David for his invaluable input, encouragement and for always being available for consultation. I thank Dr. Owino Isaac and the entire School of Mathematics, Statistics and Actuarial Science for their continuous assistance and support.

I am deeply grateful to my whole family, more so, my spouse Jasper, children Joey, Joely and Jasper, my parents and siblings for their emotional and financial support for the entire period of my studies.

I also wish to thank my friend and colleague in the pure mathematics class, Ketray whose encouragement and moral support has been great.

I thank Maseno University for giving me a chance to undertake graduate studies.

Finally, I'm forever indebted to God Almighty for His grace, mercy and faithfulness and for guiding me and making this possible.

Dedication

To my spouse Jasper, children Joey, Joely and Hope.

Abstract

Reproducing kernels for spaces of analytic functions continues to be of interest to many mathematicians. Most studies have concentrated on the analytic spaces of the unit disk. Reproducing kernels for the Bergman, Hardy and Dirichlet spaces of the unit disk have been extensively determined. There has been a growing interest on the analytic spaces of the upper half plane in the recent past. For instance, the Bergman and the Szegö kernels of the upper half plane have recently been determined. However, the theory of the Dirichlet space of the upper half plane is not well established in literature. In this study therefore, we have determined the reproducing kernel for the Dirichlet space of the upper half plane using the Cayley transform to construct an invertible isometry between the corresponding spaces of the unit disk and that of the upper half plane. By applying Cauchy-Schwarz inequality, we have established the growth condition for functions in the Dirichlet space of the upper half plane. We have then constructed an integral operator of the Cesàro type which is acting on the Dirichlet space of the upper half plane using the approach of strongly continuous semigroups of composition operators on Banach spaces. Moreover, we have determined the spectra and norm properties of the Cesàro type operator using the spectral mapping theorems as well as the Hille-Yosida theorem. Results of this study have contributed new knowledge to this area of mathematics and will advance further research on this and related areas.

Contents

Title page i
Declaration ii
Acknowledgements iii
Dedication iv
Abstract v
Table of Contents vi
Index of Notations viii

Chapter 1 Introduction **1**

1.1 Background of the study 1
1.2 Basic concepts 4
1.2.1 The unit disk $\mathbb{D}$ and Upper half plane $\mathbb{U}$ 4
1.2.2 Analytic Functions 4
1.2.3 Spaces of Analytic Functions of interest 5
1.2.4 Reproducing Kernels 9
1.2.5 Integral Operators 10
1.2.6 Spectra of Linear operators 11
1.2.7 Semigroups of Linear operators 12
1.2.8 Composition Operators and Semigroups 13
1.3 Statement of the Problem 14
1.4 Objectives of the Study 15
1.5 Significance of the Study 15
1.6 Research Methodology 16

Chapter 2 Literature Review **18**

Chapter 3 Reproducing kernel for the Dirichlet space of the upper half plane, $\mathcal{D}(\mathbb{U})$ **25**

3.1 Introduction 25

3.2 Reproducing Kernels for the Dirichlet spaces 28

3.2.1 Dirichlet space of the unit disk 29

3.2.2 Dirichlet space of the upper half plane 34

Chapter 4 Cesàro - Type operator on Dirichlet space of the upper half plane. **41**

4.1 Introduction 41

4.2 Scaling group 42

4.2.1 Cesàro - Type operator 51

Chapter 5 Summary and Recommendations **55**

5.1 Summary 55

5.2 Recommendations 56

References 58

Index of Notations

$\mathbb{C}$ Complex plane. 4

$\mathbb{D}$ Unit disk. 4

dA Area measure. 4

$\mathbb{U}$ Upper half plane. 4

$\Im$ Imaginary part of a complex number. 4

ψ Cayley transform. 4

Ω Open subset of $\mathbb{C}$. 5

$\mathcal{H}(\Omega)$ Space of holomorphic functions on Ω. 5

$H^p(\mathbb{D})$ Hardy space of the unit disk. 5

$H^p(\mathbb{U})$ Hardy space of the upper half plane. 6

$L^p_a(\mathbb{D})$ Bergman space of the unit disk. 6

$L^p_a(\mathbb{U})$ Bergman space of the upper half plane. . . . 7

$\mathcal{D}(\mathbb{D})$ Dirichlet space of the unit disk. 7

$\mathcal{D}(\mathbb{U})$ Dirichlet space of the upper half plane. . . . 8

H Hilbert space. 9

K Reproducing kernel. . . . 10

$\mathrm{dom}(T)$ Domain of T. 11

$\rho(T)$ Resolvent set of T. . . 11

$\sigma(T)$ Spectrum of T. 11

$r(T)$ Spectral radius of T. . 11

$\sigma_p(T)$ Point spectrum of T. . 12

Γ Infinitesimal generator. . . 13

$R(\lambda, \Gamma)$ Resolvent operator . 21

$\partial\mathbb{D}$ Boundary of the unit disk. 22

$\Re$ Real part of a complex number. 27

$K_\omega(z) := K(z, \omega)$ Reproducing kernel. 28

$K_{\mathbb{D}}(z, \omega)$ Reproducing kernel for the Dirichlet space of the unit disk. 32

$K_{\mathbb{U}}(z, \omega)$ Reproducing kernel for the Dirichlet space of the upper half plane. 36

$\mathcal{C}$ Cesàro - Type operator . . 51

Chapter 1

Introduction

1.1 Background of the study

The notion of the reproducing kernel was first introduced early in the 20th century by Zaremba [34] in 1907 in his work concerning boundary value problems for harmonic and biharmonic functions. Zaremba was the first to introduce, in a particular case, the kernel corresponding to a class of functions, and to state its reproducing property. He however did not develop any theory nor give a name to the kernels he introduced. Mercer [19] in 1909, examined functions which satisfy the reproducing property in the theory of integral equations. He considered continuous kernels of positive definite integral operators under the name "positive definite kernels" and this has been used by many others interested in integral equations. Mercer [19] gave a connection of the theory of integral equations to reproducing kernels and determined functions which satisfy a reproducing property in the theory of integral equations thus characterizing his kernels among all the continuous kernels of integral equations. The idea of the reproducing kernel remained untouched for several years until in 1921 when it appeared in the dissertations of three Berlin mathematicians, Szegö (1921), Bergman (1922) and Bochher (1922). In 1935, Moore [21] examined the positive definite kernels in his general analysis under the name "positive hermitian matrices" with a kind of generalization of integral equations. Moore [20] proved that for every positive Hermitian matrix, there corresponds a class of functions forming a Hilbert space with scalar product $\langle f, g \rangle$ and in which the kernel has the reproducing property $f(y) = \langle f(x), K(x, y) \rangle \forall x, y \in \mathbf{H}$. The subject was developed systematically in the early 1950's by Aronszajin [5] and Bergman [9]. In particular, Bergman [9] introduced reproducing kernels in one variable and in several variables for the class of harmonic and analytical functions and he called them kernel functions. He further noticed the reproducing property of these kernels but did not use their basic characteristic property as is done in the present. The original idea of Zaremba to apply the kernels to the solution of boundary value theorem was developed by Bergman and Schiffer [10]. In their work, the kernel was shown to be a powerful tool for solving boundary value problems of Partial Differential Equations of elliptic type. Further, by application of kernels to conformal mapping of multiply connected domains, some useful results were obtained by Bergman and Schiffer. Many important results were achieved by the use of these kernels in the theory of one and several complex variables; in conformal mapping of simply and multiply connected domains; in pseudo-conformal mappings; in the study of invariant Riemannian metrics and in other subjects. The reproducing kernel of analytic spaces of the unit disk is well captured in literature. Bergman [10] determined the reproducing kernel of the Bergman spaces on the unit disk also known as the Bergman kernel. Szegö [33] also computed the reproducing kernel of the Hardy space on the unit disk also known as the Szegö kernel. The reproducing kernel for the Dirichlet space of the unit disk is also well defined in literature [15]. The study of Reproducing kernels for the analytic spaces of the upper half plane has not been extensively done. Recently Bonyo [11] determined the reproducing kernels for the Hardy and Bergman spaces of the upper half plane. However, the reproducing kernel for the Dirichlet space of the upper half plane has not been determined. We shall work out this reproducing kernel and the corresponding growth condition for the functions in the Dirichlet space of the upper half plane. The thesis is organized as follows:

In chapter 1, we give the necessary background to the study and highlight some basic concepts needed for development of other chapters. The problem statement, study objectives, significance of the study as well as the methods employed in solving the problems of the study; are given in this chapter. In chapter 2, a review of related literature on reproducing kernels and integral operators on spaces of analytic functions is given. Known theorems that are applied in this study are also stated. In chapter 3, we compute the reproducing kernel for the Dirichlet space of the upper half plane and establish the growth condition for functions in the Dirichlet space of the upper half plane. In chapter 4, we study scaling groups on the Dirichlet space of the upper half plane. We then construct a Cesàro operator acting on the Dirichlet space of the upper half plane and investigate its properties. Finally in chapter 5, we give a summary and make recommendations for further research.

1.2 Basic concepts

1.2.1 The unit disk $\mathbb{D}$ and Upper half plane $\mathbb{U}$

Let $\mathbb{C}$ be the complex plane. The set $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is known as the (open) unit disk of the complex plane. Also let $dA(z) = \frac{1}{\pi}dxdy$ for $z = x + iy$ denote the area measure on $\mathbb{D}$ normalized so that the area of $\mathbb{D}$ is one. The set $\mathbb{U} = \{\omega \in \mathbb{C} : \Im(\omega) > 0\}$ is the upper half plane of the complex plane where $\Im(\omega)$ denotes the imaginary part of the complex number ω. Also $dA(\omega)$ shall denote the area Lebesgue measure on $\mathbb{U}$. The function $\psi(z) = \frac{i(1+z)}{1-z}$ is referred to as the Cayley transform and maps the unit disc $\mathbb{D}$ conformally onto the upper half-plane $\mathbb{U}$ and its inverse is $\psi^{-1}(\omega) = \frac{\omega-i}{\omega+i}$. For more details see [36].

1.2.2 Analytic Functions

Let $\Omega \subset \mathbb{C}$ be an open set and let $f : \Omega \to \mathbb{C}$ be a complex function. Then, f is said to be differentiable at a point $z_\circ \in \Omega$ if the limit

$$f'(z_\circ) = \lim_{\Delta z \to 0} \frac{f(z_\circ + \Delta z) - f(z_\circ)}{\Delta z}$$

exists. Also, the function f is said to be analytic in an open domain Ω if its derivative $f'(z)$ exists for all $z \in \Omega$. A function that is analytic on the whole complex plane, that is to say $\Omega = \mathbb{C}$ is called an entire function. The function f is said to be holomorphic if it is complex differentiable at every point in Ω. A biholomorphism is a map that is bijective and holomorphic with an inverse that is also holomorphic. A biholomorphic map of Ω into itself is called an automorphism. The automorphism groups denoted by $\mathrm{Aut}(\Omega)$ consists of all biholomorphic mappings of $\mathrm{Aut}(\Omega)$ with composition of maps being their group operation. For details see [23, 36].

1.2.3 Spaces of Analytic Functions of interest

Let X be a Hausdorff topological space and f a continuous complex valued function on X. If f is compact, $\|f\|$ is finite. In this case, we say a sequence f_n converges uniformly on X if $\|f_n - f_m\| \to 0$ as $n, m \to \infty, n, m \in \mathbb{N}$. We shall mean by a Fréchet space X, a topological vector space with the following properties

(i) There is countable increasing family of norms $\|\cdot\|_n$ on X which induce topology

(ii) The topology is metric and X is complete in this metric.

For an open subset Ω of $\mathbb{C}$, let $\mathcal{H}(\Omega)$ denote the Fréchet space of analytic functions $f : \Omega \to \mathbb{C}$ endowed with the topology of uniform convergence on compact subsets of Ω.

(i) **Hardy spaces** For $1 \leq p < \infty$, the Hardy spaces of the unit disk, $H^p(\mathbb{D})$, are defined as

$$H^p(\mathbb{D}) := \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_{H^p(\mathbb{D})} := \sup_{0<r<1} \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} < \infty \right\}, \tag{1.1}$$

while the Hardy spaces of the upper half plane, $H^p(\mathbb{U})$, are defined as

$$H^p(\mathbb{U}) := \left\{ f \in \mathcal{H}(\mathbb{U}) : \|f\|_{H^p(\mathbb{U})} := \sup_{y>0} \left(\int_{-\infty}^{\infty} |f(x+iy)|^p dx \right)^{\frac{1}{p}} < \infty \right\}. \tag{1.2}$$

Hardy spaces, $H^p(\cdot)$, are Banach spaces with respect to their respective norms $\| \cdot \|_{H^p(\cdot)}$. If $p = 2$, $H^2(\cdot)$ is a Hilbert space with inner product defined on the unit disk by : For each $f, g \in H^2(\mathbb{D})$,

$$\langle f, g \rangle_{H^2(\mathbb{D})} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(re^{i\theta}) \overline{g(re^{i\theta})} d\theta.$$

For $f \in H^p(\mathbb{D})$, f satisfies the growth condition

$$|f(z)| \leq \frac{C\|f\|_p}{(1-|z|)^{\frac{1}{p}}}, \tag{1.3}$$

where C is a constant and $z \in \mathbb{D}$, while for $f \in H^p(\mathbb{U})$, f satisfies the growth condition

$$|f(\omega)| \leq \frac{C_p\|f\|_p}{(\Im(\omega))^{\gamma}}, \tag{1.4}$$

where C_p is a constant, $\omega \in \mathbb{U}$ and $\gamma = \frac{1}{p}$.

(ii) **Bergman spaces** For $1 \leq p < \infty$, the Bergman spaces of the unit disk, $L^p_a(\mathbb{D})$, are defined by

$$L^p_a(\mathbb{D}) := \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_{L^p_a(\mathbb{D})} := \left(\int_{\mathbb{D}} |f(z)|^p dA(z) \right)^{\frac{1}{p}} < \infty \right\}, \tag{1.5}$$

while the Bergman spaces of the upper half plane, $L_a^p(\mathbb{U})$, are defined by

$$L_a^p(\mathbb{U}) := \left\{ f \in \mathcal{H}(\mathbb{U}) : \|f\|_{L_a^p(\mathbb{U})} := \left(\int_{\mathbb{U}} |f(\omega)|^p dA(\omega) \right)^{\frac{1}{p}} < \infty \right\}. \tag{1.6}$$

Bergman spaces, $L_a^p(\cdot)$, are also Banach spaces with respect to their respective norms $\|\cdot\|_{L_a^p(\cdot)}$ and $L_a^2(\cdot)$ is a Hilbert space with inner product defined on the unit disk given by : For each $f, g \in L_a^2(\mathbb{D})$

$$\langle f, g \rangle_{L_a^2(\mathbb{D})} = \int_{\mathbb{D}} f(z)\overline{g(z)} dA(z),$$

and on the upper half plane by : For every $f, g \in L_a^2(\mathbb{U})$

$$\langle f, g \rangle_{L_a^2(\mathbb{U})} = \int_{\mathbb{U}} f(\omega)\overline{g(\omega)} dA(\omega).$$

If $f \in L_a^p(\mathbb{D})$, then f satisfies the well known growth condition

$$|f(z)| \leq \frac{K\|g\|}{(1-|z|^2)^{\gamma}}, \tag{1.7}$$

where K is a constant, $\gamma = \frac{2}{p}$ and $z \in \mathbb{D}$, while for $f \in L_a^p(\mathbb{U})$, f satisfies the growth condition

$$|f(\omega)| \leq \frac{C_p\|f\|_p}{(\Im(\omega))^{\gamma}}, \tag{1.8}$$

where C_p is a constant, $\omega \in \mathbb{U}$ and $\gamma = \frac{2}{p}$.

(iii) **Dirichlet spaces** The Dirichlet space of the unit disk, $\mathcal{D}(\mathbb{D})$, is defined by

$$\mathcal{D}(\mathbb{D}) := \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|^2_{\mathcal{D}_1(\mathbb{D})} = \int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty \right\}, \quad (1.9)$$

with its norm given by $\|f\|^2_{\mathcal{D}(\mathbb{D})} = |f(0)|^2 + \|f\|^2_{\mathcal{D}_1(\mathbb{D})}$ and where $\|.\|_{\mathcal{D}_1(\mathbb{D})}$ is a seminorm on $\mathcal{D}(\mathbb{D})$.

The corresponding Dirichlet space of the upper half plane, $\mathcal{D}(\mathbb{U})$, is given by

$$\mathcal{D}(\mathbb{U}) := \left\{ f \in \mathcal{H}(\mathbb{U}) : \|f\|^2_{\mathcal{D}_1(\mathbb{U})} = \int_{\mathbb{U}} |f'(\omega)|^2 dA(\omega) < \infty \right\}, \quad (1.10)$$

and the norm is given by $\|f\|^2_{\mathcal{D}(\mathbb{U})} = |f(i)|^2 + \|f\|^2_{\mathcal{D}_1(\mathbb{U})}$ where $\|.\|_{\mathcal{D}_1(\mathbb{U})}$ is a seminorm on $\mathcal{D}(\mathbb{U})$. The Dirichlet space, $\mathcal{D}(\cdot)$, is a Banach space with respect to its norm $\|\cdot\|_{\mathcal{D}(\cdot)}$ and is a Hilbert space with inner product defined on the unit disk by

$$\langle f, g \rangle_{\mathcal{D}(\mathbb{D})} = \langle f(0), g(0) \rangle + \int_{\mathbb{D}} f'(z)\overline{g'(z)} dA(z), \forall f, g \in \mathcal{D}(\mathbb{D}), \quad (1.11)$$

and on the upper half plane by

$$\langle f, g \rangle_{\mathcal{D}(\mathbb{U})} = \langle f(i), g(i) \rangle + \int_{\mathbb{U}} f'(\omega)\overline{g'(\omega)} dA(\omega), \forall f, g \in \mathcal{D}(\mathbb{U}). \quad (1.12)$$

Remark 1.2.1

In fact if $f = g$, then the equations (1.11) and (1.12) coincide with the norms on $\mathcal{D}(\mathbb{D})$ and $\mathcal{D}(\mathbb{U})$ respectively. Indeed, for (1.11),

$$\langle f, f \rangle_{\mathcal{D}(\mathbb{D})} = \langle f(0), \overline{f(0)} \rangle + \int_{\mathbb{D}} f'(z)\overline{f'(z)} dA(z),$$

which yields

$$\|f\|^2_{\mathcal{D}(\mathbb{D})} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 dA(z),$$

while for (1.12),

$$\langle f, f\rangle_{\mathcal{D}(\mathbb{U})} = \langle f(i), \overline{f(i)}\rangle + \int_{\mathbb{U}} f'(\omega)\overline{f'(\omega)} dA(\omega),$$

yielding

$$\|f\|^2_{\mathcal{D}(\mathbb{U})} = |f(i)|^2 + \int_{\mathbb{U}} |f'(\omega)|^2 dA(\omega).$$

For $f \in \mathcal{D}(\mathbb{D})$, then f satisfies the growth condition

$$|f(z)| \leq c\|f\|\sqrt{\log \frac{1}{1-|z|^2}}, \tag{1.13}$$

where c is a constant and $z \in \mathbb{D}$.

The growth condition for functions in $\mathcal{D}(\mathbb{U})$ is not explicitly known in literature. We refer to [15, 24, 36] for details.

1.2.4 Reproducing Kernels

Let $\mathbf{H}$ denote a Hilbert space of functions with inner product $\langle ., .\rangle_{\mathbf{H}}$ defined on an open set $\Omega \subseteq \mathbb{C}$. A function $k : \Omega \times \Omega \to \mathbb{C}$ is a kernel if there exists a Hilbert space $\mathbf{H}$ and a function ϕ such that for any $\alpha, \beta \in \Omega$, $k(\alpha, \beta) = \langle \phi(\alpha), \phi(\beta)\rangle_{\mathbf{H}}$ given that $\forall \alpha \in \Omega, \phi(\alpha) \in \mathbf{H}$.
We call a reproducing kernel for $\mathbf{H}$ a complex function $K : \Omega \times \Omega \to \mathbb{C}$ such that if we put $K_\omega(z) = K(z, \omega)$, then the following two properties

hold;

1. For every $\omega \in \Omega$, the function K_ω belongs to $\mathbf{H}$, and,

2. For all $f \in \mathbf{H}$ and $\omega \in \Omega$, we have $f(\omega) = \langle f, K_\omega \rangle_{\mathbf{H}}$.

For comprehensive details we refer to [22, 28].

1.2.5 Integral Operators

Let X be a subset of $\mathbb{R}^d, d \in \mathbb{R}$ and $K : X \times X \to \mathbb{C}$ be a reproducing kernel satisfying the assumptions stated in subsection 1.2.4. Let dA be a probability measure on X and denote by $L^2(X)$ the space of square integrable (complex) functions with norm $\|f\|^2 = \langle f, f \rangle = \int_X |f(x)|^2 dA(x)$. We define an integral operator $L_K : L^2(X) \to L^2(X)$ by

$$(L_K f)(x) = \int_X K(x, s) f(s) dA(s). \tag{1.14}$$

K is the integral kernel or simply the symbol of the integral operator. We refer to [2, 14, 22, 13] for details.

Example 1.2.2

(i) **Volterra Type Integral Operators.** A volterra type integral operator on a space of holomorphic functions of the disk, $\mathcal{H}(\mathbb{D})$, induced by a holomorphic symbol $g : \mathbb{D} \to \mathbb{C}$ is a bounded linear operator defined by

$$V_g f(z) = \int_0^z f(\xi) g'(\xi) d\xi, \tag{1.15}$$

for $z \in \mathbb{D}$ and $f \in \mathcal{H}(\mathbb{D})$.

(ii) **Cesàro operators.** For a function $f(z)$ analytic on the unit disc, the Cesàro operator $C : \mathcal{H}(\mathbb{D}) \to \mathcal{H}(\mathbb{D})$ is a bounded linear operator defined by

$$Cf(z) = \frac{1}{z}\int_0^z \frac{f(\zeta)}{1-\zeta}d\zeta, \tag{1.16}$$

for all $z \in \mathbb{D}$ and $f \in \mathcal{H}(\mathbb{D})$. For a function f analytic on the upper half plane, the Cesàro operator is defined by

$$g(\omega) = C(f)(\omega) = \frac{1}{\omega}\int_0^\omega f(\zeta)d\zeta, \tag{1.17}$$

for $\omega \in \mathbb{U}$.

(iii) **Hankel operator.** A Hankel operator on H^2 is a bounded linear operator defined by $\mathbb{T} = PJM_\phi$ for some $\phi \in L^\infty$, where P is the orthogonal projection of L^2 to H^2, J is the operator on L^2 given by $Jf(z) = f(\bar{z})$ and M_ϕ is the multiplication operator defined as $M_\phi f(z) = \phi(z)f(z)$. In this terminology $\mathbb{T}$ is said to be induced by the symbol $\phi \in L^\infty$ and is denoted as $\mathbb{T} = \mathbb{T}_\phi$.

1.2.6 Spectra of Linear operators

Let $(X, \|.\|)$ and $(Y, \|.\|)$ be Banach spaces over $\mathbb{C}$. A linear operator $T : X \to Y$ is a linear mapping of a linear subspace dom(T) of X into Y, where dom(T) is the domain of T.

T is said to be a closed operator if its graph $\{(x, Tx)|\ x \in \text{dom}(T)\}$ is closed in $X \times Y$.

Let T be closed operator on X, the resolvent set of T, $\rho(T)$ is given by $\rho(T) = \{\lambda \in \mathbb{C} : \lambda I$ - T is invertible$\}$ and its spectrum $\sigma(T) = \mathbb{C}\backslash\rho(T)$. The spectral radius of T is defined by $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$. The point spectrum $\sigma_p(T) = \{\lambda \in \mathbb{C} : Tx = \lambda x \text{ for some } 0 \neq x \in \text{dom}(T)\}$. For $\lambda \in \rho(T)$, the operator $R(\lambda, T) := (\lambda I - T)^{-1}$ is called the resolvent operator of T or simply the resolvent operator. Given two normed vector spaces, V and W, a linear isometry is a linear map $T : V \to W$ that preserves norms:

$$\|Tv\| = \|v\|$$

$\forall v \in V$. In an inner product space, the definition reduces to

$$\langle v, v \rangle = \langle Tv, Tv \rangle$$

$\forall v \in V$, equivalent to $T^*T = I_V$ where T^* is the adjoint of T. This also implies that isometries preserve inner products, as

$$\langle Tu, Tv \rangle = \langle u, T^*Tv \rangle = \langle u, v \rangle.$$

A unitary operator is a bounded linear operator $T : \mathbf{H} \to \mathbf{H}$ that satisfies $T^*T = TT^* = I$. The condition $T^*T = I$ defines an isometry. Thus a unitary operator is a bounded linear operator that is a surjective isometry.

1.2.7 Semigroups of Linear operators

Let X be a Banach space. A one parameter family $(T_t)_{t \geq 0}$ is a semigroup of bounded linear operators on X if

(i) $T_o =$ I (Identity operator on X)

(ii) $T_{t+s} = T_t \circ T_s$ for every $t, s \geq 0$.

If $(T_t)_{t\geq 0}$ and $(T_t)_{t\leq 0}$ are both semigroups on X, we say that $(T_t)_{t\in\mathbb{R}}$ is a group on X.
The semigroup, $(T_t)_{t\geq 0}$, of bounded linear operators on X is said to be strongly continuous if $\lim_{t\to 0^+}\|T_t x - x\| = 0$ for all $x \in X$.
The **infinitesimal generator** Γ of $(T_t)_{t\geq 0}$ is defined by

$$\Gamma x = \lim_{t\to 0^+} \frac{T_t x - x}{t} = \frac{\partial}{\partial t}(T_t x)\bigg|_{t=0} \tag{1.18}$$

for each $x \in \text{dom}(\Gamma)$, where the domain of Γ is given by

$$\text{dom}(\Gamma) = \left\{ x \in X : \lim_{t\to 0^+} \frac{T_t x - x}{t} \text{ exists.} \right\}$$

A strongly continuous semigroup is also called $C_\circ$- semigroup. For details, see [16, 23, 31].

1.2.8 Composition Operators and Semigroups

Suppose that φ is a function analytic on an open subset Ω of $\mathbb{C}$ such that $\varphi : \Omega \to \Omega$, then φ is referred to as a self analytic map. For a parameter $t \geq 0$, a family $(\varphi_t)_{t\geq 0}$ of self analytic maps on Ω is a semigroup if it satisfies the following conditions;

(i) $\varphi_\circ(z) = z$, the identity map of Ω, that is $\varphi_0 = I$.

(ii) $\varphi_{t+s} = \varphi_t \circ \varphi_s$ for every $t, s \geq 0$,

(iii) The map $(t, z) \to \varphi_t(z)$ is jointly continuous on $[0, \infty) \times \Omega$.

If both $(\varphi_t)_{t\geq 0}$ and $(\varphi_t)_{t\leq 0}$ are semigroups, then $(\varphi_t)_{t\in\mathbb{R}}$ is said to be a group.

The composition operator C_φ induced by φ and is acting on $\mathcal{H}(\Omega)$ is defined by

$$C_\varphi f = f \circ \varphi, \text{ for all } f \in \mathcal{H}(\Omega).$$

On the other hand, the composition semigroup C_{φ_t} induced by the semigroup $(\varphi_t)_{t\geq 0}$ is defined by

$$C_{\varphi_t}(f) = f \circ \varphi_t \text{ for all } f \in \mathcal{H}(\Omega).$$

We refer to [29, 31] for more details.

1.3 Statement of the Problem

The study of reproducing kernels for spaces of analytic functions has mainly been considered on the analytic spaces of the unit disk as compared to their counterparts on the upper half plane. The reproducing kernels for the Hardy, Bergman and Dirichlet spaces of the unit disk are well established in literature. Recently, the reproducing kernels for the Hardy and the Bergman spaces of the upper half plane were determined. For the Dirichlet space of the upper half plane, the reproducing kernel has not been adequately determined. Moreover, the growth condition for the functions in the Dirichlet space of the upper half plane is not explicitly known. In this study therefore, we have determined this reproducing kernel and as a consequence, established the growth condition for the functions in the Dirichlet space of the upper half plane. We have further constructed an integral operator of the Cesàro type which is acting on the Dirichlet space of the upper half plane and investigated its properties.

1.4 Objectives of the Study

The main objective of this study was to determine the reproducing kernel for the Dirichlet space of the upper half plane and investigate its consequences. The specific objectives were;

(i) To determine the reproducing kernel for the Dirichlet space of the upper half plane.

(ii) To establish the growth condition for the functions in the Dirichlet space of the upper half plane.

(iii) To construct a Cesàro type operator on the Dirichlet space of the upper half plane.

(iv) To investigate the properties of the Cesàro type operator constructed in (iii) above.

1.5 Significance of the Study

Reproducing kernel spaces and integral operators have been researched on by many mathematicians. However, studies on reproducing kernel and integral operators for the Dirichlet space of the upper half plane is not vast in literature. Results of this study can be applied in the study of integral equations and in solving boundary value problems. The results have also contributed new knowledge to this area of mathematics thereby adding to the existing literature and will advance further research on this and other related areas.

1.6 Research Methodology

The reproducing kernel for the Dirichlet space of the unit disk is known in literature. Using the Cayley transform, we constructed an invertible isometry from Dirichlet space of the unit disk to the Dirichlet space of the upper half plane. The constructed invertible isometry enabled us to transform the reproducing kernel of the unit disk to the reproducing kernel of the upper half plane thereby achieving our first specific objective.
To establish the growth condition for functions of the Dirichlet space of the upper half plane, we used the obtained reproducing kernel and then applied the Cauchy-Schwarz inequality to obtain the norm bound.
We constructed an integral operator of the Cesàro type which is acting on the Dirichlet space of the upper half plane by using the approach of strongly continuous semigroups of composition operators on Banach space. Specifically, we applied the Laplace transform which gave the resolvents of the infinitesimal generator in terms of an integral of the underlying group.
To investigate the properties of the constructed integral operator, we applied spectral mapping theorem for strongly continuous groups and for resolvents and determined the spectra of the integral operator. We further applied characterization by the Hille-Yosida theorem to get the

norm property of the integral operator.

Chapter 2

Literature Review

The theory of reproducing kernels corresponding to classes of analytic functions was first introduced in 1921 by Szegö [33] in his paper dealing with typical reproducing kernels. He [33] determined the reproducing kernel for the Hardy space of the unit disk and named it as the Szegö kernel. The Szegö kernel has played a fundamental role in potential theory and complex analysis as seen by Bell in [8]. In 1922, Bergman [9] also determined the Bergman kernel, the reproducing kernel for the Bergman space of the unit disk, as part of his work in his doctoral dissertation. The reproducing property of the Bergman kernel function is a very important property that plays an important role in the intrinsic geometry of domains. The Bergman and Szegö kernels on the unit disc have been extensively studied in literature. For example, Singh [30] in 1960, determined an integral equation associated with the Szegö kernel. Hua [17] in 1963 computed the explicit forms of the Bergman kernel using the holomorphic automorphism group. Saitoh [27] in 1977 determined the relation between the magnitudes of the exact Bergman kernel and a product of two kernels of Szegö type. He turned this method to the establishment

of a positive definiteness of a period matrix of a product of two kernels of Szego type which led to some completeness theorems of such products. Bell [8] in 2000, showed the fundamental role of the Szegö kernel in potential theory and complex analysis. In particular, Bell showed that the Szegö and Bergman kernels associated to a finitely connected domain in the plane are generated by only three holomorphic functions of one complex variable. Bell further established that many other important functions of potential theory and conformal mapping theory are rational combinations of the same three basic functions. In 2012, Ahn and Park [1], on the other hand showed that the main part of the explicit form of the Bergman kernel on the unit disk is a polynomial whose coefficients are combinations of stirling numbers of the second kind. The theory of reproducing kernel for the Dirichlet space of the unit disk is vast in literature. Ross in his article [26] computed the reproducing kernel for the Dirichlet space of the unit disk and further established the growth conditions for functions of the Dirichlet space of the unit disk. Arcozzi et al, in their paper [4] noted that the reproducing kernel for the Dirichlet space of the disk satisfies some estimates which are important in applications and reveal its geometric nature.
The reproducing kernels for spaces of analytic functions on the upper half plane had not been determined until in 2020 when Bonyo [11] did. He determined the Bergman kernel and also computed the Szegö kernel for H^2 on the upper half plane of the respective spaces. He also used the computed reproducing kernels to establish a weighted Bergman projection on the weighted Bergman space and a Cauchy-Szegö projection. However, not much has been done on the Dirichlet space of the upper half plane.

Specifically, the reproducing kernel of the Dirichlet space on the upper half plane has not been determined. We shall therefore compute this reproducing kernel and as a consequence, establish the growth condition for the functions in the Dirichlet space of the upper half plane.
Much has been done concerning integral operators on analytic spaces of the unit disk. Rochberg and Zhijian [25] in 1992 studied Toeplitz operators on the Dirichlet spaces of the unit disk. They obtained necessary and sufficient conditions on the symbols for the operator to be bounded and compact. Siskasis [32] in 1996 studied semigroup operators on Dirichlet spaces. In 2009, Liankuo [35] also studied Hankel operators on the Dirichlet space of the disk and showed their relation to the Hankel operators on the Bergman space, L_a^2 and Hardy space, H^2. Brevig et al [12] studied the Volterra operator on Hardy spaces of the Dirichlet series associated with an analytic function on the unit disk. Recently, there has been interest on integral operators on analytic spaces of the upper half planes. In 2010, Arvanitidis and Siskasis in [6] studied the Cesàro operators on Hardy spaces of the upper half plane. In 2016, Ballamoole et al [7] constructed Cesàro like operators associated with strongly continuous groups of invertible isometries on the Hardy and Bergman spaces of the upper half plane. Agwang [3] in 2020 proved that the group of weighted composition operators induced by continuous automorphism groups of the upper half plane is strongly continuous on the weighted Dirichlet space of the upper half plane. However, integral operators on the Dirichlet space of the upper half plane is not well captured in literature. Therefore, in our study, we shall also construct an integral operator on the Dirichlet space of the upper half plane and investigate its properties.

The following known results from literature will be used in this study. For details, we refer to [4, 15, 16, 18, 23, 26].

Theorem 2.0.1 (Cauchy-Schwarz Inequality)

Let E be an inner product space over $\mathbb{C}$. Then for all u, $v \in E$,

$$|\langle u, v\rangle| \leq \|u\|\|v\|.$$

Theorem 2.0.2 (Minkowski's Inequality)

Let $p > 1$. Then the integral form of Minkowski's inequality is given as

$$\left(\int_a^b |f(t)+g(t)|^p dt\right)^{\frac{1}{p}} \leq \left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}} + \left(\int_a^b |g(t)|^p dt\right)^{\frac{1}{p}},$$

for continuous functions f and g on $[a, b]$.

Theorem 2.0.3 (Laplace Transform)

Let X be a Banach space and Γ be the infinitesimal generator of a strongly continuous semigroup of contractions $(T_t)_{t\geq 0} \subseteq L(X)$. Then for $\lambda > 0$ and $h \in X$, the Laplace transform is the resolvent operator, $R(\lambda, \Gamma)$ given by

$$R(\lambda, \Gamma)h = \int_0^\infty e^{-\lambda t} T_t h dt, \tag{2.1}$$

with convergence in norm.

Theorem 2.0.4 (Parseval's theorem)

Suppose f and g are two complex-valued functions on $\mathbb{R}$ of period 2π that are square integrable (with respect to the Lebesgue measure) over intervals of period length, with Fourier series

$$h(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$$

and

$$g(x) = \sum_{n=-\infty}^{\infty} \gamma_n e^{inx}$$

respectively. Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} h(x)\overline{g(x)}dx = \sum_{n=-\infty}^{\infty} C_n\bar{\gamma}_n,$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |h(x)|^2 dx = \sum_{n=-\infty}^{\infty} |C_n|^2.$$

Theorem 2.0.5 (Spectrum of an invertible and non-invertible isometry)
Let T be an arbitrary operator on a Banach space X. If T is an invertible isometry, then

$$\sigma(T) \subseteq \partial\mathbb{D}, \tag{2.2}$$

where $\partial\mathbb{D}$ denotes the boundary of the unit disk which is a unit circle. For a non-invertible isometry T,

$$\sigma(T) = \overline{\mathbb{D}}.$$

Theorem 2.0.6 (Hille-Yosida)
A linear operator Γ is the infinitesimal generator of a strongly continuous semigroup of contractions $(T_t)_{t\geq 0} \subseteq L(X)$ if and only if;
(i) Γ is closed and $\overline{\mathrm{dom}(\Gamma)} = X$.
(ii) The resolvent set $\rho(\Gamma)$ of Γ contains $\mathbb{R}^+$ and for every $\lambda \geq 0$,

$$\|R(\lambda, \Gamma)\| \leq \tfrac{1}{\lambda}.$$

Theorem 2.0.7 (Spectral mapping theorem for resolvents)
Let T be a closed operator on a Banach space X and $\lambda \in \rho(T)$. Then the following hold;

1. $\sigma(R(\lambda,T)) \setminus \{0\} = (\lambda - \sigma(T))^{-1} = \left\{\frac{1}{\lambda-\mu} : \mu \in \sigma(T)\right\}$,
2. $\sigma_p(R(\lambda,T)) \setminus \{0\} = (\lambda - \sigma_p(T))^{-1} = \left\{\frac{1}{\lambda-\mu} : \mu \in \sigma_p(T)\right\}$.

Theorem 2.0.8 (Spectral mapping theorem for semigroups)
Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup and Γ be its infinitesimal generator. Then;

$$\sigma(T_t) \supset e^{t\sigma(\Gamma)},$$

and for point spectrum;

$$e^{t\sigma_p(\Gamma)} = \sigma_p(T_t).$$

Theorem 2.0.9 (Closed graph theorem)
Let X and Y be Banach spaces. Then every closed linear mapping $T : X \to Y$ is continuous.

Theorem 2.0.10 (Fatou's lemma)
Let X be a measurable set, $\mathcal{A}$ a sigma algebra and μ a positive measure on the measurable space $(X, \mathcal{A})$. If $(X, \mathcal{A}, \mu)$ is a measure space and suppose $f_n : X \to [0, \infty)$ is measurable for all $n \in \mathbb{N}$, then;

$$\int_X \liminf f_n d\mu \leq \liminf \int_X f_n d\mu.$$

Theorem 2.0.11 (Classification theorem for the $Aut(\mathbb{U})$)
Let $\varphi : \mathbb{R} \to Aut(\mathbb{U})$ be a nontrivial continuous group homomorphism. Then exactly one of the following cases holds:

1. *There exists $k \in \mathbb{R}, k \neq 0$, and $g \in Aut(\mathbb{U})$ so that $\varphi_t(z) = g^{-1}(g(z) + kt)$ for all $z \in \mathbb{U}$ and $t \in \mathbb{R}$.*

2. *There exists* $k > 0, k \neq 0$, *and* $g \in Aut(\mathbb{U})$ *so that* $\varphi_t(z) = g^{-1}(k^t g(z))$ *for all* $z \in \mathbb{U}$ *and* $t \in \mathbb{R}$.

3. *There exists* $k > 0, k \neq 0$, *and a conformal mapping* g *of* $\mathbb{U}$ *onto* $\mathbb{D}$ *such that* $\varphi_t(z) = g^{-1}(e^{ikt} g(z))$ *for all* $z \in \mathbb{U}$ *and* $t \in \mathbb{R}$.

Chapter 3

Reproducing kernel for the Dirichlet space of the upper half plane, $\mathcal{D}(\mathbb{U})$

3.1 Introduction

In this chapter, we determine the reproducing kernel for the Dirichlet space of the upper half plane. We construct an invertible isometry from Dirichlet space of the disk, $\mathcal{D}(\mathbb{D})$ to Dirichlet space of the upper half plane, $\mathcal{D}(\mathbb{U})$ in Proposition 3.1.1 that will help in computing the reproducing kernel for the Dirichlet space of the upper half plane. Thereafter, we use the reproducing kernel for the Dirichlet space of the unit disk, $K_{\mathcal{D}(\mathbb{D})}$, as well as the obtained invertible isometry from the Dirichlet space of the unit disk to the Dirichlet space of the upper half plane, to establish the reproducing kernel for the Dirichlet space of the upper half plane, $K_{\mathcal{D}(\mathbb{U})}$. We consequently establish the growth condition for functions in the Dirichlet space of the upper half plane. We first start by proving some elementary results.

Proposition 3.1.1

For $f \in \mathcal{D}(\mathbb{U})$, $\|f\|_{\mathcal{D}(\mathbb{U})} = \|f \circ \psi\|_{\mathcal{D}(\mathbb{D})}$. In particular, $f \in \mathcal{D}(\mathbb{U})$ if and only if $f \circ \psi \in \mathcal{D}(\mathbb{D})$.

PROOF. For $f \in \mathcal{D}(\mathbb{U})$, we have we have from the definition (see section 1.2.3) that

$$\|f\|^2_{\mathcal{D}(\mathbb{U})} = |f(i)|^2 + \int_{\mathbb{U}} |f'(\omega)|^2 dA(\omega) < \infty.$$

Now by change of variables, if ψ is the Cayley transform, we let $\omega = \psi(z)$, then $dA(\omega) = |\psi'(z)|^2 dA(z)$. Therefore

$$\begin{aligned}
\|f\|^2_{\mathcal{D}(\mathbb{U})} &= |f(\psi(0))|^2 + \int_{\mathbb{D}} |f'(\psi(z))|^2 |\psi'(z)|^2 dA(z) \\
&= |(f \circ \psi)(0)|^2 + \int_{\mathbb{D}} |\psi' f'(\psi(z))|^2 dA(z) \\
&= |(f \circ \psi)(0)|^2 + \int_{\mathbb{D}} |(f \circ \psi)'(z)|^2 dA(z) \\
&= \|f \circ \psi\|^2_{\mathcal{D}(\mathbb{D})},
\end{aligned}$$

which completes the proof. □

The next two propositions give examples of analytic functions and the conditions they must satisfy to belong to $\mathcal{D}(\mathbb{D})$ and $\mathcal{D}(\mathbb{U})$ respectively. We first give the following Lemma that highlights the condition necessary for a function on the unit disk to be bounded.

Lemma 3.1.2 ([36])

Suppose $z \in \mathbb{D}$, c is real, $t > 1$, and

$$I_{c,t}(z) = \int_{\mathbb{D}} \frac{(1 - |w|^2)^t}{|1 - z\bar{w}|^{2+t+c}} dA(w).$$

If $c < 0$, then as a function of $z, I_{c,t}(z)$ is bounded above and below on $\mathbb{D}$.

Proposition 3.1.3

Let $\theta \in \mathbb{R}, \eta \in \mathbb{C}$ and $z \in \mathbb{D}$, then $(e^{i\theta} - z)^\eta \in \mathcal{D}(\mathbb{D})$ if and only if $\Re(\eta) > 0$.

Proof. Given $\theta \in \mathbb{R}, \eta \in \mathbb{C}$ and $z \in \mathbb{D}$, we recall

$$(e^{i\theta} - z)^\eta \in \mathcal{D}(\mathbb{D}) \Leftrightarrow \int_{\mathbb{D}} |((e^{i\theta} - z)^\eta)'|^2 dA(z) < \infty.$$

But

$$\begin{aligned}\int_{\mathbb{D}} |((e^{i\theta} - z)^\eta)'|^2 dA(z) &= \int_{\mathbb{D}} |\eta(e^{i\theta} - z)^{\eta-1}(-1)|^2 dA(z), \\ &= \int_{\mathbb{D}} |\eta|^2 |(e^{i\theta} - z)^{\eta-1}|^2 dA(z), \\ &= \int_{\mathbb{D}} |\eta|^2 |1 - ze^{-i\theta}|^{2\Re(\eta-1)} dA(z).\end{aligned}$$

Thus

$$\int_{\mathbb{D}} |((e^{i\theta} - z)^\eta)'|^2 dA(z) = \int_{\mathbb{D}} |\eta|^2 \frac{dA(z)}{|1 - ze^{-i\theta}|^{-2\Re(\eta-1)}}. \tag{3.1}$$

It follows from Lemma 3.1.2 with $t = 0$ and $c = -2\Re(\eta - 1) - 2$ that (3.1) is bounded if and only if $-2\Re(\eta - 1) - 2 < 0$. That is, if and only if $\Re(\eta) > 0$. □

Proposition 3.1.4

For $\lambda, \nu \in \mathbb{C}$, let $f(\omega) = (\omega - c)^\lambda(\omega + i)^\nu$.
Then $f \in \mathcal{D}(\mathbb{U})$ if and only if $0 < \Re(\lambda) < -\Re(\nu)$.

Proof. Recall from Proposition 3.1.1 that, $f \in \mathcal{D}(\mathbb{U})$ if and only if

$f \circ \psi \in \mathcal{D}(\mathbb{D})$. Now, let $g(z) = f(\psi(z))$. Then

$$\begin{aligned}
g(z) &= f(\psi(z)), \\
&= (\psi(z) - c)^{\lambda}(\psi(z) + i)^{\nu}, \\
&= \left(\frac{i(1+z)}{1-z} - c\right)^{\lambda}\left(\frac{i(1+z)}{1-z} + i\right)^{\nu}, \\
&= \left(\frac{zi + zc + i - c}{1-z}\right)^{\lambda}\left(\frac{2i}{1-z}\right)^{\nu}, \\
&= \frac{(z(c+i) - 1(c-i))^{\lambda}}{(1-z)^{\lambda+\nu}} \cdot (2i)^{\nu}, \\
&= (c+i)^{\lambda}\frac{(z - \frac{c-i}{c+i})^{\lambda}}{(1-z)^{\lambda+\nu}} \cdot (2i)^{\nu}, \\
&= k\frac{(z - \frac{c-i}{c+i})^{\lambda}}{(1-z)^{\lambda+\nu}},
\end{aligned}$$

where $k = (c+i)^{\lambda}(2i)^{\nu}$ is a constant.

Since $\left|\frac{c-i}{c+i}\right|^2 = \frac{c-i}{c+i} \cdot \frac{c+i}{c-i} = 1$, it follows that $\frac{c-i}{c+i} \in \partial\mathbb{D}$ and therefore there exists $\theta \in \mathbb{R}$ such that $\frac{c-i}{c+i} = e^{i\theta}$. It follows from Proposition 3.1.3 that $g(z) \in \mathcal{D}(\mathbb{D})$ if and only if $\Re(\lambda) > 0$ and $\Re(-\lambda - \nu) > 0$. That is, $0 < \Re(\lambda) < -\Re(\nu)$, as desired. □

3.2 Reproducing Kernels for the Dirichlet spaces

Let $\mathbf{H}$ denote a Hilbert space of functions with inner product $\langle ., . \rangle_{\mathbf{H}}$ defined on an open set $\Omega \subseteq \mathbb{C}$. We call a reproducing kernel for $\mathbf{H}$ as a complex function $K : \Omega \times \Omega \to \mathbb{C}$ such that if we put $K_{\omega}(z) := K(z, \omega)$, then the following two properties hold;

1. For every $\omega \in \Omega$, the function K_ω belongs to $\mathbf{H}$, and,

2. For all $f \in \mathbf{H}$ and $\omega \in \Omega$, we have $f(\omega) = \langle f, K_\omega \rangle_H$

It is clear that the above two properties imply that such a kernel K satisfies the identity $K(z,\omega) = \overline{K(\omega, z)}$ for all $z, \omega \in \Omega$. Indeed,

$$\begin{aligned} K(z,\omega) &= K_\omega(z) = \langle K_\omega, K_z \rangle \\ &= \overline{\langle K_z, K_\omega \rangle} = \overline{K_z(\omega)} = \overline{K(\omega, z)}. \end{aligned}$$

3.2.1 Dirichlet space of the unit disk

An analytic function f on the open unit disk $\mathbb{D}$ belongs to the classical Dirichlet space $\mathcal{D}$ if it has the finite Dirichlet integral as given in equation (1.9). In particular, the Dirichlet space of the unit disk $\mathcal{D}(\mathbb{D})$, consists of those analytic function $f \in \mathcal{H}(\mathbb{D})$ for which $\int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty$ with the norm given by

$$\|f\|^2_{\mathcal{D}(\mathbb{D})} = |f(0)|^2 + \|f\|^2_{\mathcal{D}_1(\mathbb{D})},$$

where $\|f\|^2_{\mathcal{D}_1(\mathbb{D})} := \int_{\mathbb{D}} |f'(z)|^2 dA(z)$. Clearly, $\|\cdot\|_{\mathcal{D}_1(\mathbb{D})}$ is a seminorm on $\mathcal{D}(\mathbb{D})$. Indeed, for $f \in \mathcal{D}(\mathbb{D})$,

$$\begin{aligned} \|f\|^2_{\mathcal{D}_1(\mathbb{D})} = 0 &\Leftrightarrow \int_{\mathbb{D}} |f'(z)|^2 dA(z) = 0 \\ &\Leftrightarrow |f'(z)|^2 = 0 \Leftrightarrow f'(z) = 0 \\ &\Leftrightarrow f(z) = k \end{aligned}$$

where k is any constant. Also for $\lambda \in \mathbb{C}$ and $f \in \mathcal{D}(\mathbb{D})$, we have that

$$\begin{aligned}
\|\lambda f\|^2_{\mathcal{D}_1(\mathbb{D})} &= \int_{\mathbb{D}} |(\lambda f)'(z)|^2 dA(z) \\
&= \int_{\mathbb{D}} |\lambda|^2 |f'(z)|^2 dA(z) \\
&= |\lambda|^2 \int_{\mathbb{D}} |f'(z)|^2 dA(z) \\
&= |\lambda|^2 \|f\|^2_{\mathcal{D}_1(\mathbb{D})}.
\end{aligned}$$

Finally, for the triangle inequality by application of Minkowski's inequality, (Theorem 2.0.2), have that for $f, g \in \mathcal{D}(\mathbb{D})$,

$$\|f+g\|_{\mathcal{D}_1(\mathbb{D})} \leq \|f\|_{\mathcal{D}_1(\mathbb{D})} + \|g\|_{\mathcal{D}_1(\mathbb{D})}.$$

$$\begin{aligned}
\|f+g\|_{\mathcal{D}_1(\mathbb{D})} &= \left(\int_{\mathbb{D}} |(f+g)'(z)|^2 dA(z)\right)^{\frac{1}{2}} \\
&\leq \left(\int_{\mathbb{D}} |f'(z)|^2 dA(z)\right)^{\frac{1}{2}} + \left(\int_{\mathbb{D}} |g'(z)|^2 dA(z)\right)^{\frac{1}{2}}, \\
&= \|f\|_{\mathcal{D}_1(\mathbb{D})} + \|g\|_{\mathcal{D}_1(\mathbb{D})}.
\end{aligned}$$

Therefore $\|\cdot\|_{\mathcal{D}_1(\mathbb{D})}$ is a seminorm on $\mathcal{D}(\mathbb{D})$ as claimed. Moreover, $\|\cdot\|^2_{\mathcal{D}(\mathbb{D})}$ is a norm on $\mathcal{D}(\mathbb{D})$. Indeed,

$$\begin{aligned}
\|f\|^2_{\mathcal{D}(\mathbb{D})} = 0 &\Leftrightarrow |f(0)|^2 + \|f\|^2_{\mathcal{D}_1(\mathbb{D})} = 0 \\
&\Leftrightarrow f(0) = 0 \text{ and } f'(z) = 0 \\
&\Leftrightarrow f(0) = 0 \text{ and } f(z) = k \text{ for any constant } k \\
&\Leftrightarrow f(z) = 0,
\end{aligned}$$

which together with the fact that $\|\cdot\|_{\mathcal{D}_1(\mathbb{D})}$ is a seminorm on $\mathcal{D}(\mathbb{D})$ shows that $\|\cdot\|_{\mathcal{D}(\mathbb{D})}$ is a norm on $\mathcal{D}(\mathbb{D})$ as claimed.

We now work out the formula for the function f in terms of the Taylor coefficients of f in the next proposition.

Proposition 3.2.1

Let $f \in \mathcal{H}(\mathbb{D})$ be such that $f(z) = \sum_{n\geq 0} a_n z^n$, then

$$\|f\|^2_{\mathcal{D}_1(\mathbb{D})} = \textstyle\sum_{n\geq 1} n|a_n|^2.$$

PROOF. Writing the area integral in polar co-ordinates, we have

$$\begin{aligned} \|f\|^2_{\mathcal{D}_1(\mathbb{D})} &= \int_{\mathbb{D}} \left| \sum_{n\geq 1} n a_n z^{n-1} \right|^2 dA(z) \\ &= \frac{1}{\pi} \int_0^1 \int_0^{2\pi} \left| \sum_{n\geq 1} n a_n r^{n-1} e^{i(n-1)\theta} \right|^2 d\theta r dr \end{aligned}$$

for $z = re^{i\theta}$. Parseval's theorem, (Theorem 2.0.4), with

$$h(\theta) = \sum_{n\geq 1} n a_n r^{n-1} e^{i(n-1)\theta}$$

and

$$C_n = n a_n r^{n-1},$$

implies that for each $r \in (0, 1)$,

$$\int_0^{2\pi} \left| \sum_{n\geq 1} n a_n r^{n-1} e^{i(n-1)\theta} \right|^2 d\theta = 2\pi \sum_{n\geq 1} |n a_n r^{n-1}|^2 |e^{i(n-1)\theta}|^2 = 2\pi \sum_{n\geq 1} n^2 |a_n|^2 r^{2n-2}. \tag{3.2}$$

Then

$$\begin{aligned} \|f\|^2_{\mathcal{D}_1(\mathbb{D})} &= 2\int_0^1 \sum_{n\geq 1} n^2|a_n|^2 r^{2n-2} r\,dr \\ &= 2\sum_{n\geq 1} n^2|a_n|^2 \int_0^1 r^{2n-1}dr \\ &= 2\sum_{n\geq 1} n^2|a_n|^2 \left(\frac{r^{2n}}{2n}\Big|_0^1\right) \\ &= \sum_{n\geq 1} n|a_n|^2, \end{aligned}$$

as desired. □

The next proposition gives the reproducing kernel for the Dirichlet space of the unit disk.

Proposition 3.2.2 ([15])

The reproducing kernel for the Dirichlet space of the unit disk, $\mathcal{D}(\mathbb{D})$, is given by

$$K_{\mathbb{D}}(z,\omega) = \frac{1}{z\bar{\omega}} \log \frac{1}{1 - z\bar{\omega}}, \tag{3.3}$$

where $z, \omega \in \mathbb{D}$.

As a consequence we have the growth condition for functions on the Dirichlet space of the unit disk.

Corollary 3.2.3

For every $f \in \mathcal{D}(\mathbb{D})$, we have

$$|f(z)| \leq c\|f\| \sqrt{\log \frac{1}{1 - |z|^2}}, \tag{3.4}$$

where c is a constant.

PROOF. Let $f \in \mathcal{D}(\mathbb{D})$ and $K_z(\omega) = \frac{1}{z\bar{\omega}} \log \frac{1}{1-z\bar{\omega}}$ be the reproducing kernel for $\mathcal{D}(\mathbb{D})$, then by Cauchy-Schwarz Inequality, (Theorem 2.0.1),

$$\begin{aligned} |f(z)| &= |\langle f, K_z \rangle|, \\ &\leq \|f\| \|K_z\|, \\ &= \|f\| \langle K_z, K_z \rangle^{\frac{1}{2}}, \end{aligned}$$

which implies that

$$|f(z)| \leq \|f\| K_z(z)^{\frac{1}{2}}. \tag{3.5}$$

From (3.3), we can rewrite $K_z(z)^{\frac{1}{2}}$ as

$$\begin{aligned} K_z(z)^{\frac{1}{2}} &= \left(\frac{1}{z\bar{z}} \log \frac{1}{1 - z\bar{z}} \right)^{\frac{1}{2}} &(3.6) \\ &= \left(\frac{1}{|z|^2} \log \frac{1}{1 - |z|^2} \right)^{\frac{1}{2}} &(3.7) \end{aligned}$$

and therefore equation (3.5) becomes

$$|f(z)| \leq \|f\| \left(\frac{1}{|z|^2} \log \frac{1}{1 - |z|^2} \right)^{\frac{1}{2}}$$

which can be simplified further as

$$|f(z)| \leq c\|f\| \sqrt{\log \frac{1}{1 - |z|^2}},$$

where $c = \frac{1}{|z|}$ and $\|f\| = \|f\|_{\mathcal{D}(\mathbb{D})}$. This completes the proof. □

REMARK 3.2.4

Equation (3.4) is the growth condition for functions in the Dirichlet space of the disk, $\mathcal{D}(\mathbb{D})$.

The function $K_{\mathbb{D}}$ on $\mathcal{D}(\mathbb{D})$ has been exhaustively studied in literature. See for instance [8, 30, 17, 27]. In the next section we study the Dirichlet space of the upper half plane, then compute the corresponding reproducing kernel for the Dirichlet space of the upper half plane; which has not been explicitly determined in literature. Consequently, we determine the growth condition for functions in the Dirichlet space of the upper half plane.

3.2.2 Dirichlet space of the upper half plane

Just as in the case of the unit disk, the Dirichlet space of the upper half plane $\mathcal{D}(\mathbb{U})$ consists of those analytic functions $f \in \mathcal{H}(\mathbb{U})$ satisfying

$$\|f\|^2_{\mathcal{D}_1(\mathbb{U})} = \int_{\mathbb{U}} |f'(\omega)|^2 dA(\omega) < \infty,$$

with the norm given by

$$\|f\|^2_{\mathcal{D}(\mathbb{U})} = |f(i)|^2 + \|f\|^2_{\mathcal{D}_1(\mathbb{U})},$$

while $\|\cdot\|_{\mathcal{D}_1(\mathbb{U})}$ is a seminorm on $\mathcal{D}(\mathbb{U})$. Now for $f \in \mathcal{D}(\mathbb{U})$,

$$\begin{aligned} \|f\|^2_{\mathcal{D}_1(\mathbb{U})} = 0 &\Leftrightarrow \int_{\mathbb{U}} |f'(\omega)|^2 dA(\omega) = 0 \\ &\Leftrightarrow |f'(\omega)|^2 = 0 \\ &\Leftrightarrow f'(\omega) = 0 \\ &\Leftrightarrow f(\omega) = c \end{aligned}$$

for any constant c. Thus, together with some other seminorm properties, it follows that $\|\cdot\|_{\mathcal{D}_1(\mathbb{U})}$ is seminorm on $\mathcal{D}(\mathbb{U})$ as claimed. Moreover, $\|\cdot\|_{\mathcal{D}(\mathbb{U})}$ is a norm on $\mathcal{D}(\mathbb{U})$ since for all $f \in \mathcal{D}(\mathbb{U})$,

$$\begin{aligned}
\|f\|^2_{\mathcal{D}(\mathbb{U})} = 0 \quad &\Leftrightarrow \quad |f(i)|^2 + \|f\|^2_{\mathcal{D}_1(\mathbb{U})} = 0 \\
&\Leftrightarrow \quad f(i) = 0 \text{ and } f'(\omega) = 0 \\
&\Leftrightarrow \quad f(i) = 0 \text{ and } f(\omega) = k \text{ for any constant } k \\
&\Leftrightarrow \quad f(i) = 0 \text{ and } f(\omega) = 0 \\
&\Leftrightarrow \quad f = 0,
\end{aligned}$$

which demonstrates that $\|\cdot\|_{\mathcal{D}(\mathbb{U})}$ is a norm on $\mathcal{D}(\mathbb{U})$, as claimed.

In the next section we compute the reproducing kernel for the Dirichlet space of the upper half plane using an invertible isometry from the Dirichlet space of the unit disk to the Dirichlet space of the upper half plane, then transforming $K_{\mathbb{D}}$ to $K_{\mathbb{U}}$ but first we establish the relationship between functions on $\mathcal{D}(\mathbb{U})$ and $L^2_a(\mathbb{U})$ in the following Lemma.

Lemma 3.2.5

A function $f \in \mathcal{D}(\mathbb{U})$ if and only if $f' \in L^2_a(\mathbb{U})$.

Proof. By definition of $\mathcal{D}(\mathbb{U})$, (see section 1.2.3) $f \in \mathcal{D}(\mathbb{U})$ if and only if

$$\|f\|^2_{\mathcal{D}_1(\mathbb{U})} = \int_{\mathbb{U}} |f'(\omega)|^2 dA(\omega) < \infty,$$

Also, by definition of $L^2_a(\mathbb{U})$, (see section 1.2.3) $f' \in L^2_a(\mathbb{U})$ if and only if

$$\|f'\|^2_{L^2_a(\mathbb{U})} = \int_{\mathbb{U}} |f'(\omega)|^2 dA(\omega) < \infty.$$

This implies that $f \in \mathcal{D}(\mathbb{U})$ if and only if $f' \in L^2_a(\mathbb{U})$, as desired. □

We now show that the composition operator C_ψ induced by ψ is invertible.

Proposition 3.2.6

Let $C_\psi : \mathcal{H}(\mathbb{U}) \to \mathcal{H}(\mathbb{D})$ be the composition operator induced by ψ. Then $C_\psi^{-1} = C_{\psi^{-1}}$.

Proof. Let $T = C_\psi$ be the composition by ψ and $T^{-1} = C_{\psi^{-1}}$ composition by ψ^{-1}. That is, for $f \in \mathcal{H}(\mathbb{U})$ and $g \in \mathcal{H}(\mathbb{D})$, $Tf = f \circ \psi$ and $T^{-1}g = g \circ \psi^{-1}$. Then for every $g \in \mathcal{H}(\mathbb{D})$, we have

$$T^{-1}g(z) = g(\psi^{-1}(z)).$$

But

$$T^{-1}(Tf)(z) = (Tf)(\psi^{-1}(z)) = f(\psi(\psi^{-1}(z))) = f(z),$$

Moreover

$$T(T^{-1}g)(\xi) = (T^{-1}g)(\psi(\xi)) = g(\psi^{-1}(\psi(\xi))) = g(\xi).$$

This shows that C_ψ is invertible with $C_\psi^{-1} = C_{\psi^{-1}}$, as desired. $\square$

We now compute the reproducing kernel for the Dirichlet space of the upper half plane, $\mathcal{D}(\mathbb{U})$, in the next theorem.

Theorem 3.2.7

The reproducing kernel for the Dirichlet space of the upper half plane, $\mathcal{D}(\mathbb{U})$ is given by

$$K_{\mathbb{U}}(z,\omega) = \frac{(z+i)(\bar{\omega}-i)}{(z-i)(\bar{\omega}+i)} \log \frac{i(z+i)(\bar{\omega}-i)}{2(z-\bar{\omega})}, \tag{3.8}$$

$z, \omega \in \mathbb{U}$.

PROOF. Let $K_{\mathbb{U}}$ and $K_{\mathbb{D}}$ be the reproducing kernels for the Dirichlet space of the upper half plane and the unit disk respectively. Then $K_{\mathbb{D}}$ is given by equation (3.3). We need to work out $K_{\mathbb{U}}$. The Cayley transform $\psi(z) = \frac{i(1+z)}{1-z}$ maps the unit disk conformally onto the upper half plane with inverse $\psi^{-1}(\omega) = \frac{\omega - i}{\omega + i}$. It follows from Propositions 3.1.1 and 3.2.6 that $Tf(\xi) = f(\psi(\xi))$ is an isometric isomorphism of $\mathcal{D}(\mathbb{U})$ onto $\mathcal{D}(\mathbb{D})$. Also, by Proposition 3.2.6 and from the fact that $\mathcal{D}(\cdot)$ is a Hilbert space, it then follows that T is unitary, that is, $T^* = T^{-1}$.
For every $\xi \in \mathbb{D}$, we have using the definition of $K_{\mathbb{D}}$

$$Tf(\xi) = f(\psi(\xi)) = \langle Tf, K_{\mathbb{D},\xi}\rangle_{\mathcal{D}(\mathbb{D})} = \langle f, T^{-1}K_{\mathbb{D},\xi}\rangle_{\mathcal{D}(\mathbb{U})}. \tag{3.9}$$

We now work out $T^{-1}K_{\mathbb{D},\xi}$. For $z \in \mathbb{U}$, we have

$$T^{-1}K_{\mathbb{D},\xi}(z) = K_{\mathbb{D},\xi}(\psi^{-1}(z)). \tag{3.10}$$

By equation (3.3),

$$\begin{aligned}
K_{\mathbb{D},\xi}(\psi^{-1}(z)) &= \frac{1}{(\frac{z-i}{z+i})\bar{\xi}} \log \frac{1}{1 - (\frac{z-i}{z+i})\bar{\xi}} \\
&= \frac{z+i}{(z-i)\bar{\xi}} \log \frac{z+i}{(z+i) - (z-i)\bar{\xi}} \\
&= \frac{z+i}{(z-i)\bar{\xi}} \log \frac{z+i}{z+i-z\bar{\xi}+i\bar{\xi}} \\
&= \frac{z+i}{(z-i)\bar{\xi}} \log \frac{z+i}{z(1-\bar{\xi})+i(1+\bar{\xi})} \\
&= \frac{z+i}{(z-i)\bar{\xi}} \log \frac{z+i}{(1-\bar{\xi})\left(z+i(\frac{1+\bar{\xi}}{1-\bar{\xi}})\right)} \\
&= \frac{z+i}{(z-i)\bar{\xi}} \log \frac{z+i}{(1-\bar{\xi})\left(z-\overline{\psi(\xi)}\right)}.
\end{aligned}$$

Therefore,

$$\begin{aligned} Tf(\xi) &= f(\psi(\xi)) = \left\langle f, T^{-1}K_{\mathbb{D},\xi}\right\rangle_{\mathcal{D}(\mathbb{U})}, \\ &= \left\langle f, \frac{z+i}{(z-i)\bar{\xi}} \log \frac{z+i}{(1-\bar{\xi})(z-\overline{\psi(\xi)})} \right\rangle, \end{aligned}$$

which implies that $f(\psi(\xi)) = \left\langle f, \frac{z+i}{(z-i)\bar{\xi}} \log \frac{z+i}{(1-\bar{\xi})(z-\overline{\psi(\xi)})} \right\rangle$ and thus

$$f(\omega) = \left\langle f, \frac{z+i}{(z-i)\overline{\psi^{-1}(\omega)}} \log \frac{z+i}{(1-\overline{\psi^{-1}(\omega)})(z-\bar{\omega}} \right\rangle, \tag{3.11}$$

where $\omega = \psi(\xi)$ and $\omega \in \mathbb{U}$. But $\psi^{-1}(\omega) = \frac{\omega-i}{\omega+i}$ and so

$$\begin{aligned} K_{\mathbb{U},\omega}(z) &= \frac{z+i}{(z-i)(\frac{\bar{\omega}+i}{\bar{\omega}-i})} \log \frac{z+i}{\left(1-\frac{\bar{\omega}+i}{\bar{\omega}-i}\right)(z-\bar{\omega})}, \\ &= \frac{(z+i)(\bar{\omega}-i)}{(z-i)(\bar{\omega}+i)} \log \frac{(z+i)(\bar{\omega}-i)}{(\omega-i-\omega-i)(z-\bar{\omega})}, \\ &= \frac{(z+i)(\bar{\omega}-i)}{(z-i)(\bar{\omega}+i)} \log \frac{(z+i)(\bar{\omega}-i)}{-2i(z-\bar{\omega})} \\ &= \frac{(z+i)(\bar{\omega}-i)}{(z-i)(\bar{\omega}+i)} \log \frac{i(z+i)(\bar{\omega}-i)}{2(z-\bar{\omega})}. \end{aligned}$$

This completes the proof. □

As a consequence to the Theorem 3.2.7 above, we explicitly determine the growth condition for functions in the Dirichlet space of the upper half plane.

Corollary 3.2.8

For $f \in \mathcal{D}(\mathbb{U})$, we have

$$|f(\omega)| \leq c\|f\| \sqrt{\log \frac{|\omega+i|^2}{4\Im(\omega)}}, \tag{3.12}$$

where c is a constant, and $\Im(\omega) \in \mathbb{U}$ denotes the imaginary part of $\omega \in \mathbb{U}$.

PROOF. By Cauchy-Schwarz inequality, (Theorem 2.0.1), we have that for every $f \in \mathcal{D}(\mathbb{U})$,

$$\begin{aligned} |f(\omega)| &= |\langle f, K_\omega \rangle| \\ &\leq \|f\| \|K_\omega\| \\ &= \|f\| \langle K_\omega, K_\omega \rangle^{\frac{1}{2}} \end{aligned}$$

It therefore follows that

$$|f(\omega)| \leq \|f\| K_\omega(\omega)^{\frac{1}{2}}. \tag{3.13}$$

Now

$$\begin{aligned} K_\omega(\omega)^{\frac{1}{2}} &= \left(\frac{(\omega+i)(\bar{\omega}-i)}{(\omega-i)(\bar{\omega}+i)} \log \frac{(\omega+i)(\bar{\omega}-i)}{-2i(\omega-\bar{\omega})} \right)^{\frac{1}{2}} \\ &= \left(\frac{|\omega+i|^2}{|\omega-i|^2} \log \frac{|\omega+i|^2}{-2i(\omega-\bar{\omega})} \right)^{\frac{1}{2}} \\ &= \left(\left| \frac{\omega+i}{\omega-i} \right|^2 \log \frac{|\omega+i|^2}{4\Im(\omega)} \right)^{\frac{1}{2}} \\ &= \left| \frac{\omega+i}{\omega-i} \right| \left(\log \frac{|\omega+i|^2}{4\Im(\omega)} \right)^{\frac{1}{2}} \\ &= c\sqrt{\log \frac{|\omega+i|^2}{4\Im(\omega)}}, \end{aligned}$$

where $c = \left|\frac{\omega+i}{\omega-i}\right|$ is some constant.

It follows from (3.13) that

$$|f(\omega)| \leq c\|f\|\sqrt{\log \frac{|\omega + i|^2}{4\Im(\omega)}},$$

where $\|f\| = \|f\|_{\mathcal{D}(\mathbb{U})}$. This completes the proof. $\square$

Chapter 4

Cesàro - Type operator on Dirichlet space of the upper half plane.

4.1 Introduction

In this chapter, we construct an integral operator of the Cesàro type on the Dirichlet space of the upper half plane. We first note that the automorphisms of the upper half plane were classified into three groups [7, Theorem 2.0.11], that is the scaling, the translation and the rotation groups depending on the location of their fixed points. In this study, we first consider groups of composition operators associated with the scaling group. We determine the group of composition operator on the Dirichlet space of the upper half plane $\mathcal{D}(\mathbb{U})$ associated with the scaling group and investigate if it is an isometry on $\mathcal{D}(\mathbb{U})$. We then investigate both the semigroup and spectral properties of the composition semigroup. Finally, we construct a Cesàro - type operator on $\mathcal{D}(\mathbb{U})$ that we obtain as the resolvent of the infinitesimal generator then determine the spectral and norm properties of the operator. We first start by proving some results on composition operators corresponding to the scaling group.

4.2 Scaling group

The automorphisms of this group are in the form $\varphi_t(\omega) = (k^t(\omega))$ for all $\omega \in \mathbb{U}$ and $k, t \in \mathbb{R}$ where $k > 0, k \neq 1$. We consider the self analytic map $\varphi_t : \mathbb{U} \to \mathbb{U}$ of the form $\varphi_t(\omega) = e^{-t}\omega$ for $\omega \in \mathbb{U}$. We note that φ_t is analytic on $\mathbb{U}$. Indeed, analytic maps on $\mathbb{U}$ are of the form $\psi(\omega) = \frac{a\omega+b}{c\omega+d}$ where $a, b, c, d \in \mathbb{R}$ with $ad - bc > 0$.
For $\varphi_t(\omega) = e^{-t}\omega; a = e^{-t}, b = c = 0$ and $d = 1$. Thus $ad - bc = e^{-t} - 0$ $= e^{-t} > 0$ and so φ_t is analytic on $\mathbb{U}$.
The composition semigroup induced by the scaling group and acting on $\mathcal{D}(\mathbb{U})$ is defined as

$$\begin{aligned} C_{\varphi_t} f(\omega) &= f \circ \varphi_t(\omega) \\ &= f(e^{-t}\omega), \end{aligned} \tag{4.1}$$

for all $f \in \mathcal{D}(\mathbb{U})$.
We begin by proving that the functions given by (4.1) form a group on $\mathcal{D}(\mathbb{U})$.
Proposition 4.2.1
The functions $(C_{\varphi_t})_{t\in\mathbb{R}}$ form a group on $\mathcal{D}(\mathbb{U})$ under composition.

Proof. We prove that both $(C_{\varphi_t})_{t\leq 0}$ and $(C_{\varphi_t})_{t\geq 0}$ are semigroups on $\mathcal{D}(\mathbb{U})$. Clearly, $C_{\varphi_0}(\omega) = I$ (Identity) since $f(e^0\omega) = f(\omega)$. Also, for every $f \in \mathcal{D}(\mathbb{U})$, $t, s \geq 0$,

$$\begin{aligned} C_{\varphi_t} \circ C_{\varphi_s} f(\omega) &= C_{\varphi_t}(C_{\varphi_s} f(\omega)) \\ &= C_{\varphi_t}(f(\varphi_s(\omega))) \\ &= f(\varphi_s(\varphi_t(\omega))) \\ &= f(\varphi_s(e^{-t}(\omega))) \\ &= f(e^{-s}e^{-t}\omega) \\ &= f(e^{-(s+t)}\omega) \\ &= C_{\varphi_{s+t}} f(\omega). \end{aligned}$$

Therefore $(C_{\varphi_t})_{t\geq 0}$ is a semigroup on $\mathbb{U}$. It can similarly be shown that $(C_{\varphi_t})_{t\leq 0}$ is a semigroup on $\mathcal{D}(\mathbb{U})$. Thus $(C_{\varphi_t})_{t\in\mathbb{R}}$ is a group on $\mathcal{D}(\mathbb{U})$ as desired. □

Next, we show that the operator $C_{\varphi t}$ at $t \in \mathbb{R}$ fails to be an isometry on $\mathcal{D}(\mathbb{U})$.

Proposition 4.2.2

The operator C_{φ_t} fails to be an isometry on $\mathcal{D}(\mathbb{U})$.

Proof. By norm definition,

$$\|C_{\varphi_t} f\|^2_{\mathcal{D}(\mathbb{U})} = \|f \circ \varphi_t\|^2_{\mathcal{D}(\mathbb{U})} = |f \circ \varphi_t(i)|^2 + \int_{\mathbb{U}} |(f \circ \varphi_t)'(\omega)|^2 dA(\omega). \quad (4.2)$$

But $(f \circ \varphi_t)(\omega) = f(e^{-t}\omega)$. Thus $(f \circ \varphi_t)'(\omega) = e^{-t} f'(e^{-t}\omega)$ implying that $|(f \circ \varphi_t)'(\omega)|^2 = e^{-2t}|f'(e^{-t}\omega)|^2$ and $|f \circ \varphi_t(i)|^2 = |f(e^{-t})(i)|^2$.

By change of variables, we let $z = e^{-t}\omega$, then, $\omega = e^t z$. Applying the Jacobian, $dA(z) = |\varphi_t'(\omega)|^2 dA(\omega)$. But $\varphi_t(\omega) = e^{-t}\omega$, therefore $dA(z) =$

$e^{-2t}dA(\omega)$ and $dA(\omega) = e^{2t}dA(z)$.

Substituting these in (4.2), we get

$$\begin{aligned}\|C_{\varphi_t} f\|^2_{\mathcal{D}(\mathbb{U})} &= |f(e^{-t}i)|^2 + \int_{\mathbb{U}} |(f \circ \varphi_t)'(\omega)|^2 dA(\omega). \\ &= |f(e^{-t}i)|^2 + \int_{\mathbb{U}} e^{-2t}|f'(e^{-t}\omega)|^2 dA(\omega). \\ &= |f(e^{-t}i)|^2 + \int_{\mathbb{U}} e^{-2t}|f'(z)|^2 e^{2t} dA(z). \\ &= |f(e^{-t}i)|^2 + \int_{\mathbb{U}} |f'(z)|^2 dA(z).\end{aligned}$$

But $|f(e^{-t}i)|^2 + \int_{\mathbb{U}} |f'(z)|^2 dA(z) \neq \|f\|^2_{\mathcal{D}(\mathbb{U})}$. Thus, at each $t \in \mathbb{R}$, the functions $C_{\varphi t}$ fail to be an isometry on $\mathcal{D}(\mathbb{U})$. $\square$

Remark 4.2.3

Because of Proposition 4.2.2, we consider $\mathcal{D}_\circ(\mathbb{U})$, the subspace of $\mathcal{D}(\mathbb{U})$ consisting of functions vanishing at i, $f(i) = 0$, defined as $\mathcal{D}_\circ(\mathbb{U}) = \{f \in \mathcal{D}(\mathbb{U}) : f(i) = 0\}$ with the norm defined as $\|f\|^2_{\mathcal{D}_\circ(\mathbb{U})} = \int_{\mathbb{U}} |f'(\omega)|^2 dA(\omega)$. However, $C_{\varphi_t} f(i) = f(e^{-t}i) \neq 0$, and so C_{φ_t} does not map $\mathcal{D}_\circ(\mathbb{U})$ into $\mathcal{D}_\circ(\mathbb{U})$ as expected for our semigroups. Therefore, we apply a correction factor and redefine C_{φ_t} as

$$\hat{C}_{\varphi_t} f(z) = f(e^{-t}z) - f(e^{-t}i). \tag{4.3}$$

Now $\hat{C}_{\varphi_t} f(i) = f(e^{-t}i) - f(e^{-t}i) = 0$ as desired so that indeed $\hat{C}_{\varphi_t} : \mathcal{D}_\circ(\mathbb{U}) \to \mathcal{D}_\circ(\mathbb{U})$. From now henceforth, we shall focus our attention on the semigroup $(\hat{C}_{\varphi_t})_{t \geq 0}$ and study its properties in detail.

We now show that (4.3) is a group on $\mathcal{D}_\circ(\mathbb{U})$ and investigate its semigroup properties.

Proposition 4.2.4

The functions $(\hat{C}_{\varphi_t})_{t\in\mathbb{R}}$ form a group on $\mathcal{D}_\circ(\mathbb{U})$.

PROOF. It suffices to show that both $(\hat{C}_{\varphi_t})_{t\geq 0}$ and $(\hat{C}_{\varphi_t})_{t\leq 0}$ are semigroups on $\mathcal{D}_\circ(\mathbb{U})$.

Indeed, for $f \in \mathcal{D}_\circ(\mathbb{U})$,

$$\begin{aligned} \hat{C}_{\varphi_0} f(z) &= f(e^0) - f(e^0 i) \\ &= f(z) - f(i) \\ &= f(z), \end{aligned}$$

and so $\hat{C}_{\varphi_0} = I$.

Also, for every $f \in \mathcal{D}_\circ(\mathbb{U})$, $t, s \geq 0$,

$$\begin{aligned} (\hat{C}_{\varphi_t} \circ \hat{C}_{\varphi_s}) f(z) &= \hat{C}_{\varphi_t}(\hat{C}_{\varphi_s} f(z)) \\ &= \hat{C}_{\varphi_s} f(e^{-t} z) - \hat{C}_{\varphi_s} f(e^{-t} i) \\ &= f(e^{-s} e^{-t} z) - f(e^{-s} e^{-t} i) - \big(f(e^{-s} e^{-t} i) - f(e^{-s} e^{-t} i)\big) \\ &= f(e^{-(s+t)} z) - f(e^{-(s+t)} i) \\ &= \hat{C}_{\varphi_{s+t}} f(z). \end{aligned}$$

Therefore $\hat{C}_{\varphi_t} \circ \hat{C}_{\varphi_s} = \hat{C}_{\varphi_{t+s}}$ and hence $(\hat{C}_{\varphi_t})_{t\geq 0}$ is a semigroup on $\mathcal{D}_\circ(\mathbb{U})$. We can similarly show that $(\hat{C}_{\varphi_t})_{t\leq 0}$ is also a semigroup on $\mathcal{D}_\circ(\mathbb{U})$. Thus, $(\hat{C}_{\varphi_t})_{t\in\mathbb{R}}$ is a group on $\mathcal{D}_\circ(\mathbb{U})$. □

Proposition 4.2.5

The operator $\hat{C}_{\varphi_t}$ is an isometry on $\mathcal{D}_\circ(\mathbb{U})$.

PROOF. By norm definition,

$$\begin{aligned} \|\hat{C}_{\varphi_t} f\|^2_{\mathcal{D}_\circ(\mathbb{U})} &= \int_{\mathbb{U}} |(\hat{C}_{\varphi_t} f)'(\omega)|^2 dA(\omega) \\ &= \int_{\mathbb{U}} \left| \left(f(e^{-t}\omega) - f(e^{-t}i) \right)' \right|^2 dA(\omega) \\ &= \int_{\mathbb{U}} \left| e^{-t} f'(e^{-t}\omega) \right|^2 dA(\omega). \end{aligned} \tag{4.4}$$

By change of variables, we let $z = e^{-t}\omega$, then, $\omega = e^t z$ and applying the Jacobian, $dA(z) = e^{-2t} dA(\omega)$, implying that $dA(\omega) = e^{2t} dA(z)$. Substituting them in (4.4),

$$\begin{aligned} \|\hat{C}_{\varphi_t} f\|^2_{\mathcal{D}_\circ(\mathbb{U})} &= \int_{\mathbb{U}} e^{-2t} |f'(e^{-t}\omega)|^2 dA(\omega) \\ &= \int_{\mathbb{U}} e^{-2t} |f'(z)|^2 e^{2t} dA(z) \\ &= \int_{\mathbb{U}} |f'(z)|^2 dA(z) \\ &= \|f\|^2_{\mathcal{D}_\circ(\mathbb{U})}. \end{aligned}$$

This completes our proof. □

Next, we prove that the operator $\hat{C}_{\varphi_t}$ is strongly continuous on the Dirichlet space of the upper half plane $\mathcal{D}_\circ(\mathbb{U})$.

Proposition 4.2.6

The operator $\hat{C}_{\varphi_t}$ is strongly continuous on $\mathcal{D}_\circ(\mathbb{U})$.

PROOF. It is known that $\|\hat{C}_{\varphi_t} f\|^2_{\mathcal{D}_\circ(\mathbb{U})} = \int_{\mathbb{U}} |f'(\omega)|^2 dA(\omega)$. To prove strong continuity of $(\hat{C}_{\varphi_t})_{t\in\mathbb{R}}$, it suffices to show that $\lim_{t\to 0^+} \|\hat{C}_{\varphi t} f - f\|_{\mathcal{D}_\circ(\mathbb{U})} = 0$ for all $f \in \mathcal{D}_\circ(\mathbb{U})$. That is to say that, $\int_{\mathbb{U}} |(\hat{C}_{\varphi t} f - f)'(\omega)|^2 \, dA(\omega) \to 0$ as $t \to 0^+$ which is equivalent to showing that $\lim_{t\to 0^+} \int_{\mathbb{U}} |(\hat{C}_{\varphi t} f - f)'(\omega)|^2 dA(\omega) = 0$.

Let $f \in \mathcal{D}_\circ(\mathbb{U})$ and suppose that $t_n \to 0$ in $\mathbb{R}$. Let $f_n = \hat{C}_{\varphi tn} f$, then $f_n(z) \to f(z)$ on compact subsets of $\mathbb{U}$ and $f_n' \to f'$ for each n.

Let $g_n(z) := 2(|f'|^2 + |f_n'|^2) - |f' - f_n'|^2$, then $g_n \geq 0$ and $g_n(z) \to 2^2|f'(z)|^2$ on $\mathcal{D}_\circ(\mathbb{U})$ as $n \to \infty$.

By Fatou's lemma, we have

$$\begin{aligned}
\int_{\mathbb{U}} 2^2|f'(\omega)^2 dA(\omega) &= \int_{\mathbb{U}} \liminf g_n dA(\omega) \\
&\leq \liminf \int_{\mathbb{U}} g_n dA(\omega) \\
&= \liminf \int_{\mathbb{U}} 2(|f'|^2 + |f_n'|^2) - |f' - f_n'|^2) dA(\omega) \\
&= 2\int_{\mathbb{U}} |f'|^2 dA + 2\int_{\mathbb{U}} |f_n'|^2 dA - \limsup_n \int_{\mathbb{U}} |f' - f_n'| dA(\omega) \\
&= 2^2 \int_{\mathbb{U}} |f'(z)|^2 dA - \limsup_n \int_{\mathbb{U}} |f' - f_n'| dA(\omega)
\end{aligned}$$

Thus $0 \leq -\limsup_n \int_{\mathbb{U}} |f' - f_n'|^2 dA \leq 0$, implying that $\limsup_n \int_{\mathbb{U}} |f' - f_n'|^2 dA(\omega) = 0$.

Hence $\lim_n \int_{\mathbb{U}} |f' - f_n'|^2 dA(\omega) = 0$, that is

$\lim_n \int_{\mathbb{U}} \|\hat{C}_{\varphi tn} f - f\|^2 dA(\omega) = 0$.

Therefore, $\|\hat{C}_{\varphi t} f - f\|_{\mathcal{D}_\circ(\mathbb{U})} \to 0$ as $t \to 0$, implying that $(\hat{C}\varphi_t)_{t\in\mathbb{R}}$ is strongly continuous as desired. □

We have shown that $(\hat{C}_{\varphi_t})_{t\in\mathbb{R}}$ is a strongly continuous group of isometries on $\mathcal{D}_\circ(\mathbb{U})$. We now obtain the infinitesimal generator Γ of $(\hat{C}_{\varphi_t})_{t\in\mathbb{R}}$ and investigate some of its properties. In particular, we determine the resolvent operator as an integral operator of the Cesàro type and then determine the point spectrum, spectrum and spectral radius as well as the norm of the operator.

Proposition 4.2.7

The infinitesimal generator Γ *of* $(\hat{C}_{\varphi_t})_{t\geq 0}$ *is given by*

$$\Gamma f(\omega) = -\omega f'(\omega) + i f'(i),$$

with its domain $dom(\Gamma) = \{f \in \mathcal{D}(\mathbb{U}) : \omega f'(\omega) \in \mathcal{D}(\mathbb{U})\}$.

PROOF. If $f \in \mathrm{dom}(\Gamma)$ in $\mathcal{D}(\mathbb{U})$, then growth condition in (3.12) implies that for all $\omega \in \mathbb{U}$ and $f \in \mathcal{D}(\mathbb{U})$,

$$\begin{aligned}
\Gamma f(\omega) &= \lim_{t\to 0^+} \frac{(f(e^{-t}\omega) - f(e^{-t}i)) - f(\omega)}{t} \\
&= \frac{\partial}{\partial t}(f(e^{-t}\omega) - f(e^{-t}i))\Big|_{t=0} \\
&= -e^{-t}\omega f'(e^{-t}\omega) + ie^{-t}f'(e^{-t}i)\big|_{t=0} \\
&= -\omega f'(\omega) + i f'(i)
\end{aligned}$$

This shows that $\mathrm{dom}(\Gamma) \subseteq \{f \in \mathcal{D}(\mathbb{U}) : \omega f'(\omega) \in \mathcal{D}(\mathbb{U})\}$. Conversely, let $f \in \mathcal{D}(\mathbb{U})$ such that $\omega f'(\omega) \in \mathcal{D}(\mathbb{U})$. Then for $\omega \in \mathbb{U}$, and by fundamental theorem of calculus, we have,

$$\begin{aligned}
\hat{C}_{\varphi_t} f(\omega) - f(\omega) &= \int_0^t \frac{\partial}{\partial s}(f(e^{-s}\omega) - f(e^{-s}i))ds \\
&= \int_0^t -e^{-s}\omega f'(e^{-s}\omega) + e^{-s}if'(e^{-s}i))ds \\
&= \int_0^t e^{-s}(-\omega f'(\omega) + if'(i))ds \\
&= \int_0^t \hat{C}_{\varphi_s} F(\omega)ds,
\end{aligned}$$

where $F(\omega) = -\omega f'(\omega) + if'(i)$ is a function in $\mathcal{D}(\mathbb{U})$. Thus $\lim_{t\to 0} \frac{\hat{C}_{\varphi_t} f - f}{t}$ $= \lim_{t\to 0} \frac{1}{t} \int_0^t C_{\varphi_s}(F)ds$ and strong continuity of $(\hat{C}_{\varphi_t})_{t\in\mathbb{R}}$ implies that

$\frac{1}{t}\int_0^t \|\hat{C}_{\varphi_t}F - F\|ds \to 0$ as $t \to 0$. Thus

$\text{dom}(\Gamma) \supseteq \{f \in \mathcal{D}(\mathbb{U}) : \omega f'(\omega) \in \mathcal{D}(\mathbb{U})\}$ completing the proof. $\square$

We now investigate the spectral properties of the infinitesimal generator. We start by computing the point spectrum of the infinitesimal generator but first we give the Lemma below which states the condition necessary for a function to be in Bergman space.

Lemma 4.2.8 ([7])

Let X denote the space $L^p_a(\mathbb{U}), 1 \leq p < \infty$. If $c \in \mathbb{R}$ and $\lambda, v \in \mathbb{C}$, then $f(\omega) = (\omega - c)^\lambda (w + i)^v \in X$ if and only if $\Re(\lambda + v) < -1 < \Re(\lambda)$. In particular, $(\omega - c)^\lambda \notin X$ for any $\lambda \in \mathbb{C}$, and $(\omega + i)^v \in X$ if and only if $\Re(v) < -1$.

Proposition 4.2.9

Let Γ be the infinitesimal generator of the group $(\hat{C}_{\varphi_t})_{t\in\mathbb{R}}$, then the point spectrum $\sigma_p(\Gamma)$ of Γ is empty, that is, $\sigma_p(\Gamma) = \emptyset$.

Proof. Let λ be an eigenvalue of Γ and let f be a corresponding eigenvector. The eigenvalue equation $\Gamma(f) = \lambda f$ is equivalent to the differential equation

$$-zf'(z) + if'(i) = \lambda f(z).$$

To solve the differential equation, we let $if'(i) = B$ so that

$$-zf'(z) - \lambda f(z) = -B.$$

Dividing through by z yields

$$f'(z) + \lambda \frac{f(z)}{z} = \frac{B}{z}.$$

This is a first order differential equation whose solution is obtained by using an integrating factor technique as,

$$f(z) = \frac{B}{\lambda} + Cz^{-\lambda},$$

where $B = if'(i)$ and C is an arbitrary constant.
It remains to find for which $\lambda's$ is $f \in \mathcal{D}_\circ(\mathbb{U})$ given that $f(z) = \frac{B}{\lambda} + Cz^{-\lambda}$. But $f \in \mathcal{D}_\circ(\mathbb{U})$ if and only if $f' \in L^2_a(\mathbb{U})$. By differentiation, $f'(z) = -\lambda C z^{-(\lambda+1)}$. It follows clearly from Lemma 4.2.8, that $f' \in L^2_a(\mathbb{U})$ if and only if $\Re(\lambda) < -1 < \Re(\lambda)$. No such λ exists and so $\sigma_p(\Gamma) = \emptyset$. □

We now compute the spectrum of the infinitesimal generator, $\sigma(\Gamma)$.

Proposition 4.2.10

Let Γ be the infinitesimal generator of $(\hat{C}_{\varphi_t})_{t\geq 0}$. Then $\sigma(\Gamma) \subseteq i\mathbb{R}$.

Proof. Since $(\hat{C}_{\varphi_t})$ is an invertible isometry, (Theorem 2.0.5), $\sigma(\hat{C}_{\varphi_t}) \subseteq \partial\mathbb{D}$ and by spectral mapping theorem for semigroups, (Theorem 2.0.8), $e^{t\sigma(\Gamma)} \subseteq \sigma(\hat{C}_{\varphi t})$. Thus

$$e^{t\sigma(\Gamma)} \subseteq \sigma(\hat{C}_{\varphi t}) \subseteq \partial\mathbb{D}.$$

Let $\lambda \in \sigma(\Gamma)$, then

$$|e^{\lambda t}| = 1.$$

This shows that

$$\begin{aligned} e^{t\Re(\lambda)} = 1 \quad &\Rightarrow \quad t\Re(\lambda) = 0 \\ &\Rightarrow \quad \Re(\lambda) = 0. \end{aligned}$$

So $\lambda \in i\mathbb{R}$ implying that $\sigma(\Gamma) \subseteq i\mathbb{R}$. □

4.2.1 Cesàro - Type operator

We obtain the Cesáro type operator on the Dirichlet space of the upper half plane. Since $\sigma(\Gamma) \subseteq i\mathbb{R}$, we can consider a point $\lambda = 1$ in the resolvent set, $\rho(\Gamma)$, and obtain the resolvent operator given by the Laplace transform.

Theorem 4.2.11

Let Γ be the infinitesimal generator of $(\hat{C}_{\varphi_t})_{t\in\mathbb{R}}$, then the following holds;

(a) The resolvent operator $\mathcal{C} = R(1,\Gamma)$ on $\mathcal{D}_\circ(\mathbb{U})$ is given by

$$\mathcal{C}h(z) = R(1,\Gamma)h(z) = \frac{1}{z}\int_0^z \Big(h(\omega) - h(\frac{\omega}{z}i)\Big)\,d\omega \qquad (4.5)$$

The operator $\mathcal{C}$ is a Cesàro type operator which is a difference of two Cesàro operators.

(b) $\sigma(\mathcal{C}) \subseteq \{\omega : |\omega - \frac{1}{2}| = \frac{1}{2}\}$

(c) $\|\mathcal{C}\| \leq 1$

(d) $r(\mathcal{C}) \leq 1$

Proof. To prove (a), we consider a point $\lambda = 1$. Then $\lambda \in \rho(\Gamma)$ since $\sigma(\Gamma) \subseteq i\mathbb{R}$. The resolvent operator, $R(\lambda,\Gamma)$, is therefore given by the Laplace transform, (Theorem 2.0.3),
$R(\lambda,\Gamma)h = \int_0^\infty e^{-\lambda t}\hat{C}_{\varphi_t}h dt$ with convergence in norm.

Now

$$R(\lambda,\Gamma)h(z) = \int_0^\infty e^{-\lambda t}(h(e^{-t}z) - h(e^{-t}i))dt.$$

By change of variables, we let $\omega = e^{-t}z$. Then $e^{-t} = \frac{\omega}{z}, dw = -e^{-t}zdt$, $t = 0 \Rightarrow \omega = z, t = \infty \Rightarrow \omega = 0$.

Therefore

$$\begin{aligned} R(\lambda,\Gamma)h(z) &= \int_z^0 (\frac{\omega}{z})^\lambda (h(\omega) - h(\frac{\omega}{z}i))\frac{-1}{\omega}d\omega. \\ &= \int_0^z (\frac{\omega}{z})^\lambda \left(h(\omega) - h(\frac{\omega}{z}i)\right)\frac{1}{\omega}d\omega. \end{aligned}$$

Taking $\lambda = 1$, we obtain

$$\begin{aligned} R(1,\Gamma)h(z) &= \int_0^z \frac{\omega}{z}\left(h(\omega) - h(\frac{\omega}{z}i)\right)\frac{d\omega}{\omega} \\ &= \frac{1}{z}\int_0^z \left(h(\omega) - h(\frac{\omega}{z}i)\right)d\omega, \end{aligned}$$

which is a difference of two Cesàro operators.

To prove (b), we apply the spectral mapping theorem, (Theorem 2.0.7), for the resolvents which asserts that

$$\sigma(R(\lambda,T))\backslash\{0\} = (\lambda - \sigma(T))^{-1} = \left\{\frac{1}{\lambda-\mu} : \mu \in \sigma(T)\right\} \tag{4.6}$$

Thus,

$$\sigma(R(1,\Gamma))\backslash\{0\} \subseteq \left\{\frac{1}{1-ir} : r \in \mathbb{R}\right\}. \tag{4.7}$$

Rationalizing the denominator and simplifying we get,

$$\frac{1}{1-ir} = \left\{\frac{1}{1-ir}\cdot\frac{1+ir}{1+ir} : r \in \mathbb{R}\right\}$$

$$= \left\{\frac{1+ir}{1+r^2} : r \in \mathbb{R}\right\}.$$

Letting $\omega = \frac{1+ir}{1+r^2}$ and subtracting $\frac{1}{2}$, we get

$$\begin{aligned}
w - \frac{1}{2} &= \frac{1+ir}{1+r^2} - \frac{1}{2} \\
&= \frac{2(1+ir) - (1+r^2)}{2(1+r^2)} \\
&= \frac{1+2ir-r^2}{2+2r^2} \\
&= \frac{(r-i)(-r+i)}{2(r+i)(r-i)} \\
&= \frac{-r+i}{2(r+i)}.
\end{aligned}$$

Getting the magnitude on both sides of the equation and simplifying, we get

$$\begin{aligned}
\left|\omega - \frac{1}{2}\right|^2 &= \left|\frac{-r+i}{2(r+i)}\right|^2 \\
&= \left|\frac{r^2+1}{4(r^2+1)}\right| \\
\left|\omega - \frac{1}{2}\right|^2 &= \frac{1}{4} \\
\omega - \frac{1}{2} &= \frac{1}{2} \\
\sigma(\mathcal{C}) &\subseteq \left\{\omega : \left|\omega - \frac{1}{2}\right| = \frac{1}{2}\right\}
\end{aligned}$$

For (c), we apply the Hille Yosida theorem, (Theorem 2.0.6)

$$\|R(1, \Gamma)\| \leq 1,$$

implying that

$$\|\mathcal{C}\| \leq 1. \tag{4.8}$$

For (d), we use (4.8) and the fact that $r(\mathcal{C}) \leq \|\mathcal{C}\| \leq 1$. Clearly,

$$r(\mathcal{C}) \leq 1.$$

This completes our proof. □

Chapter 5

Summary and Recommendations

5.1 Summary

In this study, we determined the reproducing kernel for the Dirichlet space of the upper half plane using the Cayley transform to construct an invertible isometry from the Dirichlet space of the disk to the Dirichlet space of the upper half plane then transformed the reproducing kernel of the disk to the reproducing kernel of the upper half plane as given in Theorem 3.2.7. We then used the reproducing kernel of the Dirichlet space of the upper half plane and applied the Cauchy-Schwarz inequality to establish the growth condition for the functions in the Dirichlet space of the upper half plane as it is shown in Corollary 3.2.8.

Using the approach of strongly continuous semigroups on Banach spaces and considering the group of composition operators associated with the scaling group, we computed the infinitesimal generator of the composition semigroup as seen in Proposition 4.2.7. We further established that the point spectrum of the infinitesimal generator is empty, see Proposition 4.2.9 and that it's spectrum is contained in the imaginary axis of the complex plane as shown in Proposition 4.2.10. Moreover, we constructed an integral operator of the Cesáro type acting on the Dirichlet space of the upper half plane. We specifically applied the Laplace transform which gave the resolvent of the infinitesimal generator in terms of a Cesáro type integral operator, see Theorem 4.2.11. Consequently, we determined the spectrum of the Cesáro type integral operator using the spectral mapping theorems for resolvents. We applied characterization by Hille-Yosida theorem to obtain the upper bound of the norm and spectral radius of the integral operator.

5.2 Recommendations

From the results of this thesis, we recommend the following for further research.

(i) In this study, we determined the reproducing kernel for the Dirichlet space of the upper half plane and established the growth conditions for the functions in the Dirichlet space of the upper half plane. We recommend an extension of the study touching on the properties of the obtained reproducing kernel such as geometric perspectives and duality relations.

(ii) We considered the group of composition operators corresponding to the self analytic maps defined on the scaling group of $\mathcal{D}_\circ(\mathbb{U})$ and studied their semigroup and spectral properties in this work.

We therefore recommend further investigation on the composition operators associated with the rotation group and translation group defined on $\mathcal{D}(\mathbb{U})$.

(iii) The spectral properties of the constructed Ces$\acute{a}$ro type integral operator, $\mathcal{C}$ was also investigated in the study but we did not completely characterize the operator based on the spectral properties. We recommend an extension of the study that will focus further on the spectral analysis of the Ces$\acute{a}$ro type operator and the infinitesimal generator.

(iv) We computed the reproducing kernel for the Dirichlet space of the upper half plane. We recommend extension of the study to the weighted Dirichlet spaces.

References

[1] H. Ahn and J.-D. Park, *The explicit forms and zeros of the Bergman kernel function for Hartogs type domains*, Journal of Functional Analysis, **262**, (2012), 3518-3547.

[2] E. Albrecht, T. L. Miller and M. M. Neumann, *Spectral properties of generalized Cesàro operators on Hardy and weighted Bergman spaces*, Archiv der Mathematik, **85**, (2005), 446459.

[3] M. O. Agwang and J. O. Bonyo, *Spectra of composition groups on the weighted Dirichlet space of the upper half plane*, Acta Mathematica Scientia, **40B(6)**, (2020), 1739-1752.

[4] N. Arcozzi, R. Rochberg, E. T. Sawyer and B. D. Wick, *The Dirichlet space: a survey*, New York Journal of Mathematics, **17a**, (2011), 46-52.

[5] N. Aronszajn, *Theory of reproducing kernels*, Transactions of the American Mathematical Society, **68(3)**, (1950), 337-404.

[6] A. G. Arvanitidis and A. G. Siskasis, *Cesàro Operators on the Hardy Spaces of the half plane*, Canadian Mathematical Bulletin, **56(2)**, (2013), 229-240.

[7] S. Ballamoole, J. O. Bonyo, T. L. Miller and V. G. Miller, *Cesàro Operators on the Hardy and Bergman Spaces of the half plane*, Complex Analysis and Operator Theory, **10**, (2016), 187-203.

[8] S. Bell, *The Fundamental Role of the Szegö Kernel in Potential Theory and Complex Analysis*, Journal für die reine und angewandte Mathematik, **525**, (2000), 1-16.

[9] S. Bergman, *The Kernel function and conformal mapping*, Mathematical Surveys and Monographs, **5**, American Mathematical Society, (1950), 21-45.

[10] S. Bergman and M. Schiffer, *Kernel functions and conformal mapping*, Compositio Mathematica, **8**, (1951), 205-249.

[11] J. O. Bonyo, *Reproducing Kernels for Hardy and Bergman spaces of the upper half plane*, Communications in Advanced Mathematical Sciences, **3(1)**, (2020), 13-23.

[12] O. F. Brevig, K. -M Perfekt and K. Seip, *Volterra operators on Hardy spaces of Dirichlet series*, Journal für die reine und angewandte Mathematik, **754**, (2019), 179-223.

[13] M. Darus and R. W. Ibrahim, *General properties of Volterra type operators in the unit disk*, ISRN Mathematical Analysis, (2011), 1-11.

[14] G. Datt and R. Aggarwal, *On a Generalization of Hankel Operators via Operator Equations*, Extracta Mathematicae, **28(2)**, (2013), 197 211.

[15] O. El-Fallah, K. Kellay, J. Mashreghi and T. Ransford, *A primer on the Dirichlet space*, Cambridge University Press, New York, **203**, (2014).

[16] K. Engel and R. Nagel, *A short course on Operator Semigroups*, Universitext, Springer, New York, **37**, (2006).

[17] L. K. Hua, *Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains*, Translations of Mathematical Monographs, **6**, (1963).

[18] K. Laursen and M. Neumann, *An Introduction to Local Spectral Theory*, London Mathematical Society Monographs, **20**, Oxford University press, (2000).

[19] J. Mercer, *Functions of positive and negative type, and their connection with the theory of integral equations*, Philosophical Transactions of the Royal Society A, **209**, (1909), 415-446.

[20] E. H. Moore, *On properly positive Hermitian matrices*, Bulletin American Mathematical Society, **23(2)**, (1916), pg 59.

[21] E. H. Moore, *General Analysis, part 1*, Bulletin of the American Mathematical Society, **42(7)**, (1936), 465-468.

[22] V. I. Paulsen and M. Raghupathi, *An Introduction to the Theory of Reproducing kernel Hilbert Spaces*, Cambridge Studies in Advanced Mathematics, **152**, Cambridge University Press, (2016).

[23] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, **44**, Springer-Verlag, New York, 1970.

[24] M. M. Peloso, *Classical Spaces of holomorphic functions*, Technical report, Universit di Milano, (2014).

[25] R. Rochberg and Z. Wu, *Toeplitz operators on Dirichlet spaces*, Integral Equations and Operator Theory, **15**, (1992), 325-342.

[26] W. T. Ross, *The Classical Dirichlet Space*, Contemporary Mathematics, **393**, (2006), 1-3.

[27] S. Saitoh, *The exact Bergman Kernel and the kernels of Szegö types*, Pacific Journal of Mathematics, **71(2)**, (1977), 545-556.

[28] S. Saitoh, *Theory of reproducing kernels and its application*, Longman Scientific and Technical, **189**, 1988.

[29] R. K. Singh and J. S. Manhas, *Composition operators on functional spaces*, Elsevier science publishers, Amsterdam, **179**, 1993.

[30] V. Singh, *An Integral Equation Associated with the Szegö Kernel Function*, Proceedings of the London Mathematical Society, **s3-10(1)**, (1960), 376-394.

[31] A. G. Siskakis, *Semigroups of composition operators on the spaces of analytic functions, a review*, Contemporary Mathematics, **213**, (1998), 229-252.

[32] A. G. Siskakis, *Semigroups of composition operators on Dirichlet spaces*, Results in Mathematics, **30**, (1996), 165-173.

[33] G. Szegö, *Über orthogonale Polynome, die zu einer gegebenen Kurve der komplexen Ebene gehören*, mathematische Zeitschrift, **9**, (1921), 218-270.

[34] S. Zaremba, *L'equation biharmonique et une clasee remarquable de fonctions fondamentales harmoniques,* Bulletin International de l'Acadmie des Sciences de Cracovie, (1907), 147-196.

[35] L. Zhau, *Hankel operators on the Dirichlet space*, Journal of Mathematic Analysis and Applications, **352(2)**, (2009), 767-772.

[36] K. Zhu, *Operator Theory in function spaces*, Mathematical Surveys and Monographs, **138**, American Mathematical Society, Providence, (2007).