ON SHIFTED FIBONACCI SEQUENCES AND THEIR POLYNOMIALS

BY

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DECLARATION

This thesis is my own work and has not been presented for a degree award in any other institution.

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foundation admired by many.

ABSTRACT

Fibonacci sequences and their polynomials have been generalized mainly by two ways: by maintaining the recurrence relation and varying the initial conditions and by varying the recurrence relation and maintaining the initial conditions. In this thesis, we maintain the recurrence relation and vary initial conditions which are taken as sum of Fibonacci numbers or polynomials. The main objective of this work was to generalize Fibonacci sequences and their polynomials by r-shift operation and to determine properties of these generalized sequences and their polynomials. The specific objectives were to generalize Finonacci sequences and their polynomials , to determine properties of r-shifted Fibonacci sequences and to determine properties of r-shifted Fibonacci polynomials. To achieve the first objective, we maintain recurrence relation and vary the initial conditions by $r-$ shift operation. To achieve the second objective we mainly use Binet's formula and generating function of r-shifted Fibonacci sequences, mathematical induction and direct proofs, and to achieve the third objective we used Binet's formula and generating function for r-shifted Fibonacci polynomials. Among results obtained in this thesis for both r-shifted Fibonacci numbers and polynomials are explicit sum formula, sum of first n terms, sum of first n terms with even indices, sum of first n terms with odd indices, Honsberger's identity, and generalized identity from which we get Catalan's identity, Cassini's identity, and d'Ocagne's identity. The results obtained in this study add to the already existing literature in this area of research and they are also of importance to researchers in Computer Science and other fields Mathematics.

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CHAPTER 1

INTRODUCTION

Generalizations of Fibonacci and Lucas numbers as well as their polynomials have many interesting properties and applications to almost every field of science and art. The beauty and rich application of these numbers and their relatives are found in Koshy's book [10].

1.1 Basic concepts

1.1.1 Fibonacci and Lucas numbers

The following definitions and selected identities involving Fibonacci and Lucas numbers were of essential use throughout this study.

Definition 1.1.1 ([10])**.** A *recursive formula* is an equation that expresses the n^{th} term of the sequence in terms of one or more of the previous terms of the sequence. Initial conditions are required to specify terms that precede the first term where the relation takes effect. Recursion is an example of an iterative procedure.

Definition 1.1.2 ([11, 18])**.** *Fibonacci numbers* are numbers in the integer sequence defined recursively as

$$f_n = f_{n-1} + f_{n-2}$$

for all $n \geq 2$ with initial conditions $f_0 = 0$ and $f_1 = 1$.

Definition 1.1.3 ([11, 18]). *Lucas numbers* are companion to Fibonacci numbers. They are numbers in the integer sequence defined by the recurrence relation

$$l_n = l_{n-1} + l_{n-2}$$

for all $n \geq 2$ with initial conditions $l_0 = 2$ and $l_1 = 1$.

Definition 1.1.4 ([10]). *Fibonacci numbers for negative subscripts* are numbers in the integer sequence defined as

$$f_{-n} = (-1)^{n+1} f_n$$

for all $n \geq 1$.

Definition 1.1.5 ([10]). *Lucas numbers for negative subscripts* are defined as

$$l_{-n} = (-1)^n l_n$$

for all $n \geq 1$.

Definition 1.1.6 ([10]). A *generating function* is a way of encoding an infinite sequence of numbers by treating them as coefficients of a power series.

The generating function for the Fibonacci numbers $F(x)$ is given by

$$F(x) = \sum_{n=0}^{\infty} f_n x^n = \frac{x}{1 - x - x^2}.$$

The generating function for the Lucas numbers $L(x)$ is given as

$$L(x) = \sum_{n=0}^{\infty} l_n x^n = \frac{2 - x}{1 - x - x^2}.$$

Definition 1.1.7 ([10]). An *explicit / closed formula* is a formula for a sequence which allows one to find the value of any term of a sequence.

Definition 1.1.8 ([10]). *Binet's formula* is an explicit formula used to find the n^{th} term of the Fibonacci sequence. If f_n is the $(n+1)^{th}$ Fibonacci number then, Binet's formula is given by

$$f_n = \frac{1}{\alpha - \beta}\left[\alpha^n - \beta^n\right], \tag{1.1}$$

where $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$.

Binet's formula for the Lucas numbers is given by

$$l_n = \alpha^n + \beta^n. \tag{1.2}$$

The following are some properties of α and β :

$$\begin{array}{ll}
\alpha + \beta = 1, & \alpha - \beta = \sqrt{5}, \\
\alpha\beta = -1, & \alpha^2 + \beta^2 = \sqrt{5} \\
\alpha - 1 = -\beta, & \beta - 1 = -\alpha \\
\alpha^2 - 1 = \alpha, & \beta^2 - 1 = \beta, \\
\alpha^2 + 1 = \sqrt{5}\alpha, & \beta^2 + 1 = -\sqrt{5}\beta, \\
\alpha^3 - 1 = 2\alpha, & \beta^3 - 1 = 2\beta
\end{array}$$

Many identities have been discovered that reveal interesting relationships between Fibonacci and Lucas numbers [10].

1.1.2 Properties of Fibonacci sequence

Some of the properties of Fibonacci numbers are:

The sum of the first n terms of Fibonacci sequence is given by

$$f_0 + f_1 + f_2 + \cdots + f_{n-1} = f_{n+1} - 1. \tag{1.3}$$

We also have the other sum formulas for Fibonacci numbers for $n \in \mathbb{N}$ as:

1. $f_1 f_2 + f_2 f_3 + f_3 f_4 + \cdots + f_{2n-1} f_{2n} = f_{2n}^2$.

2. $f_1^2 + f_2^2 + f_3^2 + \cdots + f_n^2 = f_n f_{n+1}$.

3. $f_1 + f_3 + f_5 + \cdots + f_{2n-1} = f_{2n}$.

4. $f_2 + f_4 + f_6 + \cdots + f_{2n} = f_{2n+1} - 1$.

5. $f_1 - f_2 + f_3 - f_4 + \cdots + (-1)^{n+1} f_n = (-1)^{n+1} f_{n-1} + 1$.

Honsberger identity for Fibonacci numbers is given by

$$f_{n+m} = f_{n-1} f_m + f_n f_{m+1}, \tag{1.4}$$

for $n \geq 1$, $n > m$ and $n, m \in \mathbb{Z}$

The following formulas relate Fibonacci and Lucas numbers:

1. $l_n = f_{n-1} + f_{n+1}$, for all $n \geq 1$.

2. $f_{2n} = l_n f_n$, for all $n \geq 1$.

3. $l_{m+n} = f_{m+1} l_n + f_m l_{n-1}$, for all $m > 0, n > 0$ and $m, n \in \mathbb{Z}$

The d'Ocagne's identity for Fibonacci numbers is given by

$$f_m f_{n+1} - f_{m+1} f_n = (-1)^n f_{m-n},$$

for all $n \geq 1$ and $n > m$.

Catalan's identity for Fibonacci numbers is

$$f_{n+m} f_{n-m} - f_n^2 = (-1)^{n-m+1} f_m^2, \tag{1.5}$$

for all $n \geq 1$ and $n > m$.

Setting $m = 1$ in (1.5), we get Cassini's identity:

$$f_{n-1} f_{n+1} - f_n^2 = (-1)^n,$$

for all $n \geq 1$.

Other formulas for Fibonacci numbers are

1. $f_{2n} = f_{n+1}^2 - f_{n-1}^2$, for all $n \geq 1$.

2. $f_{2n+1} = f_{n+1}^2 + f_n^2$, for all $n \geq 0$.

3. $f_{m+n+1} = f_{m+1}f_{n+1} + f_m f_n$, for $m > 0$, and $n \geq 0$.

4. $f_{-n} = (-1)^{n-1} f_n$, for all $n \geq 1$.

1.1.3 Fibonacci and Lucas polynomials

Definition 1.1.9 ([10])**.** *Fibonacci polynomials* satisfy the recurrence relation

$$f_n(x) = x f_{n-1}(x) + f_{n-2}(x), \tag{1.6}$$

for $n \geq 2$ with $f_0(x) = 0$ and $f_1(x) = 1$. Here $x \in \mathbb{N}$.

Definition 1.1.10 ([10])**.** *Lucas polynomials* are defined by

$$l_n(x) = x l_{n-1}(x) + l_{n-2}(x),$$

for $n \geq 2$ with $l_0(x) = 2$ and $l_1(x) = x$.

Generating function for Fibonacci and Lucas polynomials are

$$\sum_{n=0}^{\infty} f_n(x) t^n = \frac{t}{1 - xt - t^2},$$

and

$$\sum_{n=0}^{\infty} l_n(x) t^n = \frac{2 - xt}{1 - xt - t^2}$$

respectively.

Binet's formula of Fibonacci polynomials is given by

$$f_n(x) = \frac{\alpha(x)^n - \beta(x)^n}{\alpha(x) - \beta(x)}, \tag{1.7}$$

where $\alpha(x) = \dfrac{x + \sqrt{x^2 + 4}}{2}$ and $\beta(x) = \dfrac{x - \sqrt{x^2 + 4}}{2}$.

Note that $\alpha = \alpha(1)$ and $\beta = \beta(1)$

On the other hand, Binet's formula of Lucas polynomials is given by

$$l_n(x) = \alpha(x)^n + \beta(x)^n. \tag{1.8}$$

We also note that:

$$\begin{aligned}
\alpha(x) - \beta(x) &= \sqrt{x^2+4} \\
\alpha(x)\beta(x) &= -1 \\
\alpha(x) + \beta(x) &= x \\
\alpha(x)^2 + \beta(x)^2 &= x^2 + 2.
\end{aligned}$$

Moreover, we have

$$\alpha(x)^2 = x\alpha(x) + 1 \tag{1.9}$$

and

$$\beta(x)^2 = x\beta(x) + 1. \tag{1.10}$$

1.1.4 Properties of Fibonacci polynomials

Some of the properties of Fibonacci polynomials are highlighted below:

The sum of the first n terms of the Fibonacci polynomials is

$$\sum_{i=0}^{n-1} f_i(x) = \frac{f_n(x) + f_{n-1}(x) - 1}{x}. \tag{1.11}$$

Honsberger's identity for Fibonacci polynomials is given by:

$$f_{n+m}(x) = f_{n-1}(x)f_m(x) + f_n(x)f_{m+1}(x). \tag{1.12}$$

We also have the formulas that connect Fibonacci and Lucas polynomials given by following relations:

1. $l_n(x) = f_{n-1}(x) + f_{n+1}(x)$, for all $n \geq 1$.

2. $f_{2n}(x) = l_n(x)f_n(x)$, for all $n \geq 1$.

3. $l_{m+n}(x) = f_{m+1}(x)l_n(x) + f_m(x)l_{n-1}(x)$, for all $m > 0, n > 0$.

1.2 Statement of the problem

Generalizations of Fibonacci sequences and their polynomials have been done using various approaches. Some researchers have maintained the recurrence relation and changed the initial conditions, while others have maintained the initial conditions and changed the recurrence relation, but the initial conditions have never been taken as a sum of Fibonacci numbers or polynomials themselves. In this work we introduce $r-$ Fibonacci numbers and polynomials which generalize Fibonacci numbers and polynomials. We obtain properties of these numbers and their polynomials.

1.3 Objectives of the study

1.3.1 General Objective

The purpose of this study was to generalize Fibonacci sequences and their polynomials by r-shift operation and to determine properties of these generalized sequences and their polynomials.

1.3.2 Specific Objectives

The specific objectives of the study were:

1. To generalize Fibonacci numbers and polynomials.
2. To determine the properties of r-shifted Fibonacci sequences.
3. To determine the properties of r-shifted Fibonacci polynomials.

1.4 Significance of the study

Fibonacci sequence originated from rabbit breeding problem, introduced by Italian mathematician Fibonacci in 1202. The sequence also occur in Pascal's triangle, Pythagorean triples, computer algorithms, combinatorics, graph theory and quasi crystals. They are also found in a variety of other fields such as art, music and architecture. Fibonacci polynomials are of great importance in algebra, geometry, approximation theory, statistics and number theory itself. Thus the results obtained in this research add to the existing literature in this area of study. The results are also of significance to computer scientists and researchers in other fields of mathematics.

1.5 Methodology

To achieve the first objective we maintain the recurrence relation of $r-$ shifted Fibonacci numbers and polynomials and vary their initial conditions which are considered as sum of Fibonacci numbers and Fibonacci polynomials.

To achieve the second objective of determining the properties of r-shifted Fibonacci sequences, we use closed formula for the n^{th} term of r-shifted Fibonacci sequence (Binet's formula), generating functions, mathematical induction and direct proofs.

For the third objective of determining properties of r-shifted polynomials, we employ Binet's formula and generating functions for $r-$ shifted Fibonacci polynomials and standard identities relating Fibonacci and Lucas numbers and their polynomials.

CHAPTER 2

LITERATURE REVIEW

Fibonacci sequence is generated by the recursive formula $f_n = f_{n-1} + f_{n-2}$, for $n \geq 2$ with $f_0 = 0$ and $f_1 = 1$. The sequence has many interesting properties. For example, the ratio $\frac{f_{n+1}}{f_n}$ converges to the golden ratio, $\frac{1+\sqrt{5}}{2}$, as n goes to infinity. These sequences were first studied in 1202 by Italian mathematician Fibonacci, in his book *Liber Abasi*. Over the centuries, many authors have studied these sequences and presented different versions of their generalizations.

Horadam [6] introduced and obtained various properties of generalized Fibonacci sequence H_n defined by the recurrence relation $H_n = H_{n-1} + H_{n-2}$, for $n \geq 3$ with $H_1 = p$ and $H_2 = p + q$ where p and q are arbitrary integers. Some of the properties he obtained are Catalan's identity and Cassini's identity for this sequence.

In 1965, Horadam [7] again generalized Fibonacci sequence, w_n, defined by the recurrence relation $w_n = Pw_{n-1} + qw_{n-2}$, for $n \geq 2$ with $w_0 = a$ and $w_1 = b$ where a, b, p and q are arbitrary integers. He obtained Binet's formula but used mainly direct proof in proving identities of this sequence.

Later on, Kalma and Dena [9] generalized Fibonacci sequence as $f_n = af_{n-1} + bf_{n-2}$, for $n \geq 2$ with $f_0 = 0$ and $f_1 = 1$ and obtained properties of this sequence such as Fibonacci connections to pythagorean triples and the greatest common divisor function.

Falcon and Plaza [3] introduced k^{th} Fibonacci numbers, $f_{k,n}$, with $n \in \mathbb{N}$ and obtained properties of the numbers by use of elementary matrix algebra. For any positive integer $k \geq 1$, the k^{th} Fibonacci number is defined by $f_{k,0} = 0$ and $f_{k,1} = 1$ and $f_{k,n+1} = kf_{k,n} + f_{k,n-1}$, for $n \geq 1$.

Singh, Sikhwal and Bhatnagar [18], in 2010, defined Fibonacci-like sequence by the recurrence relation $S_n = S_{n-1} + S_{n-2}$ for all $n \geq 2$ with $S_0 = 2$ and $S_1 = 2$. The associated initial conditions S_0 and S_1 are the sum of initial conditions of Fibonacci sequence and Lucas sequence, i.e., $S_0 = f_0 + L_0$ and $S_1 = f_1 + L_1$. They derived Binet's formula and generating function of the Fibonacci-like sequence and used Binet's formula and mathematical induction to prove the properties of the sequence. They also showed that $S_n = 2f_{n+1}$, where f_n is the classical Fibonacci number.

In 2013, Singh, Bhatnagar and Sikhwal [20] defined Fibonacci-like sequence by the recurrence relation $H_n = 2H_{n-1} + H_{n-2}$, where $n \geq 2$ with $H_0 = 2$ and $H_1 = 1$. The associated initial conditions $H_0 = 2$ and $H_1 = 1$ are the difference of initial conditions of pell-lucas and pell sequence respectively. The said authors derived Binet's formula and generating function and used Binet's formula to prove the connection formulae and properties of Fibonacci-like sequence.

Panwar and Singh [13], in 2014, obtained properties of generalized Fibonacci sequence defined recursively as $f_k = pf_{k-1} + qf_{k-2}$ for $k \geq 2$, with $f_0 = a$ and $f_1 = b$. Here, p, q, a and b are positive integers. The authors used the principle of mathematical induction as well as Binet's formula to derive several identities involving the generalized Fibonacci sequence.

In [23], Singh, Sikhwal and Gupta defined generalized Fibonacci-like sequence by the recurrence relation $M_n = M_{n-1} + M_{n-2}$, for all $n \geq 2$ with $M_0 = 2$ and $M_1 = s + 1$ where s is a fixed integer. Various identities of these sequences were derived using Binet's formula and generating functions.

Singh, Sikhwal and Gupta [21], in 2014, studied generalized Fibonacci-Lucas sequence defined by $B_n = B_{n-1} + B_{n+1}$ for $n \geq 2$ with $B_0 = 2b$ and $B_1 = s$, where b and s are integers. Determinant identities of this sequence were determined. Other identities were proved by mathematical induction and Binet's formula.

In 2014, Gupta, Singh and Sikhwal [5] defined generalized Fibonacci-like sequence by the recurrence relation: $B_n = B_{n-1} + B_{n-2}$, for $n \geq 2$ with $B_0 = 2s$ and $B_1 = s + 1$, such that s is a fixed integer. The associated initial conditions B_0 and B_1 are the sum of initial condition of generalized Fibonacci-like sequence given by $B_0 = f_0 + sL_0$ and $B_1 = f_1 + sL_1$. The said authors determined determinant identities, Binet's formula and generating function. They used Binet's formula to prove identities of the generalized Fibonacci-like sequence.

Wani, Rathore and Sisodiya [26] introduced Fibonacci-like sequence defined by the recurrence relation $T_n = T_{n-1} + T_{n-2}$, $n \geq 2$ with initial conditions $T_0 = m$ and $T_1 = m$ where m is a fixed integer. They established properties of Fibonacci-like sequence mainly by Binet's formula and generating functions.

In 2016, Rathore, Sikhwal and Choundry [14] defined generalized Fibonacci-like sequence by the recurrence relation, $R_n = R_{n-1} + R_{n-2}$, for $n \geq 2$ with initial conditions $R_0 = 2b$ and $R_1 = a + b$, where a and b are nonzero real numbers. Here, the initial conditions are $R_0 = af_1 + bL_0$ and $R_1 = af_1 + bL_1$. They further determined determinant identities and connection formulae. The authors used Binet's formula and generating functions to prove identities of the Fibonacci-like sequence. If $a = b = 1$, then the sequence becomes conventional Fibonacci-like sequence $2, 2, 4, 6, 10, 16, 26, \ldots$ and if $a = 2$, $b = 1$, then it becomes Fibonacci sequence $2, 3, 5, 8, 13, \ldots$.

Sikhwal and Vyas [17] defined Fibonacci-type sequence by the recurrence relation $Y_{n+2} = Y_{n+1} + aY_n$, for $n \geq 0$, with initial conditions $Y_0 = 2$ and $Y_1 = 2 + b$, where a and b are integers. Further, they established determinant identities and proved standard identities of the Fibonacci-type sequence by means of Binet's formula and mathematical induction.

From the reviewed literature, Fibonacci-like sequences where initial conditions are a sum of Fibonacci numbers themselves, have not been studied.

On the other hand, Fibonacci and Lucas polynomials posses wonderful and amazing properties just like Fibonacci and Lucas numbers. They have been applied in every branch of mathematics and studied on a more advanced level by many mathematicians.

Horzum and Kocer [8], in 2009, studied some properties of Horadam polynomials defined by the recurrence relation $h_n(x) = pxh_{n-1}(x) + qh_{n-2}(x)$ for $n \geq 3$ with initial conditions $h_1(x) = a$ and $h_2(x) = bx$. Summation formulas of these polynomials are derived by Binet's formula.

Years later, Singh, Bhatnagar and Sikhwal [19] obtained properties of Fibonacci polynomials defined by the recurrence relation $S_n(x) = xS_{n-1}(x) + S_{n-2}(x)$, for $n \geq 2$ with initial conditions $S_0(x) = 2$ and $S_1(x) = 2x$. They obtained properties of the Fibonacci-like polynomials such as explicit sum formula, sum of first n terms and other basic identities by means of Binet's formula, generating function and matrix method.

In 2014, Singh, Gupta and Sikhwal [22], defined generalized Fibonacci-like polynomials by $M_n(x) = xM_{n-1}(x) + M_{n-2}(x)$, for $n \geq 2$ with initial conditions $M_0(x) = 2s$ and $M_1(x) = 1 + s$ where s is an integer. They obtained properties of the polynomials by means of Binet's formula and generating function.

Singh, Sikhwal and Gupta [24], again in 2014, generalized Fibonacci-Lucas polynomials defined by the recurrence relation $b_n(x) = xb_{n-1}(x) + b_{n-2}(x)$, for $n \geq 2$ with initial conditions $b_0(x) = 2b$ and $b_1(x) = s$, where b and s are integers. Further, they obtained identities of these polynomials by means of generating function.

In 2015, Godase and Dhakne [4] generalized Fibonacci-like polynomials by the recurrence relation $M_{n+1}(x) = K(x)M_n(x) + M_{n-1}(x)$ for $n \geq 2$ with initial conditions $M_0(x) = 2$ and $M_1(x) = M(x) + K(x)$ where $K(x)$ and $M(x)$ are polynomials with real coefficients. They used Binet's formula, mathematical induction and matrix method to prove the properties of these polynomials such as sum formulas, Catalan's identity and d'Ocagne's identity.

Sikhwal and Vyas [16], in 2015, studied generalized Fibonacci polynomials defined by $u_n(x) = xu_{n-1}(x) + u_{n-2}(x)$, for $n \geq 2$ with $u_0(x) = a$ and $u_1(x) = 2a + 1$ where a is any integer. They obtained Binet's formula and generating function of the polynomials. Moreover, they obtained explicit sum formula, sum formulas and generalized identity for the polynomials.

In 2016, Bhatnagar and Sikhwal [2] generalized Fibonacci polynomials defined recursively as $g_n(x) = pxg_{n-1}(x) + qg_{n-2}(x)$ for $n \geq 2$ with $g_0(x) = a$ and $g_1(x) = (a + b)x$, where a,b,p and q are integers. The authors obtained the properties of these polynomials such as explicit sum formula, sum of first n terms and generalized identity by means of Binet's formula and generating function.

Rathore, Sikwal and Choundry [15], in 2016, generalized Fibonacci polynomials by the recurrence relation $W_n(x) = xW_{n-1}(x) + W_{n-2}(x)$, for $n \geq 2$ with $W_0(x) = 2b$ and $W_1(x) = a + b$ where a and b are constants. They derived properties of these polynomials by use of generating functions.

Fibonacci-type polynomials were generalized by Panwar, Rathore and Chawla [12] in 2018. The polynomials are defined by the recurrence relation $y_n(x) = y_{n-1}(x) + 2xy_{n-2}(x)$, where $n \geq 2$ with initial conditions being $y_0(x) = 2$ and $y_1(x) = 2$. Identities of these polynomials were derived using Binet's formula.

Generally, Fibonacci sequences and their polynomials have been generalized in two ways; by varying the recurrence relation and maintaining the initial conditions and varying the initial conditions and maintaining recurrence relation.

In this study we have maintained the recurrence relation, similar to those of Fibonacci numbers and Fibonacci polynomials, and vary the initial conditions. These initial conditions are given as sums of Fibonacci numbers (or polynomials) and are different depending on a shift r. We have obtained equivalent Binet's formula and generating functions for these sequences and their polynomials. We have also obtained various properties of these sequences and their polynomials. The results are obtained by means of direct proofs, Binet's formula, generating functions and induction. Our results generalize known results for Fibonacci numbers and their polynomials.

CHAPTER 3

SHIFTED FIBONACCI SEQUENCES

3.1 Introduction

In this chapter and in the next chapter, we present our results. In Neil Sloane's On-Line Encyclopedia of Integer Sequences [25], Fibonacci sequence is given by A000045. Various generalizations of the Fibonacci sequence have been studied since it was first discovered by Fibonacci in the 13^{th} century. Fibonacci sequence has been generalized mainly by two ways: by maintaining the recurrence relation and varying the initial conditions [3, 6, 7, 9, 13, 17, 18], and by varying the recurrence relation and maintaining the initial conditions [5, 7, 14, 17, 20, 21, 23, 26]. Some of the properties that have been obtained by various researchers are not limited to finding a closed form for the n^{th} term of the sequence, sum of the first n terms of the sequence, sum of the first n terms with odd (or even) indices of the sequence, explicit sum formula, Catalan's identity, Cassini's identity, d'Ocagne's identity, Honsberger's identity, determinant identities, and generating function among many others.

We now introduce r-shifted Fibonacci number. Let $r > 0$ and f_n be the $(n+1)^{th}$ term of Fibonacci sequence.

Definition 3.1.1. The $(n+1)^{th}$ term of *r-shifted Fibonacci sequence*, $h_{n,r}$, is given by

$$h_{n,r} = f_{n+1} + f_{n+2} + \cdots + f_{n+r}. \tag{3.1}$$

Using Definition 3.1.1, it follows that for all $r > 0$, the first term given as

$$h_{0,r} = f_1 + f_2 + \cdots + f_r = f_{r+2} - 1$$

and the second term

$$h_{1,r} = f_2 + f_3 + \cdots + f_{r+1} = f_{r+3} - 2$$

. As with Fibonacci sequence, the $r-$shifted Fibonacci sequence satisfies the recurrence relation

$$h_{n,r} = h_{n-1,r} + h_{n-2,r}, \tag{3.2}$$

with initial conditions $h_{0,r} = f_{r+2} - 1$ and $h_{1,r} = f_{r+3} - 2$, for all $r > 0$.

Few entries of $h_{n,r}$ are given in Table 3.1 below.

Table 3.1: r-shifted Fibonacci numbers

r	$h_{0,r}$	$h_{1,r}$	$h_{2,r}$	$h_{3,r}$	$h_{4,r}$	$h_{5,r}$	$h_{6,r}$	$h_{7,r}$	$h_{8,r}$	$h_{9,r}$
1	1	1	2	3	5	8	13	21	34	55
2	2	3	5	8	13	21	34	55	89	144
3	4	6	10	16	26	42	68	110	178	288
4	7	11	18	29	47	76	123	199	322	521
5	12	19	31	50	81	131	212	343	555	898
6	20	32	52	84	136	220	356	576	932	1508

When $r = 1, 2$, we get Fibonacci sequences with different initial conditions. For $r \geq 3$, we get Fibonacci-like numbers. We also note that when $r = 4$, we obtain Lucas numbers.

This chapter is organized as follows: Some basic properties of $h_{n,r}$ are given in Section 3.2. In Section 3.3, we obtain Binet's formula and generating function for these numbers. Further properties of these numbers are presented in Section 3.4. Moreover, determinant identities are presented in Section 3.5.

3.2 Preliminary results

We start off, with these basic properties:

Lemma 3.2.1. *For $n \geq 0$, we have $h_{n,3} = 2h_{n,2}$.*

Proof. From (3.1), we have

$$\begin{aligned} h_{n,3} &= f_{n+1} + f_{n+2} + f_{n+3} \\ &= f_{n+1} + f_{n+2} + f_{n+1} + f_{n+2} \\ &= 2(f_{n+1} + f_{n+2}) \\ &= 2h_{n,2}. \end{aligned}$$

□

Proposition 3.2.2. *The $(n+1)^{th}$ term of $r-$shifted Fibonacci number, $h_{n,r}$, can be expressed as $h_{n,r} = f_{r+n+2} - f_{n+2}$, for all $r > 0$.*

Proof. From recurrence relation (3.2) and equation (3.1), we have

$$\begin{aligned} h_{n,r} &= h_{n-1,r} + h_{n-2,r} \\ &= (f_n + f_{n+1} + \cdots + f_{n+r-1}) + (f_{n-1} + f_n + \cdots + f_{n+r-2}) \\ &= [(f_0 + f_1 + \cdots + f_{n+r-1} - (f_0 + f_1 + \cdots + f_{n-1})] \\ &+ [(f_0 + f_1 + \cdots + f_{n+r-2}) - (f_0 + f_1 + \cdots + f_{n-2})]\,. \end{aligned}$$

By equation (1.3), we get

$$\begin{aligned} h_{n,r} &= [(f_{n+r+1} - 1) - (f_{n+1} - 1)] + [(f_{n+r} - 1) - (f_n - 1)] \\ &= f_{n+r+1} - f_{n+1} + f_{n+r} - f_n \\ &= f_{n+r+2} - f_{n+2.} \end{aligned}$$

□

Proposition 3.2.3.

$$h_{n,r} = f_r f_{n+3} + f_{n+2} \sum_{i=0}^{r-3} f_i.$$

Proof. Using Proposition 3.2.2, we have that

$$h_{n,r} = f_{n+r+2} - f_{n+2}. \tag{3.3}$$

Now, from Honsberger's identity (1.4), we have

$$f_{n+r+2} = f_r f_{n+3} + f_{r-1} f_{n+2}.$$

Substituting this sum in (3.3), we obtain

$$\begin{aligned} h_{n,r} &= f_r f_{n+3} + f_{r-1} f_{n+2} - f_{n+2} \\ &= f_r f_{n+3} + f_{n+2}(f_{r-1} - 1). \end{aligned}$$

Since $\sum_{i=0}^{n-1} f_i = f_{n+1} - 1$, then $h_{n,r} = f_r f_{n+3} + f_{n+2} \sum_{i=0}^{r-3} f_i$. □

Theorem 3.2.4. *The numbers, $h_{n,r}$, can be expressed in terms of Fibonacci and Lucas numbers as:*

$$h_{n,r} = \begin{cases} \sum_{i=1}^{m} l_{n+4i} & \text{if} \quad r = 4m, \\ \sum_{i=1}^{m} l_{n+4i} + f_{n+4m+1} & \text{if} \quad r = 4m+1, \\ \sum_{i=1}^{m} l_{n+4i} + f_{n+4m+3} & \text{if} \quad r = 4m+2, \\ \sum_{i=1}^{m} l_{n+4i} + 2f_{n+4m+3} & \text{if} \quad r = 4m+3. \end{cases}$$

Proof. If $r = 4m$, then

$$\begin{aligned} h_{n,r} &= f_{n+1} + f_{n+2} + \cdots + f_{n+4m} \\ &= f_{n+3} + f_{n+5} + \cdots + f_{n+4m+1} \\ &= l_{n+4} + l_{n+8} + \cdots + l_{n+4m} \\ &= \sum_{i=1}^{m} l_{n+4i}. \end{aligned}$$

If $r = 4m+1$, then

$$\begin{aligned} h_{n,r} &= f_{n+1} + f_{n+2} + \cdots + f_{n+4m+1} \\ &= f_{n+3} + f_{n+5} + \cdots + f_{n+4m-1} + f_{n+4m+1} + f_{n+4m+1} \\ &= l_{n+4} + l_{n+8} + \cdots + l_{n+4m} + f_{n+4m+1} \\ &= \sum_{i=1}^{m} l_{n+4i} + f_{n+4m+1}. \end{aligned}$$

If $r = 4m + 2$, then

$$\begin{aligned}
h_{n,r} &= f_{n+1} + f_{n+2} + f_{n+3} + \cdots + f_{n+4m+2} \\
&= f_{n+3} + f_{n+5} + \cdots + f_{n+4m+3} \\
&= l_{n+4} + l_{n+8} + \cdots + l_{n+4m} + f_{n+4m+3} \\
&= \sum_{i=1}^{m} l_{n+4i} + f_{n+4m+3}.
\end{aligned}$$

If $r = 4m + 3$, then

$$\begin{aligned}
h_{n,r} &= f_{n+1} + f_{n+2} + \cdots + f_{n+4m+3} \\
&= f_{n+3} + f_{n+5} + \cdots + f_{n+4m+1} + f_{n+4m+3} + f_{n+4m+3} \\
&= l_{n+4} + l_{n+8} + \cdots + l_{n+4m} + 2f_{n+4m+3} \\
&= \sum_{i=1}^{m} l_{n+4i} + 2f_{n+4m+3}.
\end{aligned}$$

□

Remark 3.2.5. We note that:

1. For $r = 1, 2$, the $r-$shifted Fibonacci numbers, $h_{n,r}$, are themselves Fibonacci numbers.

2. We have $h_{n,3}$ as a sum of Fibonacci numbers for all $n \geq 0$.

3. The numbers, $h_{n,4}$, are Lucas numbers for all integers $n \geq 0$.

4. The numbers, $h_{n,4m}$, are sums of Lucas numbers for all integers $m \geq 1$ and $n \geq 0$.

5. For all $m \in \mathbb{N}$ and $n \geq 0$, we have that $h_{n,4m+1}$, $h_{n,4m+2}$, and $h_{n,4m+3}$ are sums of Fibonacci and Lucas numbers.

Proposition 3.2.6. *Let $m \geq 1$. Then the $(n+1)^{th}$ term of $4m-$shifted Fibonacci sequence, $h_{n,4m}$, satisfies the relation $h_{n,4m} = f_{2m} l_{n+2m+2}$.*

Proof. Using Binet's formulas for Fibonacci numbers (1.1) and Lucas numbers (1.2) and by equation (3.3), we obtain

$$\begin{aligned} h_{n,4m} &= f_{n+4m+2} - f_{n+2} \\ &= \frac{1}{\alpha-\beta}(\alpha^{n+4m+2} - \beta^{n+4m+2}) - \frac{1}{\alpha-\beta}(\alpha^{n+2} - \beta^{n+2}) \\ &= \frac{1}{\alpha-\beta}(\alpha^{n+4m+2} - \alpha^{n+2} - \beta^{n+4m+2} + \beta^{n+2}). \end{aligned}$$

Using $\alpha\beta = -1$ then, $(\alpha\beta)^{2m} = 1$, and

$$\begin{aligned} h_{n,4m} &= \frac{1}{\alpha-\beta}(\alpha^{n+4m+2} - (\alpha\beta)^{2m}\alpha^{n+2} - \beta^{n+4m+2} + (\alpha\beta)^{2m}\beta^{n+2}) \\ &= \frac{1}{\alpha-\beta}(\alpha^{n+4m+2} - \beta^{2m}\alpha^{n+2m+2} - \beta^{n+4m+2} + \alpha^{2m}\beta^{n+2m+2}) \\ &= \frac{1}{\alpha-\beta}(\alpha^{n+2m+2}(\alpha^{2m} - \beta^{2m}) + \beta^{n+2m+2}(\alpha^{2m} - \beta^{2m})) \\ &= \frac{1}{\alpha-\beta}(\alpha^{2m} - \beta^{2m})(\alpha^{n+2m+2} + \beta^{n+2m+2}) \\ &= f_{2m} l_{n+2m+2}. \end{aligned}$$

□

Setting $m = 1$ in Proposition 3.2.6, we get the following Corollary.

Corollary 3.2.7. $h_{n,4} = l_{n+4}$, *for all* $n \geq 0$.

Proposition 3.2.8.

$$\sum_{k=1}^{n} h_{k,r}^2 = h_{n,r}h_{n+1,r} - h_{0,r}h_{1,r}.$$

Proof. Using the relation (3.2), we obtain

$$h_{n,r}^2 = h_{n,r}h_{n+1,r} - h_{n-1,r}h_{n,r}.$$

Now, we have

$$\begin{aligned}
h_{1,r}^2 &= h_{1,r}h_{2,r} - h_{0,r}h_{1,r} \\
h_{2,r}^2 &= h_{2,r}h_{3,r} - h_{1,r}h_{2,r} \\
h_{3,r}^2 &= h_{3,r}h_{4,r} - h_{2,r}h_{3,r} \\
&\vdots \\
h_{n-1,r}^2 &= h_{n-1,r}h_{n,r} - h_{n-2,r}h_{n-1,r} \\
h_{n,r}^2 &= h_{n,r}h_{n+1,r} - h_{n-1,r}h_{n,r}.
\end{aligned}$$

Adding up these equations, we get

$$h_{1,r}^2 + h_{2,r}^2 + h_{3,r}^2 + \cdots + h_{n-1,r}^2 + h_{n,r}^2 = h_{n,r}h_{n+1,r} - h_{0,r}h_{1,r}.$$

Hence the proof. □

Proposition 3.2.9. *For every positive integer $n \geq 2$,*

$$h_{n,r}^2 - h_{n-1,r}^2 = h_{n+1,r}h_{n-2,r}.$$

Proof. Since

$$h_{n-1,r}^2 = h_{n-1,r}h_{n,r} - h_{n-1,r}h_{n-2,r}$$

then,

$$\begin{aligned}
h_{n,r}^2 - h_{n-1,r}^2 &= h_{n,r}^2 - h_{n-1,r}h_{n,r} + h_{n-1,r}h_{n-2,r} \\
&= h_{n,r}(h_{n,r} - h_{n-1,r}) + h_{n-1,r}h_{n-2,r} \\
&= h_{n,r}h_{n-2,r} + h_{n-1,r}h_{n-2,r} \\
&= h_{n-2,r}(h_{n,r} + h_{n-1,r}) \\
&= h_{n+1,r}h_{n-2,r}.
\end{aligned}$$

□

3.3 Binet's formula and generating function for $r-$shifted Fibonacci sequence

Theorem 3.3.1 (Binet's Formula for $r-$shifted Fibonacci sequence)**.** *The* $(n+1)^{th}$ *term of r-shifted Fibonacci sequence,* $h_{n,r}$*, is given by*

$$h_{n,r} = \frac{1}{\alpha - \beta}[(h_{1,r} - \beta h_{0,r})\alpha^n - (h_{1,r} - \alpha h_{0,r})\beta^n], \tag{3.4}$$

where $\alpha = \dfrac{1+\sqrt{5}}{2}$ *and* $\beta = \dfrac{1-\sqrt{5}}{2}$.

Proof. Let $n \geq 2$, then r-shifted Fibonacci numbers are defined by the recurrence relation

$$h_{n,r} = h_{n-1,r} + h_{n-2,r},$$

with initial conditions $h_{0,r} = f_{r+2} - 1$ and $h_{1,r} = f_{r+3} - 2$, for all $r > 0$. The characteristic equation of the recurrence relation is $\lambda^2 - \lambda - 1 = 0$. We solve this equation to get its roots as α and β. These roots are real and distinct and thus the solution of the recurrence relation is of the form

$$h_{n,r} = A\alpha^n + B\beta^n, \tag{3.5}$$

where A and B are constants.

Setting $n = 0$ and $n = 1$ in (3.5), we obtain

$$A + B = h_{0,r}$$

and

$$A\alpha + B\beta = h_{1,r}$$

respectively.

Solving these equations simultaneously, we get

$$A = \frac{h_{1,r} - \beta h_{0,r}}{\alpha - \beta}$$

and

$$B = \frac{\alpha h_{0,r} - h_{1,r}}{\alpha - \beta}.$$

Thus the result. □

Corollary 3.3.2. *The* $(n+1)^{th}$ *term of the* $r-$*shifted Fibonacci sequence satisfies the equation* $h_{n,r} = h_{1,r} f_n + h_{0,r} f_{n-1}$.

Proof. From Binet's formula (3.4), we get

$$h_{n,r} = \frac{1}{\alpha - \beta}\Big[h_{1,r}(\alpha^n - \beta^n) - h_{0,r}(\alpha\beta)(\alpha^{n-1} - \beta^{n-1})\Big].$$

Using $\alpha\beta = -1$, then

$$\begin{aligned} h_{n,r} &= \frac{1}{\alpha - \beta}\Big[h_{1,r}(\alpha^n - \beta^n) + h_{0,r}(\alpha^{n-1} - \beta^{n-1})\Big] \\ &= h_{1,r} f_n + h_{0,r} f_{n-1}. \end{aligned}$$

□

The following formula is rediscovered immediately upon setting $r = 1$ in (3.4).

Corollary 3.3.3 (Binet's formula for Fibonacci sequence)**.** *The* $(n+2)^{th}$ *Fibonacci number,* f_{n+1}*, is given explicitly as*

$$f_{n+1} = \frac{1}{\alpha - \beta}\Big[\alpha^{n+1} - \beta^{n+1}\Big].$$

Corollary 3.3.4. *The sequence of ratio of successive r-shifted Fibonacci numbers* $\frac{h_{n+1,r}}{h_{n,r}}$ *converges to the golden ratio, i.e.,* $\lim_{n\to\infty} \frac{h_{n+1,r}}{h_{n,r}} = \frac{1+\sqrt{5}}{2}$.

Proof. From Binet's formula (3.4), we get

$$\lim_{n\to\infty} \frac{h_{n+1,r}}{h_{n,r}} = \lim_{n\to\infty} \frac{\frac{1}{\alpha-\beta}\left[(h_{1,r} - \beta h_{0,r})\alpha^{n+1} - (h_{1,r} - \alpha h_{0,r})\beta^{n+1}\right]}{\frac{1}{\alpha-\beta}\left[(h_{1,r} - \beta h_{0,r})\alpha^{n} - (h_{1,r} - \alpha h_{0,r})\beta^{n}\right]}.$$

Factorizing α^n, we obtain

$$\lim_{n\to\infty}\frac{h_{n+1},r}{h_{n,r}}=\lim_{n\to\infty}\frac{\frac{1}{\alpha-\beta}\alpha^n\left[(h_{1,r}-\beta h_{0,r})\alpha-(h_{1,r}-\alpha h_{0,r})\alpha^{-n}\beta^{n+1}\right]}{\frac{1}{\alpha-\beta}\alpha^n\left[(h_{1,r}-\beta h_{0,r})-(h_{1,r}-\alpha h_{0,r})\alpha^{-n}\beta^{n}\right]},$$

which simplifies to

$$\lim_{n\to\infty}\frac{h_{n+1,r}}{h_{n,r}}=\lim_{n\to\infty}\frac{(h_{1,r}-\beta h_{0,r})\alpha-(h_{1,r}-\alpha h_{0,r})\left(\frac{\beta}{\alpha}\right)^n\beta}{(h_{1,r}-\beta h_{0,r})-(h_{1,r}-\alpha h_{0,r})\left(\frac{\beta}{\alpha}\right)^n}.$$

Since $|\frac{\beta}{\alpha}|<1$, we have $\lim_{n\to\infty}\left(\frac{\beta}{\alpha}\right)^n=0$ so that

$$\lim_{n\to\infty}\frac{h_{n+1,r}}{h_{n,r}}=\lim_{n\to\infty}\frac{(h_{1,r}-\beta h_{0,r})\alpha}{(h_{1,r}-\beta h_{0,r})}=\alpha=\frac{1+\sqrt{5}}{2}.$$

□

We now obtain the generating function for r-shifted Fibonacci sequence.

Theorem 3.3.5. *Let $H_r(t)$ be the generating function for $r-$shifted Fibonacci sequence, then*

$$H_r(t)=\frac{h_{0,r}+t(h_{1,r}-h_{0,r})}{1-t-t^2}, \tag{3.6}$$

where $|t+t^2|<1$.

Proof. Let $H_r(t)=\sum_{n=0}^{\infty}h_{n,r}t^n$ be the generating function for $r-$shifted Fibonacci numbers, then from $h_{n,r}=h_{n-1,r}+h_{n-2,r}$, we get

$$\sum_{n\geq 2}h_{n,r}t^n=\sum_{n\geq 2}h_{n-1,r}t^n+\sum_{n\geq 2}h_{n-2,r}t^n.$$

This is the same as

$$\sum_{n\geq 0}h_{n,r}t^n-h_{1,r}t-h_{0,r}=t\sum_{n\geq 1}h_{n,r}t^n+t^2\sum_{n\geq 0}h_{n,r}t^n$$

or

$$\sum_{n\geq 0} h_{n,r}t^n - h_{1,r}t - h_{0,r} = t\left(\sum_{n\geq 0} h_{n,r}t^n - h_{0,r}\right) + t^2 \sum_{n\geq 0} h_{n,r}t^n.$$

Substituting $H_r(t) = \sum_{n=0}^{\infty} h_{n,r}t^n$ we get,

$$H_r(t) - h_{1,r}t - h_{0,r} = t(H_r(t) - h_{0,r}) + t^2 H_r(t).$$

Thus,

$$H_r(t) = \frac{h_{0,r} + t(h_{1,r} - h_{0,r})}{1 - t - t^2}.$$

□

3.4 Properties of $r-$shifted Fibonacci numbers

In this section, we obtain properties of $r-$shifted Fibonacci numbers.

Proposition 3.4.1 (Sum of first n terms)**.** *The sum of first n terms of r-shifted Fibonacci numbers is given by $h_{n+1,r} - h_{1,r}$.*

Proof. Using Binet's formula (3.4), we obtain

$$\begin{aligned}
\sum_{k=0}^{n-1} h_{k,r} &= \frac{1}{\alpha - \beta}[(h_{1,r} - \beta h_{0,r})\alpha^0 - (h_{1,r} - \alpha h_{0,r})\beta^0 + (h_{1,r} - \beta h_{0,r})\alpha^1 \\
&\quad - (h_{1,r} - \alpha h_{0,r})\beta^1 + \cdots + (h_{1,r} - \beta h_{0,r})\alpha^{n-1} - (h_{1,r} - \alpha h_{0,r})\beta^{n-1}] \\
&= \frac{1}{\alpha - \beta}\left[(h_{1,r} - \beta h_{0,r})\left(1 + \alpha + \cdots + \alpha^{n-1}\right)\right. \\
&\quad \left. -(h_{1,r} - \alpha h_{0,r})\left(1 + \beta + \cdots + \beta^{n-1}\right)\right] \\
&= \frac{1}{\alpha - \beta}\left[(h_{1,r} - \beta h_{0,r})\frac{\alpha^n - 1}{\alpha - 1} - (h_{1,r} - \alpha h_{0,r})\frac{\beta^n - 1}{\beta - 1}\right].
\end{aligned}$$

Since $\alpha - 1 = -\beta$ and $\beta - 1 = -\alpha$, we get

$$\sum_{k=0}^{n-1} h_{k,r} = \frac{1}{\alpha - \beta}\left[\frac{(h_{1,r} - \beta h_{0,r})(\alpha^n - 1)\alpha - (h_{1,r} - \alpha h_{0,r})(\beta^n - 1)\beta}{-\alpha\beta}\right].$$

Using $-\alpha\beta = 1$, we obtain

$$\begin{aligned}\sum_{k=0}^{n-1} h_{k,r} &= \frac{1}{\alpha-\beta}\Big[(h_{1,r}-\beta h_{0,r})(\alpha^{n+1}-\alpha)-(h_{1,r}-\alpha h_{0,r})(\beta^{n+1}-\beta)\Big]\\ &= \frac{(h_{1,r}-\beta h_{0,r})\alpha^{n+1}-(h_{1,r}-\alpha h_{0,r})\beta^{n+1}}{\alpha-\beta}-\frac{(h_{1,r}-\beta h_{0,r})\alpha-(h_{1,r}-\alpha h_{0,r})\beta}{\alpha-\beta}\\ &= h_{n+1,r}-h_{1,r}.\end{aligned}$$

□

Proposition 3.4.2 (Sum of first n terms with odd indices)**.** *The sum of the first n terms with odd indices of $r-$shifted Fibonacci numbers is given by $h_{2n,r}-h_{0,r}$.*

Proof. Using Binet's formula (3.4), we get

$$\begin{aligned}\sum_{k=0}^{n-1} h_{2k+1,r} &= \frac{1}{\alpha-\beta}[(h_{1,r}-\beta h_{0,r})\alpha^1-(h_{1,r}-\alpha h_{0,r})\beta^1+(h_{1,r}-\beta h_{0,r})\alpha^3\\ &-(h_{1,r}-\alpha h_{0,r})\beta^3+\cdots+(h_{1,r}-\beta h_{0,r})\alpha^{2n-1}-(h_{1,r}-\alpha h_{0,r})\beta^{2n-1}]\\ &= \frac{1}{\alpha-\beta}[(h_{1,r}-\beta h_{0,r})(\alpha+\alpha^3+\cdots+\alpha^{2n-1})\\ &-(h_{1,r}-\alpha h_{0,r})(\beta+\beta^3+\cdots+\beta^{2n-1})]\\ &= \frac{1}{\alpha-\beta}\left[\frac{(h_{1,r}-\beta h_{0,r})(\alpha^{2n+1}-\alpha)}{\alpha^2-1}-\frac{(h_{1,r}-\alpha h_{0,r})(\beta^{2n+1}-\beta)}{\beta^2-1}\right].\end{aligned}$$

Using $\alpha^2-1=\alpha$ and $\beta^2-1=\beta$, obtain

$$\begin{aligned}\sum_{k=0}^{n-1} h_{2k+1,r} &= \frac{1}{\alpha-\beta}\Big[(h_{1,r}-\beta h_{0,r})(\alpha^{2n}-1)-(h_{1,r}-\alpha h_{0,r})(\beta^{2n}-1)\Big]\\ &= \frac{(h_{1,r}-\beta h_{0,r})\alpha^{2n}-(h_{1,r}-\alpha h_{0,r})\beta^{2n}}{\alpha-\beta}-\frac{(h_{1,r}-\beta h_{0,r})-(h_{1,r}-\alpha h_{0,r})}{\alpha-\beta}\\ &= h_{2n,r}-h_{0,r}.\end{aligned}$$

,

□

Proposition 3.4.3 (Sum of first n terms with even indices)**.** *The sum of the first n terms with even indices of $r-$shifted Fibonacci numbers is given by $h_{2n-1,r}-h_{1,r}+h_{0,r}$.*

Proof. From Binet's formula (3.4), obtain

$$\begin{aligned}\sum_{k=0}^{n-1} h_{2k,r} &= \frac{1}{\alpha-\beta}[(h_{1,r}-\beta h_{0,r})\alpha^0 - (h_{1,r}-\alpha h_{0,r})\beta^0 + (h_{1,r}-\beta h_{0,r})\alpha^2 \\ &- (h_{1,r}-\alpha h_{0,r})\beta^2 + \cdots + (h_{1,r}-\beta h_{0,r})\alpha^{2n-2} - (h_{1,r}-\alpha h_{0,r})\beta^{2n-2}] \\ &= \frac{1}{\alpha-\beta}[(h_{1,r}-\beta h_{0,r})(1+\alpha^2+\cdots+\alpha^{2n-2}) \\ &- (h_{1,r}-\alpha h_{0,r})(1+\beta^2+\cdots+\beta^{2n-2})] \\ &= \frac{1}{\alpha-\beta}\left[(h_{1,r}-\beta h_{0,r})\frac{\alpha^{2n}-1}{\alpha^2-1} - (h_{1,r}-\alpha h_{0,r})\frac{\beta^{2n}-1}{\beta^2-1}\right].\end{aligned}$$

Using $\alpha^2-1=\alpha$ and $\beta^2-1=\beta$, we get

$$\begin{aligned}\sum_{k=0}^{n-1} h_{2k,r} &= \frac{1}{\alpha-\beta}\left[(h_{1,r}-\beta h_{0,r})(\alpha^{2n-1}-\alpha^{-1}) - (h_{1,r}-\alpha h_{0,r})(\beta^{2n-1}-\beta^{-1})\right] \\ &= \frac{(h_{1,r}-\beta h_{0,r})\alpha^{2n-1} - (h_{1,r}-\alpha h_{0,r})\beta^{2n-1}}{\alpha-\beta} \\ &- \frac{(h_{1,r}-\beta h_{0,r})\alpha^{-1} - (h_{1,r}-\alpha h_{0,r})\beta^{-1}}{\alpha-\beta} \\ &= h_{2n-1,r} - h_{-1,r} \\ &= h_{2n-1,r} - h_{1,r} + h_{0,r}.\end{aligned}$$

□

Proposition 3.4.4. *For every positive integer n,*

$$h_{1,r} + h_{4,r} + h_{7,r} + \cdots + h_{3n-2,r} = \frac{1}{2}(h_{3n,r} - h_{0,r}).$$

Proof. From Binet's formula (3.4), we get

$$\begin{aligned}\sum_{k=1}^{n} h_{3k-2,r} &= \frac{1}{\alpha-\beta}[(h_{1,r}-\beta h_{0,r})\alpha - (h_{1,r}-\alpha h_{0,r})\beta + (h_{1,r}-\beta h_{0,r})\alpha^4 \\ &- (h_{1,r}-\alpha h_{0,r})\beta^4 + \cdots + (h_{1,r}-\beta h_{0,r})\alpha^{3n-2} - (h_{1,r}-\alpha h_{0,r})\beta^{3n-2}]. \\ &= \frac{1}{\alpha-\beta}[(h_{1,r}-\beta h_{0,r})(\alpha+\alpha^4+\cdots+\alpha^{3n-2}) \\ &- (h_{1,r}-\alpha h_{0,r})(\beta+\beta^4+\cdots+\beta^{3n-2})] \\ &= \frac{1}{\alpha-\beta}\left[(h_{1,r}-\beta h_{0,r})\frac{\alpha^{3n+1}-\alpha}{\alpha^3-1} - (h_{1,r}-\alpha h_0)\frac{\beta^{3n+1}-\beta}{\beta^3-1}\right].\end{aligned}$$

Using $\alpha^3 - 1 = 2\alpha$ and $\beta^3 - 1 = 2\beta$, the above equation simplifies to

$$\begin{aligned}
\sum_{k=1}^{n} h_{3k-2,r} &= \frac{1}{2(\alpha-\beta)}\Big[(h_{1,r} - \beta h_{0,r})(\alpha^{3n} - 1) - (h_{1,r} - \alpha h_{0,r})(\beta^{3n} - 1)\Big] \\
&= \frac{1}{2} \cdot \frac{(h_{1,r} - \beta h_{0,r})\alpha^{3n} - (h_{1,r} - \alpha h_{0,r})\beta^{3n}}{\alpha - \beta} \\
&- \frac{1}{2} \cdot \frac{(h_{1,r} - \beta h_{0,r}) - (h_{1,r} - \alpha h_{0,r})}{\alpha - \beta} \\
&= \frac{1}{2}(h_{3n,r} - h_{0,r}).
\end{aligned}$$

□

Proposition 3.4.5. *For every positive integer n,*

$$h_{2,r} + h_{5,r} + h_{8,r} + \cdots + h_{3n-1,r} = \frac{1}{2}(h_{3n+1,r} - h_{1,r}).$$

Proof. From Binet's formula (3.4), we get

$$\begin{aligned}
\sum_{k=1}^{n} h_{3k-1,r} &= \frac{1}{\alpha-\beta}[(h_{1,r} - \beta h_{0,r})\alpha^2 - (h_{1,r} - \alpha h_{0,r})\beta^2 + (h_{1,r} - \beta h_{0,r})\alpha^5 \\
&- (h_{1,r} - \alpha h_{0,r})\beta^5 + \cdots + (h_{1,r} - \beta h_{0,r})\alpha^{3n-1} - (h_{1,r} - \alpha h_{0,r})\beta^{3n-1}] \\
&= \frac{1}{\alpha-\beta}[(h_{1,r} - \beta h_{0,r})(\alpha^2 + \alpha^5 + \cdots + \alpha^{3n-1}) \\
&- (h_{1,r} - \alpha h_{0,r})(\beta^2 + \beta^5 + \cdots + \beta^{3n-1})] \\
&= \frac{1}{\alpha-\beta}\left[(h_{1,r} - \beta h_{0,r})\frac{\alpha^{3n+2} - \alpha^2}{\alpha^3 - 1} - (h_{1,r} - \alpha h_{0,r})\frac{\beta^{3n+2} - \beta^2}{\beta^3 - 1}\right].
\end{aligned}$$

Using $\alpha^3 - 1 = 2\alpha$ and $\beta^3 - 1 = 2\beta$, then

$$\begin{aligned}
\sum_{k=1}^{n} h_{3k-1,r} &= \frac{1}{2(\alpha-\beta)}\Big[(h_{1,r} - \beta h_{0,r})(\alpha^{3n+1} - \alpha) - (h_{1,r} - \alpha h_{0,r})(\beta^{3n+1} - \beta)\Big] \\
&= \frac{1}{2} \cdot \frac{(h_{1,r} - \beta h_{0,r})\alpha^{3n+1} - (h_{1,r} - \alpha h_{0,r})\beta^{3n+1}}{\alpha - \beta} \\
&- \frac{1}{2} \cdot \frac{(h_{1,r} - \beta h_{0,r})\alpha - (h_{1,r} - \alpha h_{0,r})\beta}{\alpha - \beta} \\
&= \frac{1}{2}(h_{3n+1,r} - h_{1,r}).
\end{aligned}$$

□

Proposition 3.4.6. *For every positive integer n,*

$$h_{3,r} + h_{6,r} + h_{9,r} + \cdots + h_{3n,r} = \frac{1}{2}(h_{3n+2,r} - h_{2,r}).$$

Proof. From Binet's formula (3.4), we obtain

$$\sum_{k=1}^{n} h_{3k,r} = \frac{1}{\alpha-\beta}[(h_{1,n}-\beta h_{0,r})\alpha^3 - (h_{1,r}-\alpha h_{0,r})\beta^3 + (h_{1,r}-\beta h_{0,r})\alpha^6$$
$$- (h_{1,r}-\alpha h_{0,r})\beta^6 + \cdots + (h_{1,r}-\beta h_{0,r})\alpha^{3n} - (h_{1,r}-\alpha h_{0,r})\beta^{3n}]$$
$$= \frac{1}{\alpha-\beta}[(h_{1,r}-\beta h_{0,r})(\alpha^3+\alpha^6+\cdots+\alpha^{3n})$$
$$- (h_{1,r}-\alpha h_{0,r})(\beta^3+\beta^6+\cdots+\beta^{3n})]$$
$$= \frac{1}{\alpha-\beta}\left[(h_{1,r}-\beta h_{0,r})\frac{\alpha^{3n+3}-\alpha^3}{\alpha^3-1} - (h_{1,r}-\alpha h_{0,r})\frac{\beta^{3n+3}-\beta^3}{\beta^3-1}\right]$$

Using $\alpha^3-1=2\alpha$ and $\beta^3-1=2\beta$, we get

$$\sum_{k=1}^{n} h_{3k,r} = \frac{1}{2(\alpha-\beta)}\left[(h_{1,r}-\beta h_{0,r})(\alpha^{3n+2}-\alpha^2) - (h_{1,r}-\alpha h_{0,r})(\beta^{3n+2}-\beta^2)\right]$$
$$= \frac{1}{2}\cdot\frac{(h_{1,r}-\beta h_{0,r})\alpha^{3n+2} - (h_{1,r}-\alpha h_{0.r})\beta^{3n+2}}{\alpha-\beta}$$
$$- \frac{1}{2}\cdot\frac{(h_{1,r}-\beta h_{0,r})\alpha^2 - (h_{1,r}-\alpha h_{0,r})\beta^2}{\alpha-\beta}$$
$$= \frac{1}{2}(h_{3n+2,r}-h_{2,r}).$$

□

Proposition 3.4.7 (Alternating sum formula for $r-$shifted Fibonacci sequence)**.** *For every positive integer n,*

$$\sum_{k=1}^{n}(-1)^{k+1}h_{k,r} = (-1)^{n+1}h_{n-1,r} + h_{1,r} - h_{0,r}.$$

Proof. From Binet's formula (3.4), we get

$$\sum_{k=1}^{n}(-1)^{k+1}h_{k,r}$$
$$= \frac{1}{\alpha-\beta}[(h_{1,r}-\beta h_{0,r})\alpha - (h_{1,r}-\alpha h_{0,r})\beta - ((h_{1,r}-\beta h_{0,r})\alpha^2$$
$$- (h_{1,r}-\alpha h_{0,r})\beta^2) + \cdots + (-1)^{n+1}((h_{1,r}-\beta h_{0,r})\alpha^n - (h_{1,r}-\alpha h_{0,r})\beta^n)]$$
$$= \frac{1}{\alpha-\beta}[(h_{1,r}-\beta h_{0,r})(\alpha-\alpha^2+\cdots+(-1)^{n+1}\alpha^n)$$
$$- ((h_{1,r}-\alpha h_{0,r})(\beta-\beta^2+\cdots+(-1)^{n+1}\beta^n)]$$
$$= \frac{1}{\alpha-\beta}\left[(h_{1,r}-\beta h_{0,r})\frac{\alpha((-\alpha)^n-1)}{-\alpha-1} - (h_{1,r}-\alpha h_{0,r})\frac{\beta((-\beta)^n-1)}{-\beta-1}\right].$$

Using $-\alpha - 1 = -\alpha^2$ and $-\beta - 1 = -\beta^2$, we get

$$\begin{aligned}
&\sum_{k=1}^{n}(-1)^{k+1}h_{k,r} \\
&\quad = \frac{1}{\alpha-\beta}\left[(h_{1,r}-\beta h_{0,r})\frac{(-1)^n\alpha^{n+1}-\alpha}{-\alpha^2} - (h_{1,r}-\alpha h_{0,r})\frac{(-1)^n\beta^{n+1}-\beta}{-\beta^2}\right] \\
&\quad = (-1)^{n+1}\frac{1}{\alpha-\beta}\left[(h_{1,r}-\beta h_{0,r})\alpha^{n-1} - (h_{1,r}-\alpha h_{0,r})\beta^{n-1}\right] \\
&\quad + \frac{1}{\alpha-\beta}\left[(h_{1,r}-\beta h_{0,r})\alpha^{-1} - (h_{1,r}-\alpha h_{0,r})\beta^{-1}\right] \\
&\quad = (-1)^{n+1,r}h_{n-1,r} + \frac{1}{\alpha-\beta}\left[\frac{(h_{1,r}-\beta h_{0,r})}{\alpha} - \frac{(h_{1,r}-\alpha h_{0,r})}{\beta}\right].
\end{aligned}$$

Rearranging we obtain,

$$\begin{aligned}
\sum_{k=1}^{n}(-1)^{k+1}h_{k,r} &= (-1)^{n+1}h_{n-1,r} + \frac{1}{\alpha\beta(\alpha-\beta)}[(h_{1,r}-\beta h_{0,r})\beta - (h_{1,r}-\alpha h_{0,r})\alpha] \\
&= (-1)^{n+1}h_{n-1,r} + (-1)\left[h_{0,r}\left(\frac{\alpha^2-\beta^2}{\alpha-\beta}\right) - h_{1,r}\left(\frac{\alpha-\beta}{\alpha-\beta}\right)\right].
\end{aligned}$$

Using $\alpha - \beta = \sqrt{5}$ and $\alpha^2 - \beta^2 = \sqrt{5}$, we get

$$\sum_{k=1}^{n}(-1)^{k+1}h_{k,r} = (-1)^{n+1}h_{n-1,r} + h_{1,r} - h_{0,r}.$$

□

Proposition 3.4.8. *For every positive integer n,*

$$h_{2n,r} = \sum_{k=0}^{n}\binom{n}{k}h_{k,r}.$$

Proof. From Binet's formula (3.4), we get

$$h_{2n,r} = \frac{1}{\alpha-\beta}[(h_{1,r}-\beta h_{0,r})\alpha^{2n} - (h_{1,r}-\alpha h_{0,r})\beta^{2n}].$$

Using $\alpha^2 = 1 + \alpha$ and $\beta^2 = 1 + \beta$, we get

$$h_{2n,r} = \frac{1}{\alpha-\beta}[(h_{1,r}-\beta h_{0,r})(1+\alpha)^n - (h_{1,r}-\alpha h_{0,r})(1+\beta)^n].$$

Using $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$, we have

$$
\begin{aligned}
h_{2n,r} &= \frac{1}{\alpha - \beta}\left[(h_{1,r} - \beta h_{0,r}) \sum_{k=0}^{n} \binom{n}{k} \alpha^k - (h_{1,r} - \alpha h_{0,r}) \sum_{k=0}^{n} \binom{n}{k} \beta^k\right] \\
&= \sum_{k=0}^{n} \binom{n}{k} \left[\frac{(h_{1,r} - \beta h_{o,r})\alpha^k - (h_{1,r} - \alpha h_{0,r})\beta^k}{\alpha - \beta}\right] \\
&= \sum_{k=0}^{n} \binom{n}{k} h_{k,r}.
\end{aligned}
$$

□

Proposition 3.4.9 (Explicit sum formula for $r-$shifted Fibonacci sequence)**.** *For every positive integer n,*

$$
h_{n,r} = h_{0,r} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} + (h_{1,r} - h_{0,r}) \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k}, \tag{3.7}
$$

where $\lfloor n \rfloor$ is the greatest integer less than or equal to n.

Proof. Using the generating function (3.6), we get

$$
\begin{aligned}
\sum_{n=0}^{\infty} h_{n,r} t^n &= \frac{h_{0,r} + t(h_{1,r} - h_{0,r})}{1 - t - t^2} \\
&= [h_{0,r} + t(h_{1,r} - h_{0,r})](1 - t - t^2)^{-1} \\
&= [h_{0,r} + t(h_{1,r} - h_{0,r})][1 - (t + t^2)]^{-1} \\
&= [h_{0,r} + t(h_{1,r} - h_{0,r})] \sum_{n=0}^{\infty} t^n (1+t)^n \\
&= [h_{0,r} + t(h_{1,r} - h_{0,r})] \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} \binom{n}{k} t^k \\
&= [h_{0,r} + t(h_{1,r} - h_{0,r})] \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} t^{n+k}.
\end{aligned}
$$

Replacing n by $n+k$, we get

$$
\sum_{n=0}^{\infty} h_{n,r} t^n = [h_{0,r} + t(h_{1,r} - h_{0,r})] \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(n+k)!}{k!n!} t^{n+2k}.
$$

Now, replacing n by $n-2k$, we obtain

$$\sum_{n=k}^{\infty} h_{n,r} t^n = [h_{0,r} + t(h_{1,r} - h_{0,r})] \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n-k)!}{k!(n-2k)!} t^n$$
$$= h_{0,r} \left[\sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n-k)!}{k!(n-2k)!} \right] t^n + (h_{1,r} - h_{0,r}) \sum_{n=0}^{\infty} \left[\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n-k)!}{k!(n-2k)!} \right] t^{n+1}.$$

Equating the coefficients of t^n, we get

$$h_{n,r} = h_{0,r} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} + (h_{1,r} - h_{0,r}) \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k}.$$

Hence the proof. □

Proposition 3.4.10. *For every positive integer n,*

$$h_{-n,r} = (-1)^n (h_{0,r} f_{n+1} - h_{1,r} f_n).$$

Proof. From Binet's formula (3.4), we obtain

$$h_{-n,r} = \frac{1}{\alpha - \beta} \left[(h_{1,r} - \beta h_{0,r}) \alpha^{-n} - (h_{1,r} - \alpha h_{0,r}) \beta^{-n} \right]$$
$$= \frac{1}{\alpha - \beta} \left[(h_{1,r} - \beta h_{0,r}) \frac{1}{\alpha^n} - (h_{1,r} - \alpha h_{0,r}) \frac{1}{\beta^n} \right].$$

Using $\frac{1}{\alpha} = -\beta$ and $\frac{1}{\beta} = -\alpha$, we get

$$h_{-n,r} = \frac{1}{\alpha - \beta} [(h_{1,r} - \beta h_{0,r})(-1)^n \beta^n - (h_{1,r} - \alpha h_{0,r})(-1)^n \alpha^n]$$
$$= \frac{(-1)^n}{\alpha - \beta} [(h_{1,r} - \beta h_{0,r}) \beta^n - (h_{1,r} - \alpha h_{0,r}) \alpha^n]$$
$$= \frac{(-1)^n}{\alpha - \beta} \left[h_{1,r} \beta^n - h_{0,r} \beta^{n+1} - h_{1,r} \alpha^n + h_{0,r} \alpha^{n+1} \right]$$
$$= \frac{(-1)^{n+1}}{\alpha - \beta} \left[h_{1,r} (\alpha^n - \beta^n) - h_{0,r} (\alpha^{n+1} - \beta^{n+1}) \right]$$
$$= (-1)^{n+1} \left[\frac{h_{1,r}(\alpha^n - \beta^n)}{\alpha - \beta} - \frac{h_{0,r}(\alpha^{n+1} - \beta^{n+1})}{\alpha - \beta} \right]$$
$$= (-1)^{n+1} (h_{1,r} f_n - h_{0,r} f_{n+1})$$
$$= (-1)^n (h_{0,r} f_{n+1} - h_{1,r} f_n).$$

□

Proposition 3.4.11 (Honsberger's identity for $r-$shifted Fibonacci sequence). *If $n > m$ then*

$$h_{n+m,r} = h_{n-1,r}f_m + h_{n,r}f_{m+1},$$

for all $m \geq 0$ and $n > 0$.

Proof. From Corollary 3.3.2, with n replaced by $n+m$ we obtain

$$h_{n+m,r} = h_{1,r}f_{n+m} + h_{0,r}f_{n+m-1},$$

then using Honsberger's identity for Fibonacci numbers (1.4), we get

$$\begin{aligned} h_{n+m,r} &= h_{1,r}(f_{n-1}f_m + f_n f_{m+1}) + h_{0,r}(f_{n-2}f_m + f_{n-1}f_{m+1}) \\ &= f_m(h_{1,r}f_{n-1} + h_{0,r}f_{n-2}) + f_{m+1}(h_{1,r}f_n + h_{0,r}f_{n-1}). \end{aligned}$$

Applying Corollary 3.3.2 again we obtain, $h_{n+m,r} = h_{n-1,r}f_m + h_{n,r}f_{m+1}$. □

Corollary 3.4.12. *The following identities hold:*

(i.) $h_{2n,r} = h_{n-1,r}f_n + h_{n,r}f_{n+1}$.

(ii.) $h_{2n-1,r} = h_{n-1,r}f_{n-1} + h_{n,r}f_n$.

(iii.) $h_{2n-2,r} = h_{n-1,r}f_{n-2} + h_{n,r}f_{n-1}$.

(iv.) $h_{2n-k,r} = h_{n-1,r}f_{n-k} + h_{n,r} + f_{n-k+1}$.

Proof. The results follows from Proposition 3.4.11 upon setting $m = n$, $m = n-1$, $m = n-2$, and $m = n-k$ in that order. □

Proposition 3.4.13. *For every $n \geq 1$, we obtain*

$$h_{0,r}h_{1,r} + h_{1,r}h_{2,r} + \cdots + h_{2n-1,r}h_{2n,r} = h_{2n,r}^2 - h_{0,r}^2.$$

Proof. We induct on n. For base case, $n = 1$:

The left hand side gives

$$h_{0,r}h_{1,r} + h_{1,r}h_{2,r} = h_{1,r}(h_{0,r} + h_{2,r})$$

while the right hand side gives

$$(h_{2,r} - h_{0,r})(h_{2,r} + h_{0,r}) = h_{1,r}(h_{0,r} + h_{2,r}).$$

Since the left hand side equals to the right hand side, the base case holds.

For the induction step, we will assume the formula holds true for n and prove that it holds true for $n+1$.

Since by inductive hypothesis

$$h_{0,r}h_{1,r} + h_{1,r}h_{2,r} + \cdots + h_{2n-1,r}h_{2n,r} = h_{2n,r}^2 - h_{0,r}^2,$$

then

$$h_{0,r}h_{1,r} + h_1h_{2,r} + \cdots + h_{2n-1,r}h_{2n,r} + h_{2n,r}h_{2n+1,r} + h_{2n+1,r}h_{2n+2,r}$$

$$\begin{aligned}
&= h_{2n,r}^2 - h_{0,r}^2 + h_{2n+1,r}h_{2n+2,r} + h_{2n,r}h_{2n+1,r} \\
&= h_{2n,r}^2 + h_{2n+1,r}h_{2n+2,r} + h_{2n,r}h_{2n+1,r} - h_{0,r}^2 \\
&= h_{2n,r}(h_{2n,r} + h_{2n+1,r}) + h_{2n+1,r}h_{2n+2,r} - h_{0,r}^2 \\
&= h_{2n,r}h_{2n+2,r} + h_{2n+1,r}h_{2n+2,r} - h_{0,r}^2 \\
&= h_{2n+2,r}(h_{2n,r} + h_{2n+1,r}) - h_{0,r}^2 \\
&= h_{2n+2,r}^2 - h_{0,r}^2.
\end{aligned}$$

By the principle of mathematical induction, the result follows. □

Lemma 3.4.14. *The $(n+1)^{th}$ Fibonacci number, f_n, is given by*

$$f_n = \frac{h_{1,r}h_{n,r} - h_{0,r}h_{n+1,r}}{h_{1,r}^2 - h_{0,r}h_{2,r}}.$$

Proof. Using Binet's formula (3.4), we obtain

$$\begin{aligned}
h_{1,r}h_{n,r} - h_{0,r}h_{n+1,r} &= h_{1,r}\left[\frac{(h_{1,r} - \beta h_{0,r})\alpha^n}{\alpha - \beta} + \frac{(\alpha h_{0,r} - h_{1,r})\beta^n}{\alpha - \beta}\right] \\
&- h_{0,r}\left[\frac{(h_{1,r} - \beta h_{0,r})\alpha^{n+1}}{\alpha - \beta} + \frac{(\alpha h_{0,r} - h_{1,r})\beta^{n+1}}{\alpha - \beta}\right] \\
&= h_{1,r}\left[h_{1,r}\frac{(\alpha^n - \beta^n)}{\alpha - \beta} + h_{0,r}\frac{(\alpha^{n-1} - \beta^{n-1})}{\alpha - \beta}\right] \\
&- h_{0,r}\left[h_{0,r}\frac{(\alpha^{n+1} - \beta^{n+1})}{\alpha - \beta} + h_{0,r}\frac{(\alpha^n - \beta^n)}{\alpha - \beta}\right] \\
&= \frac{\alpha^n - \beta^n}{\alpha - \beta}\left[h_{1,r}^2 - h_{0,r}^2\right] - \frac{h_{0,r}h_{1,r}}{\alpha - \beta}\left[\alpha^{n-1} - \beta^{n-1} - \alpha^{n+1} + \beta^{n+1}\right] \\
&= \frac{\alpha^n - \beta^n}{\alpha - \beta}\left[h_{1,r}^2 - h_{0,r} - h_{0,r}h_{1,r}\right].
\end{aligned}$$

Now,

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{h_{1,r}h_{n,r} - h_{0,r}h_{n+1,r}}{h_{1,r}^2 - h_{0,r}h_{2,r}}.$$

□

Theorem 3.4.15 (Generalized identity for $r-$shifted Fibonacci sequence)**.** *Let $h_{n,r}$ be the $(n+1)^{th}$ term of $r-$shifted Fibonacci sequence, then*

$$\begin{aligned}
&h_{m,r}h_{n,r} - h_{m-k,r}h_{n+k,r} \\
&= \frac{(-1)^{m-k}}{h_{1,r}^2 - h_{0,r}h_{2,r}}[(h_{1,r}h_{k,r} - h_{0,r}h_{k+1,r})(h_{1,r}h_{n-m+k,r} - h_{0,r}h_{n-m+k+1,r})], \qquad (3.8)
\end{aligned}$$

where $n \geq m$ and $k \geq 1$.

Proof. From Binet's formula (3.4), we obtain

$$h_{n,r} = A\alpha^n + B\beta^n$$

where $A = \dfrac{h_{1,r} - \beta h_{0,r}}{\alpha - \beta}$, $B = \dfrac{\alpha h_{0,r} - h_{1,r}}{\alpha - \beta}$, $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$.

Now,

$$\begin{aligned} h_{m,r}h_{n,r} - h_{m-k,r}h_{n+k,r} &= (A\alpha^m + B\beta^m)(A\alpha^n + B\beta^n) \\ &\quad - \left(A\alpha^{m-k} + B\beta^{m-k}\right)\left(A\alpha^{n+k} + B\beta^{n+k}\right) \\ &= AB\left(\alpha^k - \beta^k\right)\left[\frac{\alpha^m\beta^n}{\alpha^k} - \frac{\alpha^n\beta^m}{\beta^k}\right] \\ &= AB(-1)^{-k}\left(\alpha^k - \beta^k\right)(\alpha^m\beta^m)\left(\beta^{n-m+k} - \alpha^{n-m+k}\right) \\ &= -AB(-1)^{m-k}\left(\alpha^k - \beta^k\right)\left(\alpha^{n-m+k} - \beta^{n-m+k}\right). \end{aligned}$$

Using $-AB = \dfrac{h_{1,r}^2 - h_{0,r}h_{2,r}}{(\alpha-\beta)^2}$, we obtain upon substitution

$$\begin{aligned} h_{m,r}h_{n,r} - h_{m-k,r}h_{m+k,r} &= \frac{h_{1,r}^2 - h_{0,r}h_{2,r}}{(\alpha-\beta)^2}(-1)^{m-k}\left[(\alpha^k - \beta^k)(\alpha^{n-m+k} - \beta^{n-m+k})\right] \\ &= \left(h_{1,r}^2 - h_{0,r}h_{2,r}\right)(-1)^{m-k}\left[\frac{\alpha^k - \beta^k}{\alpha-\beta}\left(\frac{\alpha^{n-m+k} - \beta^{n-m+k}}{\alpha-\beta}\right)\right]. \end{aligned}$$

Using Lemma 3.4.14, we get

$$f_k = \frac{\alpha^k - \beta^k}{\alpha - \beta} = \frac{h_{1,r}h_{k,r} - h_{0,r}h_{k+1,r}}{h_{1,r}^2 - h_{0,r}h_{2,r}}$$

and

$$f_{n-m+k} = \frac{\alpha^{n-m+k} - \beta^{n-m+k}}{\alpha - \beta} = \frac{h_{1,r}h_{n-m+k,r} - h_{0,r}h_{n-m+k+1,r}}{h_{1,r}^2 - h_{0,r}h_{2,r}}.$$

So

$$\begin{aligned} &h_{m,r}h_{n,r} - h_{m-k,r}h_{n+k,r} \\ &\quad = (-1)^{m-k}\left[\frac{(h_{1,r}h_{k,r} - h_{0,r}h_{k+1,r})(h_{1,r}h_{n-m+k,r} - h_{0,r}h_{n-m+k+1})}{h_{1,r}^2 - h_{0,r}h_{2,r}}\right]. \end{aligned}$$

Hence the proof. □

Corollary 3.4.16 (Catalan's identity for $r-$shifted Fibonacci sequence)**.** *If $m = n$ in the generalized identity* (3.8)*, then*

$$h_{n,r}^2 - h_{n-k,r}h_{n+k,r} = \frac{(-1)^{n-k}}{h_{1,r}^2 - h_{0,r}h_{2,r}}[h_{1,r}h_{k,r} - h_{0,r}h_{k+1,r}]^2, \qquad (3.9)$$

for all $n > k \geq 1$.

Corollary 3.4.17 (Cassini's identity for $r-$shifted Fibonacci sequence). *If $m = n$ and $k = 1$ in the generalized identity* (3.8)*, then*

$$h_{n,r}^2 - h_{n-1,r}h_{n+1,r} = (-1)^{n-1}\left(h_{1,r}^2 - h_{0,r}h_{2,r}\right), \tag{3.10}$$

for all $n \geq 1$.

Corollary 3.4.18 (d'Ocagne's identity for $r-$shifted Fibonacci sequence). *If $n = m$, $m = n + 1$ and $k = 1$ in the generalized identity* (3.8)*, then*

$$h_{n+1,r}h_{m,r} - h_{n,r}h_{m+1,r} = (-1)^n[h_{1,r}h_{m-n,r} - h_{0,r}h_{m-n+1,r}], \tag{3.11}$$

where $m > n \geq 0$.

3.5 Determinant identities for $r-$shifted Fibonacci sequence

In this section, we give determinant identities for $r-$shifted Fibonacci numbers.

Proposition 3.5.1. *For every positive integer n,*

$$\begin{vmatrix} h_{n+1,r} & h_{n+2,r} & h_{n+3,r} \\ h_{n+4,r} & h_{n+5,r} & h_{n+6,r} \\ h_{n+7,r} & h_{n+8,r} & h_{n+9,r} \end{vmatrix} = 0.$$

Proof. Applying $C_1 + C_2 \longrightarrow C_1$ to the matrix, we get that two columns are identical and the result follows. □

Proposition 3.5.2. *For every positive integer n,*

$$\begin{vmatrix} h_{n,r} + h_{n+1,r} & h_{n+1,r} + h_{n+2,r} & h_{n+2,r} + h_{n,r} \\ h_{n+2,r} & h_{n,r} & h_{n+1,r} \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

Proof. Applying $R_1 + R_2 \longrightarrow R_1$, we get that the determinant of the matrix is

$$\begin{vmatrix} 2h_{n+2,r} & 2h_{n+2,r} & 2h_{n+2,r} \\ h_{n+2,r} & h_{n,r} & h_{n+1,r} \\ 1 & 1 & 1 \end{vmatrix} = 2h_{n+2,r} \begin{vmatrix} 1 & 1 & 1 \\ h_{n+2,r} & h_{n,r} & h_{n+1,r} \\ 1 & 1 & 1 \end{vmatrix}.$$

Since two rows are identical, the determinant is zero. □

Proposition 3.5.3. *Let n be a positive integer, then*

$$\begin{vmatrix} h_{n,r} & f_n & 1 \\ h_{n+1,r} & f_{n+1} & 1 \\ h_{n+2,r} & f_{n+2} & 1 \end{vmatrix} = f_n h_{n+1,r} - f_{n+1} h_{n,r}.$$

Proof. Applying $R_2 - R_1 \longrightarrow R_1$ and $R_3 - R_2 \longrightarrow R_2$, we get that

$$\begin{vmatrix} h_{n,r} & f_n & 1 \\ h_{n+1,r} & f_{n+1} & 1 \\ h_{n+2,r} & f_{n+2} & 1 \end{vmatrix} = \begin{vmatrix} h_{n+1,r} - h_{n,r} & f_{n+1} - f_n & 0 \\ h_{n,r} & f_n & 0 \\ h_{n+2,r} & f_{n+2} & 1 \end{vmatrix}.$$

The result is thus immediate. □

Proposition 3.5.4. *For every positive integer n,*

$$\begin{vmatrix} h_{n,r} & l_n & 1 \\ h_{n+1,r} & l_{n+1} & 1 \\ h_{n+2,r} & l_{n+2} & 1 \end{vmatrix} = l_n h_{n+1,r} - l_{n+1} h_{n,r}.$$

Proof. The proof follows as in the proof of Proposition 3.5.3. □

Proposition 3.5.5. *For every positive integer n,*

$$\begin{vmatrix} 1 + h_{n,r} & h_{n+1,r} & \cdots & h_{n+p,r} \\ h_{n,r} & 1 + h_{n+1,r} & \cdots & h_{n+p,r} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n,r} & h_{n+1,r} & \cdots & 1 + h_{n+p,r} \end{vmatrix} = 1 + h_{n,r} + h_{n+1,r} + \cdots + h_{n+p,r}.$$

Proof. The proof follows by induction on n and making use of column reductions.

□

Proposition 3.5.6. *Let n be a positive integer, then*

$$\begin{vmatrix} h_{n,r} & h_{n+1,r} & h_{n+2,r} \\ h_{n+2,r} & h_{n,r} & h_{n+1,r} \\ h_{n+1,r} & h_{n+2,r} & h_{n,r} \end{vmatrix} = 2(h_{n,r}^3 + h_{n+1,r}^3).$$

Proof. Computing the determinant directly, we obtain

$$\begin{vmatrix} h_{n,r} & h_{n+1,r} & h_{n+2,r} \\ h_{n+2,r} & h_{n,r} & h_{n+1,r} \\ h_{n+1,r} & h_{n+2,r} & h_{n,r} \end{vmatrix} = h_{n,r}(h_{n,r}^2 - h_{n+1,r}h_{n+2,r}) + h_{n+1,r}(h_{n+1,r}^2 - h_{n,r}h_{n+2,r})$$
$$+ h_{n+2,r}(h_{n+2,r}^2 - h_{n,r}h_{n+1,r})$$
$$= h_{n,r}^3 + h_{n+1,r}^3 + h_{n+2,r}^3 - 3h_{n,r}h_{n+1,r}h_{n+2,r}.$$

Substituting $h_{n+2,r} = h_{n,r} + h_{n+1,r}$ and expanding, we obtain the desired result.

□

CHAPTER 4

SHIFTED FIBONACCI POLYNOMIALS

4.1 Introduction

In this chapter, we present our results on Fibonacci polynomials. These polynomials are defined recursively by (1.6).

We now introduce r-shifted Fibonacci polynomials. Let $f_n(x)$ be the $(n+1)^{th}$ term of Fibonacci polynomials.

Definition 4.1.1. The $(n+1)^{th}$ term, $h_{n,r}(x)$, of *r-shifted Fibonacci polynomial* is given by

$$h_{n,r}(x) = f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+r}(x). \quad (4.1)$$

From definition 4.1.1, we have for $r > 0$ that the first term

$$h_{0,r}(x) = f_1(x) + f_2(x) + \cdots + f_r(x) = \frac{f_{r+1}(x) + f_r(x) - 1}{x}$$

and the second term

$$h_{1,r}(x) = f_2(x) + f_3(x) + \cdots + f_{r+1}(x) = \frac{f_{r+2}(x) + f_{r+1}(x) - x - 1}{x}.$$

The r-shifted Fibonacci polynomials satisfy the recurrence relation,

$$h_{n,r}(x) = xh_{n-1,r}(x) + h_{n-2,r}(x), \tag{4.2}$$

for all $n \geq 2$, with initial conditions

$$h_{0,r}(x) = \frac{f_{r+1}(x) + f_r(x) - 1}{x}$$

and

$$h_{1,r}(x) = \frac{f_{r+2}(x) + f_{r+1}(x) - x - 1}{x},$$

for all $r > 0$.

Table 4.1 shows a few entries of r-shifted polynomials.

r	$h_{0,r}(x)$	$h_{1,r}(x)$	$h_{2,r}(x)$
1	1	x	x^2+1
2	x+1	x^2+x+1	x^3+x^2+2x+1
3	x^2+x+2	x^3+x^2+3x+1	x^4+x^3+4x^2+2x+2
4	x^3+x^2+3x+2	x^4+x^3+4x^2+3x+2	x^5+x^4+5x^3+4x^2+5x+2
5	x^4+x^3+4x^2+3x+3	x^5+x^4+5x^3+4x^2+6x+2	x^6+x^5+6x^4+5x^3+10x^2+5x+3
6	x^5+x^4+5x^3+4x^2+6x+3	x^6+x^5+6x^4+5x^3+10x^2+6x+3	x^7+x^6+7x^5+6x^4+15x^3+10x^2+9x+3

Table 4.1: r-shifted Fibonacci polynomials

Setting $x = 1$ in the r-shifted polynomials (4.1), we obtain r-shifted Fibonacci numbers.

In Section 4.2, we obtain Binet's formula for the r-shifted Fibonacci polynomials. Generating function, relation between these polynomials and Chebyshevs polynomials of the first and second kinds, and hypergeometric series are presented in Section 4.3. Some of the properties that have been obtained by various researchers for Fibonacci and Fibonacci like polynomials include sum of the first n terms of the sequence, sum of the first n terms with odd (or even) indices of the sequence, explicit sum formula, Honsberger's identity, Catalan's identity, Cassini's identity and d'Ocagne's identity among many others. These properties are derived in Section 4.4.

4.2 Binet's formula for $r-$ shifted Fibonacci polynomials and other preliminary results

Proposition 4.2.1. *The $(n+1)^{th}$ term of $r-$shifted Fibonacci polynomials, $h_{n,r}(x)$, for all $r > 0$ is given by*

$$h_{n,r}(x) = \frac{f_{n+r+1}(x) + f_{n+r}(x) - f_{n+1}(x) - f_n(x)}{x}. \tag{4.3}$$

Proof. From recurrence relation (4.2) and equation (4.1), we obtain

$$\begin{aligned}
h_{n,r}(x) &= xh_{n-1,r}(x) + h_{n-2,r}(x) \\
&= x(f_n(x) + f_{n+1}(x) + \cdots + f_{n+r-1}(x)) \\
&+ (f_{n-1}(x) + f_n(x) + \cdots + f_{n+r-2}(x)) \\
&= x\left[(f_0(x) + f_1(x) + \cdots + f_{n+r-1}(x)) - (f_0(x) + f_1(x) + \cdots + f_{n-1}(x))\right] \\
&+ \left[(f_0(x) + f_1(x) + \cdots + f_{n+r-2}(x)) - (f_0(x) + f_1(x) + \cdots + f_{n-2}(x))\right].
\end{aligned}$$

By equation (1.11), we get

$$\begin{aligned}
h_{n,r}(x) &= \frac{1}{x}\left[x((f_{n+r-1}(x) + f_{n+r}(x) - 1) - (f_n(x) + f_{n-1}(x) - 1))\right] \\
&+ \frac{1}{x}\left[(f_{n+r-1}(x) + f_{n+r-2}(x) - 1) - (f_{n-2}(x) + f_{n-1}(x) - 1)\right] \\
&= \frac{xf_{n+r}(x) + xf_{n+r-1}(x) + f_{n+r-1}(x) + f_{n+r-2}(x)}{x} \\
&- \frac{xf_n(x) + xf_{n-1}(x) + f_{n-1}(x) + f_{n-2}(x)}{x} \\
&= \frac{f_{n+r+1}(x) + f_{n+r}(x) - f_{n+1}(x) - f_n(x)}{x}.
\end{aligned}$$

□

Theorem 4.2.2. *The polynomials $h_{n,r}(x)$ are expressed in terms of Lucas and Fibonacci* *polynomials as:*

$$h_{n,r}(x) = \begin{cases} \sum_{i=1}^{m}(l_{n+4i-2}(x) + l_{n+4i-1}(x)) & \textit{if} \quad r = 4m, \\ \sum_{i=1}^{m}(l_{n+4i-2}(x) + l_{n+4i-1}(x)) + f_{n+4m+1}(x) & \textit{if} \quad r = 4m+1, \\ \sum_{i=1}^{m}(l_{n+4i-2}(x) + l_{n+4i-1}(x)) + f_{n+4m+1}(x) + f_{n+4m+2}(x) & \textit{if} \quad r = 4m+2, \\ \sum_{i=1}^{m}(l_{n+4i-2}(x) + l_{n+4i-1}(x)) + l_{n+4m+2}(x) + f_{n+4m+2}(x) & \textit{if} \quad r = 4m+3. \end{cases}$$

Proof. If $r = 4m$, then

$$\begin{aligned} h_{n,r}(x) &= f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+4m}(x) \\ &= l_{n+2}(x) + l_{n+3}(x) + l_{n+6}(x) + l_{n+7}(x) + l_{n+10}(x) + l_{n+11}(x) \\ &+ \cdots + l_{n+4m-2}(x) + l_{n+4m-1}(x) \\ &= \sum_{i=1}^{m}(l_{n+4i-1}(x) + l_{n+4i-2}(x)). \end{aligned}$$

If $r = 4m+1$, then

$$\begin{aligned} h_{n,r}(x) &= f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+4m+1}(x) \\ &= l_{n+2}(x) + l_{n+3}(x) + l_{n+6}(x) + l_{n+7}(x) + l_{n+10}(x) + l_{n+11}(x) \\ &+ \cdots + l_{n+4m-2}(x) + l_{n+4m-1}(x) + f_{n+4m+1}(x) \\ &= \sum_{i=1}^{m}(l_{n+4i-1}(x) + l_{n+4i-2}(x)) + f_{n+4m+1}(x). \end{aligned}$$

If $r = 4m+2$, then

$$\begin{aligned} h_{n,r}(x) &= f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+4m+2}(x) \\ &= l_{n+2}(x) + l_{n+3}(x) + l_{n+6}(x) + l_{n+7}(x) + l_{n+10}(x) + l_{n+11}(x) \\ &+ \cdots + l_{n+4m-2}(x) + l_{n+4m-1}(x) + f_{n+4m+1}(x) + f_{n+4m+2}(x) \\ &= \sum_{i=1}^{m}(l_{n+4i-1}(x) + l_{n+4i-2}(x)) + f_{n+4m+1}(x) + f_{n+4m+2}(x). \end{aligned}$$

If $r = 4m+3$, then

$$
\begin{aligned}
h_{n,r}(x) &= f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+4m+2}(x) \\
&= l_{n+2}(x) + l_{n+3}(x) + l_{n+6}(x) + l_{n+7}(x) + l_{n+10}(x) + l_{n+11}(x) \\
&+ \cdots + l_{n+4m-2}(x) + l_{n+4m-1}(x) + f_{n+4m+1}(x) + f_{n+4m+2}(x) + f_{n+4m+3}(x) \\
&= \sum_{i=1}^{m} (l_{n+4i-1}(x) + l_{n+4i-2}(x)) + l_{n+4m+2}(x) + f_{n+4m+2}(x).
\end{aligned}
$$

□

Remark 4.2.3. We remark that:

1. For all $r = 1, 2$, the $r-$shifted Fibonacci polynomials, $h_{n,r}(x)$, are themselves Fibonacci polynomials.
2. The polynomials, $h_{n,4}(x)$, are Lucas polynomials for all integers $n \geq 0$.
3. The polynomials, $h_{n,4m}(x)$, are sums of Lucas polynomials for all integers $m \geq 1$ and $n \geq 0$.
4. For all $m \geq 0$ and $n \geq 0$, we have that $h_{n,4m+1}$, $h_{n,4m+2}$, and $h_{n,4m+3}$ are sums of Fibonacci and Lucas polynomials.

Proposition 4.2.4. *Let $m \geq 1$. The $(n+1)^{th}$ term of $4m-$shifted Fibonacci polynomial, $h_{n,4m}(x)$, satisfies the equation*

$$h_{n,4m}(x) = \frac{f_{2m}(x)(l_{n+2m+1}(x) + l_{n+2m}(x))}{x}.$$

Proof. Using Binet's formulas for Fibonacci polynomial (1.7) and Lucas polynomial (1.8) and by equation (4.3), we get

$$
\begin{aligned}
h_{n,4m}(x) &= \frac{f_{n+4m+1}(x) + f_{n+4m}(x) - f_{n+1}(x) - f_n(x)}{x} \\
&= \frac{1}{x(\alpha(x) - \beta(x))}(\alpha(x)^{n+4m+1} - \beta(x)^{n+4m+1} + \alpha(x)^{n+4m} - \beta(x)^{n+4m}) \\
&- \frac{1}{x(\alpha(x) - \beta(x))}(\alpha(x)^{n+1} - \beta(x)^{n+1} + \alpha(x)^{n} - \beta(x)^{n}) \\
&= \frac{1}{x(\alpha(x) - \beta(x))}[\alpha(x)^{n+4m+1} + \alpha(x)^{n+4m} - \alpha(x)^{n+1} - \alpha(x)^{n} \\
&- \beta(x)^{n+4m+1} - \beta(x)^{n+4m} + \beta(x)^{n+1} + \beta(x)^{n}].
\end{aligned}
$$

Since $\alpha(x)\beta(x) = -1$ then, $(\alpha(x)\beta(x))^{2m} = 1$, and

$$h_{n,4m}(x) = \frac{1}{x(\alpha(x)-\beta(x))}\Big[\alpha(x)^{n+4m+1} - (\alpha(x)\beta(x))^{2m}\alpha(x)^{n+1} + \alpha(x)^{n+4m} - (\alpha(x)\beta(x))^{2m}\alpha(x)^{n} - \beta(x)^{n+4m+1} + (\alpha(x)\beta(x))^{2m}\beta(x)^{n+1} - \beta(x)^{n+4m} + (\alpha(x)\beta(x))^{2m}\beta(x)^{n}\Big],$$

which simplifies to

$$\begin{aligned} h_{n,4m}(x) &= \frac{1}{x(\alpha(x)-\beta(x))}(\alpha(x)^{2m} - \beta(x)^{2m})\Big[\alpha(x)^{n+2m+1} + \beta(x)^{n+2m+1} \\ &\quad + \alpha(x)^{n+2m} + \beta(x)^{n+2m}\Big] \\ &= \frac{f_{2m}(x)(l_{n+2m+1}(x) + l_{n+2m}(x))}{x}. \end{aligned}$$

□

Setting $m = 1$ in Proposition 4.2.4, we get that

Corollary 4.2.5. $h_{n,4}(x) = l_{n+2}(x) + l_{n+3}(x)$, *for all* $n \geq 0$.

Proposition 4.2.6.

$$\sum_{k=1}^{n} h_{k,r}^2(x) = \frac{h_{n,r}(x)h_{n+1,r}(x) - h_{0,r}(x)h_{1,r}(x)}{x}.$$

Proof. Since $xh_{n,r}(x) = h_{n+1,r}(x) - h_{n-1,r}(x)$ then,

$$xh_{n,r}^2(x) = h_{n,r}(x)h_{n+1,r}(x) - h_{n-1,r}(x)h_{n,r}(x).$$

Now, we have

$$\begin{aligned} xh_{1,r}^2(x) &= h_{1,r}(x)h_{2,r}(x) - h_{0,r}(x)h_{1,r}(x) \\ xh_{2,r}^2(x) &= h_{2,r}(x)h_{3,r}(x) - h_{1,r}(x)h_{2,r}(x) \\ xh_{3,r}^2(x) &= h_{3,r}(x)h_{4,r}(x) - h_{2,r}(x)h_{3,r}(x) \\ &\vdots \\ xh_{n-1,r}^2(x) &= h_{n-1,r}(x)h_{n,r}(x) - h_{n-2,r}(x)h_{n-1,r}(x) \\ xh_{n,r}^2(x) &= h_{n,r}(x)h_{n+1,r}(x) - h_{n-1,r}(x)h_{n,r}(x). \end{aligned}$$

Adding up these equations, we get

$$x(h_{1,r}^2(x) + h_{2,r}^2(x) + h_{3,r}^2(x) + \cdots + h_{n-1,r}^2(x) + h_{n,r}^2(x))$$
$$= h_{n,r}(x)h_{n+1}(x) - h_{0,r}(x)h_{1,r}(x).$$

Hence the proof. □

Theorem 4.2.7 (Binet's formula for $r-$ Shifted Fibonacci polynomials)**.** *The* $(n+1)^{th}$ *term of r-shifted Fibonacci polynomials is given by*

$$h_{n,r}(x) = \frac{1}{\alpha(x) - \beta(x)} \left[h_{1,r}(x)(\alpha(x)^n - \beta(x)^n) + h_{0,r}(x)(\alpha(x)^{n-1} - \beta(x)^{n-1}) \right], \tag{4.4}$$

where $\alpha(x) = \dfrac{x + \sqrt{x^2+4}}{2}$ *and* $\beta(x) = \dfrac{x - \sqrt{x^2+4}}{2}$.

Proof. Let $n \geq 2$, then the r-shifted Fibonacci polynomials are defined by the recurrence formula $h_{n,r}(x) = xh_{n-1,r}(x) + h_{n-2,r}(x)$, with initial conditions

$$h_{0,r}(x) = \frac{f_{r+1}(x) + f_r(x) - 1}{x}$$

and

$$h_{1,r}(x) = \frac{f_{r+2}(x) + f_{r+1}(x) - x - 1}{x},$$

for all $r > 0$.

The characteristic equation of the recurrence relation (4.2) is $t^2 - xt - 1 = 0$. We solve this equation to get its roots as $\alpha(x)$ and $\beta(x)$. Since the roots are real and distinct, the solution of the recurrence relation is of the form

$$h_{n,r}(x) = C\alpha(x)^n + D\beta(x)^n, \tag{4.5}$$

where C and D are constants.

Setting $n = 0$ and $n = 1$, we get

$$h_{0,r}(x) = C + D$$
$$h_{1,r}(x) = C\alpha(x) + D\beta(x).$$

Solving these equations simultaneously, we get

$$C = \frac{h_{1,r}(x) - \beta(x)h_{0,r}(x)}{\alpha(x) - \beta(x)} \quad \text{and} \quad D = \frac{\alpha(x)h_{0,r}(x) - h_{1,r}(x)}{\alpha(x) - \beta(x)}.$$

Thus the result. □

Remark 4.2.8. If $C = \dfrac{h_{1,r}(x) - \beta(x)h_{0,r}(x)}{\alpha(x) - \beta(x)}$ and $D = \dfrac{\alpha(x)h_{0,r}(x) - h_{1,r}(x)}{\alpha(x) - \beta(x)}$ then the following hold:

$$C + D = h_{0,r}(x), \tag{4.6}$$

$$C\alpha(x) + D\beta(x) = h_{1,r}(x), \tag{4.7}$$

$$C\beta(x) + D\alpha(x) = xh_{0,r}(x) - h_{1,r}(x) and \tag{4.8}$$

$$C\beta(x)^2 + D\alpha(x)^2 = x^2 h_{0,r}(x) - xh_{1,r}(x) + h_{0,r}(x) \tag{4.9}$$

Corollary 4.2.9. *The $(n+1)^{th}$ term of r-shifted Fibonacci polynomials satisfies the equation*

$$h_{n,r}(x) = h_{1,r}(x)f_n(x) + h_{0,r}(x)f_{n-1}(x).$$

Proof. Since the $(n+1)^{\text{th}}$ Fibonacci polynomial is

$$f_n(x) = \frac{1}{\alpha(x) - \beta(x)}(\alpha(x)^n - \beta(x)^n)$$

then the result follows from the Binet's formula (4.5). □

Setting $r = 1$ in (4.5) and making use of equations (1.9) and (1.10), we rediscover Binet's formula for Fibonacci polynomials (1.7).

Corollary 4.2.10 (Binet's formula for Fibonacci polynomials)**.** *The $(n+2)^{th}$ Fibonacci polynomial is given by*

$$f_{n+1}(x) = \frac{1}{\alpha(x) - \beta(x)}\Big[\alpha^{n+1}(x) - \beta^{n+1}(x)\Big].$$

Corollary 4.2.11. *For $n \geq 1$, we have*

$$h_{n,r}(x) = \frac{h_{1,r}(x)}{2^{n-1}} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} x^{n-2k-1}(x^2+4)^k + \frac{h_{0,r}(x)}{2^{n-2}} \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-1}{2k+1} x^{n-2k-2}(x^2+4)^k,$$

where $\lfloor n \rfloor$ is the greatest integer less than or equal to n.

Proof. We have

$$\alpha(x)^n - \beta(x)^n = 2^{-n} \left[\left(x + \sqrt{x^2+4}\right)^n - \left(x + \sqrt{x^2-4}\right)^n \right].$$

By Binomial Theorem, we get

$$\begin{aligned} \alpha(x)^n - \beta(x)^n &= 2^{-n} \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \left[\left(\sqrt{x^2+4}\right)^k - \left(-\sqrt{x^2+4}\right)^k \right] \\ &= 2^{-n+1} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} x^{n-2k-1} \left(\sqrt{x^2+4}\right)^{2k+1}. \end{aligned}$$

Using $\alpha(x) - \beta(x) = \sqrt{x^2+4}$, we have

$$\frac{\alpha(x)^n - \beta(x)^n}{\alpha(x) - \beta(x)} = 2^{-n+1} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} x^{n-2k-1} \left(x^2+4\right)^k,$$

and hence the result follows from the Binet's formula (4.4). □

4.3 Generating function and its hypergeometric representation

Theorem 4.3.1. *Let $H_r(x,t)$ be the generating function of r-shifted Fibonacci polynomials, then*

$$H_r(x,t) = \frac{h_{0,r}(x) + t(h_{1,r}(x) - xh_{0,r}(x))}{1 - xt - t^2}, \tag{4.10}$$

where $|t+t^2| < 1$.

Proof. Let $H_r(x,t) = \sum_{n=0}^{\infty} h_{n,r}(x)t^n$ be the generating function of r-shifted Fibonacci polynomials, then from the recurrence relation (4.2), we get

$$\sum_{n\geq 2} h_{n,r}(x)t^n = x\sum_{n\geq 2} h_{n-1}(x)t^n + \sum_{n\geq 2} h_{n-2,r}(x)t^n.$$

Now we have

$$\sum_{n\geq 0} h_{n,r}(x)t^n - h_{1,r}(x)t - h_{0,r}(x) = xt\left(\sum_{n\geq 0} h_{n,r}(x)t^n - h_{0,r}(x)\right) + t^2\sum_{n\geq 0} h_{n,r}(x)t^n$$

or

$$H_r(x,t) - h_{1,r}(x)t - h_{0,r}(x) = xt(H_r(x,t) - h_{0,r}(x)) + t^2 H_r(x,t).$$

Thus

$$H_r(x,t) = \frac{h_{0,r}(x) + t(h_{1,r}(x) - xh_{0,r}(x))}{1 - xt - t^2}.$$

□

We now express (4.10) in terms of Chebyshev polynomials.

Chebyshev polynomial of the second kind, $U_n(z)$, is given by

$$U_n(z) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n-k}{k} (2z)^{n-2k},$$

with its generating function being

$$\sum_{n=0}^{\infty} U_n(z)y^n = \frac{1}{1 - 2zy + y^2}.$$

Now, letting $y = it$ and $z = \frac{x}{2i}$, we get

$$\sum_{n=0}^{\infty} i^n U_n\left(\frac{x}{2i}\right) t^n = \frac{1}{1 - xt - t^2} \tag{4.11}$$

or

$$\sum_{n=0}^{\infty} i^n U_n\left(\frac{x}{2i}\right) t^{n+1} = \frac{t}{1 - xt - t^2}. \tag{4.12}$$

Using the generating function (4.10), we obtain

$$\sum_{n=0}^{\infty} h_{n,r}(x)t^n = h_{0,r}(x)\frac{1}{1-xt-t^2} + (h_{1,r}(x) - xh_{0,r}(x))\frac{t}{1-xt-t^2}$$
$$= h_{0,r}(x)\sum_{n=0}^{\infty} i^n U_n\left(\frac{x}{2i}\right)t^n + (h_{1,r}(x) - xh_{0,r}(x))\sum_{n=0}^{\infty} i^n U_n\left(\frac{x}{2i}\right)t^{n+1} \tag{4.13}$$

from equations (4.11) and (4.12).

If we extract the coefficient of t^n on both sides of equation (4.13), we get the following corollary which expresses r-shifted polynomials in terms of Chebyshev polynomial of the second kind.

Corollary 4.3.2. *For $n \geq 1$,*

$$h_{n,r}(x) = i^n h_{0,r}(x)U_n\left(\frac{x}{2i}\right) + i^{n-1}\left(h_{1,r}(x) - xh_{0,r}(x)\right)U_{n-1}\left(\frac{x}{2i}\right).$$

Similarly, Chebyshev polynomial of the first kind, $T_n(z)$, is defined by

$$T_n(z) = \frac{n}{2}\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k}{n-k}\binom{n-k}{k}(2z)^{n-2k}.$$

The generating function for these polynomials is

$$\sum_{n=0}^{\infty} T_n(z)y^n = \frac{1-zy}{1-2zy+y^2}.$$

Again doing the substitutions $y = it$ and $z = \frac{x}{2i}$ we get

$$\sum_{n=0}^{\infty} i^n T_n\left(\frac{x}{2i}\right)t^n = \frac{1-\frac{tx}{2}}{1-xt-t^2}. \tag{4.14}$$

Now, we write the generating function (4.10) as

$$\sum_{n=0}^{\infty} h_{n,r}(x)t^n = h_{0,r}(x)\cdot\frac{1-\frac{tx}{2}}{1-xt-t^2} + \left(h_{1,r}(x) - \frac{x}{2}\cdot h_{0,r}(x)\right)\frac{t}{1-xt-t^2}$$
$$= h_{0,r}(x)\sum_{n=0}^{\infty} i^n T_n\left(\frac{x}{2i}\right)t^n + \left(h_{1,r}(x) - \frac{x}{2}\cdot h_{0,r}(x)\right)\sum_{n=0}^{\infty} i^n U_n\left(\frac{x}{2i}\right)t^{n+1} \tag{4.15}$$

by equations (4.12) and (4.14). Extracting the coefficient of t^n in (4.15) we get,

Corollary 4.3.3. *For $n \geq 1$,*

$$h_{n,r}(x) = i^n h_{0,r}(x) T_n\left(\frac{x}{2i}\right) + i^{n-1}\left(h_{1,r}(x) - \frac{x}{2}\cdot h_{0,r}(x)\right) U_{n-1}\left(\frac{x}{2i}\right).$$

Almost all of the most common special functions in mathematics are particular cases of the *Gauss hypergeometric series* defined by

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}$$

where the *rising factorial* $(a)_k$ is defined by $(a)_0 = 1$ and $(a)_k = a(a+1)\cdots(a+k-1)$ for all $(k \geq 1)$ and for arbitrary $a \in \mathbb{C}$.

Corollary 4.3.4. *Hypergeometric representation of the generating function for r-shifted Fibonacci polynomial is given by*

$$\sum_{n=0}^{\infty} \frac{h_{n,r}(x)}{n!} t^n = ((1-xt)h_{0,r}(x) + h_{1,r}(x)t)e^{xt}{}_2F_1\left(n+1,1;1;t^2\right).$$

Proof. Using the generating function (4.10), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} h_{n,r}(x)t^n &= \frac{(1-xt)h_{0,r}(x) + h_{1,r}(x)t}{1-tx-t^2} \\
&= [(1-xt)h_{0,r}(x) + h_{1,r}(x)t][1-(x+t)t]^{-1} \\
&= [(1-xt)h_{0,r}(x) + h_{1,r}(x)t] \sum_{n=0}^{\infty} (x+t)^n t^n \\
&= [(1-xt)h_{0,r}(x) + h_{1,r}(x)t] \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} \binom{n}{k} x^{n-k} t^k \\
&= [(1-xt)h_{0,r}(x) + h_{1,r}(x)t] \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^{n-k} t^{n+k} \\
&= [(1-xt)h_{0,r}(x) + h_{1,r}(x)t] \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(n+k)!}{k!n!} x^n t^{n+2k} \\
&= [(1-xt)h_{0,r}(x) + h_{1,r}(x)t] \sum_{n=0}^{\infty} \frac{(xt)^n}{n!} \sum_{k=0}^{\infty} \frac{(n+k)!}{k!} t^{2k}.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{h_{n,r}(x)}{n!} t^n &= [(1-xt)h_{0,r}(x) + h_{1,r}(x)t]e^{xt} \sum_{k=0}^{\infty} \frac{(n+k)!}{n!} \frac{(t^2)^k}{k!} \\
&= [(1-xt)h_{0,r}(x) + h_{1,r}(x)t]e^{xt} \sum_{k=0}^{\infty} (n+1)_k \frac{(1)_k}{(1)_k} \frac{(t^2)^k}{k!}.
\end{aligned}$$

The solution is thus immediate. □

4.4 Properties of r-shifted Fibonacci polynomials

In this section, we obtain properties of r-shifted Fibonacci polynomials.

Proposition 4.4.1 (Explicit sum formula for $r-$shifted Fibonacci polynomials). *Let $h_{n,r}(x)$ be the $(n+1)^{th}$ r-shifted Fibonacci polynomial, then*

$$h_{n,r}(x) = h_{0,r}(x) \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} x^{n-2k} + (h_{1,r}(x) - xh_{0,r}(x)) \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} x^{n-2k-1}.$$

Proof. From the generating function (4.10), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} h_{n,r}(x)t^n &= [(1-xt)h_{0,r}(x) + h_{1,r}(x)t](1-xt-t^2)^{-1} \\
&= [h_{0,r}(x) + t(h_{1,r}(x) - xh_{0,r}(x))]\,[1-(x+t)t]^{-1} \\
&= [h_{0,r}(x) + t(h_{1,r}(x) - xh_{0,r}(x))] \sum_{n=0}^{\infty} (x+t)^n t^n \\
&= [h_{0,r}(x) + t(h_{1,r}(x) - xh_{0,r}(x))] \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} \binom{n}{k} x^{n-k} t^k \\
&= [h_{0,r}(x) + t(h_{1,r}(x) - xh_{0,r}(x))] \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^{n-k} t^{n+k}.
\end{aligned}$$

Replacing n with $n+k$, we get

$$\begin{aligned}
\sum_{k=0}^{\infty} h_{n,r}(x)t^n &= [h_{0,r}(x) + t(h_{1,r}(x) - xh_{0,r}(x))] \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(n+k)!}{k!n!} x^n t^{n+2k} \\
&= [h_{0,r}(x) + t(h_{1,r}(x) - xh_{0,r}(x))] \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n-k)!}{k!(n-2k)!} x^{n-2k} t^n.
\end{aligned}$$

Equating the coefficient of t^n on both sides of the equation gives

$$h_{n,r}(x) = h_{0,r}(x) \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} x^{n-2k} + (h_{1,r}(x) - xh_{0,r}(x)) \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} x^{n-2k-1}.$$

□

Proposition 4.4.2 (Sum of first n terms). *The sum of first n terms of r-shifted Fibonacci polynomials is given by*

$$\sum_{k=0}^{n-1} h_{k,r}(x) = \frac{h_{n,r}(x) + h_{n-1,r}(x) + xh_{0,r}(x) - h_{1,r}(x) - h_{0,r}(x)}{x}.$$

Proof. Using Binet's formula (4.5), we obtain

$$\sum_{k=0}^{n-1} h_{k,r}(x) = \sum_{k=0}^{n-1} \left(C\alpha(x)^k + D\beta(x)^k \right),$$

where $C = \dfrac{h_{1,r}(x) - \beta(x)h_{0,r}(x)}{\alpha(x) - \beta(x)}$ and $D = \dfrac{\alpha(x)h_{0,r}(x) - h_{1,r}(x)}{\alpha(x) - \beta(x)}$.

It follows that,

$$\begin{aligned}
\sum_{k=0}^{n-1} h_{k,r}(x) &= C\sum_{k=0}^{n-1} \alpha(x)^k + D\sum_{k=0}^{n-1} \beta(x)^k \\
&= \frac{C\left(\alpha(x)^n - 1\right)}{\alpha(x) - 1} + \frac{D\left(\beta(x)^n - 1\right)}{\beta(x) - 1} \\
&= \frac{C + D - (C\beta(x) + D\alpha(x)) - (C\alpha(x)^n + D\beta(x)^n)}{\alpha(x)\beta(x) - \alpha(x) - \beta(x) + 1} \\
&+ \frac{\alpha(x)\beta(x)(C\alpha(x)^{n-1} + D\beta(x)^{n-1})}{\alpha(x)\beta(x) - \alpha(x) - \beta(x) + 1}.
\end{aligned}$$

Since $\alpha(x) + \beta(x) = x$ and $\alpha(x)\beta(x) = -1$ and using (4.5), (4.6) and (4.8), we get

$$\sum_{k=0}^{n-1} h_{k,r}(x) = \frac{h_{n,r}(x) + h_{n-1,r}(x) + xh_{0,r}(x) - h_{1,r}(x) - h_{0,r}(x)}{x}.$$

□

Proposition 4.4.3 (Sum of first n terms with odd indices)**.** *The sum of first n terms of r-shifted Fibonacci polynomials with odd indices is given by*

$$\frac{h_{2n+1,r}(x) - h_{2n-1,r}(x) - xh_{0,r}(x)}{x^2}.$$

Proof. Using Binet's formula (4.5), we get

$$\begin{aligned}
\sum_{k=0}^{n-1} h_{2k+1,r}(x) &= \sum_{k=0}^{n-1} \left(C\alpha(x)^{2k+1} + D\beta(x)^{2k+1} \right) \\
&= C\sum_{k=0}^{n-1} \alpha(x)^{2k+1} + D\sum_{k=0}^{n-1} \beta(x)^{2k+1} \\
&= \frac{C\left(\alpha(x)^{2n+1} - \alpha(x)\right)}{\alpha(x)^2 - 1} + \frac{D\left(\beta(x)^{2n+1} - \beta(x)\right)}{\beta(x)^2 - 1}.
\end{aligned}$$

Thus,

$$\sum_{k=0}^{n-1} h_{2k+1,r}(x) = \frac{C\alpha(x) + D\beta(x) - \alpha(x)\beta(x)(C\beta(x) + D\alpha(x))}{(\alpha(x)\beta(x))^2 - \alpha(x)^2 - \beta(x)^2 + 1}$$
$$- \frac{C\alpha(x)^{2n+1} + D\beta(x)^{2n+1} + (\alpha(x)\beta(x))^2(C\alpha(x)^{2n-1} + D\beta(x)^{2n-1})}{(\alpha(x)\beta(x))^2 - \alpha(x)^2 - \beta(x)^2 + 1}.$$

Since $\alpha(x)\beta(x) = -1$ and $\alpha(x)^2 + \beta(x)^2 = x^2 + 2$, then using (4.5), (4.7) and (4.8), we obtain

$$\sum_{k=0}^{n-1} h_{2k+1,r}(x) = \frac{h_{2n+1,r} - h_{2n-1,r}(x) - xh_{0,r}(x)}{x^2}.$$

□

Proposition 4.4.4 (Sum of first n terms with even indices)**.** *The sum of first n terms of r-shifted Fibonacci sequences with even indices is given by*

$$\sum_{k=0}^{n-1} h_{2k,r}(x) = \frac{h_{2n,r}(x) - h_{2n-2,r}(x) - xh_{1,r}(x) + x^2 h_{0,r}(x)}{x^2}.$$

Proof. Using Binet's formula (4.5), we obtain

$$\begin{aligned}
\sum_{k=0}^{n-1} h_{2k,r}(x) &= \sum_{k=0}^{n-1} \left(C\alpha(x)^{2k} + D\beta(x)^{2k}\right) \\
&= C\sum_{k=0}^{n-1} \alpha(x)^{2k} + D\sum_{k=0}^{n-1} \beta(x)^{2k} \\
&= \frac{C\left(\alpha(x)^{2n} - 1\right)}{\alpha(x)^2 - 1} + \frac{D\left(\beta(x)^{2n} - 1\right)}{\beta(x)^2 - 1}.
\end{aligned}$$

Hence,

$$\sum_{k=0}^{n-1} h_{2k,r}(x) = \frac{C + D - (C\beta(x)^2 + D\alpha(x)^2) - (C\alpha(x)^{2n} + D\beta(x)^{2n})}{(\alpha(x)\beta(x))^2 - \alpha(x)^2 - \beta(x)^2 + 1}$$
$$+ \frac{(\alpha(x)\beta(x))^2(C\alpha(x)^{2n-2} + D\beta(x)^{2n-2})}{(\alpha(x)\beta(x))^2 - \alpha(x)^2 - \beta(x)^2 + 1}.$$

Since $\alpha(x)^2 + \beta(x)^2 = x^2 + 2$, and $\alpha(x)\beta(x) = -1$, then using (4.5), (4.6) and (4.9), we obtain

$$\sum_{k=0}^{n-1} h_{2k,r}(x) = \frac{h_{2n,r}(x) - h_{2n-2,r}(x) - xh_{1,r}(x) + x^2 h_{0,r}(x)}{x^2}.$$

□

Proposition 4.4.5. *For every positive integer n,*

$$\sum_{k=1}^{n} h_{3k-2,r}(x) = \frac{h_{3n+1,r}(x) + h_{3n-2,r}(x) + (x-1)h_{1,r}(x) - (x^2+1)h_{0,r}(x)}{x^3+3x}.$$

Proof. From Binet's formula (4.5), we get

$$\begin{aligned}\sum_{k=1}^{n} h_{3k-2,r}(x) &= \sum_{k=1}^{n} \left(C\alpha(x)^{3k-2} + D\beta(x)^{3k-2}\right) \\ &= C\sum_{k=1}^{n} \alpha(x)^{3k-2} + D\sum_{k=1}^{n} \beta(x)^{3k-2} \\ &= \frac{C\alpha(x)\left(\alpha(x)^{3n}-1\right)}{\alpha(x)^3-1} + \frac{D\beta(x)\left(\beta(x)^{3n}-1\right)}{\beta(x)^3-1}.\end{aligned}$$

We have,

$$\begin{aligned}\sum_{k=1}^{n} h_{3k-2,r}(x) &= \frac{C\alpha(x) + D\beta(x) + C\beta(x)^2 + D\alpha(x)^2}{(\alpha(x)\beta(x))^3 - \alpha(x)^3 - \beta(x)^3 + 1} \\ &\quad - \frac{C\alpha(x)^{3n-2} + D\beta(x)^{3n-2} + C\alpha(x)^{3n+1} + D\beta(x)^{3n+1}}{(\alpha(x)\beta(x))^3 - \alpha(x)^3 - \beta(x)^3 + 1}.\end{aligned}$$

Since $\alpha(x)^3 + \beta(x)^3 = x^3 + 3x$, and $\alpha(x)\beta(x) = -1$, then making use of (4.5), (4.6) and (4.9), we obtain

$$\sum_{k=1}^{n} h_{3k-2,r}(x) = \frac{h_{3n+1,r}(x) + h_{3n-2,r}(x) - (1-x)h_{1,r}(x) - (x^2+1)h_{0,r}(x)}{x^3+3x}.$$

□

Proposition 4.4.6. *For every positive integer n,*

$$\sum_{k=1}^{n} h_{3k-1,r}(x) = \frac{h_{3n+2,r}(x) + h_{3n-1,r}(x) - (x+1)h_{1,r}(x) + (x-1)h_{0,r}(x)}{x^3+3x}.$$

Proof. From Binet's formula (4.5), we get

$$\begin{aligned}\sum_{k=1}^{n} h_{3k-1,r}(x) &= \sum_{k=1}^{n} \left(C\alpha(x)^{3k-1} + D\beta(x)^{3k-1}\right) \\ &= \frac{C\alpha(x)^2\left(\alpha(x)^{3n}-1\right)}{\alpha(x)^3-1} + \frac{D\beta(x)^2\left(\beta(x)^{3n}-1\right)}{\beta(x)^3-1}.\end{aligned}$$

Hence,

$$\sum_{k=1}^{n} h_{3k-1,r}(x) = \frac{C\alpha(x)^2 + D\beta(x)^2 - (C\beta(x) + D\alpha(x))}{(\alpha(x)\beta(x))^3 - \alpha(x)^3 - \beta(x)^3 + 1} - \frac{C\alpha(x)^{3n-1} + D\beta(x)^{3n-1} + C\alpha(x)^{3n+2} + D\beta(x)^{3n+2}}{(\alpha(x)\beta(x))^3 - \alpha(x)^3 - \beta(x)^3 + 1}.$$

Since $\alpha(x)^3 + \beta(x)^3 = x^3 + 3x$, and $\alpha(x)\beta(x) = -1$, using (4.5), (4.6) and (4.9) , we obtain

$$\sum_{k=1}^{n} h_{3k-1,r}(x) = \frac{h_{3n+2,r}(x) + h_{3n-1,r}(x) - h_{2,r}(x) - h_{1,r}(x) + xh_{0,r}(x)}{x^3 + 3x}.$$

Thus the result. □

Proposition 4.4.7. *For every positive integer n,*

$$\sum_{k=1}^{n} h_{3k,r}(x) = \frac{h_{3n+3,r}(x) + h_{3n,r}(x) - (x^2+1)h_{1,r}(x) - 2h_{0,r}(x)}{x^3 + 3x}.$$

Proof. Using Binet's formula (4.5), we get

$$\sum_{k=1}^{n} h_{3k,r}(x) = \sum_{k=1}^{n} \left(C\alpha(x)^{3k} + D\beta(x)^{3k}\right) = \frac{C\alpha(x)^3\left(\alpha(x)^{3n} - 1\right)}{\alpha(x)^3 - 1} + \frac{D\beta(x)^3\left(\beta(x)^{3n} - 1\right)}{\beta(x)^3 - 1}.$$

This sum equals

$$\frac{C + D + C\alpha(x)^3 + D\beta(x)^3 - (C\alpha(x)^{3n} + D\beta(x)^{3n}) - (C\alpha(x)^{3n+3} + D\beta(x)^{3n+3})}{(\alpha(x)\beta(x))^3 - \alpha(x)^3 - \beta(x)^3 + 1}.$$

Since $\alpha(x)^3 + \beta(x)^3 = x^3 + 3x$, and $\alpha(x)\beta(x) = -1$, then by equations (4.5), (4.6) and (4.9) , we get

$$\sum_{k=1}^{n} h_{3k,r}(x) = \frac{h_{3n+3,r}(x) + h_{3n,r}(x) - h_{3,r}(x) - h_{0,r}(x)}{x^3 + 3x}.$$

The result follows. □

Proposition 4.4.8 (Alternating sum formula for $r-$shifted Fibonacci polynomials)**.** *For every positive integer n,*

$$\sum_{k=1}^{n} (-1)^{k+1} h_{k,r}(x) = \frac{(-1)^{n+1}(h_{n+1,r}(x) - h_{n,r}(x)) + h_{1,r}(x) - h_{0,r}(x)}{x}.$$

Proof. From Binet's formula (4.5), we get

$$\sum_{k=1}^{n}(-1)^{k+1}h_{k,r}(x) = \sum_{k=1}^{n}(-1)^{k+1}\left(C\alpha(x)^k + D\beta(x)^k\right)$$
$$= \frac{C\alpha(x)\left((-\alpha(x))^n - 1\right)}{-\alpha(x)-1} + \frac{D\beta(x)\left((-\beta(x))^n - 1\right)}{-\beta(x)-1}.$$

Thus

$$\sum_{k=1}^{n}(-1)^{k+1}h_{k,r}(x) = \frac{(-1)^{n+1}(C\alpha(x)^{n+1} + D\beta(x)^{n+1}) + (-1)^n(C\alpha(x)^n + D\beta(x)^n)}{\alpha(x)\beta(x) + \alpha(x) + \beta(x) + 1}$$
$$+ \frac{C\alpha(x) + D\beta(x) - (C+D)}{\alpha(x)\beta(x) + \alpha(x) + \beta(x) + 1}.$$

Since $\alpha(x) + \beta(x) = x$, and $\alpha(x)\beta(x) = -1$, then by Binet's formula (4.5) we have

$$\sum_{k=1}^{n}(-1)^{k+1}h_{k,r}(x) = \frac{(-1)^{n+1}(h_{n+1,r}(x) - h_{n,r}(x)) + h_{1,r}(x) - h_{0,r}(x)}{x}.$$

□

Proposition 4.4.9. *For every positive integer n,*

$$h_{2n,r}(x) = \sum_{k=0}^{n}\binom{n}{k}h_{k,r}(x)x^k.$$

Proof. From Binet's formula (4.5), we get

$$h_{2n,r} = \frac{1}{\alpha(x)-\beta(x)}[(h_{1,r}(x) - \beta h_{0,r}(x))\alpha(x)^{2n} - (h_{1,r}(x) - \alpha(x)h_{0,r}(x))\beta(x)^{2n}].$$

Using $\alpha(x)^2 = 1 + x\alpha(x)$ and $\beta(x)^2 = 1 + x\beta(x)$, then

$$h_{2n,r}(x)$$
$$= \frac{1}{\alpha(x)-\beta(x)}[(h_{1,r}(x) - \beta h_{0,r}(x))(1 + x\alpha(x))^n - (h_{1,r}(x) - \alpha h_{0,r}(x))(1 + x\beta(x))^n].$$

Since $(1 + x\alpha)^n = \sum_{k=0}^{n}\binom{n}{k}\alpha^k x^k$, we obtain

$$h_{2n,r}(x) = \frac{1}{\alpha(x)-\beta(x)}\Bigg[(h_{1,r}(x) - \beta(x)h_{0,r}(x))\sum_{k=0}^{n}\binom{n}{k}\alpha(x)^k x^k$$
$$- (h_{1,r}(x) - \alpha(x)h_{0,r}(x))\sum_{k=0}^{n}\binom{n}{k}\beta(x)^k x^k\Bigg]$$
$$= \sum_{k=0}^{n}\binom{n}{k}x^k\left[\frac{(h_{1,r}(x) - \beta(x)h_{0,r}(x))\alpha(x)^k - (h_{1,r}(x) - \alpha(x)h_{0,r}(x))\beta(x)^k}{\alpha(x)-\beta(x)}\right]$$
$$= \sum_{k=0}^{n}\binom{n}{k}h_{k,r}(x)x^k.$$

□

Proposition 4.4.10. *For every positive integer n,*

$$h_{-n,r}(x) = (-1)^n (h_{0,r}(x) f_{n+1}(x) - h_{1,r}(x) f_n(x)).$$

Proof. Using Binet's formula (4.5), we obtain

$$\begin{aligned} & h_{-n,r}(x) \\ & = \frac{1}{\alpha(x)-\beta(x)} \left[(h_{1,r}(x) - \beta(x) h_{0,r}(x)) \alpha(x)^{-n} - (h_{1,r}(x) - \alpha(x) h_{0,r}(x)) \beta(x)^{-n} \right] \\ & = \frac{1}{\alpha(x)-\beta(x)} \left[(h_{1,r}(x) - \beta(x) h_{0,r}(x)) \frac{1}{\alpha(x)^n} - (h_{1,r}(x) - \alpha(x) h_{0,r}(x)) \frac{1}{\beta(x)^n} \right]. \end{aligned}$$

Using $\dfrac{1}{\alpha(x)} = -\beta(x)$ and $\dfrac{1}{\beta(x)} = -\alpha(x)$, we get

$$\begin{aligned} & h_{-n,r}(x) \\ & = \frac{(-1)^n}{\alpha(x)-\beta(x)} [(h_{1,r}(x) - \beta(x) h_{0,r}(x)) \beta(x)^n - (h_{1,r}(x) - \alpha(x) h_{0,r}(x)) \alpha(x)^n] \\ & = \frac{(-1)^n}{\alpha(x)-\beta(x)} \left[h_{1,r}(x) \beta(x)^n - h_{0,r}(x) \beta(x)^{n+1} - h_{1,r}(x) \alpha(x)^n + h_{0,r}(x) \alpha^{n+1} \right] \\ & = \frac{(-1)^{n+1}}{\alpha(x)-\beta(x)} \left[h_{1,r}(x)(\alpha(x)^n - \beta(x)^n) - h_{0,r}(x)(\alpha(x)^{n+1} - \beta(x)^{n+1}) \right] \\ & = (-1)^{n+1} \left[\frac{h_{1,r}(x)(\alpha(x)^n - \beta(x)^n)}{\alpha(x)-\beta(x)} - \frac{h_{0,r}(x)(\alpha(x)^{n+1} - \beta(x)^{n+1})}{\alpha(x)-\beta(x)} \right] \\ & = (-1)^{n+1} (h_{1,r}(x) f_n(x) - h_{0,r}(x) f_{n+1}(x)) \\ & = (-1)^n (h_{0,r}(x) f_{n+1}(x) - h_{1,r}(x) f_n(x)). \end{aligned}$$

□

Proposition 4.4.11 (Honsberger's identity for $r-$ shifted Fibonacci polynomials)**.** *If $m \geq 0$ and $n > 0$, then*

$$h_{m+n,r}(x) = h_{n-1,r}(x) f_m(x) + h_{n,r}(x) f_{m+1}(x).$$

Proof. From Corollary 4.2.9, we obtain

$$h_{n+m,r}(x) = h_{1,r}(x) f_{m+n}(x) + h_{0,r}(x) f_{m+n-1}(x).$$

Using Honsberger's identity for Fibonacci polynomials (1.12), we get

$$\begin{aligned} h_{n+m,r}(x) &= h_{1,r}(x)(f_{n-1}(x)f_m(x) + f_n(x)f_{m+1}(x)) \\ &\quad + h_{0,r}(x)(f_{n-2}(x)f_m(x) + f_{n-1}(x)f_{m+1}(x)). \end{aligned}$$

Thus,

$$\begin{aligned} h_{n+m,r}(x) &= f_m(x)(h_{1,r}(x)f_{n-1}(x) + h_{0,r}(x)f_{n-2}(x)) \\ &\quad + f_{m+1}(x)(h_{1,r}(x)f_n(x) + h_{0,r}(x)f_{n-1}(x)) \\ &= h_{n-1}(x)f_m(x) + h_{n,r}(x)f_{m+1}(x). \end{aligned}$$

□

Setting $m = n - k$ in Proposition 4.4.11, we get the following corollary:

Corollary 4.4.12. *For $n \geq 0$ and $k \leq n$ we obtain*

$$h_{2n-k,r}(x) = h_{n-1,r}(x)f_{n-k}(x) + h_{n,r}(x)f_{n-k+1}(x).$$

Lemma 4.4.13. *For all $n \geq 0$, the $(n+1)^{th}$ Fibonacci polynomials $f_n(x)$ is given by*

$$f_n(x) = \frac{h_{1,r}(x)h_{n,r}(x) - h_{0,r}(x)h_{n+1,r}(x)}{h_{1,r}^2(x) - h_{0,r}(x)h_{2,r}(x)}.$$

Proof. From Binet's formula (4.5), we get

$$\begin{aligned} & h_{1,r}(x)h_{n,r}(x) - h_{0,r}(x)h_{n+1,r}(x) \\ &\quad = h_{1,r}(x)\left(\frac{(h_{1,r}(x) - \beta(x)h_{0,r}(x))\alpha(x)^n}{\alpha(x)-\beta(x)} + \frac{(\alpha(x)h_{0,r}(x) - h_{1,r}(x))\beta(x)^n}{\alpha(x)-\beta(x)}\right) \\ &\quad - h_{0,r}(x)\left(\frac{(h_{1,r}(x) - \beta(x)h_{0,r}(x))\alpha(x)^{n+1}}{\alpha(x)-\beta(x)} + \frac{(\alpha(x)h_{0,r}(x) - h_{1,r}(x))\beta(x)^{n+1}}{\alpha(x)-\beta(x)}\right) \\ &\quad = \frac{\alpha(x)^n - \beta(x)^n}{\alpha(x)-\beta(x)}\left(h_{1,r}^2(x) - h_{0,r}^2(x)\right) - \frac{h_{0,r}(x)h_{1,r}(x)}{\alpha(x)-\beta(x)}(\alpha(x)^n - \beta(x)^n) \\ &\quad = \frac{\alpha(x)^n - \beta(x)^n}{\alpha(x)-\beta(x)}\left(h_{1,r}^2(x) - h_{0,r}^2(x) - xh_{0,r}(x)h_{1,r}(x)\right). \end{aligned}$$

Thus,

$$f_n = \frac{\alpha(x)^n - \beta(x)^n}{\alpha(x)-\beta(x)} = \frac{h_{1,r}(x)h_{n,r}(x) - h_{0,r}(x)h_{n+1,r}(x)}{h_{1,r}^2(x) - h_{0,r}(x)h_{2,r}(x)}.$$

□

Theorem 4.4.14 (Generalized identity for $r-$ shifted Fibonacci polynomials)**.** *Let $h_{n,r}(x)$ be the $(n+1)^{th}$ term of r-shifted Fibonacci polynomials, then*

$$h_{m,r}(x)h_{n,r}(x)-h_{m-k,r}(x)h_{n+k,r}(x)=$$
$$(-1)^{m-k}\frac{(h_{1,r}(x)h_{k,r}(x)-h_{0,r}(x)h_{k+1,r}(x))(h_{1,r}(x)h_{n-m+k,r}(x)-h_{0,r}(x)h_{n-m+k+1,r}(x))}{h_{1,r}^2(x)-h_{0,r}(x)h_{2,r}(x)}, \tag{4.16}$$

where $n>m\geq k\geq 1$.

Proof. Using Binet's formula (4.5), we obtain

$$\begin{aligned}
&h_{m,r}(x)h_{n,r}(x)-h_{m-k,r}(x)h_{n+k,r}(x)\\
&\quad=(C\alpha(x)^m+D\beta(x)^m)(C\alpha(x)^n+D\beta(x)^n)\\
&\quad-(C\alpha(x)^{m-k}+D\beta(x)^{m-k})(C\alpha(x)^{n+k}+D\beta(x)^{n+k})\\
&\quad=CD\Big(\alpha(x)^k-\beta(x)^k\Big)\left(\frac{\alpha(x)^m\beta(x)^n}{\alpha(x)^k}-\frac{\alpha(x)^n\beta(x)^m}{\beta(x)^k}\right)\\
&\quad=CD\frac{(\alpha(x)^k-\beta(x)^k)}{(\alpha(x)\beta(x))^k}\Big(\alpha(x)^m\beta(x)^{n+k}-\alpha(x)^{n+k}\beta(x)^m\Big)\\
&\quad=CD\left(\alpha(x)^k-\beta(x)^k\right)\right)(-1)^{-k}\left(\alpha(x)\beta(x)\right)^m\left(\beta(x)^{n+k-m}-\alpha(x)^{n+k-m}\right)\\
&\quad=-CD(-1)^{m-k}\Big(\alpha(x)^k-\beta(x)^k\Big)\Big(\alpha(x)^{n-m+k}-\beta(x)^{n-m+k}\Big).
\end{aligned}$$

Using $-CD=\dfrac{h_{1,r}^2(x)-h_{0,r}(x)h_{2,r}(x)}{(\alpha(x)-\beta(x))^2}$, then

$$\begin{aligned}
&h_{m,r}(x)h_{n,r}(x)-h_{m-k,r}(x)h_{n+k,r}(x)\\
&=\Big(h_{1,r}^2(x)-h_{0,r}(x)h_{2,r}(x)\Big)(-1)^{m-k}\left(\frac{\alpha(x)^k-\beta(x)^k}{\alpha(x)-\beta(x)}\right)\left(\frac{\alpha(x)^{n-m+k}-\beta(x)^{n-m+k}}{\alpha(x)-\beta(x)}\right).
\end{aligned}$$

From Lemma (4.4.13), we get

$$f_k=\frac{\alpha(x)^k-\beta(x)^k}{\alpha(x)-\beta(x)}=\frac{h_{1,r}(x)h_{k,r}(x)-h_{0,r}(x)h_{k+1,r}(x)}{h_{1,r}^2(x)-h_{0,r}(x)h_{2,r}(x)}$$

and

$$f_{n-m+k}=\frac{\alpha(x)^{n-m+k}-\beta(x)^{n-m+k}}{\alpha(x)-\beta(x)}=\frac{h_{1,r}(x)h_{n-m+k,r}(x)-h_{0,r}(x)h_{n-m+k+1,r}(x)}{h_{1,r}^2(x)-h_{0,r}(x)h_{2,r}(x)}.$$

So,

$$h_{m,r}(x)h_{n,r}(x) - h_{m-k,r}(x)h_{n+k,r}(x) =$$
$$(-1)^{m-k}\frac{(h_{1,r}(x)h_{k,r}(x) - h_{0,r}(x)h_{k+1,r}(x))(h_{1,r}(x)h_{n-m+k,r}(x) - h_{0,r}(x)h_{n-m+k+1}(x))}{h_{1,r}^2(x) - h_{0,r}(x)h_{2,r}(x)}.$$

□

Corollary 4.4.15 (Catalan's identity for $r-$shifted Fibonacci polynomials). *If* $m = n$ *in the generalized identity* (4.16)*, we get*

$$\begin{aligned} h_{n,r}^2(x) &- h_{n-k,r}(x)h_{n+k,r}(x) \\ &= \frac{(-1)^{n-k}}{h_{1,r}^2(x) - h_{0,r}(x)h_{2,r}(x)}[h_{1,r}(x)h_{k,r}(x) - h_{0,r}(x)h_{k+1,r}(x)]^2, \end{aligned} \tag{4.17}$$

where $n > k \geq 1$.

Corollary 4.4.16 (Cassini's identity for $r-$ shifted Fibonacci polynomials). *If* $m = n$ *and* $k = 1$ *in the generalized identity* (4.16)*, we obtain*

$$h_{n,r}^2(x) - h_{n-1,r}(x)h_{n+1,r}(x) = (-1)^{n-1}\Big(h_{1,r}^2(x) - h_{0,r}(x)h_{2,r}(x)\Big), \tag{4.18}$$

for all $n \geq 1$.

Corollary 4.4.17 (d'Ocagne's identity for $r-$ shifted Fibonacci polynomials). *If* $n = m$, $m = n + 1$ *and* $k = 1$ *in the generalized identity* (4.16)*, we get*

$$h_{m,r}(x)h_{n+1,r}(x) - h_{m+1,r}(x)h_{n,r}(x) = (-1)^n(h_{1,r}(x)h_{m-n,r}(x) - h_{0,r}(x)h_{m-n+1,r}(x)), \tag{4.19}$$

where $m > n \geq 1$.

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

5.1 Conclusions

In this thesis, we have introduced r-shifted Fibonacci sequences (Definition 3.1.1). We then obtained Binet's formula (Theorem 3.3.1) and generating function (Theorem 3.3.5) for these sequences and also derived various properties of the $r-$ shifted Fibonacci sequences; such as sum of first n terms (Proposition 3.4.1), sum of first n terms with odd indices (Proposition 3.4.2), sum of first n terms with even indices (Proposition 3.4.3), alternating sum of r-shifted Fibonacci sequence (Proposition 3.4.7), explicit sum formula (Proposition 3.4.9), Honsberger's identity (Proposition 3.4.11), determinant identities, and generalized identity (Theorem 3.4.15), which gives Catalan's identity (3.9), Cassini's identity (3.10) and d'Ocagne's identity (3.11). Further, we introduced r-shifted Fibonacci polynomials (Definition 4.1.1) and obtained their Binet's formula (Theorem 3.2.2) and generating function (Theorem 4.3.1). We then represented r-shifted Fibonacci polynomials in terms of Chebyshevs polynomial of the second kind (Corollary 4.3.2) and in terms of both Chebyshev's polynomials of the first and second kinds (Corollary 4.3.3). The hypergeometric representation of r-shifted polynomials is obtained in Corollary 4.3.4. Analogous properties of these polynomials, obtained in this thesis, include explicit sum formula (Proposition 4.4.1), sum of first n terms (Proposition 4.4.2), sum of first n terms with odd indices (Proposition 4.4.3), sum of first n terms with even indices (Proposition 4.4.4), alternating sum of the first n terms (Proposition 4.4.8), Honsberger identity (Proposition 4.4.11) and generalized identity (Theorem 4.4.14) from which one obtains Catalan's identity (4.17), Cassini's identity (4.18) and d'Ocagne's identity (4.19).

5.2 Recommendations

Fibonacci sequences and their polynomials have been generalized either by varying the recurrence relation and maintaining initial conditions or by varying initial conditions and maintaining recurrence relations. Properties of r-shifted Tribonacci numbers and their polynomials have not been studied as well as the properties of r-shifted Fibonacci sequences and polynomials where both recurrence relation and initial conditions have been varied. We therefore, recommend that further study be done on these areas.

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