GENERALIZED BLOCH SPACES OF THE UPPER HALF-PLANE AND THEIR COMPOSITION SEMIGROUPS

BY

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Declaration

This thesis is my own work and has not been presented for a degree award in any other institution.

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This thesis has been submitted for examination with our approval as the university supervisors.

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Dedication

This thesis is dedicated to my daughters Levine and Verna

Abstract

Banach space structure of the Bloch space of the unit disc $\mathcal{B}(\mathbb{D})$ has been studied widely by many Mathematicians. Cima, Anderson, among others have proved that the Bloch space of the unit disc, $\mathcal{B}(\mathbb{D})$ and the little Bloch space of the unit disc, $\mathcal{B}_0(\mathbb{D})$ are Banach spaces with respect to the Bloch norm. Boundedness, compactness, as well as semigroup properties have been studied on the Bloch spaces of the unit disc. Zhu among other scholars have studied the generalized little Bloch space of the unit disc $\mathcal{B}^{\alpha}_{\circ}(\mathbb{D})$, as closed, separable subspace of the generalized Bloch space of the unit disc $\mathcal{B}^{\alpha}(\mathbb{D})$. On the other hand, there is little and much less complete literature on Bloch space of other domains. On the upper half plane, $\mathbb{U}$, the properties of the generalized Bloch spaces as Banach spaces are not known. In our study therefore, we have investigated the properties of the generalized Bloch space of the upper half plane, $\mathcal{B}^{\alpha}(\mathbb{U})$. Specifically, we have proved that $\mathcal{B}^{\alpha}(\mathbb{U})$ and the generalized little Bloch space of the upper half plane, $\mathcal{B}^{\alpha}_{\circ}(\mathbb{U})$ are Banach spaces. Cayley transform has been employed in getting equivalent representation of functions from $\mathcal{B}^{\alpha}(\mathbb{D})$ to $\mathcal{B}^{\alpha}(\mathbb{U})$. By use of classification theorem for the automorphisms of $\mathbb{U}$, we have established that automorphism groups of $\mathbb{U}$ generate strongly continuous semigroups on $\mathcal{B}^{\alpha}_{\circ}(\mathbb{U})$. We applied the theory of linear operators in the study of semigroup properties of the composition semigroups. The results of this study have contributed to the existing knowledge and enhanced further research in this field of study.

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Chapter 1

Introduction

1.1 Background of the study

Banach spaces of analytic functions have played a prominent role in both classical and modern analysis. Most studies on the classical analytic function spaces such as Hardy spaces, Bergman spaces, Bloch spaces, Analytic spaces of Bounded Mean Oscillations (BMOA), among others are based on the unit disc. K. Zhu among other authors [7, 12, 22, 28] proved that the Bloch space of the unit disc is a Banach space with respect to its norm. He proved that the little Bloch space of the unit disc is a separable, closed, nowhere dense subspace of the Bloch space of the unit disc. In addition, it has been proved that the little Bloch space of the unit disc is identical with the closure of the polynomials in the Bloch norm [28]. He further introduced generalized Bloch spaces of the unit disc and investigated their properties where he established that the generalized Bloch spaces of the unit disc are Banach spaces with respect to their norm. He also obtained the corresponding generalized little Bloch spaces of the unit disc and proved that the latter are closed, separable subspaces of the generalized Bloch spaces. He later established the generalized little Bloch spaces as the closure of the set of polynomials in the norm topology of the generalized Bloch spaces of the unit disc. Vast majority of the literature on Bloch spaces [8, 14] focus on the properties of these spaces of analytic functions on the unit disc. On the other hand, there is very little literature on Bloch space of other domains and in particular, of the upper half plane, see for instance [29] and references therein. We therefore studied properties of the generalized Bloch spaces in the setting of the upper half plane.

1.2 Basic concepts and Notation

1.2.1 Unit disk and upper half plane

Let $\mathbb{C}$ be the complex plane. The set $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is called the *open unit disc*. On the other hand, the set $\mathbb{U} := \{\omega \in \mathbb{C} : \Im(\omega) > 0\}$ denotes the upper half of the complex plane $\mathbb{C}$, where $\Im(\omega)$ is the imaginary part of $\omega \in \mathbb{C}$. The function $\psi(z) = \frac{i(1+z)}{1-z}$ is referred to as the *Cayley transform* and maps the unit disc $\mathbb{D}$ conformally onto the upper half-plane $\mathbb{U}$, with the inverse $\psi^{-1}(\omega) = \frac{\omega-i}{\omega+i}$ mapping the upper half plane $\mathbb{U}$, onto the unit disc, $\mathbb{D}$. We refer to [27] for details.

1.3 Linear fractional transformations

Let $\Omega \subset \mathbb{C}$ be an open set. A function $f : \Omega \to \mathbb{C}$ is said to be complex differentiable at a point $z_0 \in \Omega$ if the limit

$$\lim_{\triangle z \to 0} \frac{f(z_0 + \triangle z) - f(z_0)}{\triangle z}, \tag{1.1}$$

exists. A *holomorphic function* is a complex-valued function of one or more complex variables that is, at every point of its domain, complex differentiable in a neighbourhood of the point. A function φ is known as analytic self map of Ω if φ is analytic and $\varphi(\Omega) \subseteq \Omega$. A mapping $\phi : \Omega \to \mathbb{C}$ is biholomorphic if

(i) ϕ is one to one and onto,

(ii) ϕ is holomorphic, and

(iii) ϕ^{-1} is holomorphic.

Linear fractional transformations (LFTs) are mappings of the form

$$\varphi(z) = \frac{az + b}{cz + d}, \text{ where } z \in \mathbb{C},\ a, b, c, d \in \mathbb{R} \text{ and } ad - bc \neq 0 \tag{1.2}$$

Each linear fractional transformation is a one to one holomorphic map of a domain to itself. We denote the set of all LFTs from a domain Ω to itself by $\mathrm{LFT}(\Omega, \Omega)$. Moreover, if $z_0 \in \mathbb{C}$ is such that $\varphi(z_0) = z_0$ then z_0 is a fixed point of φ.

Linear fractional transformations of the unit disc, $\mathbb{D}$ are maps of the form

$$f(z) = \frac{az+b}{\bar{b}z+\bar{a}} \tag{1.3}$$

where $z \in \mathbb{C}$, $a, b \in \mathbb{C}$ and $|a|^2 - |b|^2 = 1$. On the upper half plane $\mathbb{U}$, linear fractional transformations are of the form

$$f(z) = \frac{az+b}{cz+d} \tag{1.4}$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$.

A *homomorphism* is a structure preserving map between two algebraic structures of the same type. A bijective homomorphism of an object to itself is called an *automorphism*, that is $\mathrm{Aut}(\Omega) = \mathrm{LFT}(\Omega, \Omega)$. The set $(\mathrm{Aut}(\Omega), \circ)$ of all automorphisms of Ω forms a group under composition operator $\circ$. We verify that $(\mathrm{Aut}(\Omega), \circ)$ indeed forms a group. To prove closure, let $f, g \in \mathrm{Aut}(\Omega)$, and consider the composition $g \circ f$. Since g and f are bijective, it follows that $g \circ f$ is bijective. Moreover, for $z_1, z_2 \in \Omega$

$$\begin{aligned}
(g \circ f)(z_1 z_2) &= g(f(z_1 z_2)) \\
&= g(f(z_1)f(z_2)) \\
&= g(f(z_1))g(f(z_2)) \\
&= (g \circ f)(z_1)(g \circ f)(z_2)
\end{aligned}$$

hence $g \circ f \in \mathrm{Aut}(\Omega)$ as desired.

Secondly, we need to show that $\circ$ is associative. Let $f, g, h \in \mathrm{Aut}(\Omega)$.

Then $\forall z \in \Omega$, we have

$$\begin{aligned}(h \circ g) \circ f(z) &= (h \circ g) f(z) \\ &= h(g(f(z)) \\ &= h(g \circ f)(z) \\ &= h \circ (g \circ f)(z).\end{aligned}$$

Thus $\circ$ is associative.

Thirdly, we need to check that there is an identity element in $\text{Aut}(\Omega)$. Let $e : \Omega \to \Omega$ be defined by $e(z) = z$ for all $z \in \Omega$. Consider $g \in \text{Aut}(\Omega)$. For any $z \in \Omega$, we have

$$g \circ e(z) = g(z) = e \circ g(z).$$

Hence e is the identity element. Lastly, we need to prove that $f \in \text{Aut}(\Omega)$ has an inverse for $\circ$. Consider the inverse function f^{-1}. Clearly

$$f^{-1} \circ f = e \text{ and } f \circ f^{-1} = e.$$

Since every inverse function must necessarily be bijective, we now prove that the bijection f^{-1} is an automorphism. Let $z_1, z_2 \in \Omega$. By definition, there exist $\omega_1, \omega_2 \in \Omega$ such that $f(\omega_1) = z_1$ and $f(\omega_2) = z_2$. Hence

$$\begin{aligned}f^{-1}(z_1 z_2) &= f^{-1}(f(\omega_1) f(\omega_2)) \\ &= f^{-1}(f(\omega_1 \omega_2)) \\ &= \omega_1 \omega_2.\end{aligned}$$

Similarly,

$$\begin{aligned} f^{-1}(z_1)f^{-1}(z_2) &= f^{-1}(f(\omega_1))f^{-1}(f(\omega_2)) \\ &= \omega_1\omega_2. \end{aligned}$$

So $f^{-1}(z_1z_2) = f^{-1}(z_1)f^{-1}(z_2)$, and therefore $f^{-1} \in \text{Aut}(\Omega)$. Thus $(\text{Aut}(\Omega), \circ)$ satisfies all the axioms of a group and hence, is a group.

1.4 Spaces of Analytic functions

Consider $\mathcal{H}(\Omega)$ as the Fréchet space of analytic functions $f : \Omega \to \mathbb{C}$ endowed with the topology of uniform convergence on compact subsets of Ω. Some of the analytic spaces considered in this study are:

(i) **Bloch space of the unit disc**

A function $f \in \mathcal{H}(\mathbb{D})$ is in the Bloch space of the unit disc $\mathcal{B}(\mathbb{D})$ if

$$\|f\|_{\mathcal{B}_1(\mathbb{D})} := \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty$$

and in the little Bloch space of the unit disc $\mathcal{B}_0(\mathbb{D})$ if

$$\lim_{|z|\longrightarrow 1}(1-|z|^2)|f'(z)| = 0.$$

For $f \in \mathcal{B}(\mathbb{D})$, we define the norm on $\mathcal{B}(\mathbb{D})$ by

$$\|f\|_{\mathcal{B}(\mathbb{D})} := |f(0)| + \|f\|_{\mathcal{B}_1(\mathbb{D})},$$

where $\|.\|_{\mathcal{B}_1(\mathbb{D})}$ is a seminorm on $\mathcal{B}(\mathbb{D})$.

The space $\mathcal{B}(\mathbb{D})$ is a Banach space with respect to the norm $\|.\|_{\mathcal{B}(\mathbb{D})}$. If X is a Banach space and $Y \subseteq X$ be its subspace, then we say that Y is dense in X if its closure is the whole of X, that is, $\overline{Y} = X$. As noted in [27], the little Bloch space of the unit disc, $\mathcal{B}_0(\mathbb{D})$ is a closed subspace of $\mathcal{B}(\mathbb{D})$ and it's therefore a Banach space with respect to the norm $\|.\|_{\mathcal{B}(\mathbb{D})}$. Moreover, the set of polynomials is dense in $\mathcal{B}_0(\mathbb{D})$. For comprehensive account of the theory of the Bloch and the little Bloch spaces of the unit disc $\mathbb{D}$, we refer to [4, 14, 29].

(ii) **Bloch space of the upper half plane**

Bloch space of the upper half plane $\mathcal{B}(\mathbb{U})$ is a set of analytic functions $f \in \mathcal{H}(\mathbb{U})$ such that

$$\|f\|_{\mathcal{B}_1(\mathbb{U})} := \sup_{\omega \in \mathbb{U}} \Im(\omega)|f'(\omega)| < \infty.$$

For $f \in \mathcal{B}(\mathbb{U})$, we define the norm on $\mathcal{B}(\mathbb{U})$ by

$$\|f\|_{\mathcal{B}(\mathbb{U})} := |f(i)| + \|f\|_{\mathcal{B}_1(\mathbb{U})},$$

where $\|.\|_{\mathcal{B}_1(\mathbb{U})}$ is a seminorm on $\mathcal{B}(\mathbb{U})$. Indeed, $\|f\|_{\mathcal{B}(\mathbb{U})}$ defines a norm on $\mathcal{B}(\mathbb{U})$. We note that,

$$\|f\|_{\mathcal{B}(\mathbb{U})} = 0 \Leftrightarrow |f(i)| + \|f\|_{\mathcal{B}_1(\mathbb{U})} = 0.$$

Therefore, we have

$$\|f\|_{\mathcal{B}(\mathbb{U})} = 0 \Leftrightarrow \left(|f(i)| + \sup_{\omega \in \mathbb{U}} \Im(\omega) |f'(\omega)| \right) = 0,$$

which is equivalent to

$$|f(i)| = 0 \text{ and } \sup_{\omega \in \mathbb{U}} \Im(\omega) |f'(\omega)| = 0.$$

Since $\Im(\omega) > 0$ and f is holomorphic we have

$$|f'(\omega)| = 0 \Leftrightarrow f \text{ is a constant.}$$

Now $|f(i)| = 0 \Leftrightarrow f(i) = 0$. Hence $f = 0$, as desired. Other norm axioms are clear.

The space $\mathcal{B}(\mathbb{U})$ is a Banach space with respect to the norm $\|.\|_{\mathcal{B}(\mathbb{U})}$.

The little Bloch space of the upper half plane $\mathcal{B}_0(\mathbb{U})$ is defined by

$$\mathcal{B}_\circ(\mathbb{U}) := \{f \in \mathcal{H}(\mathbb{U}) : \lim_{\Im(\omega) \longrightarrow 0} \Im(\omega) |f'(\omega)| = 0\}$$

with the same norm as $\mathcal{B}(\mathbb{U})$. It is also known that $\mathcal{B}_0(\mathbb{U})$ is a Banach space with respect to the norm $\|.\|_{\mathcal{B}(\mathbb{U})}$. See [28, 29] for details.

(iii) **Generalized Bloch Space of the unit disc**

Let $\alpha > 0$ be a real number, we define the generalized Bloch space of the unit disc, $\mathcal{B}^\alpha(\mathbb{D})$ as the space of all functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}_1^{\alpha}(\mathbb{D})} := \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^{\alpha} |f'(z)| < \infty$$

For $f \in \mathcal{B}^{\alpha}(\mathbb{D})$, we define the norm on $\mathcal{B}^{\alpha}(\mathbb{D})$ by

$$\|f\|_{\mathcal{B}^{\alpha}(\mathbb{D})} := |f(0)| + \|f\|_{\mathcal{B}_1^{\alpha}(\mathbb{D})}. \tag{1.5}$$

We also define the corresponding generalized little Bloch space of the unit disc as the space of all functions $f \in \mathcal{H}(\mathbb{D})$ for which

$$\lim_{|z|\to 1} \left(1-|z|^2\right)^{\alpha} |f'(z)| = 0,$$

with the same norm given by (1.5). Here, $\mathcal{B}^{\alpha}(\mathbb{D})$ and $\mathcal{B}_{\circ}^{\alpha}(\mathbb{D})$ are both Banach spaces with respect to the norm $\|.\|_{\mathcal{B}^{\alpha}(\mathbb{D})}$. The generalized little Bloch space of the unit disc, $\mathcal{B}_{\circ}^{\alpha}(\mathbb{D})$ is the closure of the set of polynomials in the norm topology of $\mathcal{B}^{\alpha}(\mathbb{D})$. For more details we refer to [28, 29].

(iv) **Generalized Bloch Space of the upper half plane**

A function $f \in \mathcal{H}(\mathbb{U})$ belongs to the generalized Bloch space of the upper half plane, $\mathcal{B}^{\alpha}(\mathbb{U})$ if

$$\|f\|_{\mathcal{B}_1^{\alpha}(\mathbb{U})} := \sup_{\omega\in\mathbb{U}} \Im(\omega)^{\alpha} |f'(\omega)| < \infty$$

with the norm given by

$$\|f\|_{\mathcal{B}^{\alpha}(\mathbb{U})} := |f(i)| + \|f\|_{\mathcal{B}_1^{\alpha}(\mathbb{U})}.$$

The corresponding generalized little Bloch space of the upper half plane, $\mathcal{B}_0^{\alpha}(\mathbb{U})$ is defined as

$$\mathcal{B}_{\circ}^{\alpha}(\mathbb{U}) := \{f \in \mathcal{H}(\mathbb{U}) : \lim_{\Im(\omega)\longrightarrow 0} \Im(\omega)^{\alpha} |f'(\omega)| = 0\}$$

having the same norm as $\mathcal{B}^{\alpha}(\mathbb{U})$. There is little literature on the properties of the generalized Bloch space of the upper half plane as Banach spaces.

(v) Let X be a Banach space, $1 \leq q, r \leq \infty$ and $s \in \mathbb{R}$. The Besov space $\mathcal{B}_{q,r}^{s}(\mathbb{R}^{\mathbb{N}}, X)$ is the space of all $f \in S'(\mathbb{R}^{\mathbb{N}}, X)$ for which

$$\|f\|_{\mathcal{B}_{q,r}^{s}(\mathbb{R}^{\mathbb{N}},X)} := \|2^{ks}(\varphi_k^{v} * f)_{k=0}^{\infty}\|_{l_r(L_q(X))}.$$

For more details we refer to [13]

1.4.1 Semigroups of Linear Operators

Let X be a Banach space. A one-parameter family $(T_t)_{t\geq 0}$ is a *semigroup* of bounded linear operators on X, if

(i) $T_o = I$ (Identity operator on X), and

(ii) $T_{t+s} = T_t \circ T_s$ for every $t, s, \geq 0$ (Semigroup property).

A semigroup $(T_t)_{t\geq 0}$ of bounded linear operators on X is *strongly continuous* if

$$\lim_{t\to 0^+} \|T_t x - x\| = 0 \text{ for all } x \in X.$$

The *infinitesimal generator* denoted by Γ of $(T_t)_{t\geq 0}$ is defined by

$$\Gamma x := \lim_{t\to 0^+} \frac{T_t x - x}{t} = \frac{\partial}{\partial t}(T_t x)\bigg|_{t=0} \quad \text{for each } x \in \text{dom}(\Gamma),$$

where $\text{dom}(\Gamma)$ denotes the domain of Γ given by

$$\text{dom}(\Gamma) = \left\{ x \in X : \lim_{t\to 0^+} \frac{T_t x - x}{t} \text{ exists} \right\}.$$

We define a group of bounded linear operators as

$$(T_t)_{t\in\mathbb{R}} = \begin{cases} T_t, & t \geq 0, \\ T_{-t}, & t \geq 0. \end{cases}$$

if both $(T_t)_{t\geq 0}$ and $(T_{-t})_{t\geq 0}$ are semigroups on X. For more details see [10, 11, 16].

1.4.2 Composition Operators and Semigroups

Suppose $\varphi : \Omega \to \Omega$ is a self analytic map. The composition operator induced by φ on $\mathcal{H}(\Omega)$ is defined as

$$C_\varphi(f) = f\, o\, \varphi,$$

for all $f \in \mathcal{H}(\Omega)$. On the other hand, given $t \geq 0$ we define a semigroup as a family $(\varphi_t)_{t\geq 0}$ of self analytic maps on Ω satisfying the following properties

(i) $\varphi_0(z) = z$ (Identity map on Ω).

(ii) $\varphi_{t+s} = \varphi_t \circ \varphi_s, \forall\, t, s \geq 0$ (Semigroup property).

(iii) $\varphi_t \to \varphi_0$ uniformly on compact subsets of Ω as $t \to 0$.

Composition semigroup induced by φ_t on $\mathcal{H}(\Omega)$ is defined as

$$C_{\varphi_t}(f) = f o\, \varphi_t, \text{ for all } f \in \mathcal{H}(\Omega).$$

We refer to [5, 6, 11, 16, 28] for more details on semigroups.

1.5 Statement of the Problem

Extensive research has been done on the properties of the Bloch spaces as well as operators defined on them. Most of the studies are based on the Bloch spaces of the open unit disc. The immense interest in generalizing these spaces has partly succeeded. For instance, the generalized Bloch spaces including the little Bloch spaces of the unit disc have been proved to be Banach spaces with respect to their norms, among other properties. However, their counterparts on the upper half plane have hardly been studied in literature. In this study, we have considered the generalized Bloch spaces of the upper half plane and studied their properties. Moreover, we have defined the composition semigroups on the generalized Bloch spaces of the upper half plane and determined their semigroup properties.

1.6 Objective of the Study

The main objective of this study was to investigate the properties of the generalized Bloch spaces of the upper half-plane, $\mathcal{B}^{\alpha}(\mathbb{U})$, as well as composition semigroups defined on them. The specific objectives were to

(i) Investigate the properties of the generalized Bloch spaces of the upper half plane.

(ii) Determine the composition semigroups on the generalized Bloch spaces of the upper half-plane.

(iii) Investigate the semigroup properties of composition semigroups determined in (ii) above.

1.7 Significance of the Study

The study of the semigroups of composition operators has wide application in applied mathematics. In particular, evolution equations arise in many disciplines of science. An abstract way to study and dissect these equations is through semigroups. For instance, solution of the heat equation is given by a semigroup. Using semigroups is advantageous as the associated theory is quite rich. Studying semigroups, as we have for this study, heightens our awareness of their prevalence throughout applied mathematics.

1.8 Research methodology

In this section, we outline the methods used to achieve the objectives of the study.

To investigate the properties of the generalized Bloch spaces of the upper half plane $\mathcal{B}^{\alpha}(\mathbb{U})$, we used Cayley transform to obtain the equivalent representations of functions on the generalized Bloch spaces of the unit disk to functions on the corresponding spaces of the upper half plane. We then considered the known properties of the Bloch space of the disc and investigated the corresponding properties of the Bloch space of the upper half plane. We then extended these properties to the setting of generalized Bloch spaces of the upper half plane.

To determine composition semigroups on the generalized Bloch spaces of the upper half plane, we used the classification theorem for the automorphisms of the upper half plane, $\mathbb{U}$ and from the definition of composition operators, we obtained three distinct composition semigroups on the generalized Bloch spaces of the upper half plane as induced by the automorphism groups.

Finally, we used the theory of semigroups of linear operators on Banach spaces to determine the semigroup properties of the obtained composition semigroups on the generalized Bloch spaces of the upper half plane.

Chapter 2

Literature Review

In 1974, J. Anderson [9] obtained various characterizations of Bloch functions on $\mathcal{B}(\mathbb{D})$, where he presented basic theory of Bloch functions with emphasis given to connections, which Bloch functions provide between function theory and harmonic analysis. In 1979, J. Cima [8] extended Anderson's work where he established that a Bloch function is an analytic function on the unit disc $\mathbb{D}$, whose derivative grows no faster than a constant times the reciprocal of the distance from a point z to the boundary of the disc. He also proved the Bloch theorem and gave equivalence conditions for a function holomophic on the Bloch space of the disc. J. Cima [8] established that bounded holomorphic functions are Bloch functions. He further proved basic analytic facts concerning Bloch functions. On the Banach space structure, J. Cima noted that a set of Bloch functions is a complex vector space which when equipped with a norm becomes a Banach space. He further proved the analytic space of bounded mean oscillation (BMOA) as a subspace of the Bloch space and collected facts concerning the geometry of the space. In 1991, K. Zhu [29] proved the Bloch space, $\mathcal{B}(\mathbb{D})$, as a Banach space with its semi norm, $\|f\|_{\mathcal{B}_1(\mathbb{D})} = \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|\,\forall f \in \mathcal{B}(\mathbb{D})$, being complete and invariant under the action of an automorphism. He proved that the little Bloch space, $\mathcal{B}_\circ(\mathbb{D})$ is a closed subspace of $\mathcal{B}(\mathbb{D})$ and that the set of polynomials is dense in $\mathcal{B}_\circ(\mathbb{D})$ [29]. The space $\mathcal{B}(\mathbb{D})$ has been studied by many authors because of its intrinsic interest since its introduction [1, 4, 14, 17, 21, 25, 29]. In 1993, K. Zhu [28] defined the generalized Bloch spaces of the open unit disc, $\mathcal{B}^\alpha(\mathbb{D})$ and proved that they are Banach spaces with respect to their norm. Zhu established generalized little Bloch spaces of the unit disc $\mathcal{B}_\circ(\mathbb{D})$, as closed, separable subspaces of $\mathcal{B}^\alpha(\mathbb{D})$. There is little literature on the properties of the generalized Bloch spaces of the upper half plane $\mathcal{B}^\alpha(\mathbb{U})$, including whether they are Banach spaces.

On the Bloch space of the unit disc, boundedness and compactness of composition operators is well captured in the literature. For instance, in 1995, Madigan and Matheson [15] gave sufficient and necessary conditions for composition operators to be compact on $\mathcal{B}(\mathbb{D})$ and the corresponding $\mathcal{B}_\circ(\mathbb{D})$. In 1997, A. Siskakis [19] initiated the study of semigroups of composition operators in the framework of analytic spaces of bounded mean oscillation (BMOA) and the Bloch space of the unit disc $\mathcal{B}(\mathbb{D})$. On strong continuity of composition semigroups, he [19] proved that no non trivial composition semigroups are strongly continuous on the Bloch space of the unit disc $\mathcal{B}(\mathbb{D})$. In 2000, Shi and Luo [20] studied composition operators on the Bloch space of several complex variables. This study [20] was then extended in the year 2001 by Ohno and Zhao [24] who examined compactness and boundedness of weighted composition operators on the Bloch space of several complex variables. In 2003, on the generalized Bloch spaces of the disc, B. Macluer [14] obtained the essential norms of composition operators between the generalized Bloch spaces of the unit disc. He [14] further obtained estimates for the essential norm of the composition operator mapping the standard Bloch space into the weighted generalized Bloch spaces of the disc. In Siskakis' review [19], he established strong continuity of composition semigroups on the little Bloch space. Further research on compactness was done in 2017 by the author in [17] who studied compact composition operators on the Bloch space and the growth space of the upper half plane. In 2019, M. Bagasa [2] studied spectral properties of semigroups of weighted composition operators on the little Bloch space $\mathcal{B}_\circ(\mathbb{D})$, which were obtained as adjoints of composition semigroups defined on the nonreflexive Bergman space using the duality relations. For a comprehensive theory of composition operators on the Bloch space, we refer the reader to monographs; [7, 17, 21, 26]. Evidently, the study of composition semigroups defined on generalized Bloch spaces of the upper half plane has not yet been exhausted. In this study therefore, we have investigated the properties of the generalized Bloch spaces of the upper half plane as Banach spaces and extended the study of semigroups of composition operators to the setting of the generalized Bloch spaces of the upper half plane. The following theorem has been useful in this study

Theorem 2.0.1 (Classification theorem for Aut($\mathbb{U}$) [3])
Let $\varphi : \mathbb{R} \longrightarrow Aut(\mathbb{U})$ be a nontrivial continuous group homomorphism. Then exactly one of the following cases holds:

1. There exists $k > 0$, $k \neq 1$, and $g \in \text{Aut}(\mathbb{U})$ so that $\varphi_t(z) = g^{-1}(k^t g(z))$ for all $z \in \mathbb{U}$ and $t \in \mathbb{R}$.

2. There exists $k \in \mathbb{R}$, $k \neq 0$, and $g \in \text{Aut}(\mathbb{U})$ so that $\varphi_t(z) = g^{-1}(g(z) + kt)$ for all $z \in \mathbb{U}$ and $t \in \mathbb{R}$.

3. There exists $k \in \mathbb{R}$, $k \neq 0$, and a conformal mapping g of $\mathbb{U}$ onto $\mathbb{D}$ such that $\varphi_t(z) = g^{-1}(e^{ikt} g(z))$ for all $z \in \mathbb{U}$ and $t \in \mathbb{R}$.

Chapter 3

Generalized Bloch spaces of the upper half plane

In this chapter, we study properties of the generalized Bloch spaces as Banach spaces. We also relate functions in the generalized Bloch space of the upper half plane $\mathbb{U}$ to their counterparts in the unit disc $\mathbb{D}$, but first, we state the following propositions that are readily available in literature.

Proposition 3.0.1 ([29])

$\mathcal{B}^{\alpha}(\mathbb{D})$ is a Banach space with respect to the norm $\|.\|_{\mathcal{B}^{\alpha}(\mathbb{D})}$

Proposition 3.0.2 ([29])

$\mathcal{B}_0^{\alpha}(\mathbb{D})$ is a Banach space with respect to the norm $\|.\|_{\mathcal{B}^{\alpha}(\mathbb{D})}$

In the next proposition, we state density of polynomials in $\mathcal{B}^{\alpha}(\mathbb{D})$.

Proposition 3.0.3 ([28])

The set of analytic polynomials $\mathbb{C}[z] := \left\{ \sum_{n=0}^{\infty} a_n z^n \ : z \in \mathbb{C} \right\}$ is dense in $\mathcal{B}_0^{\alpha}(\mathbb{D})$.

In the following theorem, we establish the completeness of $\mathcal{B}^{\alpha}(\mathbb{U})$ with respect to the norm $\|.\|_{\mathcal{B}^{\alpha}(\mathbb{U})}$.

Theorem 3.0.4

$\mathcal{B}^{\alpha}(\mathbb{U})$ *is a Banach space with respect to the norm* $\|.\|_{\mathcal{B}^{\alpha}(\mathbb{U})}$

Proof. We first note that $\mathcal{B}^{\alpha}(\mathbb{U})$ is a vector space under the pointwise operations given as: For $f, g \in \mathcal{B}^{\alpha}(\mathbb{U})$ and $\lambda \in \mathbb{C}$, we have for every $z \in \mathbb{U}$

$$(f+g)(z) = f(z) + g(z)$$

and

$$(\lambda f) = \lambda f(z).$$

To check that $\mathcal{B}^{\alpha}(\mathbb{U})$ is a normed space, we need to verify that the definition of $\|.\|_{\mathcal{B}^{\alpha}(\mathbb{U})}$ indeed defines a norm on $\mathcal{B}^{\alpha}(\mathbb{U})$. We note that, $\forall f \in \mathcal{B}^{\alpha}(\mathbb{U})$, $\|f\|_{\mathcal{B}^{\alpha}(\mathbb{U})} > 0$ since $|f(i)| > 0$, $\Im(\omega) > 0$ and $|f'(\omega)| > 0$ by definition for $f \neq 0$.
Moreover,

$$\begin{aligned} \|f\|_{\mathcal{B}^{\alpha}(\mathbb{U})} = 0 \quad &\Leftrightarrow \quad |f(i)| + \|f\|_{\mathcal{B}_1^{\alpha}(\mathbb{U})} = 0 \\ &\Leftrightarrow \quad \left(|f(i)| + \sup_{\omega \in \mathbb{U}} \Im(\omega)^{\alpha} |f'(\omega)| \right) = 0, \end{aligned}$$

which is equivalent to

$$|f(i)| = 0 \text{ and } \sup_{\omega \in \mathbb{U}} \Im(\omega)^{\alpha} |f'(\omega)| = 0.$$

Since $\Im(\omega) > 0$ and f is holomorphic we have

$$|f'(\omega)| = 0 \Leftrightarrow f \text{ is a constant.}$$

Now $|f(i)| = 0 \Leftrightarrow f(i) = 0$. Hence $f = 0$, as desired.

Now, for $\lambda \in \mathbb{C}$ and $f \in \mathcal{B}^{\alpha}(\mathbb{U})$, we have

$$\begin{aligned}
\|\lambda f\|_{\mathcal{B}^{\alpha}(\mathbb{U})} &= |(\lambda f)(i)| + \|\lambda f\|_{\mathcal{B}^{\alpha}_{1}(\mathbb{U})} \\
&= |\lambda f(i)| + \sup_{\omega \in \mathbb{U}} \Im(\omega)^{\alpha} |(\lambda f)'(\omega)| \\
&= |\lambda||f(i)| + \sup_{\omega \in \mathbb{U}} \Im(\omega)^{\alpha} |\lambda f'(\omega)| \\
&= |\lambda||f(i)| + \sup_{\omega \in \mathbb{U}} \Im(\omega)^{\alpha} |\lambda||f'(\omega)| \\
&= |\lambda| \left(|f(i)| + \sup_{\omega \in \mathbb{U}} \Im(\omega)^{\alpha} |f'(\omega)| \right) \\
&= |\lambda| \|f\|_{\mathcal{B}^{\alpha}(\mathbb{U})}, \quad \text{as desired.}
\end{aligned}$$

Finally, for the triangle inequality property of the norm, we have for $f, g \in \mathcal{B}^{\alpha}(\mathbb{U})$,

$$\begin{aligned}
\|f+g\|_{\mathcal{B}^{\alpha}(\mathbb{U})} &= |(f+g)(i)| + \|f+g\|_{\mathcal{B}^{\alpha}_{1}(\mathbb{U})} \\
&= |f(i)+g(i)| + \sup_{\omega \in \mathbb{U}} \Im(\omega)^{\alpha} |(f+g)'(\omega)| \\
&= |f(i)+g(i)| + \sup_{\omega \in \mathbb{U}} \Im(\omega)^{\alpha} |f'(\omega)+g'(\omega)| \\
&\leq |f(i)| + |g(i)| + \sup_{\omega \in \mathbb{U}} \Im(\omega)^{\alpha} (|f'(\omega)| + |g'(\omega)|) \\
&\leq |f(i)| + |g(i)| + \sup_{\omega \in \mathbb{U}} \Im(\omega)^{\alpha} |f'(\omega)| + \sup_{\omega \in \mathbb{U}} \Im(\omega)^{\alpha} |g'(\omega)| \\
&= \left(|f(i)| + \sup_{\omega \in \mathbb{U}} \Im(\omega)^{\alpha} |f'(\omega)| \right) + \left(|g(i)| + \sup_{\omega \in \mathbb{U}} \Im(\omega)^{\alpha} |g'(\omega)| \right) \\
&= \|f\|_{\mathcal{B}^{\alpha}(\mathbb{U})} + \|g\|_{\mathcal{B}^{\alpha}(\mathbb{U})}.
\end{aligned}$$

Therefore, $\left(\mathcal{B}^{\alpha}(\mathbb{U}), \|.\|_{\mathcal{B}^{\alpha}(\mathbb{U})}\right)$ is a normed space.
Next, we prove that the space $\mathcal{B}^{\alpha}(\mathbb{U})$ is complete in $\|.\|_{\mathcal{B}^{\alpha}(\mathbb{U})}$.
Let $(f_k)_k$ denote a Cauchy sequence in $\mathcal{B}^{\alpha}(\mathbb{U})$. For $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|f_k - f_l\|_{\mathcal{B}^{\alpha}(\mathbb{U})} < \epsilon, \ \forall k, l > N$. Hence by the definition

of the norm, we have for all $\forall\, k, l > N$,

$$|f_k(i) - f_l(i)| + \sup_{\omega \in \mathbb{U}} \Im\,(\omega)^\alpha\,|f_k'(\omega) - f_l'(\omega)| < \epsilon,$$

which means that

$$|f_k(i) - f_l(i)| < \epsilon \ \text{ and } \ (\Im(\omega))^\alpha\,|f_k'(\omega) - f_l'(\omega)| < \epsilon,$$

for $\omega \in \mathbb{U}$. So, $(f_k(i))_{k\in\mathbb{N}}$ is Cauchy in $\mathbb{C}$. By the completeness of $\mathbb{C}$, $(f_k(i))_k$ converges to a limit, say u_0. Similarly, $(f_k'(\omega))_{k\in\mathbb{N}}$ is Cauchy in $\mathbb{C}$ and therefore converges to a limit, say g.
Since $|f_k'(\omega) - f_l'(\omega)| < \frac{\epsilon}{\Im(\omega)^\alpha}$ and $f_k'(\omega) \to g$ uniformly on compact subsets of $\mathbb{U}$, then $g \in \mathcal{H}(\mathbb{U})$.
Now, take f such that $f'(\omega) = g(\omega)\,\forall\,\omega\, \in \mathbb{U}$ and $f(i) = u_0$.
Thus, $\forall\,\epsilon > 0$, $\exists\, N$ such that

$$\Im\,(\omega)^\alpha\,|f_k'(\omega) - f_l'(\omega)| < \epsilon,\ \forall\,\omega\, \in \mathbb{U}.$$

Taking limits as $l \to \infty$, we obtain

$$\Im\,(\omega)^\alpha\,|f_k'(\omega) - f'(\omega)| < \epsilon,\ \forall\,\omega\, \in \mathbb{U}.$$

It follows that

$$\|f_k - f\|_{\mathcal{B}^\alpha(\mathbb{U})} = |f_k(i) - f(i)| + \sup_{\omega\in\mathbb{U}} \Im\,(\omega)^\alpha\,|f_k'(\omega) - f'(\omega)| < \epsilon$$

and so $\|f_k - f\|_{\mathcal{B}^\alpha(\mathbb{U})} \to 0\,\text{as}\,k \to \infty$.

Now, it remains to show that $f \in \mathcal{B}^{\alpha}(\mathbb{U})$. We have

$$\begin{aligned} \Im(\omega)^{\alpha}|f'(\omega)| &= \Im(\omega)^{\alpha}|f'(\omega) - f'_k(\omega) + f'_k(\omega)| \\ &\leq \Im(\omega)^{\alpha}|f'(\omega) - f'_k\omega| + \Im(\omega)^{\alpha}|f'_k(\omega)| \\ &< \epsilon + \Im(\omega)^{\alpha}|f'_k(\omega)| < \infty \end{aligned}$$

since $(f_k)_k \subset \mathcal{B}^{\alpha}(\mathbb{U})$.

Now, taking supremum over all $\omega \in \mathbb{U}$ in the above equation, we have that

$$\sup_{\omega\in\mathbb{U}} \Im(\omega)^{\alpha}|f'(\omega)| < \infty$$

which implies that $f \in \mathcal{B}^{\alpha}(\mathbb{U})$, as desired. □

As an immediate consequence, we have

Corollary 3.0.5

$\mathcal{B}(\mathbb{U})$ is a Banach space with respect to the norm $\| \, . \, \|_{\mathcal{B}(\mathbb{U})}$

Proof. Follows immediately by taking $\alpha = 1$ in Theorem (**??**). □

Under the norm $\| \, . \, \|_{\mathcal{B}^{\alpha}(\mathbb{U})}$, the space $\mathcal{B}_0^{\alpha}(\mathbb{U})$ also becomes a Banach space as in the following theorem,

Theorem 3.0.6

$\mathcal{B}_0^{\alpha}(\mathbb{U})$ is a Banach space with respect to the norm $\| \, . \, \|_{\mathcal{B}^{\alpha}(\mathbb{U})}$.

Proof. Following Theorem (**??**), we need to show that every sequence in $\mathcal{B}_0^{\alpha}(\mathbb{U})$ convergent in $\mathcal{B}^{\alpha}(\mathbb{U})$ has its limit in $\mathcal{B}_0^{\alpha}(\mathbb{U})$.

Let $(f_n) \subset \mathcal{B}_0^{\alpha}(\mathbb{U})$ and $g \in \mathcal{B}^{\alpha}(\mathbb{U})$ be such that $f_n \to g$ as $n \to \infty$. We need to prove that $g \in \mathcal{B}_0^{\alpha}(\mathbb{U})$. Since f_n, g are holomorphic on compact subsets of $\mathbb{U}$, and $f_n \to g$, we have $f'_n \to g'$ uniformly. Now that $f_n \subset \mathcal{B}_0^{\alpha}(\mathbb{U})$, we have

$$\lim_{\Im(\omega)\to 0} (\Im(\omega))^{\alpha} |f_n'(\omega)| = 0, \forall\, n. \tag{3.1}$$

Since $\lim_{n\to\infty} f_n' = g'$, we have

$$\lim_{\Im(\omega)\to 0} (\Im(\omega))^{\alpha} |g'(\omega)| = \lim_{\Im(\omega)\to 0} (\Im(\omega))^{\alpha} |\lim_{n\to\infty} f_n'(\omega)|$$

which is equivalent to

$$\lim_{\Im(\omega)\to 0} (\Im(\omega))^{\alpha} |g'(\omega)| = \lim_{n\to\infty} \left(\lim_{\Im(\omega)\to 0} (\Im(\omega))^{\alpha} |f_n'(\omega)| \right).$$

Following equation (3.1), we see that

$$\lim_{\Im(\omega)\to 0} (\Im(\omega))^{\alpha} |g'(\omega)| = 0.$$

So, $g \in \mathcal{B}_0^{\alpha}(\mathbb{U})$, completing the proof. □

As a consequence, we have the following,

Corollary 3.0.7

$\mathcal{B}_0(\mathbb{U})$ is a Banach space with respect to the norm $\|.\|_{\mathcal{B}^{\alpha}(\mathbb{U})}$

PROOF. Follows immediately by taking $\alpha = 1$ in Theorem (**??**). □

In the next results, we generate a relationship between functions in the generalized Bloch space of the upper half plane $\mathbb{U}$ and their counterparts in the unit disc $\mathbb{D}$

Proposition 3.0.8

Let $f \in \mathcal{B}^{\alpha}(\mathbb{U})$ and ψ be the Cayley transform, then

$$\|f\|_{\mathcal{B}_1^{\alpha}(\mathbb{U})} = \frac{1}{2^{\alpha}} |\psi'(z)^{\alpha-1}| \|f \circ \psi\|_{\mathcal{B}_1^{\alpha}(\mathbb{D})}.$$

Proof. Let f be a function in $\mathcal{B}^\alpha(\mathbb{U})$. Then by definition,

$$\|f\|_{\mathcal{B}_1^\alpha(\mathbb{U})} = \sup_{\omega\in\mathbb{U}} \Im(\omega)^\alpha |f'(\omega)| < \infty.$$

Now, by changing variables, let $\omega = \psi(z)$, where ψ is the Cayley transform. Then

$$\begin{aligned} \Im(\omega) &= \frac{\omega - \overline{\omega}}{2i} \\ &= \frac{\psi(z) - \overline{\psi(z)}}{2i}. \end{aligned}$$

Using $\psi(z) = \frac{i(1+z)}{1-z}$ and $\overline{\psi(z)} = \frac{-i(1+\overline{z})}{1-\overline{z}}$, we have

$$\begin{aligned} \Im(\omega) &= \frac{\frac{i(1+z)}{1-z} - \frac{-i(1+\overline{z})}{1-\overline{z}}}{2i} \\ &= \frac{i(1+z)(1-\overline{z}) + i(1+\overline{z})(1-z)}{2i(1-z)(1-\overline{z})} \\ &= \frac{i(2-2\overline{z}z)}{2i(1-z)(1-\overline{z})} \\ &= \frac{1-|z|^2}{|1-z|^2}. \end{aligned}$$

We get the absolute of $\psi'(z) = \frac{2i}{(1-z)^2}$ as

$$|\psi'(z)| = \frac{2}{|1-z|^2}. \tag{3.2}$$

Now, by definition we have

$$\|f\|_{\mathcal{B}_1^\alpha(\mathbb{U})} = \sup_{z\in\mathbb{D}} \left(\frac{1-|z|^2}{|1-z|^2}\right)^\alpha |f'(\psi(z))|.$$

From equation (3.2), we have $|1-z|^2 = \frac{2}{|\psi'(z)|}$, therefore

$$\|f\|_{\mathcal{B}_1^\alpha(\mathbb{U})} = \frac{1}{2^\alpha}\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|\psi'(z)|^\alpha|f'(\psi(z))|.$$

Since, $(f\circ\psi)(z)' = f'(\psi(z))\psi'(z)$, we have
$|\psi'(z)|^\alpha|f'(\psi(z))| = |(f\circ\psi)'(z)\psi'(z)^{\alpha-1}|$ and hence

$$\begin{aligned}\|f\|_{\mathcal{B}_1^\alpha(\mathbb{U})} &= \frac{1}{2^\alpha}\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|(f\circ\psi)'(z)\psi'(z)^{\alpha-1}|.\\ &= \frac{1}{2^\alpha}|\psi'(z)^{\alpha-1}|\|f\circ\psi\|_{\mathcal{B}_1^\alpha(\mathbb{D})}, \text{ as desired.}\end{aligned}$$

□

An immediate consequence is the following,

Corollary 3.0.9

Let $f\in\mathcal{B}(\mathbb{U})$ and ψ be the Cayley transform, then

$$\|f\|_{\mathcal{B}_1(\mathbb{U})} = \frac{1}{2}\|f\circ\psi\|_{\mathcal{B}_1(\mathbb{D})} \tag{3.3}$$

In particular, a function $f\in\mathcal{B}(\mathbb{U})$ if and only if $f\circ\psi\in\mathcal{B}(\mathbb{D})$.

Proof. From Theorem (**??**), we have that for $\alpha = 1$,

$$\|f\|_{\mathcal{B}_1(\mathbb{U})} = \frac{1}{2}\|f\circ\psi\|_{\mathcal{B}_1(\mathbb{D})}.$$

But from the definition, $f\in\mathcal{B}(\mathbb{U})$ if and only if $\|f\|_{\mathcal{B}_1(\mathbb{U})} < \infty$. Equivalently, by equation (3.3) we conclude that $f\in\mathcal{B}(\mathbb{U})$ if and only if $\|f\circ\psi\|_{\mathcal{B}_1(\mathbb{D})} < \infty$, which completes the proof. □

Another result that relates functions in the generalized little Bloch space of the upper half plane and their counterparts in the unit disc $\mathbb{D}$ is the following

Proposition 3.0.10

A function $f \in \mathcal{B}_0^\alpha(\mathbb{U})$ *if and only if* $f \circ \psi \in \mathcal{B}_0^\alpha(\mathbb{D})$

Proof. From the definition of $\mathcal{B}_0^\alpha(\mathbb{U})$, we have

$$f \in \mathcal{B}_0^\alpha(\mathbb{U}) \Leftrightarrow \lim_{\Im(\omega)\to 0} (\Im(\omega))^\alpha \, |f'(\omega)| = 0.$$

From equation (3.3) and (3.2), we have

$$\Im(\omega) = \frac{1-|z|^2}{|1-z|^2} \text{ and } |\psi'(z)| = \frac{2}{|1-z|^2},$$

respectively.

Now, by changing variables, let $\omega = \psi(z)$, so that we obtain

$$\lim_{|z|\to 1} \left(\frac{1-|z|^2}{|1-z|^2}\right)^\alpha |f'(\psi(z))| = 0. \tag{3.4}$$

Substituting $|1-z|^2 = \frac{2}{|\psi'(z)|}$ in equation (3.4), we have

$$\lim_{|z|\to 1} (1-|z|^2)^\alpha |f'(\psi(z))| \frac{|\psi'(z)|^\alpha}{2^\alpha} = 0.$$

Simplifying and rearranging, we get

$$\frac{1}{2^\alpha} \lim_{|z|\to 1} (1-|z|^2)^\alpha |\psi'(z)|^{\alpha-1} |(f\circ\psi)'(z)| = 0.$$

Since $|\psi'(z)|^{\alpha-1}$ does not converge to 0 as $|z| \to 1$, it follows that

$$\lim_{|z|\to 1} (1-|z|^2)^\alpha |(f\circ\psi)'(z)| = 0.$$

Equivalently, $f \circ \psi \in \mathcal{B}_0^\alpha(\mathbb{D})$, and hence $f \in \mathcal{B}_0^\alpha(\mathbb{U}) \Leftrightarrow f \circ \psi \in \mathcal{B}_0^\alpha(\mathbb{D})$, as claimed. $\square$

As a consequence, we give the following result

Corollary 3.0.11

A function $f \in \mathcal{B}_0(\mathbb{U})$ if and only if $f \circ \psi \in \mathcal{B}_0(\mathbb{D})$.

PROOF. Follows immediately by taking $\alpha = 1$ in Proposition 3.0.10. $\square$

Chapter 4

Composition semigroups on the generalized little Bloch space of the upper half plane

4.1 Introduction

Following Theorem 2.0.1, the non trivial automorphisms of the upper half plane $\mathbb{U}$ were classified according to the location of their fixed points into three distinct classes namely; scaling, translation and rotation groups. In this chapter, we determine composition semigroups induced by these automorphism groups of the upper half plane $\mathbb{U}$, on the generalized Bloch space of the upper half plane $\mathcal{B}^{\alpha}(\mathbb{U})$. We then employ the theory of linear operators on Banach spaces to investigate the semigroup properties of the induced composition semigroup. For any given semigroup φ_t, the induced operator semigroup C_{φ_t} is known to be strongly continuous on the little Bloch space. On the other hand, no non trivial composition semigroup is strongly continuous on the big Bloch space. See [18]. Therefore, we shall determine the composition semigroup induced by these automorphism groups on the generalized little Bloch space of the upper half plane, $\mathcal{B}_0^\alpha(\mathbb{U})$. In particular, in section 4.1 and 4.2, we show that composition semigroups induced by scaling and translation groups respectively, are strongly continuous on $\mathcal{B}_0^\alpha(\mathbb{U})$. In section 4.3, we determine composition semigroups induced by rotation group on $\mathcal{B}_0^\alpha(\mathbb{D})$. The infinitesimal generator is identified and its domain stated. We start by defining the scaling group in the next section.

4.2 Scaling group

The automorphisms of this group are of the form $\varphi_t(z) = k^t z$, where $z \in \mathbb{U}$ and $k, t \in \mathbb{R}$ with $k \neq 0$. As noted in [3], the semigroup properties of the induced composition operators will differ significantly depending on whether $0 < k < 1$ or $k > 1$. Thus for $0 < k < 1$, we consider without loss of generality, the analytic self maps $\varphi_t : \mathbb{U} \longrightarrow \mathbb{U}$ of the form

$$\varphi_t(z) = e^{-t} z, \; z \in \mathbb{U}. \tag{4.1}$$

The composition semigroup induced by (4.1) on $\mathcal{B}_0^\alpha(\mathbb{U})$ is given by

$$\begin{aligned} C_{\varphi_t} f(z) &= (f \circ \varphi_t)(z) \\ &= f\left(e^{-t} z\right). \end{aligned} \tag{4.2}$$

In the following proposition, we prove that (4.2) defines a group on $\mathcal{B}_0^\alpha(\mathbb{U})$.

Proposition 4.2.1

$(C_{\varphi_t})_{t\in\mathbb{R}}$ *is a group on* $\mathcal{B}_0^\alpha(\mathbb{U})$.

PROOF. It suffices to show that both $(C_{\varphi_t})_{t\geq 0}$ and $\left(C_{\varphi_{-t}}\right)_{t\geq 0}$ are semigroups on $\mathcal{B}_0^\alpha(\mathbb{U})$.

Indeed $C_{\varphi_0} = I$ and for every $f \in \mathcal{B}_0^\alpha(\mathbb{U})$, we have

$$\begin{aligned} C_{\varphi_t} \circ C_{\varphi_s} f(z) &= C_{\varphi_t}\left(C_{\varphi_s} f(z)\right) \qquad (4.3) \\ &= C_{\varphi_t} f(\varphi_s(z)) \\ &= f\left(\varphi_t(\varphi_s(z))\right) \\ &= f\left(\varphi_t(e^{-s}z)\right) \\ &= f\left(e^{-t}e^{-s}z\right) \\ &= f\left(e^{-(t+s)}z\right) \\ &= C_{\varphi_{t+s}} f(z), \end{aligned}$$

as desired. Therefore $(C_{\varphi_t})_{t\geq 0}$ is a semigroup on $\mathcal{B}_0^\alpha(\mathbb{U})$. Similarly, it can be shown that $\left(C_{\varphi_{-t}}\right)_{t\geq 0}$ is also a semigroup on $\mathcal{B}_0^\alpha(\mathbb{U})$. Thus $(C_{\varphi_t})_{t\in\mathbb{R}}$ is a group on $\mathcal{B}_0^\alpha(\mathbb{U})$. $\square$

In what follows, we prove that the composition semigroup given by (4.2) fails to be an isometry on $\mathcal{B}_0^\alpha(\mathbb{U})$.

Proposition 4.2.2

The operator C_{φ_t} *fails to be an isometry on* $\mathcal{B}_0^\alpha(\mathbb{U})$

PROOF. By the definition of the norm, we have for all $f \in \mathcal{B}_0^\alpha(\mathbb{U})$

$$\begin{aligned} \|C_{\varphi_t} f\|_{\mathcal{B}^\alpha(\mathbb{U})} &= |C_{\varphi_t} f(i)| + \sup_{\omega\in\mathbb{U}} \Im(\omega)^\alpha \left|\left(C_{\varphi_t} f\right)'(\omega)\right| \\ &= |f(e^{-t}i)| + \sup_{\omega\in\mathbb{U}} \Im(\omega)^\alpha |e^{-t} f'(e^{-t}\omega)|. \end{aligned}$$

Now by change of variables:

Let $z = e^{-t}\omega$, then $\omega = e^{t}z$, and $\Im(\omega) = e^{t}\Im(z)$. Therefore,

$$\begin{aligned}
\|C_{\varphi_t}f\|_{\mathcal{B}^\alpha(\mathbb{U})} &= |f(e^{-t}i)| + \sup_{z\in\mathbb{U}} e^{t\alpha}\Im(z)^\alpha|e^{-t}f'(z)| \\
&= |f(e^{-t}i)| + e^{(\alpha-1)t}\sup_{z\in\mathbb{U}} \Im(z)^\alpha|f'(z)| \\
&\neq |f(i)| + \sup_{z\in\mathbb{U}} \Im(z)^\alpha|f'(z)| = \|f\|_{\mathcal{B}^\alpha(\mathbb{U})},
\end{aligned}$$

which completes the proof. □

Next, we prove that the operator C_{φ_t} given by (4.2) is strongly continuous on $\mathcal{B}_0^\alpha(\mathbb{U})$.

Theorem 4.2.3

$(C_{\varphi_t})_{t\in\mathbb{R}}$ *is strongly continuous on* $\mathcal{B}_0^\alpha(\mathbb{U})$.

Proof. To prove strong continuity of $(C_{\varphi_t})_{t\in\mathbb{R}}$, it suffices to show that $\|C_{\varphi_t}f - f\|_{\mathcal{B}^\alpha(\mathbb{U})} \to 0\,\text{as}\,t \to 0$. That is, $|\left(C_{\varphi_t}f - f\right)(i)| + \|C_{\varphi_t}f - f\|_{\mathcal{B}_1^\alpha(\mathbb{U})} \to 0\,\text{as}\,t \to 0$. This is equivalent to $|\left(C_{\varphi_t}f - f\right)(i)| \to 0$ and $\|C_{\varphi_t}f - f\|_{\mathcal{B}_1^\alpha(\mathbb{U})} \to 0,\ \text{as}\,t \to 0$. For the former, we have

$$\begin{aligned}
|\left(C_{\varphi_t}f - f\right)(i)| &= |C_{\varphi_t}f(i) - f(i)| \qquad (4.4)\\
&= |f(\varphi_t(i)) - f(i)| \\
&= |f(e^{-t}i) - f(i)| \to 0\,\text{as}\,t \to 0,
\end{aligned}$$

as desired. We now prove that $\|C_{\varphi_t}f - f\|_{\mathcal{B}_1^\alpha(\mathbb{U})} \to 0\,\text{as}\,t \to 0$. Recall that $\psi : \mathbb{D} \to \mathbb{U}$, $\varphi_t : \mathbb{U} \to \mathbb{U}\,\text{and}\,\psi^{-1} : \mathbb{U} \to \mathbb{D}$. We can therefore have $\mathbb{D} \xrightarrow{\psi} \mathbb{U} \xrightarrow{\varphi_t} \mathbb{U} \xrightarrow{\psi^{-1}} \mathbb{D}$. Now, let $\mathcal{X}_t = \psi^{-1}\circ\varphi_t\circ\psi : \mathbb{D} \to \mathbb{D}$. If $(\varphi_t)_{t\geq 0}$ is an automorphism of the upper half plane $\mathbb{U}$, then $(\mathcal{X}_t)_{t\geq 0}$ is an automorphism of the unit disc $\mathbb{D}$. Since $\mathcal{X}_t = \psi^{-1}\circ\varphi_t\circ\psi$, it follows that

$\|C_{\varphi_t} f - f\|_{\mathcal{B}_1^\alpha(\mathbb{U})} \to 0 \,\text{as}\, t \to 0$ if and only if $\|C_{\mathcal{X}_t} f^* - f^*\|_{\mathcal{B}^\alpha(\mathbb{D})} \to 0 \,\text{as}\, t \to 0$ Cayley transform is given by $\psi(z) = \frac{i(1+z)}{1-z}$. We therefore have

$$\begin{aligned}
\psi^{-1} \circ \varphi_{-t} \circ \psi(z) &= \psi^{-1}\left(\varphi_t\left(\psi(z)\right)\right). \\
&= \psi^{-1}\left(\varphi_t\left(\frac{i(1+z)}{1-z}\right)\right) \\
&= \psi^{-1}\left(e^{-t}\left(\frac{i(1+z)}{1-z}\right)\right).
\end{aligned}$$

Substituting $\psi^{-1}(z) = \frac{z-i}{z+i}$, we obtain

$$\psi^{-1} \circ \varphi_{-t} \circ \psi(z) = \frac{e^{-t}(\frac{i(1+z)}{1-z}) - i}{e^{-t}(\frac{i(1+z)}{1-z}) + i}.$$

Simplifying the fraction, we have

$$\begin{aligned}
\psi^{-1} \circ \varphi_{-t} \circ \psi(z) &= \frac{e^{-t}\left(i(1+z)\right) - i(1-z)}{e^{-t}\left(i(1+z)\right) + i(1-z)}. \\
&= \frac{e^{-t}\left(-(1+z)\right) + (1-z)}{e^{-t}\left(-(1+z)\right) - (1-z)}. \\
&= \frac{e^{-t}(1+z) - (1-z)}{e^{-t}(1+z) + (1-z)}. \\
&= \frac{z + e^{-t}z - 1 + e^{-t}}{-z + e^{-t}z + 1 + e^{-t}}.
\end{aligned}$$

Now, by factorizing z and dividing both the numerator and denominator by $(1 + e^{-t})$, we obtain

$$\psi^{-1} \circ \varphi_{-t} \circ \psi(z) = \frac{z - \frac{(1-e^{-t})}{(1+e^{-t})}}{1 - \frac{(1-e^{-t})}{1+e^{-t}} z}.$$

Let $b_t = \frac{1-e^{-t}}{1+e^{-t}}$, and substitute to obtain

$$\begin{aligned} \psi^{-1} \circ \varphi_{-t} \circ \psi(z) &= \frac{z - b_t}{1 - b_t z} \\ &:= \mathcal{X}_t(z). \end{aligned}$$

Next, we apply density of polynomials in $\mathcal{B}_0^\alpha(\mathbb{D})$ to prove that for $f^* \in \mathcal{B}_0^\alpha(\mathbb{D})$, we have $\|C_{\mathcal{X}t} f^* - f^*\|_{\mathcal{B}_1^\alpha(\mathbb{D})} \to 0$ as $t \to 0$.

By the definition of the norm, we have

$$\lim_{t\to 0^+} \|C_{\mathcal{X}t} f^* - f^*\|_{\mathcal{B}^\alpha(\mathbb{D})} = \lim_{t\to 0^+} \left(|(C_{\mathcal{X}t} f^* - f^*)(0)| + \sup_{z\in\mathbb{D}} \left(1 - |z|^2\right)^\alpha |(C_{\mathcal{X}t} f^* - f^*)'(z)| \right).$$

Let $f^*(z) = z^n$ and $z \in \mathbb{D}$.

We need to show that $\| \left(C_{\mathcal{X}_t} f^* - f^*\right) \|_{\mathcal{B}_1^\alpha(\mathbb{D})} \to 0, \text{ as} \, t \to 0.$

Since

$$C_{\mathcal{X}_t} z^n - z^n = (\mathcal{X}_t(z))^n - z^n, n \geq 1,$$

differentiating $(\mathcal{X}_t(z))^n - z^n$ with respect to z, we obtain

$$\begin{aligned} (C_{\mathcal{X}_t} f^* - f^*)'(z) &= n(\mathcal{X}_t(z))^{n-1} \mathcal{X}_t'(z) - n z^{n-1} \\ &= n[(\mathcal{X}_t(z))^{n-1} \mathcal{X}_t'(z) - z^{n-1}]. \end{aligned}$$

Substituting for

$$\mathcal{X}_t(z) = \frac{z - b_t}{1 - b_t z}$$

and

$$
\begin{aligned}
\mathcal{X}_t'(z) &= \frac{(1-b_tz)1-(z-b_t)(-b_t)}{(1-b_tz)^2} \\
&= \frac{(1-b_t^2)}{(1-b_tz)^2},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
(C_{\mathcal{X}_t}f^*-f^*)'(z) &= n\left[\left(\frac{z-b_t}{1-b_tz}\right)^{n-1}\frac{(1-b_t^2)}{(1-b_tz)^2}-z^{n-1}\right] \\
&= n\left[\frac{(z-b_t)^{n-1}(1-b_t^2)}{(1-b_tz)^{n-1}(1-b_tz)^2}-z^{n-1}\right] \\
&= n\left[\frac{(z-b_t)^{n-1}(1-b_t^2)-z^{n-1}(1-b_tz)^{n+1}}{(1-b_tz)^{n+1}}\right]
\end{aligned}
$$

It therefore follows that $\lim_{t\to 0^+}\|C_{\mathcal{X}t}f^*-f^*\|_{\mathcal{B}_1^\alpha(\mathbb{D})}$ is equivalent to

$$
\lim_{t\to 0^+}\left((\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha\left|n\left[\frac{(z-b_t)^{n-1}(1-b_t^2)-z^{n-1}(1-b_tz)^{n+1}}{(1-b_tz)^{n+1}}\right]\right|\right).
$$

Now, let $b_t\to 0$ as $t\to 0$, we obtain

$$
\begin{aligned}
\lim_{t\to 0^+}\|C_{\mathcal{X}t}f^*-f^*\|_{\mathcal{B}_1^\alpha(\mathbb{D})} &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha\left|n[z^{n-1}-z^{n-1}]\right| \\
&= 0.
\end{aligned}
$$

Since $\lim_{t\to 0^+}\|(C_{\mathcal{X}_t}f^*-f^*\|_{\mathcal{B}_1^\alpha(\mathbb{D})}=0$, it follows that

$$
\lim_{t\to 0^+}\left(\|C_{\varphi_t}f-f\|_{\mathcal{B}_1^\alpha(\mathbb{U})}\right)=0.
$$

Therefore $\|C_{\varphi_t}f-f\|_{\mathcal{B}^\alpha(\mathbb{U})}=|\varphi_tf(i))-f(i)|+\|C_{\varphi_t}f-f\|_{\mathcal{B}_1^\alpha(\mathbb{U})}\to 0$ as t $\to$ 0, as desired. □

In the next proposition, we compute the infinitesimal generator and determine the domain of the composition semigroup in equation (4.2).

Proposition 4.2.4

The infinitesimal generator Γ *of* $(C_{\varphi_t})_{t\geq 0}$ *on* $\mathcal{B}_0^\alpha(\mathbb{U})$ *is given by* $\Gamma f(z) = -zf'(z)$ *with the domain dom* $(\Gamma) = \{f \in \mathcal{B}_0^\alpha(\mathbb{U}) : zf'(z) \in \mathcal{B}_0^\alpha(\mathbb{U})\}$.

Proof. Using the definition of the infinitesimal generator Γ of $(C_{\varphi_t})_{t\geq 0}$, for $f \in \mathcal{B}_0^\alpha(\mathbb{U})$ we have

$$\begin{aligned}
\Gamma f(z) &= \lim_{t\to 0^+} \frac{C_{\varphi_t} f(z) - f(z)}{t} \\
&= \lim_{t\to 0^+} \frac{f(\varphi_t(z)) - f(z)}{t} \\
&= \lim_{t\to 0^+} \frac{f(e^{-t}z) - f(z)}{t} \\
&= \frac{\partial}{\partial t} f(e^{-t}z)\Big|_{t=0} \\
&= -e^{-t} z f'(e^{-t}z)\big|_{t=0} \\
&= -zf'(z).
\end{aligned}$$

This implies that $\Gamma f(z) = -zf'(z)$ and therefore $\mathrm{dom}(\Gamma) \subseteq \{f \in \mathcal{B}_0^\alpha(\mathbb{U}) : zf' \in \mathcal{B}_0^\alpha(\mathbb{U})\}$. To prove reverse inclusion, we let $f \in \mathcal{B}_0^\alpha(\mathbb{U})$ be such that

$zf' \in \mathcal{B}_0^\alpha(\mathbb{U})$. Then for $z \in \mathbb{U}$,

$$\begin{aligned}
\frac{C_{\varphi_t}f(z) - f(z)}{t} &= \frac{1}{t}\int_0^t \frac{\partial}{\partial s}(C_{\varphi_s}f(z))ds \\
&= \frac{1}{t}\int_0^t \frac{\partial}{\partial s}(f(e^{-s}z))ds \\
&= \frac{1}{t}\int_0^t (-e^{-s}zf'(e^{-s}z))ds \\
&= \frac{1}{t}\int_0^t -e^{-s}zf'(e^{-s}z))ds \\
&= \frac{1}{t}\int_0^t C_{\varphi_s}F(z)ds, \text{ where } F(z) = -zf'(z).
\end{aligned}$$

Since $F(z)$ is a function in $\mathcal{B}_0^\alpha(\mathbb{U})$, it remains to show that the limit of $F(z)$ exist in $\mathcal{B}_0^\alpha(\mathbb{U})$. Thus

$$\lim_{t\to 0^+} \frac{C_{\varphi_s}f(z) - f(z)}{t} = \lim_{t\to 0^+} \frac{1}{t}\int_0^t C_{\varphi_s}F(z)ds.$$

By strong continuity of $(C_{\varphi_s})_{s\geq 0}$ we have

$$\frac{1}{t}\int_0^t \|C_{\varphi_s}F - F\|ds \to 0 \,\text{as}\, t \to 0^+. \tag{4.5}$$

Hence

$$\{f \in \mathcal{B}_0^\alpha(\mathbb{U}) : zf' \in \mathcal{B}_0^\alpha(\mathbb{U})\} \subseteq \text{dom}(\Gamma).$$

This completes the proof. □

4.3 Translation group

In this case the automorphisms are of the form $\varphi_t(z) = z + kt$, where $z \in \mathbb{U}$ and $k, t \in \mathbb{R}$ with $k \neq 0$. As noted in [3], we can consider the self analytic maps of $\mathbb{U}$ of the form

$$\varphi_t(z) = z + t. \tag{4.6}$$

The composition semigroup induced by (4.6) defined on $\mathcal{B}_0^\alpha(\mathbb{U})$ is given by

$$C_{\varphi_t} f(z) = f(z+t). \tag{4.7}$$

Next, we show that $(C_{\varphi_t})_{t\geq 0}$ given by (4.7) defines a group on $\mathcal{B}_0^\alpha(\mathbb{U})$.

Proposition 4.3.1

$(C_{\varphi_t})_{t\in\mathbb{R}}$ is a group on $\mathcal{B}_0^\alpha(\mathbb{U})$.

PROOF. It suffices to prove that both $(C_{\varphi_t})_{t\geq 0}$ and $\left(C_{\varphi_{-t}}\right)_{t\geq 0}$ are semigroups on $\mathcal{B}_0^\alpha(\mathbb{U})$.
Indeed $C_{\varphi_0} = I$ and for every $f \in \mathcal{B}_0^\alpha(\mathbb{U})$, we have

$$\begin{aligned} C_{\varphi_t} \circ C_{\varphi_s} f(z) &= C_{\varphi_t}\left(C_{\varphi_s} f(z)\right) \\ &= C_{\varphi_t} f(\varphi_s(z) \\ &= f\left(\varphi_t(\varphi_s(z))\right) \\ &= f\left(\varphi_t(z+s)\right) \\ &= f\left(t+s+z\right) \\ &= f\left((t+s)+z\right) \\ &= C_{\varphi_{t+s}} f(z), \end{aligned}$$

as desired. Therefore $(C_{\varphi_t})_{t\geq 0}$ is a semigroup on $\mathcal{B}_0^\alpha(\mathbb{U})$. Similarly, it can be shown that $\left(C_{\varphi_{-t}}\right)_{t\geq 0}$ is also a semigroup on $\mathcal{B}_0^\alpha(\mathbb{U})$. Thus, $(C_{\varphi_t})_{t\in\mathbb{R}}$ is a group on $\mathcal{B}_0^\alpha(\mathbb{U})$, as desired. $\square$.

In the next proposition, we prove that the composition semigroup (4.7), fails to be an isometry on the generalized little Bloch space of the upper half plane $\mathcal{B}_0^\alpha(\mathbb{U})$.

Proposition 4.3.2

The operator C_{φ_t} fails to be an isometry on $\mathcal{B}_0^\alpha(\mathbb{U})$.

PROOF. By norm definition, we have

$$\begin{aligned}\|C_{\varphi_t} f\|_{\mathcal{B}^\alpha(\mathbb{U})} &= |C_{\varphi_t} f(i)| + \sup_{z\in\mathbb{U}} \Im(z)^\alpha \left| \left(C_{\varphi_t} f\right)'(z)\right| \\ &= |f(i+t)| + \sup_{z\in\mathbb{U}} \Im(z)^\alpha |f'(z+t)|.\end{aligned}$$

Now by change of variables: Let $z+t=\omega$ then $z=\omega-t$, and $\Im(z)=\Im(\omega)$. Therefore,

$$\|C_{\varphi_t} f\|_{\mathcal{B}^\alpha(\mathbb{U})} = |f(i+t)| + \sup_{\omega\in\mathbb{U}} \Im(\omega)^\alpha |f'(\omega)| \tag{4.8}$$

The right hand side of equation (4.8) is not equal to the norm $\|f\|_{\mathcal{B}^\alpha(\mathbb{U})}$ for any $t>0$. This implies that (4.7) is not an isometry on $\mathcal{B}_0^\alpha(\mathbb{U})$. This completes the proof. □

We prove in the next proposition that the composition semigroup (4.7) is strongly continuous on the generalized little Bloch space of the upper half plane, $\mathcal{B}_0^\alpha(\mathbb{U})$.

Proposition 4.3.3

The operator C_{φ_t} is strongly continuous on $\mathcal{B}_0^\alpha(\mathbb{U})$.

PROOF. We need to show that $\|C_{\varphi_t} f - f\|_{\mathcal{B}^\alpha(\mathbb{U})} \to 0 \,\text{as}\, t \to 0$. This approach is similar to (4.5). We omit the details. We compute the automorphism of the unit disc $\mathbb{D}$, denoted by $\mathcal{X}_t$ as follows

$$\begin{aligned}\mathcal{X}_t(z) &= \psi^{-1}\left(\varphi_t\left(\psi(z)\right)\right)\\ &= \psi^{-1}\left(\varphi_t\left(\frac{i(1+z)}{1-z}\right)\right)\\ &= \psi^{-1}\left(\frac{i(1+z)}{1-z}+t\right).\end{aligned}$$

Since the inverse of Cayley transform is given by $\psi^{-1} = \frac{z-i}{z+i}$, we substitute to obtain

$$\begin{aligned}\mathcal{X}_t &= \frac{\frac{i(1+z)}{1-z} - t - i}{\frac{i(1+z)}{1-z} - t + i}\\ &= \frac{\frac{i(1+z)}{1-z} - (t+i)}{\frac{i(1+z)}{1-z} + (i-t)}.\end{aligned}$$

We simplify further by multiplying both the numerator and denominator by $(1-z)$ to obtain

$$\begin{aligned}\mathcal{X}_t(z) &= \frac{i(1+z)+(t-i)(1-z))}{i(1+z)+(t+i)(1-z)}\\ &= \frac{i+iz+(t-tz-i+iz)}{i+iz+(t-tz+i-iz)}\\ &= \frac{2iz-tz-t}{2i-tz+t}\\ &= \frac{(2i-t)z-t}{(2i+t)-tz}.\end{aligned}$$

By dividing both the numerator and denominator by $2i-t$, we get

$$\mathcal{X}_t = \frac{z+\frac{t}{2i-t}}{\frac{2i+t}{2i-t} - \frac{t}{2i-t}z}.$$

Letting $k_t = \frac{t}{2i-t}$ and $m_t = \frac{2i+t}{2i-t}$. We have

$$\mathcal{X}_t = \frac{z + k_t}{m_t - k_t z}.$$

Next, we apply density of polynomials in $\mathcal{B}_0^\alpha(\mathbb{D})$ to prove that for $f^* \in \mathcal{B}_0^\alpha(\mathbb{D})$, we have $\|C_{\mathcal{X}t}f^* - f^*\|_{\mathcal{B}_1^\alpha(\mathbb{D})} \to 0$ as $t \to 0$. By norm definition, we have

$$\lim_{t\to 0^+} \|C_{\mathcal{X}t}f^* - f^*\|_{\mathcal{B}_1^\alpha(\mathbb{D})} = \lim_{t\to 0^+} \left(\sup_{z\in\mathbb{D}} \left(1 - |z|^2\right)^\alpha |(C_{\mathcal{X}t}f^* - f^*)'(z)| \right).$$

Using density of polynomials in $\mathcal{B}_0^\alpha(\mathbb{D})$, let $f^*(z) = z^n$ and $z \in \mathbb{D}$ be such that

$$C_{\mathcal{X}t}z^n - z^n = (\mathcal{X}_t(z))^n - z^n, n \geq 1. \tag{4.9}$$

Now, differentiating $(\mathcal{X}_t(z))^n - z^n$ with respect to z, we get

$$\begin{aligned}(C_{\mathcal{X}t}f^* - f^*)'(z) &= n(\mathcal{X}_t(z))^{n-1}\mathcal{X}_t'(z) - nz^{n-1} \\ &= n[(\mathcal{X}_t(z))^{n-1}\mathcal{X}_t'(z) - z^{n-1}]. \end{aligned} \tag{4.10}$$

We also differentiate $\mathcal{X}_t = \frac{z+k_t}{m_t - k_t z}$ by quotient rule to obtain

$$\begin{aligned}\mathcal{X}_t'(z) &= \frac{(m_t - k_t z)1 - (z + k_t)(-k_t)}{(m_t - k_t z)^2} \\ &= \frac{(m_t - k_t z)1 - (-k_t z - k_t^2)}{(m_t - k_t z)^2} \\ &= \frac{m_t - k_t z + k_t z + k_t^2}{(m_t - k_t z)^2} \\ &= \frac{m_t + k_t^2}{(m_t - k_t z)^2}.\end{aligned}$$

Substituting for $\mathcal{X}_t = \frac{z+k_t}{m_t+k_t z}$ and $\mathcal{X}_t'(z) = \frac{m_t - k_t^2}{(m_t - k_t z)^2}$ in equation (4.10) we have

$$
\begin{aligned}
(C_{\mathcal{X}t}f^* - f^*)'(z) &= n[(\mathcal{X}_t(z))^{n-1}\mathcal{X}_t'(z) - z^{n-1}] \\
&= n\left[\left(\frac{z+k_t}{m_t - k_t z}\right)^{n-1} \frac{m_t - k_t^2}{(m_t - k_t z)^2} - z^{n-1}\right] \\
&= n\left[\frac{(z+k_t)^{n-1}(m_t - k_t^2)}{(m_t - k_t z)^{n-1}(m_t - k_t z)^2} - z^{n-1}\right] \\
&= n\left[\frac{(z+k_t)^{n-1}(m_t - k_t z^2)}{(m_t - k_t z)^{n+1}} - z^{n-1}\right]. \\
&= n\left[\frac{(z+k_t)^{n-1}(m_t - k_t z^2) - z^{n-1}(m_t - k_t z)^{n+1}}{(m_t - k_t z)^{n+1}}\right]
\end{aligned}
$$

It therefore follows that as $t \to 0$, we have

$$
\begin{aligned}
\|C_{\mathcal{X}t}f^* - f^*\|_{\mathcal{B}^\alpha(\mathbb{D})} &= (|(\mathcal{X}_t(0))^n - 0|) \\
&+ (\sup_{z\in\mathbb{D}} \left(1 - |z|^2\right)^\alpha \left|n[(\mathcal{X}_t(z))^{n-1}\mathcal{X}_t'(z) - z^{n-1}]\right| = 0.
\end{aligned}
$$

Therefore $\|C_{\varphi_t}f - f\|_{\mathcal{B}^\alpha(\mathbb{U})} = |\varphi_t f(i)) - f(i)| + \|C_{\varphi_t}f - f\|_{\mathcal{B}_1^\alpha(\mathbb{U})} \to 0$ as t $\to$ 0, as desired. This completes the proof. □

In the next theorem, we obtain the infinitesimal generator of the strongly continuous composition semigroup given in equation (4.7).

Theorem 4.3.4

The infinitesimal generator Γ *of* $(C_{\varphi_t})_{t\geq 0}$ *on* $\mathcal{B}_0^\alpha(\mathbb{U})$ *is given by* $\Gamma f(z)$=$f'(z)$ *with the domain dom*$(\Gamma) = \{f \in \mathcal{B}_0^\alpha(\mathbb{U}) : f'(z) \in \mathcal{B}_0^\alpha(\mathbb{U})\}$.

Proof. Using the definition of the infinitesimal generator Γ, for $f \in$

$\mathcal{B}_0^\alpha(\mathbb{U})$, we have;

$$\begin{aligned}\Gamma f(z) &= \lim_{t\to 0^+}\frac{f(z+t)-f(z)}{t}\\ &= \frac{\partial}{\partial t}f(z+t)\Big|_{t=0}\\ &= f'(z).\end{aligned}$$

This means that $\mathrm{dom}(\Gamma)\subset\{f\in\mathcal{B}_0^\alpha(\mathbb{U}) : f'(z)\in\mathcal{B}_0^\alpha(\mathbb{U})\}$.

It remains to prove the reverse inclusion. Let $f\in\mathcal{B}_0^\alpha(\mathbb{U})$ be such that $f'(z)\in\mathcal{B}_0^\alpha(\mathbb{U})$.

Then for $z\in\mathbb{U}$, we have;

$$\begin{aligned}C_{\varphi_t}f(z)-f(z) &= \int_0^t \frac{\partial}{\partial s}f(z+s)ds\\ &= \int_0^t f'(z)ds.\end{aligned}$$

Letting $F(z)=f'(z)$, we obtain

$$C_{\varphi_t}f(z)-f(z)=\int_0^t F(z)ds.$$

This implies that $F(z)=f'(z)$ is a function of $\mathcal{B}_0^\alpha(\mathbb{U})$. It remains to show that the limit of $F(z)$ exists in $\mathcal{B}_0^\alpha(\mathbb{U})$. Since

$$\frac{C_{\varphi_t}f(z)-f(z)}{t}=\frac{1}{t}\int_0^t F(z)ds,$$

we now take limits as $t\to 0^+$ and invoke strong continuity of $(C_{\varphi_s})_{s\geq 0}$ to obtain

$$\lim_{t\to 0^+}\frac{1}{t}\int_0^t \|C_{\varphi_s}Fds-F\|=0.$$

Hence $\mathrm{dom}(\Gamma) \supseteq \{f \in \mathcal{B}_0^\alpha(\mathbb{U}) : f'(z) \in \mathcal{B}_0^\alpha(\mathbb{U})\}$ which completes the proof. $\square$

4.4 Rotation group

The induced composition semigroups for rotation group are defined on the analytic spaces of the unit disk. We shall therefore generate composition semigroups induced by rotation group on the generalized little Bloch space of the disc. The results obtained can then be mapped onto the upper half plane by use of Cayley transform. In this case, the self analytic maps of $\mathbb{D}$ are of the form $\varphi_t(z) = e^{ikt}z$. We consider the composition semigroup induced by the rotation group on $\mathcal{B}_0^\alpha(\mathbb{D})$ given by

$$\begin{aligned} C_{\varphi_t}f(z) &= (f \circ \varphi_t)(z) \\ &= f\left(e^{it}z\right), \end{aligned} \tag{4.11}$$

for all $f \in \mathcal{B}_0^\alpha(\mathbb{D})$.

Proposition 4.4.1

$(C_{\varphi_t})_{t\in\mathbb{R}}$ *defines a group on* $\mathcal{B}_0^\alpha(\mathbb{D})$.

Proof. We need to show that both $(C_{\varphi_t})_{t\geq 0}$ and $\left(C_{\varphi_{-t}}\right)_{t\geq 0}$ are semigroups on $\mathcal{B}_0^\alpha(\mathbb{D})$. Clearly, $C_{\varphi_0} = I$ and for every $f \in \mathcal{B}_0^\alpha(\mathbb{D})$, we have

$$\begin{aligned}
C_{\varphi_t} \circ C_{\varphi_s} f(z) &= C_{\varphi_t}\left(C_{\varphi_s} f(z)\right) \\
&= C_{\varphi_t} f(\varphi_s(z)) \\
&= f\left(\varphi_t(\varphi_s(z))\right) \\
&= f\left(\varphi_t(e^{is}z)\right) \\
&= f\left(e^{it}e^{is}z\right) \\
&= f\left(e^{i(t+s)}z\right) \\
&= C_{\varphi_{(t+s)}} f(z)
\end{aligned}$$

as desired. Therefore $(C_{\varphi_t})_{t\geq 0}$ is a semigroup on $\mathcal{B}_0^\alpha(\mathbb{D})$. Equivalently, $\left(C_{\varphi_{-t}}\right)_{t\geq 0}$ is also a semigroup on $\mathcal{B}_0^\alpha(\mathbb{D})$. Therefore, $(C_{\varphi_t})_{t\in\mathbb{R}}$ is a group on $\mathcal{B}_0^\alpha(\mathbb{D})$. □

Moreover, this group is an isometry, as we prove in the next proposition.

Proposition 4.4.2

The operator C_{φ_t} given by (4.11) is an isometry on $\mathcal{B}_0^\alpha(\mathbb{D})$.

Proof. We shall prove that for each $t \in \mathbb{R}$, the group $(C_{\varphi_t})_{t\in\mathbb{R}}$ is an isometry on $\mathcal{B}_0^\alpha(\mathbb{D})$. It suffices to prove that

$$\|C_{\varphi_t} f\|_{\mathcal{B}^\alpha(\mathbb{D})} = \|f\|_{\mathcal{B}^\alpha(\mathbb{D})}.$$

It follows from the definition that

$$\begin{aligned}\|C_{\varphi_t}f\|_{\mathcal{B}^\alpha(\mathbb{D})} &= |C_{\varphi_t}f(0)| + \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha |(C_{\varphi_t}f)'(z)| \\ &= |(e^{it})f(0)| + \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha |e^{it}f'(e^{it}z)| \\ &= |f(0)| + \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha |f'(e^{it}z)|.\end{aligned}$$

Now, let $\omega = e^{it}z$ so that $z = e^{-it}\omega$. Then;

$$\begin{aligned}\|C_{\varphi_t}f\|_{\mathcal{B}^\alpha(\mathbb{D})} &= |f(0)| + \sup_{\omega\in\mathbb{D}}\left(1-|e^{-it}\omega|^2\right)^\alpha |f'(\omega)|) \\ &= |f(0)| + \sup_{\omega\in\mathbb{D}}(1-|\omega|^2)^\alpha |f'(\omega)| \\ &= \|f\|_{\mathcal{B}^\alpha(\mathbb{D})}.\end{aligned}$$

□

Theorem 4.4.3

The operator C_{φ_t} given by (4.11) is strongly continuous on $\mathcal{B}_0^\alpha(\mathbb{D})$.

Proof. Since polynomials are dense in $\mathcal{B}_0^\alpha(\mathbb{D})$, it suffices to show that $(C_{\varphi_t})_{t\in\mathbb{R}}$ is strongly continuous on $\mathcal{B}_0^\alpha(\mathbb{D})$ that is, for a polynomial $(z^n)_{n\geq 0}$ where $z \in \mathbb{D}$ we obtain

$$\lim_{t\to 0^+} \|C_{\varphi_t}z^n - z^n\|_{\mathcal{B}^\alpha(\mathbb{D})} = 0.$$

Clearly,

$$\lim_{t\to 0^+} \|C_{\varphi_t}z^n - z^n\|_{\mathcal{B}^\alpha(\mathbb{D})} = \lim_{t\to 0^+} |C_{\varphi_t}f(0) - f(0)| + \left(\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |(C_{\varphi_t}z^n - z^n)'|)\right).$$

But

$$C_{\varphi_t}z^n - z^n = (e^{int}-1)z^n.$$

So its derivative is given by

$$(C_{\varphi_t} z^n - z^n)' = n(e^{int} - 1)z^{n-1},$$

implying that

$$\lim_{t\to 0^+} \|C_{\varphi_t} z^n - z^n\|_{\mathcal{B}^\alpha(\mathbb{D})} = \lim_{t\to 0^+} |e^{it} f(0) - f(0)| + \left(\sup_{z\in\mathbb{D}} (1 - |z|^2)^\alpha |nz^{n-1}| |(e^{int} - 1)| \right).$$

Hence,

$$\lim_{t\to 0^+} \|C_{\varphi_t} z^n - z^n\|_{\mathcal{B}^\alpha(\mathbb{D})} = 0 \,\text{as desired}\,.$$

□

Proposition 4.4.4

The infinitesimal generator Γ *of* (C_{φ_t}) *is given by* $\Gamma f(z) = izf'(z)$ *with the domain dom*$(\Gamma) = \{f \in \mathcal{B}_0^\alpha(\mathbb{D}) \, : \, zf'(z) \in \mathcal{B}_0^\alpha(\mathbb{D})\}$.

Proof. We obtain the infinitesimal generator as follows

$$\begin{aligned} \Gamma f(z) &= \lim_{t\to 0^+} \frac{C_{\varphi t}(z) - f(z)}{t} \\ &= \lim_{t\to 0^+} \frac{f(\varphi_t(z)) - f(z)}{t} \\ &= \lim_{t\to 0^+} \frac{f(e^{it} z)) - f(z)}{t} \\ &= \frac{\partial}{\partial t} f(e^{it} z)\Big|_{t=0} \\ &= ie^{it} z f'(e^{it} z)\big|_{t=0} \\ &= izf'(z). \end{aligned}$$

It therefore follows that $\text{dom}(\Gamma) \subseteq \{f \in \mathcal{B}_0^\alpha(\mathbb{D})\} \, : \, zf'(z) \in \mathcal{B}_0^\alpha(\mathbb{D})\}$. On the other hand, let $f \in \mathcal{B}_0^\alpha(\mathbb{D})\}$ be such that $zf'(z) \in \mathcal{B}_0^\alpha(\mathbb{D})\}$, then for $z \in \mathbb{D}$ we have by the Fundamental theorem of Calculus,

$$\begin{aligned} C_{\varphi_t}f(z) - f(z) &= \int_0^t \frac{\partial}{\partial s}(C_{\varphi_s}f(z))ds \\ &= \int_0^t \frac{\partial}{\partial s}f(e^{is}z))ds \\ &= \int_0^t ie^{is}zf'(e^{is}z)ds \\ &= \int_0^t C_{\varphi_s}F(z)ds, \end{aligned}$$

where $F(z) = izf'(z)$ is a function in $\mathcal{B}_0^\alpha(\mathbb{D})$. Thus $\lim_{t\to 0^+} \frac{C_{\varphi_t}f - f}{t} = \lim_{t\to 0^+} \frac{1}{t}\int_0^t C_{\varphi_s}Fds$ and strong continuity of $(C_{\varphi_s})_{s\geq 0}$ implies that $\|\frac{1}{t}\int_0^t C_{\varphi_s}Fds - F\| \leq \frac{1}{t}\int_0^t \|C_{\varphi_s}F - F\|ds \to 0^+$ as $t \to 0^+$. Thus $\text{dom}(\Gamma) \supseteq \{f \in \mathcal{B}_0^\alpha(\mathbb{D}) : zf'(z) \in \mathcal{B}_0^\alpha(\mathbb{D})\}$, as desired. □

Chapter 5

Summary and Recommendations

5.1 Summary

In this work, we investigated the properties of the generalized Bloch spaces of the upper half plane $\mathcal{B}^{\alpha}(\mathbb{U})$, as Banach spaces as well as those of composition semigroups. We employed the approach used by K. Zhu to study the Banach space properties of $\mathcal{B}^{\alpha}(\mathbb{U})$. We established generalized Bloch space of the upper half plane $\mathcal{B}^{\alpha}(\mathbb{U})$ and its closed subspace $\mathcal{B}^{\alpha}_0(\mathbb{U})$ to be Banach spaces, see Theorem 3.0.4 and 3.0.6. Using Cayley transform, we obtained equivalent representations of functions on the generalized Bloch space of the unit disc $\mathcal{B}^{\alpha}(\mathbb{D})$ to their counterparts on the upper half plane $\mathbb{U}$, see Proposition 3.0.8
Following Theorem 2.0.1, we obtained composition semigroups generated by scaling, translation and rotation groups on the generalized little Bloch space of the upper half plane, $\mathcal{B}^{\alpha}_0(\mathbb{U})$ as given in equations (4.2), (4.7) and (4.11). The theory of linear operators on Banach spaces enabled us

to investigate the semigroup properties of the composition semigroups defined on $\mathcal{B}_0^\alpha(\mathbb{U})$ as given in Propositions 4.4.2, 4.4.3 and 4.4.4. Density of polynomials in $\mathcal{B}_0^\alpha(\mathbb{D})$ aided in establishing strong continuity of the composition semigroups on $\mathcal{B}_0^\alpha(\mathbb{U})$. This thesis therefore completes a comprehensive analysis of the generalized Bloch spaces of the upper half plane as Banach spaces as well as composition semigroups defined on them.

5.2 Recommendations

From the results obtained in this study, we recommend the following for further research:

1. We considered composition semigroups on $\mathcal{B}_0^\alpha(\mathbb{U})$ and determined their semigroup properties. It would be interesting to consider an investigation of spectral properties of these composition semigroups on the generalized little Bloch space of the upper half plane $\mathcal{B}_0^\alpha(\mathbb{U})$.
2. In this work, we examined the concept of strong continuity of composition semigroups on $\mathcal{B}_0^\alpha(\mathbb{U})$, we suggest further research on strong continuity of the weighted composition operators.
3. We considered composition semigroups on the generalized little Bloch space of the upper half plane, we strongly advocate for an extension of the same to other spaces of analytic functions like Besov spaces, where the study of composition semigroups has never been considered.

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