ANALYTIC SOLUTION OF A NONLINEAR BLACK-SCHOLES PARTIAL DIFFERENTIAL EQUATION

by

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ABSTRACT

The assumptions under which the standard Black-Scholes equation has been derived are restrictive (e.g. liquid and frictionless markets). When illiquidity and market friction are introduced into the market, financial models based on these assumptions fail. Nonlinear equations for modelling illiquid markets have been solved numerically. Numerical techniques give approximate solutions. Recently, Lie group symmetry analysis has been used to solve the same. Although Lie group symmetry analysis is very useful in determining all the solutions of a given nonlinear equation, it has been established that any small perturbation of an equation disturbs the group admitted by it. This in effect reduces the practical use of symmetry group analysis. Our objective is to find an analytic solution of a nonlinear Black-Scholes equation for modelling illiquid markets. The methodology involved transformation of the nonlinear Black-Scholes equation into a groundwater equation. This yields Ordinary Differential Equations which have been solved. Using substitutions and integration led to an analytic solution of the nonlinear Black-Scholes equation. In a real market situation, this solution may help in finding how typical prices of derivatives can be described hence contributing significantly to the field of Financial Mathematics.

Chapter 1

Introduction

This thesis is outlined as follows: Chapter 1 is an introductory chapter. Chapter 2 addresses literature that is relevant in modelling both liquid and illiquid markets. This includes the information on the physical phenomena which are connected to the study of nonlinear Partial Differential Equations (PDEs). Basic concepts are presented in Chapter 3. Theory of the linear and the nonlinear Black-Scholes option valuation models is discussed in Chapter 4. Chapter 5 considers both linear and nonlinear equations. The solution of the Korteweg-de Vries equation is presented in Section 5. The main results and their discussion are presented in Chapter 6. These results are formulated in form of a theorem (Theorem 6.1.3). The general conclusions of this thesis and the recommendations for further research come after Chapter 6.

1.1 Background Information

Financial Mathematics is a collection of mathematical techniques applied in finance. An example of these applications is in asset pricing where derivative securities such as options are valued. Another application is in hedging and risk management. The two main modelling approaches used in Financial Mathematics are Partial Differential Equations, and Probability and Stochastic Processes.

The standard Black-Scholes equation which gave rise to the field of Financial Mathematics has been derived under restrictive assumptions such as liquid and frictionless markets. However, market liquidity has recently become an issue of high concern in managing risks. From literature, it has been shown empirically and theoretically that large traders move the underlying asset's price. In addition, financial markets are markets with friction since transaction costs are incurred when a financial asset is traded. As a result, financial models based on the assumptions of *frictionless* and *perfectly liquid* markets may fail when *transaction costs* are introduced into the market and when *market liquidity* vanishes from the market.

Our focus in this research is to use dynamic hedging to study liquidity of derivative securities in the presence of transaction costs. We studied an illiquid market model where the implementation of a dynamic hedging strategy affects the underlying asset's price process.

In this study we built on the work of Cetin *et al.* [11] where a model for European options that takes into consideration illiquidities arising from transaction costs is formulated. Perfect hedging strategies here are characterized by a nonlinear Black-Scholes Partial Differential Equation.

An analytic solution to the nonlinear Black-Scholes equation via a solitary wave solution is currently unknown.

The purpose of this research was to solve analytically by direct integration the nonlinear Black-Scholes equation arising from transaction costs in order to have a better understanding of illiquid markets for derivative assets. This was done by differentiating the equation twice with respect to the spatial variable S. After substitutions and transformations, we got a nonlinear *groundwater equation* that admits a solitary wave solution. Assuming a traveling wave solution to the nonlinear groundwater equation reduced the nonlinear Black-Scholes equation to Ordinary Differential Equations (ODEs). The parameter "gamma" (i.e. u_{SS}) ends up being a solitary wave solution since it decays to zero at large distances. We obtained the solution to the nonlinear Black-Scholes equation via the solitary wave solution by integrating u_{SS} twice with respect to the spatial variable S.

A subset of the data from the Nairobi Stock Exchange (NSE) for the Kenya Electricity Generating Company (KenGen) and the Kenya Power and Lighting Company (KPLC) for the periods between 2nd January 2007 – 24th December 2007 and 3rd January 2003 – 2nd January 2004 respectively, i.e. one year for each company, was used to test whether the solution to the nonlinear Black-Scholes equation is applicable in a real life situation.

Recent studies have focussed on derivative hedging in illiquid markets. In this study we mention the contributions of Bank and Baum [2], Bordag and Chmakova [5], Bordag and Frey [6], Cetin *et al.* [11, 12], Frey [21, 22], Frey and Patie [23], Frey and Polte [24], Frey and Stremme [25], Platen and Schweizer [46], Schönbucher and Wilmott [53], Papanicolaou and

Sircar [44].

1.2 Statement of the Problem

Variants of a nonlinear Black-Scholes Partial Differential Equation have only been solved numerically and by Lie group symmetry. Use of numerical techniques gives approximate solutions. Any small perturbation of an equation using Lie group symmetry disturbs the group admitted by it which in effect reduces the practical use of symmetry group analysis. If all these problems have to be solved, then the equation has to be solved analytically by direct integration.

1.3 Objective of the Study

The main objective of this study is to solve a nonlinear Black-Scholes Partial Differential Equation analytically.

1.4 Research Methodology

In this study we consider European options only. The linear Black-Scholes Partial Differential Equation is utilized in developing a picture of its nonlinear version.

The nonlinear Black-Scholes equation resulting from transaction costs is transformed into a nonlinear groundwater equation that admits a solitary wave solution. A wave solution is assumed before transforming the groundwater equation into Ordinary Differential Equations. This led to the analytic solution of the nonlinear Black-Scholes equation by assuming localized boundary conditions after the transformation.

1.5 Significance of the Study

Given a real market situation, the analytic solution may help in finding how typical prices of financial derivatives can be described hence contributing significantly to the field of Financial Mathematics.

Since the solution is theoretic, it may give option hedgers guidance on how to hedge risks on a real market situation.

Chapter 2

Literature Review

In this chapter we review the development of stock price modelling right from the time Brownian motion was first investigated up to the time the nonlinear Black-Scholes equations were derived and solved. We mention some of the studies that have so far been done on the nonlinear (illiquid market) models and their relevance together with their shortcomings.

Since this research is on Black-Scholes option valuation, most of the equations for modelling financial markets have been deferred until Chapter 4 where they will be discussed in details.

2.1 Brownian Motion

Brownian motion was first used by Scottish botanist Robert Brown to observe the irregular motion of pollen grains suspended in a liquid. Brown himself admitted to not having any scientific explanation for the observed phenomena (see Brown [8]). It was further studied by Albert Einstein [16] in 1905. Einstein's theory was based on the assumption that Brownian motion process exists. The definition of a $\mathbb{P}$-Brownian motion process is the modern statement of Einstein's postulates (see Khoshnevisan [30]). Bachelier [1] discussed its theory in his thesis where he applied it to model stock prices. However, the Bachelier's model allows the stock price to take both positive and negative values yet stock prices can only be positive. This was further developed by Norbert Wiener in 1923. The validity of Einstein's assumption that Brownian motion process exists was proved by Wiener in 1923 (see Khoshnevisan [30]).

The prototype for diffusion processes is Brownian motion or Wiener process (see Feller [19]). The standard Wiener process can be used to model asset returns. The main problem with it is that its mean is zero. This means that the growth rate of returns is zero whereas for instance a company's stock normally grows at some rate - and from history the prices are expected to rise because of inflation (see Baxter and Rennie [3]). To avoid such growth rate of returns, we extend Brownian motion to a generalized Brownian motion, i.e. Brownian motion with non-zero mean.

More recent work suggests that the process resulting from the generalized Brownian motion applies rather to relative prices (i.e. stock returns) in steady markets (see for instance Black and Scholes [4], and Merton [39]). In this geometric Brownian motion process as was first introduced by Samuelson [51], the stock price S_t is positive.

2.2 Merton-Black-Scholes Model

The modern application of Brownian motion to model financial markets began between late 1960s and early 1970s. Geometric Brownian motion was applied by Black and Scholes [4], Merton [39], and Paul Samuelson [51, 52] among others. In 1973, Black and Scholes [4] derived an option valuation model. This was extended by Robert Merton [39] the same year to include dividends and then coined the term *Black-Scholes theory of option pricing* (see Merton [39]). That is why the model is sometimes called the Merton-Black-Scholes model. Black and Scholes [4], and Merton [39] assume that stock returns follow a Brownian motion.

2.3 Diffusion Processes and Stochastic Integrals

Although Brownian motion is continuous everywhere, it has been shown in Theorem VII of Paley *et al.* [43] that it is nowhere differentiable almost surely. The same theorem has also been proved by Khoshnevisan [30] (see Theorem 9.13 in [30], pp. 168 and its proof on pp. 169).

Due to this notion of nowhere differentiability, the ordinary rules of calculus fail in a stochastic environment. What becomes useful is stochastic calculus, also called Itô calculus in honor of Kiyoshi Itô (see [15, 28]).

Diffusion processes are solutions to Stochastic Differential Equations (SDEs). Itô's lemma plays a very important role in stochastic calculus since it is used in solving stochastic integrals.

2.4 Standard Option Valuation Theory

The use of linear Black-Scholes equation started in 1973 when Fischer Black and Myron Scholes came up with an option valuation formula by considering a non-dividend-paying stock (see Black and Scholes [4]).

Stock prices were taken to be lognormally distributed. From the resulting Black-Scholes model, the Black-Scholes PDE was obtained and was solved to get the Black-Scholes formulae for the call and put options.

Put-call parity was first described by Professor Hans Stoll in 1969 but it had been known earlier (see Knoll [32]). It was first used in the Black-Scholes option valuation (see equation (25) of [4]) to compute the value of a put option from a call option's value easily.

2.5 Risk Parameters

The risk parameter delta for a European call option was first used by Black and Scholes [4] for hedging European call options (see equation (14) of [4]). Since then, other risk parameters such as theta, gamma, speed, vega, and rho have been computed from Black-Scholes formulae (see for instance Wilmott [61]).

A financial institution selling options faces the problem of hedging risks and the use of the risk parameters addresses this problem.

2.6 Nonlinear Black-Scholes Option Valuation Theory

Use of Black-Scholes formulae in the standard option valuation to derive option values rests on the assumption of frictionless and perfectly liquid markets. Owing to liquidity constraints, the trade of the underlying asset induced by dynamic hedging can certainly affect market prices. Existence of market frictions and market illiquidity renders the standard option valuation models unrealistic (see Frey [22]) hence the need for use of nonlinear Black-Scholes models.

A modelling philosophy that trading large amounts moves the price of an asset has been studied by Frey [21], Frey and Stremme [25], Papanicolaou and Sircar [44], Platen and Schweizer [46], Schönbucher and Wilmott [53]. The PDE for Frey [21] is given by

$$\begin{aligned} 0 &= \phi_t + \tfrac{1}{2}\sigma^2 f^2 \left(1 + 2\rho\tfrac{\psi_\alpha}{\psi_f}\phi_f\right)\phi_{ff} \\ &+ \tfrac{\sigma^2}{\psi_f}\phi_f\left(f\psi_f - \psi_t + \tfrac{f^2}{2}\psi_{ff} + \rho\phi_f(f^2\psi_{\alpha f} + f\psi_\alpha) + (\rho\phi_f)^2\tfrac{f^2}{2}\psi_{\alpha\alpha}\right), \end{aligned} \tag{2.1}$$

where $\phi = \phi(t, f)$ is a smooth function that solves equation (2.1), α is a martingale, f is the value of some fundamental state variable process $(F_t)_{0\leq t\leq T}$, ρ is market weight of the large trader, $\psi = \psi(t, F_t, \alpha)$ is the discounted stock price at time t, $\psi_\alpha = \frac{\partial}{\partial\alpha}\psi$, $\psi_f = \frac{\partial}{\partial f}\psi$, $\phi_t = \frac{\partial}{\partial t}\phi$, $\phi_f = \frac{\partial}{\partial f}\phi$, $\phi_{ff} = \frac{\partial^2}{\partial f^2}\phi$, $\psi_t = \frac{\partial}{\partial t}\psi$, $\psi_{ff} = \frac{\partial^2}{\partial f^2}\psi$, $\psi_{\alpha f} = \frac{\partial^2}{\partial\alpha\partial f}\psi$, $\psi_{\alpha\alpha} = \frac{\partial^2}{\partial\alpha^2}\psi$.

The trading strategy used by Frey and Stremme [25], and Papanicolaou and Sircar [44] is of the form $\alpha = \rho\phi(t, S)$ for α shares and for the smooth function ϕ.

By using numerical techniques, Platen and Schweizer [46] quantitatively substantiated the idea that feedback effects resulting from hedging strategies can induce option price distortions. Sergeeva [54] has solved the equation that was derived in [46] using Lie group symmetry.

Schönbucher and Wilmott [53] analysed the influence of dynamic trading strategies on the prices in financial markets. The nonlinear effect and the feedback arising from prices in the trading strategy $f(S,t)$ are analysed. A nonlinear PDE for an option replication strategy is derived and this PDE is given by

$$p_t + rSp_S + \tfrac{1}{2}\lambda^2 p_{SS} - rp = 0,$$

where $p(S,t)$ is the put option price, r is the continuously compounded risk-free interest rate, $p_t = \frac{\partial p}{\partial t}$, $p_S = \frac{\partial p}{\partial S}$, $p_{SS} = \frac{\partial^2 p}{\partial S^2}$, and the volatility is given by

$$\lambda(S,t) = -\frac{\frac{\partial}{\partial W}\chi(S,W,t)}{\frac{\partial}{\partial S}\chi(S,W,t)+\frac{\partial}{\partial S}f(S,t)},$$

where χ is the excess demand in the market and W is Brownian motion.

The stock price in the models in [21, 25, 44, 46, 53] responds instantaneously to the amount of stock a single large trader holds. These models are for price impact effects. In these models, the price impact effect would make price dynamics to be history-dependent and also to be dependent on the past trading decisions of agents. These price impact models are unsuitable as models of illiquidity since if the action of an agent affects the price, logically the actions of all agents affect the price and the resulting analysis of the inter-related behaviour of the many agents participating in the market becomes impossibly cumbersome (see Rogers and Singh [49]).

A diffusion model for stock price dynamics whose coefficients depend on the large investor's trading strategy is considered by Cuoco and Cvitanić [14]. In this model the feedback is rather indirect since only the drift and volatility coefficients depend on the trading strategy of the large investor.

Transaction costs caused by illiquidity in a continuous-time model are studied by Bank and Baum [2] and further studied by Cetin *et al.* [11] who eliminates the path dependency condition in their structure which excludes manipulation of the market and allows use of classical arbitrage pricing theory. The market dependency condition is that under market manipulation, the price process of the security can depend on the investor's current trade and the entire history of their past trades. The analysis of Cetin *et al.* [11] avoids market impact costs.

When an individual sells (or buys) huge quantities of shares this pushes the price down (or up) due to the high (or low) volumes/supply. The difference in price between the original and the new price is called *impact cost*.

The model used in Bank and Baum [2] is specified in terms of semimartingales that are parameter-dependent. The difference in the modelling approaches in [2] and [11, 12] is that in Cetin *et al.* [11, 12] the price effect of an order ends the very moment the order is placed in the market while in Bank and Baum [2] the effect of the order will extend into the next order hence making prices of the asset follow a possibly different dynamics.

The framework used by Cetin *et al.* [11] is a continuous-time trading strategy while Cetin *et al.* [12] investigates valuing of derivatives through discrete time trading strategies with the assumption that the underlying asset is not perfectly liquid.

Stock price dynamics are studied by Frey [22]. A model where implementation of a hedging strategy affects the price process of the underlying security is considered. The price process is driven by a Brownian motion and by a representative agent's strategy of hedging derivatives. Using a feedback strategy of a large trader leads to a nonlinear Black-Scholes equation. Numerical simulations of the equation are done in [22]. The same equation is solved using Lie group symmetry in [5, 6].

The analysis of Frey [22] is implemented by Frey and Patie [23] where an extensive simulation study is carried out to have a better understanding of the implications of market illiquidity for derivative asset analysis. The equation that was studied by Frey and Patie [23] is modified through introducing a liquidity profile $\lambda(S)$ to give

$$u_t + \frac{1}{2}\frac{\sigma^2}{(1-\rho\lambda(S)Su_{SS})^2}S^2u_{SS} = 0, \quad u(S_T,T) = h(S_T), \quad S_T \geq 0, \qquad (2.2)$$

where ρ is a liquidity parameter, $u_t = \frac{\partial u}{\partial t}$, $u_{SS} = \frac{\partial^2 u}{\partial S^2}$, and $h(S_T)$ is the payoff of the terminal value claim.

Equation (2.2) has been solved numerically (see [23]). The same equation has also been solved through Lie group symmetry by Bordag and Frey [6].

Modelling using numerical methods and Lie group symmetry analysis has its own disadvantages. Numerical techniques give approximate solutions. Although Lie group symmetry analysis is very useful in determining all the solutions of a given nonlinear equation, it has been established that any small perturbation of an equation disturbs the group admitted by it which in effect reduces the practical use of symmetry group analysis (see Ongati [41]).

2.7 Physical Phenomena Connected to Nonlinear Partial Differential Equations

Most nonlinear PDEs which arise in physical applications are practical and this leads to physically meaningful solutions. The study of nonlinear PDEs has shown that *solitons* are essential physical phenomena which are connected with nonlinear equations.

The solitary wave solution represents a localized wave travelling with unchanged shape. In 1834, John Scott Russell, a British engineer, observed the waves. He recounts how such a solitary wave was generated by the sudden motion of a large barge along an Edinburg canal and then chasing it on horseback for several miles (see [34, 57]). Much later, the proper nonlinear surface wave model given by

$$u_t + 6uu_x + u_{xxx} = 0 \quad \text{in} \quad \mathbb{R} \times (0, \infty), \tag{2.3}$$

where $u(t, x)$ is the speed of the traveling wave of permanent form, $u_x = \frac{\partial u}{\partial x}$ and $u_{xxx} = \frac{\partial^3 u}{\partial x^3}$, was written down by Korteweg and deVries. This model is valid for a one-way wave in shallow water. Zabusky and Kruskal got curious after some time and rederived the Korteweg-de Vries equation (KdV) (2.3) as a continuum limit of a model of nonlinear mass-spring chains that were studied by Fermi *et al.* [20] and also discovered that numerical solutions to the Korteweg-de Vries equation have remarkable properties (see Zabusky and Kruskal [62]). It is for this reason that these solutions have been given a special name- *soliton*, a word that first appeared in Zabusky and Kruskal [62] who called solitary waves solitons where the ending 'on' is a Greek word for particle (see Munteanu and Donescu [40], pp. 82).

However, none of the nonlinear Black-Scholes equations discussed in Section 2.6 has ever been solved using direct integration although some nonlinear equations have been solved using this method. An example of these equations is the Korteweg-de Vries equation (2.3) that has been solved analytically via direct integration to obtain a solitary wave solution.

Chapter 3

Basic Concepts

Black-Scholes model is a model which describes mathematically financial markets and derivative instruments. This model was used in Black and Scholes [4] to get the *Black-Scholes Partial Differential Equation* which was solved to give *Black-Scholes formulae.* The formulae are widely used in valuing European-style options.

3.1 Brownian Motion

When we talk of a continuous process, we mean the following: firstly, the value can change any time and from one moment to another; secondly, actual values taken can be expressed in arbitrarily fine fractions- a value can be any real number; lastly, the process changes continuously- no instantaneous jumps are made by the value (see Baxter and Rennie [3]).

3.1.1 Random Walk

For a positive integer n and time t, a *random walk* is a binomial process $W_n(t)$ if the following conditions hold.

1. $W_n(0) = 0$.
2. The layer spacing is $1/n$.
3. The up and down jumps are equal and of size $1/\sqrt{n}$.
4. The measure $\mathbb{P}$ resulting from the up and down probabilities everywhere equals to $1/2$.

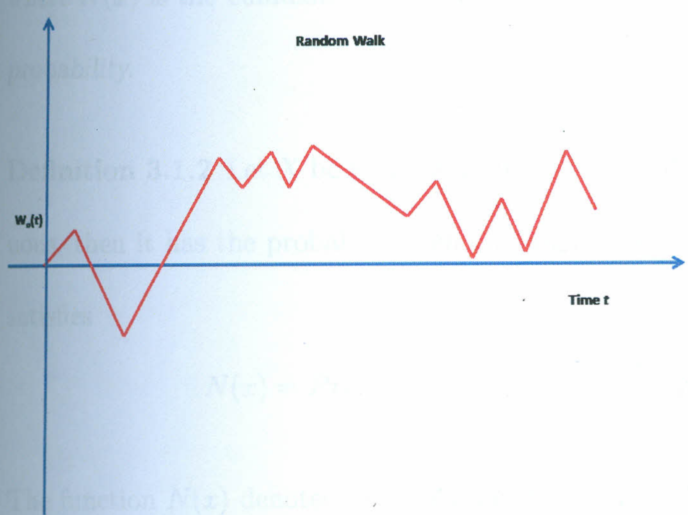

Figure 3.1: The steps of a random walk W_n.

This means that, if $X_1, X_2, \ldots$ is a sequence of independent binomial random variables taking values ± 1 with equal probability, then, at the ith step, the value of W_n is defined by

$$W_n(\tfrac{i}{n}) = W_n(\tfrac{i-1}{n}) + \tfrac{X_i}{\sqrt{n}} \quad \text{for all} \quad i \geq 1$$

Theorem 3.1.1 ***(Central Limit Theorem)*** *Let the random variables $X_1, X_2, \cdots, X_n$ form a random sample whose size is n from a probability distribution with mean and standard deviation μ and σ respectively. Then for all x*

$$\lim_{n\to\infty} Pr\left(\tfrac{\sqrt{n}(\overline{X}-\mu)}{\sigma} \leq x\right) = N(x),$$

where $N(x)$ is the cumulative distribution function *of x and Pr denotes probability.*

Definition 3.1.2 Let X be a random variable and $x \in \mathbb{R}$. If X is continuous, then it has the probability density function $f : \mathbb{R} \mapsto [0, \infty)$ which satisfies

$$N(x) = Pr(X \leq x) = \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}}\, dy. \tag{3.1}$$

The function $N(x)$ denotes the probability that the value of a standardized normal variable X is less than or equal to x. This function is represented in Figure 3.2.

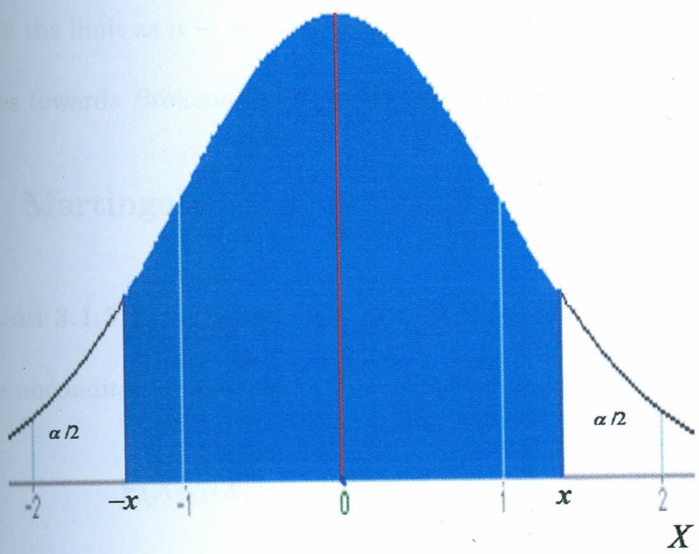

Figure 3.2: A normal distribution curve.

From the central limit theorem, the limit of these binomial distributions is that as n gets larger, the distribution of $W_n(1)$ tends to the standardized normal distribution, i.e. $\lim_{n\to\infty} W_n(1) \sim N(0,1)$. In fact

$$W_n(t) = \sqrt{t}\left(\frac{\sum_{i=1}^{nt} X_i}{\sqrt{nt}}\right).$$

By the central limit theorem, the distribution of the ratio in brackets tends to a standardized normal random variable. This means that the distribution of $W_n(t)$ tends to a normal distribution, i.e. $\lim_{n\to\infty} W_n(t) \sim N(0,t)$.

Hence, in the limit as $n \to \infty$, the distribution of the random walk $W_n(t)$ converges towards *Brownian motion* W_t (see Baxter and Rennie [3]).

3.1.2 Martingale

Definition 3.1.3 A *martingale* is a stochastic process $X = \{X_i\}_{i=1}^{\infty}$ such that the unconditional expected value is always finite and that

$$\mathbb{E}(X_{t+1}|X_t, \cdots, X_1) = X_t \quad \text{for all} \quad t.$$

The process X is a *submartingale* if

$$\mathbb{E}(X_{t+1}|X_t, \cdots, X_1) \geq X_t \quad \text{for all} \quad t$$

and it is said to be a *supermartingale* if

$$\mathbb{E}(X_{t+1}|X_t, \cdots, X_1) \leq X_t \quad \text{for all} \quad t.$$

The process X is a martingale if it is both a super- and a submartingale; it becomes a *semimartingale* if it can be written as $X_t = Y_t + Z_t$ where the process $\{Y_i\}_{i=1}^{\infty}$ is a martingale and the process $\{Z_i\}_{i=1}^{\infty}$ is a *bounded-variation process*; i.e., $Z_t = U_t - \tilde{U}_t$ where $U_1 \leq U_2 \leq \cdots$ and

$\tilde{U}_1 \leq \tilde{U}_2 \leq \cdots$ are integrable adapted processes.

3.1.3 Markov Process

A *Markov process* is a process whereby the distribution of its future values, conditional on its past and present values, depends only on the present and not the past value.

Using the fact that $W_t - W_s$ is independent of W_s, where s is a time variable, we get

$$\begin{aligned}\mathbb{E}(W_t|W_s) &= \mathbb{E}(W_t - W_s|W_s) + \mathbb{E}(W_s|W_s)\\ &= \mathbb{E}(W_t - W_s) + W_s\\ &= W_s.\end{aligned}$$

We call the expression

$$\mathbb{E}(W_t|W_s) = W_s$$

the *martingale property*.

Definition 3.1.4 (Lévy Process) A process $X = \{X_t : t \geq 0\}$ is said to be a *Lévy process* if it possesses the following properties.

1. The paths of the process X are $\mathbb{P}$-a.s. right-continuous with left limits.

2. $\mathbb{P}(X_0 = 0) = 1$.

3. For $0 \leq s \leq t$, the distributions of $X_t - X_s$ and X_{t-s} are equal, i.e. the process X has stationary increments.

4. For $0 \leq s \leq t$, $X_t - X_s$ is independent of $\{X_\tau : \tau \leq s\}$, where τ is another time variable.

Definition 3.1.5 The *quadratic variation* of the process $X = \{X_t : t \geq 0\}$ is defined to be

$$\begin{aligned} \langle X \rangle t &= \int_0^t ds \\ &= t. \end{aligned}$$

Theorem 3.1.6 ***(Lévy, one dimension)*** *Let the process $X = \{X_t : t \geq 0\}$ be a martingale. Assume that $X_0 = 0$, X_t has continuous paths, and the quadratic variation $\langle X \rangle t = t$ for all $t \geq 0$. Then X_t is a Brownian motion.*

Definition 3.1.7 Suppose that X_t is a stochastic process with time $t \in [0, T]$ for some $T > 0$. If P is a partition of the time interval $[0, T]$ such that

$$P = \{0 = t_0 \leq t_1 \leq \cdots \leq t_m = T\},$$

then the quadratic variation of the process X_t can be defined along the partition P by

$$\langle X \rangle^P \equiv \sum_{k=1}^{m} (X_{t_k} - X_{t_{k-1}})^2.$$

Definition 3.1.8 A stochastic process $W = \{W_t, t \geq 0\}$ is a $\mathbb{P}$-*Brownian motion process* if

1. $t \mapsto W_t$ is a continuous random path with probability one,

2. $W_0 = 0$,

3. $\{W_t, t \geq 0\}$ has stationary independent increments. This means that for any $0 < s < t$, $W_t - W_s$ is independent of $\{W_\tau\}_{0 \leq \tau \leq s}$. Thinking of s as the current time, this condition says that "given the value of W at the present time, the future value is independent of the past value." We call this property the *Markov property.*

4. W_t is normally distributed with mean zero and variance t under $\mathbb{P}$, i.e. $W_t \sim N(0, t)$.

Since $\sigma = 1$ for the process W_t we often call this process *Brownian motion or Wiener process.* The growth rate of returns in a Wiener process is zero since W_t has zero mean. Hence, we have to extend it to Brownian motion with non-zero mean (drift or trend) by adding the drift artificially to get the stock price process S_t (see Baxter and Rennie [3]). The extension models the dynamics of the prices in a steady market (Onyango [42]).

When $\sigma \neq 1$ and the drift is added to the Wiener process W_t we get the stochastic process $\{S_t, t \geq 0\}$ which is a Brownian motion with constant drift coefficient $\mu \in \mathbb{R}$. We write this as

$$S_t = \mu t + \sigma W_t, \tag{3.2}$$

where $W_t = \varepsilon\sqrt{t}$ and ε is a *Brownian motion* process with mean zero and variance one, i.e. $\varepsilon \sim N(0,1)$. Hence, during the time-step dt we get the SDE

$$dS_t = \mu dt + \sigma dW_t \tag{3.3}$$

from equation (3.2). The process represented in equation (3.3) is referred to as *Bachelier* or *Arithmetic Brownian motion* and it assumes that the stock price S_t follows a stochastic process (see Onyango [42]). Using the model (3.3), stock prices can take positive or negative values (see Figure 3.3) though stock prices cannot be negative.

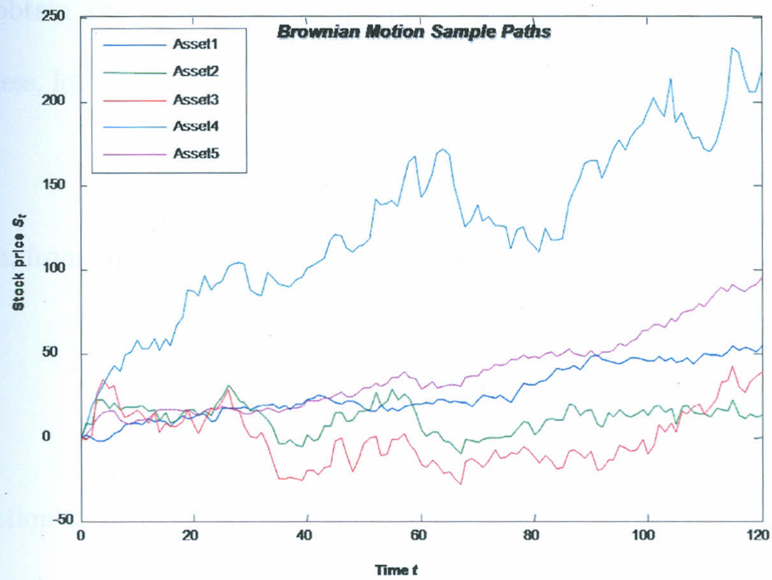

Figure 3.3: Possible realizations of Brownian motions.

3.2 Geometric Brownian Motion or Samuelson's Model

If Brownian motion is given by $\{X_t, t \geq 0\}$, we call the process $\{Y_t, t \geq 0\}$ defined by

$$\begin{aligned} Y_t &= e^{X_t} \\ &= e^{\mu t + \sigma W_t} \end{aligned} \tag{3.4}$$

a *geometric Brownian motion* or *Samuelson's model*. We also refer to this type of a stochastic process as a *geometric Wiener process*, or *economic geometric Brownian motion* (see Samuelson [52]).

To obtain the stock price model under a geometric Brownian motion process, let

$$\frac{dS_t}{S_t} = d\ln Y_t. \tag{3.5}$$

Then, from equations (3.4) and (3.5) we get

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t. \tag{3.6}$$

Equation (3.6) is called a *generalized Wiener process.* The process assumes that stock price returns $\frac{dS_t}{S_t}$ follow a stochastic process. Since S_t is the price of a stock at time t, then $\frac{dS_t}{S_t}$ is the rate of return on the asset over the next instant and its solution on integrating both sides of equation (3.6) is given by

$$\ln\left(\frac{S_t}{S_0}\right) = \mu t + \sigma \int_0^t dW_s, \quad S_0 > 0, \tag{3.7}$$

where S_0 is the initial stock price.

Hence, rearranging equation (3.7) gives the stock price as

$$S_t = S_0 e^{\mu t + \sigma \int_0^t dW_s} > 0, \quad S_0 > 0. \tag{3.8}$$

Equation (3.8) shows that the stock price S_t is positive at all time t since $S_0 > 0$. Since stock prices can never take negative values as seen in Figure 3.4, this makes geometric Brownian motion important in modelling stock prices.

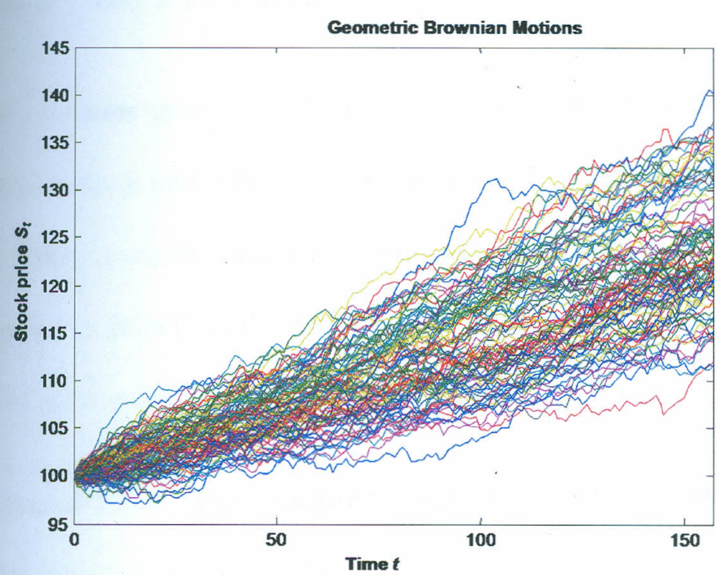

Figure 3.4: Possible realizations of geometric Brownian motions.

3.3 Diffusion Processes and Stochastic Integrals

Although the Wiener process W_t is continuous everywhere, it is (with probability one) differentiable nowhere (see for instance [3, 43]). Standard rules used in calculus are therefore inapplicable in a stochastic environment since the Wiener process W_t is nowhere differentiable. We therefore generalize the approach of Section 3.2 for us to be able to solve SDEs.

3.3.1 Itô Processes

Itô processes generalize Brownian motion in equation (3.6) by letting parameters μ and σ be functions of the underlying variable S_t and time t. Thus, a one-dimensional generalized Itô process is given in the Itô's lemma or Itô's formula stated below. This lemma is the cornerstone of stochastic calculus.

Lemma 3.3.1 ***(Itô's Lemma)*** *Suppose that the random variable S_t is described by the Itô process*

$$dS_t = \nu(S_t, t)dt + \lambda(S_t, t)dW_t, \tag{3.9}$$

where dW_t is a normal random variable. Suppose the random variable $h(S_t) = u(S_t, t)$. Then h is described by the following Itô process

$$dh = \left(\nu(S_t, t)u_S + u_t + \frac{1}{2}\left(\lambda(S_t, t)\right)^2 u_{SS}\right) dt + \lambda(S_t, t)u_S dW_t, \quad (3.10)$$

where the hedge ratio $u_S = \frac{\partial u}{\partial S}$.

Before we prove the Itô process for h given in equation (3.10) we need to define Taylor expansion.

Definition 3.3.2 Taylor expansion is an expression of the value of a function f near x in terms of the value of f and its derivatives at x and is given by

$$f(x+h) = f(x) + hf'(x) + \tfrac{1}{2}h^2 f''(x) + \tfrac{1}{3!}h^3 f'''(x) + \cdots .$$

Proof 3.3.3 (Itô's Lemma) Taylor series expansion of $u(S_{t+\Delta t}, t + \Delta t)$ around (S_t, t) results in

$$
\begin{aligned}
u(S_{t+\Delta t}, t+\Delta t) = {} & u(S_t, t) + u_t(S_t, t)\Delta t + u_S(S_t, t)(S_{t+\Delta t} - S_t) \\
& + \tfrac{1}{2}u_{tt}(S_t, t)(\Delta t)^2 + \tfrac{1}{2}u_{SS}(S_t, t)(S_{t+\Delta t} - S_t)^2 \\
& + \tfrac{1}{2}u_{St}(S_t, t)\Delta t(S_{t+\Delta t} - S_t) + O\left((\Delta t)^2\right)(S_{t+\Delta t} - S_t) \\
& + O\left((\Delta t)(S_{t+\Delta t} - S_t)^2\right) + O\left((S_{t+\Delta t} - S_t)^3\right) + O(\Delta t)^3 + \cdots,
\end{aligned}
$$

where $u_{tt} = \frac{\partial^2 u}{\partial t^2}$, and $u_{St} = \frac{\partial^2 u}{\partial t \partial S}$. We need the informal rules

$$dt \cdot dt = 0, \quad dt \cdot dW = 0, \quad dW \cdot dW = dt \tag{3.11}$$

to reduce the Taylor series expansion above further.

Taking the limit $\Delta t \to 0$ and then applying the informal rules in (3.11) to the Taylor series expansion above yields

$$dh = u_t dt + u_S dS_t + \tfrac{1}{2}u_{SS} dS_t^2 \tag{3.12}$$

since $h = u$. Since S_t is an Itô process, it follows from (3.9) and (3.11) that

$$\begin{aligned} dS_t^2 &= (\nu(S_t,t)dt + \lambda(S_t,t)dW_t)^2 \\ &= \left(\nu(S_t,t)\right)^2(dt)^2 + 2\nu(S_t,t)\lambda(S_t,t)dtdW_t + \left(\lambda(S_t,t)\right)^2 dW_t^2 \qquad (3.13) \\ &= \left(\lambda(S_t,t)\right)^2 dt. \end{aligned}$$

Hence, substituting (3.9) and (3.13) into (3.12) gives

$$\begin{aligned} dh &= u_t dt + u_S\left(\nu(S_t,t)dt + \lambda(S_t,t)dW_t\right) + \tfrac{1}{2}u_{SS}\left((\lambda(S_t,t)\right)^2 dt \\ &= \left(u_t + \nu(S_t,t)u_S + \tfrac{1}{2}\left(\lambda(S_t,t)\right)^2 u_{SS}\right)dt + \lambda(S_t,t)u_S dW_t. \end{aligned}$$

Equation (3.9) is a SDE for the process S_t. It is a generalized *diffusion (Itô) process* (see [15, 28, 42]). This SDE for the process S_t is called a *diffusion process.* Its integration gives

$$S_t = S_0 + \int_0^t \nu(\tau, S_\tau)d\tau + \int_0^t \lambda(\tau, S_\tau)dW_\tau. \qquad (3.14)$$

The last integral in equation (3.14) is called a *stochastic (Itô) integral.*

We now use Itô's lemma to solve equation (3.6) as follows.

Let

$$u(S,t) = \ln S, \quad \nu(S,t) = \mu S \quad \text{and} \quad \lambda(S,t) = \sigma S.$$

Hence,

$$u_S = \tfrac{1}{S}, \quad u_{SS} = -\tfrac{1}{S^2} \quad \text{and} \quad u_t = 0.$$

Substituting these expressions into equation (3.10) gives

$$dh = (\mu - \tfrac{1}{2}\sigma^2)dt + \sigma dW_t.$$

Therefore, applying Itô calculus gives the geometric Brownian motion process in equation (3.6) under the transformation $u = \ln S$ as

$$\begin{aligned} du &= \tfrac{dS_t}{S_t} \\ &= (\mu - \tfrac{1}{2}\sigma^2)dt + \sigma dW_t. \end{aligned} \tag{3.15}$$

Equation (3.15) is integrable using Itô calculus. Integrating both sides of the last equality of this equation gives the solution to equation (3.6) as

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}, \quad S_0 > 0.$$

3.3.2 Arbitrage-Free Pricing

In finance and economics, the practice of taking advantage of differences in prices of an asset between two or more markets is called *arbitrage.* The profit is the difference in the market prices. An individual engaging in arbitrage is called an *arbitrageur.* The term arbitrage is commonly applied in trading of financial instruments like bonds, stocks, derivatives and currencies.

If market prices do not allow for profitable arbitrage, they are said to constitute arbitrage-free market or an arbitrage equilibrium.

Arbitrage possibilities can arise when any of the following is true.

1. Two assets with equal cash flows are not marketed at the same price.
2. Prices of a particular asset are different on different markets.
3. The asset's price today is not the discount price (at the risk-free interest rate) of the known future price of the asset. This is because an asset, for example grains has an appreciable storage costs. This condition is not true for a security.

Prices in different markets tend to converge to the same price due to the effect of arbitrage [37].

Theorem 3.3.4 *Consider a discrete-time financial-market model with finitely many possible random outcomes. If there exists a martingale measure with positive probabilities, then the market is arbitrage-free. Conversely, if the market is arbitrage-free, then there exists a martingale measure with positive probabilities (see Theorem 6.2 of [15]).*

The fundamental theorem of finance/arbitrage (see Theorem 3.3.4) in a general sense relates arbitrage opportunities with risk-neutral measures equivalent to the original probability measures.

The fundamental theorem of arbitrage free pricing in a finite state market can be broken down into two parts which state that (see Harrison and Pliska [26])

1. there is no arbitrage if and only if a risk-neutral measure equivalent to the original probability measure exists, and

2. a market is complete if and only if there is a unique risk-neutral measure equivalent to the original probability measure.

The fundamental theorem of pricing is therefore a way of converting the concept of arbitrage to a question about whether a risk-neutral measure exists or not.

Chapter 4

Black-Scholes Option Pricing Theory

In this chapter Black-Scholes option pricing will be broken down into *standard option valuation* and *modified option valuation.* We will discuss the theory behind the two types of option valuation and present the models used.

4.1 Standard Option Valuation Theory

Financial instruments can be divided into *basic securities* and their 'derivatives'. Basic securities are further subdivided into *fixed income* (i.e. bonds, bank accounts, etc), and *equities* (i.e. stocks). Derivatives are subdivided into options (i.e. calls, puts and exotic options), swaps, futures and forwards (see Cvitanić and Zapatero [15]).

Derivatives are contracts which depend on a fundamental asset, i.e. a basic security. In the absence of the security there could be no future claims. The random nature of the underlying security filters through to the 'derivatives' (see Baxter and Rennie [3]). Derivative contracts are also termed as *contingent claims.* The contracts can reduce risk for example by fixing the price of a future transaction now or they can magnify it.

The results of the linear Black-Scholes equation which gave rise to the field of Financial Mathematics were obtained by considering an option maturing at time T for a non-dividend-paying stock. In this classical (or standard option valuation) theory, there is no change in the price of a security for any order size, i.e. the trader does not move the market. In addition the price process of a security is independent of the past. This means that an investor's trading strategy has a temporary impact on the price process.

A *call (put)* option is a contract where at a prescribed time in future, known as the *expiry date* T, the holder of the option may buy (sell) a prescribed asset known as the *underlying asset* for a prescribed amount known as the *exercise/strike price* K. The opposite party has the obligation to sell (buy) the asset if the holder chooses to buy (sell) it. An option's value is therefore a function of various parameters in the contract, such as the time to expiry T and strike price K. It also depends on the asset's properties such as its drift μ and volatility σ, its current market price S_t and time t, and the continuously compounded risk-free interest rate r. The option's value can therefore be written as $u(S_t, t; \sigma, \mu; K, T; r)$.

The following assumptions are used for modelling the financial market described above.

Assumption 4.1.1

1. The price S_t of an underlying asset (i.e. stock) follows a geometric Brownian motion. The stock volatility σ and the drift μ are constant for $0 \le t \le T$ and are known in advance.
2. The risk-free interest rate r is a known constant for $0 \le t \le T$.
3. No dividends are paid in the period $0 \le t \le T$.
4. The option is of European type.

5. No transaction costs (including taxes) are incurred in buying or selling either the stock or the option.

6. The price of the underlying security is divisible so that any fraction of the share of the security can be traded at the risk-free interest rate r.

7. There are no arbitrage opportunities.

8. Delta hedging is done continuously.

The market is said to be *complete* under the assumptions above. This means that any asset and any derivative can be hedged or replicated with the portfolio of other assets in the market.

4.1.1 Linear Black-Scholes Model

The first assumption in Assumption 4.1.1 above means that

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad \mu > 0, \quad \sigma > 0. \tag{4.1}$$

The SDE (4.1) is called the *Black-Scholes model.*

4.1.2 Linear Black-Scholes Partial Differential Equation

We now let Π_t be a portfolio's value of one long option position [i.e. $u(S,t)$] and a short position in some quantity Δ, delta, of the underlying asset S. Hence,

$$\Pi_t = u(S,t) - \Delta S. \tag{4.2}$$

From Itô's lemma we have

$$\lambda(S_t,t)dW_t = dS_t - \nu(S_t,t)dt.$$

Substituting this expression into (3.10) gives

$$du = (u_t + \tfrac{1}{2}\sigma^2 S^2 u_{SS})dt + u_S dS \tag{4.3}$$

since $\lambda(S,t) = \sigma S$. Hence, the portfolio changes by

$$d\Pi_t = (u_t + \tfrac{1}{2}\sigma^2 S^2 u_{SS})dt + (u_S - \Delta)dS \tag{4.4}$$

as Δ is constant during the time-step dt. The right hand side of equation (4.4) is the sum of the deterministic and random terms, i.e. the terms with dt and dS respectively.

The risk in our portfolio is the random terms. We can reduce or even eliminate the risk by carefully choosing Δ. This is done by *delta hedging*. We can *delta hedge* by choosing

$$\Delta = u_S. \tag{4.5}$$

This leaves us with a portfolio whose value changes by the amount

$$d\Pi_t = (u_t + \tfrac{1}{2}\sigma^2 S^2 u_{SS})dt. \tag{4.6}$$

The change in equation (4.6) is completely riskless as the equation does not have random components. The security Π_t in equation (4.6) is therefore said to be "risk-free" as its dynamics do not have stochastic components after delta hedging. This means that

$$d\Pi_t = r\Pi_t dt, \quad \Pi_0 = 1. \tag{4.7}$$

Integrating equation (4.7) and simplifying gives

$$\Pi_t = e^{rt}$$

since $\Pi_0 = 1$. This is an example of the *no-arbitrage* principle. The absence of arbitrage means that contingent claims that can be replicated through a trading strategy could be priced using expectations under a risk-neutral probability measure $\mathbb{Q}$.

To obtain the linear Black-Scholes PDE, substitute equation (4.5) into (4.2) to get

$$\Pi_t = u - Su_S.$$

Then, plug into the right hand side of equation (4.7) to get

$$d\Pi_t = r(u - Su_S)dt, \quad \Pi_0 = 1. \tag{4.8}$$

Equating the right hand side of equations (4.6) and (4.8) we get

$$(u_t + \tfrac{1}{2}\sigma^2 S^2 u_{SS})dt = r(u - Su_S)dt.$$

Divide through by dt and rearrange to get

$$u_t + \frac{1}{2}\sigma^2 S^2 u_{SS} + rSu_S - ru = 0 \quad \text{in} \quad \mathbb{R} \times [0, T]. \tag{4.9}$$

This is the famous linear *Black-Scholes Partial Differential Equation.*

4.1.3 Black-Scholes Option Pricing Formulae

To specify the values of a derivative at the boundaries where possible values of the stock price S_t and time t lie, we use boundary conditions. For a European call option whose price at time t is $c_t = c(t, S_t)$, the boundary conditions are

1. $c(t, 0) = 0$ for $0 \leq t \leq T$,

2. $c(t, S_t) \sim S_t - Ke^{-r(T-t)}$ as $S_t \to \infty$,

where $T - t$ is time to maturity. The *payoff function* for a call option is given by the terminal condition

$$c(T, S_T) = (S_T - K)^+ = \max\{S_T - K, 0\} \quad \text{for} \quad 0 \leq S_T \tag{4.10}$$

since the call option can only be exercised if $S_T > K$. The second condition has to be understood as

$$\lim_{S_t \to \infty} \frac{c(t,S_t)}{S_t - Ke^{-r(T-t)}} = 1$$

uniformly for $0 \leq t \leq T$.

At maturity the expected value of the payoff function is $\mathbb{E}_{\mathbb{Q}}(\max\{S_T - K, 0\})$ where $\mathbb{Q}$ is the martingale measure for the discounted stock whose value at time t is given by

$$\psi_t = e^{-rt} S_t. \tag{4.11}$$

At time $t = T$, using (4.10) and (4.11) gives the call option price as

$$\begin{aligned} c_t &= e^{rt}\mathbb{E}_{\mathbb{Q}}(e^{-rT}c_T|S_t) \\ &= e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}(c_T|S_t) \\ &= e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left((S_T - K)^+\right), \quad 0 \leq t \leq T, \end{aligned} \tag{4.12}$$

where $c_T = (S_T - K)^+$ is the terminal claim (see Figure 4.1). Equation (4.12) tells us that at time $t = 0$,

$$c_0 = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left((S_T - K)^+\right), \quad 0 \leq T. \tag{4.13}$$

Equation (4.13) gives the value of the replicating strategy at time $t = 0$.

The value $(S_T - K)^+$ depends only on the stock price at expiry time T.

This means that we need only to find the marginal distribution of S_T under $\mathbb{Q}$ in order to find the expectation of the terminal claim c_T.

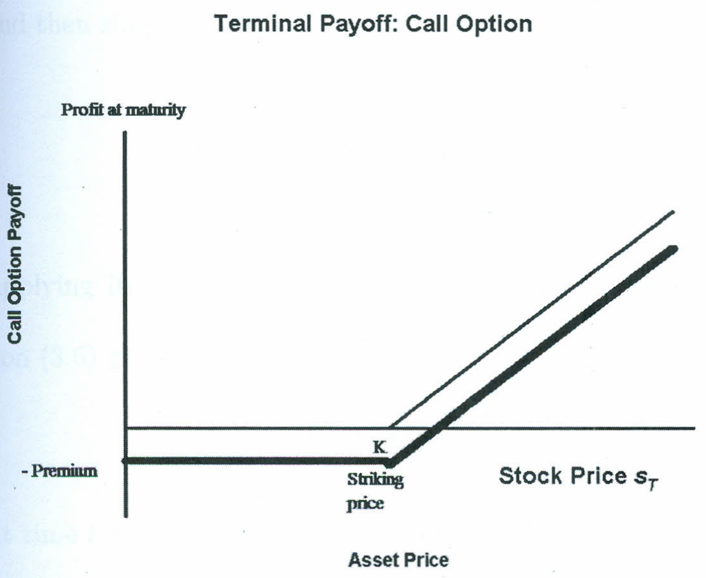

Figure 4.1: Call option terminal payoff.

To do that, we look at the process S_t that is written in terms of a $\mathbb{Q}$-Brownian motion.

Define

$$\tilde{W}_t = W_t + \frac{(\mu - r)t}{\sigma}, \tag{4.14}$$

where W_t is a $\mathbb{P}$-Brownian motion (see Definition 3.1.8). By Lévy's theorem (see Theorem 3.16 in Karatzas and Shreve [29], pp. 157), $\tilde{W}_t$ is a $\mathbb{Q}$-Brownian motion. Rearrange equation (4.14) and substitute into (4.1) and then simplify to get

$$dS_t = rS_t dt + \sigma S_t d\tilde{W}_t, \quad r > 0, \quad \sigma > 0. \tag{4.15}$$

Applying Ito's lemma to equation (4.15) the same way we did to equation (3.6) gives

$$S_t = S_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma \tilde{W}_t}. \tag{4.16}$$

At time $t = T$, equation (4.16) becomes

$$S_T = S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma \tilde{W}_T}. \tag{4.17}$$

Therefore, the marginal distribution for S_T is S_0 times the exponential of a normal *probability density function* with mean $(r - \frac{1}{2}\sigma^2)T$ and variance $\sigma^2 T$.

In Figure 3.2 the total area under the curve is 1 since this is a standard normal distribution. Since $N(x) = Pr(X \leq x)$, this means that $N(x)$ is represented by the area under the curve excluding the unshaded area on the upper tail of the distribution, i.e. $\frac{\alpha}{2}$. Hence,

$$N(x) = 1 - \tfrac{\alpha}{2}. \tag{4.18}$$

The normal distribution is symmetric about the mean zero and hence the two unshaded areas on the lower and upper tails are equal.

By Definition 3.1.2, the function

$$N(-x) = Pr(X \leq -x)$$

represents the unshaded area on the lower tail. Hence,

$$N(-x) = \tfrac{\alpha}{2}. \tag{4.19}$$

From equations (4.18) and (4.19) we get

$$N(x) + N(-x) = 1.$$

Therefore,

$$N(-x) = 1 - N(x). \tag{4.20}$$

We now calculate c_0, the value of the replicating strategy at time $t = 0$ as follows:

Let

$$Y \sim N\left(-\tfrac{1}{2}\sigma^2(T-t), \sigma^2(T-t)\right). \tag{4.21}$$

At time $t = 0$ equation (4.21) gives a distribution of the form

$$Z \sim N(-\tfrac{1}{2}\sigma^2 T, \sigma^2 T). \tag{4.22}$$

From equations (4.17) and (4.22) we get

$$S_T = S_0 e^{rT+Z}.$$

Thus, the value of the claim c_0 in equation (4.13) can be written as

$$c_0 = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left((Se^{rT+z} - K)^+\right), \quad 0 \leq T, \tag{4.23}$$

where $z \in \mathbb{R}$ is drawn from the continuous random variable Z and S is the value of the stock.

From the boundary condition $c(t, 0) = 0$ in Subsection 4.1.3 the terminal condition in (4.23) gives

$$(Se^{rT+z} - K)^+ = 0.$$

Therefore, if we use

$$Se^{rT+z} - K = 0, \tag{4.24}$$

the equation (4.24) simplifies to give

$$z = \ln\left(\tfrac{K}{S}\right) - rT.$$

We will now take the limit x in equation (3.1) to be $z = \ln(K/S) - rT$ when computing the replicating strategy at time $t = 0$ (i.e c_0).

The value of the payoff function in equation (4.23) can now be written as

$$(S_T - K)^+ = (Se^{rT+z} - K)^+ = \begin{cases} Se^{rT+z} - K & \text{if} \quad S_T > K, \\ 0 & \text{if} \quad S_T \leq K. \end{cases} \tag{4.25}$$

When $S_T > K$ the call option is exercised and when $S_T \leq K$ the call option is not exercised. Using equations (4.23) and (4.25) gives c_0 as follows:

$$
\begin{aligned}
c_0 &= e^{-rT}\int_{-\infty}^{+\infty}(Se^{rT+y}-K)^{+}\frac{1}{\sqrt{2\pi\sigma^2 T}}e^{-\frac{\left(y+\frac{1}{2}\sigma^2 T\right)^2}{2\sigma^2 T}}dy \\
&= e^{-rT}\int_{-\infty}^{z}\left(0\times\frac{1}{\sqrt{2\pi\sigma^2 T}}e^{-\frac{\left(y+\frac{1}{2}\sigma^2 T\right)^2}{2\sigma^2 T}}\right)dy \\
&+ e^{-rT}\int_{z}^{+\infty}(Se^{rT+y}-K)\frac{1}{\sqrt{2\pi\sigma^2 T}}e^{-\frac{\left(y+\frac{1}{2}\sigma^2 T\right)^2}{2\sigma^2 T}}dy \\
&= e^{-rT}\int_{z}^{+\infty}(Se^{rT+y}-K)\frac{1}{\sqrt{2\pi\sigma^2 T}}e^{-\frac{\left(y+\frac{1}{2}\sigma^2 T\right)^2}{2\sigma^2 T}}dy.
\end{aligned} \tag{4.26}
$$

Let

$$\frac{y+\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = x. \tag{4.27}$$

Then, the lower limit of the last integral of equation (4.26) changes from $z = \ln(K/S) - rT$ to

$$\varepsilon = \frac{\ln(K/S)-(r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \tag{4.28}$$

since z is a limit of y in equation (4.26). From (4.27) we get $dy = \sigma\sqrt{T}dx$. Plugging the new limit and variables to equation (4.26) gives c_0 as follows:

$$
\begin{aligned}
c_0 &= e^{-rT}\int_{\varepsilon}^{+\infty}\left(Se^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}x}-K\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx \\
&= Se^{-\frac{\sigma^2 T}{2}}\int_{\varepsilon}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{\sigma\sqrt{T}x-\frac{x^2}{2}}\,dx - Ke^{-rT}\left[1-\int_{-\infty}^{\varepsilon}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx\right] \\
&= Se^{-\frac{\sigma^2 T}{2}}\int_{\varepsilon}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{\sigma\sqrt{T}x-\frac{x^2}{2}}\,dx - Ke^{-rT}[1-N(\varepsilon)] \\
&= Se^{-\frac{\sigma^2 T}{2}}\int_{\varepsilon}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{\sigma\sqrt{T}x-\frac{x^2}{2}}\,dx - Ke^{-rT}N(-\varepsilon)
\end{aligned}
\tag{4.29}
$$

using equations (3.1) and (4.20). Completing the squares and simplifying equation (4.29) we have

$$
c_0 = S\int_{\varepsilon}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-\sigma\sqrt{T})^2}{2}}\,dx - Ke^{-rT}N(-\varepsilon). \tag{4.30}
$$

To simplify equation (4.30) further, let

$$
x-\sigma\sqrt{T}=w.
$$

Then

$$
dx = dw.
$$

The lower limit of equation (4.30) changes from ε to $\varepsilon - \sigma\sqrt{T}$ due to the change of variables from x to w. Equation (4.30) now becomes

$$\begin{aligned} c_0 &= S\int_{\varepsilon-\sigma\sqrt{T}}^{+\infty} \tfrac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} dw - Ke^{-rT}N(-\varepsilon) \\ &= S\left[1 - \int_{-\infty}^{\varepsilon-\sigma\sqrt{T}} \tfrac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} dw\right] - Ke^{-rT}N(-\varepsilon) \\ &= S\left[1 - N(\varepsilon - \sigma\sqrt{T})\right] - Ke^{-rT}N(-\varepsilon) \\ &= SN(-\varepsilon + \sigma\sqrt{T}) - Ke^{-rT}N(-\varepsilon) \end{aligned} \tag{4.31}$$

using equations (3.1) and (4.20). Substituting equation (4.28) into (4.31) gives

$$c_0 = SN\left(\tfrac{\ln(S/K)+(r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) - Ke^{-rT}N\left(\tfrac{\ln(S/K)+(r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right).$$

Since c_0 is the value of the replicating strategy at time $t = 0$, using equation (4.12), the formula for a replicating strategy at time $t = T$ becomes

$$c_t = SN(d_1) - Ke^{-r(T-t)}N(d_2), \tag{4.32}$$

where

$$d_1 = \tfrac{\ln(S/K)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \tag{4.33}$$

and

$$d_2 = \frac{\ln(S/K)+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \tag{4.34}$$

Reciprocally, for a put option whose value at time t is $p(t, S_t)$, the terminal condition is given by

$$p(T, S_T) = (K - S_T)^+ = \max\{K - S_T, 0\} \quad \text{for} \quad 0 \leq S_T$$

as the option can only be exercised if $K > S_T$. The terminal claim $(K - S_T)^+$ is shown in Figure 4.2 below. Its boundary conditions are

1. $p(t, 0) = Ke^{-r(T-t)}$ for $0 \leq t \leq T$,
2. $p(t, S_t) \to 0$ as $S_t \to \infty$.

The value of a put option can be found as we did for the call option or by using *put-call parity.* The put-call parity is the result that relates the prices of the European call and put options and is given by

$$c(t, S) + Ke^{-r(T-t)} = p(t, S) + S. \tag{4.35}$$

Equation (4.35) means that the sum of the call price $c(t, S)$ and the present value of K currency units (such as dollars) in the bank equals to the sum of the put price $p(t, S)$ and the stock price S (see Cvitanić and Zapatero [15]). When we substitute equation (4.32) into (4.35) and then rearrange, we get

$$p(t, S) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1) \tag{4.36}$$

using equation (4.20), where d_1 and d_2 are as defined in equations (4.33) and (4.34) respectively.

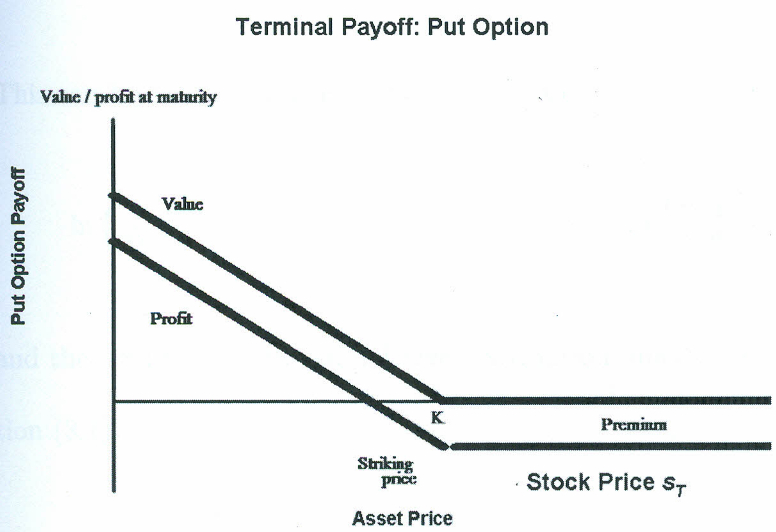

Figure 4.2: Put option terminal payoff.

4.1.4 The Greeks

When a financial institution sells an option, the only problem it is facing is that of hedging risks (see Hull [28]). The *Greeks* or *risk parameters* are used to address this problem.

We can use the following fact to simplify the option Greeks for linear option valuation theory:

$$SN'(d_1) - Ke^{-r(T-t)}N'(d_2) = 0. \tag{4.37}$$

This can be proved by considering the relation

$$\ln\left(\frac{SN'(d_1)}{Ke^{-r(T-t)}N'(d_2)}\right) = \ln(S/K) + r(T-t) + \ln\left(\frac{N'(d_1)}{N'(d_2)}\right) \tag{4.38}$$

and the definition of the cumulative distribution function given in equation (3.1).

From equation (3.1), we get

$$N'(d_1) = \frac{1}{\sqrt{2\pi}}e^{-d_1^2/2}, \tag{4.39}$$

and

$$N'(d_2) = \frac{1}{\sqrt{2\pi}} e^{-d_2^2/2}. \tag{4.40}$$

Substituting equations (4.39) and (4.40) into the last logarithm of equation (4.38) gives

$$\begin{aligned} \ln\left(\frac{N'(d_1)}{N'(d_2)}\right) &= \ln\left(\frac{(1/\sqrt{2\pi})e^{-d_1^2/2}}{(1/\sqrt{2\pi})e^{-d_2^2/2}}\right) \\ &= -\frac{1}{2}(d_1^2 - d_2^2). \end{aligned} \tag{4.41}$$

Using equations (4.33) and (4.34), we can write

$$d_1^2 - d_2^2 = 2\ln(S/K) + 2r(T-t). \tag{4.42}$$

Therefore, from equations (4.41) and (4.42) we get

$$\ln\left(\frac{N'(d_1)}{N'(d_2)}\right) = -\ln(S/K) - r(T-t).$$

Plugging the last expression into equation (4.38) proves the relation

$$\ln\left(\frac{SN'(d_1)}{Ke^{-r(T-t)}N'(d_2)}\right) = 0,$$

which is equivalent to (4.37).

Similarly,

$$SN'(-d_1) - Ke^{-r(T-t)}N'(-d_2) = 0. \tag{4.43}$$

We now use equations (4.37) and (4.43) to simplify the options Greeks as follows.

Theta

The rate at which the price of the option changes with time with all else remaining the same is called *theta* and it is denoted by Θ. It is sometimes referred to as the *time decay* of the option. The name 'time decay' is used since theta measures the rate at which the option value changes with time if the asset price doesn't move (see Wilmott [61]). Hence, the theta of a call option is obtained by differentiating both sides of equation (4.32) with respect to t as follows:

$$\begin{aligned}\Theta_{\text{call}} &= \tfrac{\partial c_t}{\partial t} \\ &= \tfrac{\partial}{\partial t}\left[SN(d_1) - Ke^{-r(T-t)}N(d_2)\right] \\ &= -\tfrac{\sigma SN'(d_1)}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2)\end{aligned}$$

by using equations (4.33) and (4.34) and then applying equation (4.37).

For a put option, we differentiate both sides of equation (4.36) with respect to t to get theta as follows:

$$\begin{aligned}\Theta_{\text{put}} &= \frac{\partial p}{\partial t} \\ &= \frac{\partial}{\partial t}\left(Ke^{-r(T-t)}N(-d_2) - SN(-d_1)\right) \\ &= -\frac{\sigma S N'(-d_1)}{2\sqrt{T-t}} + rKe^{-r(T-t)}N(-d_2)\end{aligned}$$

by using equations (4.33) and (4.34) and then applying equation (4.43).

Delta

The *delta*, Δ, of an option is the sensitivity of the option to the underlying asset's price S. It is the rate at which the option value changes with respect to the asset's price and is therefore the first derivative of the value of the option with respect to the asset price. It is the slope of the curve which relates the price of the option to the price of the underlying asset. Hence, to get the delta for a call option we differentiate both sides of equation (4.32) with respect to S to get

$$\begin{aligned}\Delta_{\text{call}} &= \frac{\partial c_t}{\partial S} \\ &= N(d_1)\end{aligned} \tag{4.44}$$

on applying equations (4.33), (4.34), and (4.37).

For a put option, we differentiate both sides of equation (4.36) with respect to S as follows:

$$\begin{aligned}\Delta_{\text{put}} &= \frac{\partial p}{\partial S} \\ &= \frac{\partial}{\partial S}\left(Ke^{-r(T-t)}N(-d_2) - SN(-d_1)\right) \\ &= N(d_1) - 1\end{aligned} \tag{4.45}$$

by applying equations (4.33), (4.34), and (4.43) and then using equation (4.20).

Gamma

An option's *gamma*, Γ, is the rate at which the delta of the option changes with respect to the underlying asset's price. It is therefore the second partial derivative of the option position with respect to the price of the underlying asset S. For a European call option, Γ is given by

$$\begin{aligned} \Gamma_{\text{call}} &= \frac{\partial N(d_1)}{\partial S} \\ &= \frac{N'(d_1)}{\sigma S\sqrt{T-t}} \end{aligned} \tag{4.46}$$

by using equations (4.33) and (4.44).

For a put option, gamma is obtained by differentiating both sides of equation (4.45) once with respect to S as follows:

$$\begin{aligned} \Gamma_{\text{put}} &= \frac{\partial}{\partial S}\left[N(d_1) - 1\right] \\ &= \frac{N'(d_1)}{\sigma S\sqrt{T-t}} \end{aligned}$$

by using (4.33).

Speed

The rate at which gamma changes with respect to the price of the stock S is called the option's *speed.* Hence,

$$\text{Speed} = \frac{\partial \Gamma}{\partial S}.$$

The speed for a call option can be obtained by differentiating equation (4.46) with respect to S as follows:

$$\text{Speed} = \frac{\partial}{\partial S}\left(\frac{N'(d_1)}{\sigma S\sqrt{T-t}}\right). \tag{4.47}$$

We can use the fact that

$$N''(d_1) = -d_1 N'(d_1) \tag{4.48}$$

to simplify equation (4.47). We prove the relation in equation (4.48) as follows.

Differentiating both sides of (4.39) with respect to d_1 gives

$$\begin{aligned} N''(d_1) &= -d_1 \frac{1}{\sqrt{2\pi}} e^{-d_1^2/2} \\ &= -d_1 N'(d_1). \end{aligned}$$

Hence, from equation (4.47), we get

$$\begin{aligned} \text{Speed} &= -\frac{N'(d_1)}{\sigma S^2\sqrt{T-t}} + \frac{N''(d_1)}{\sigma^2 S^2(T-t)} \\ &= -\frac{N'(d_1)}{\sigma^2 S^2(T-t)}\left(d_1 + \sigma\sqrt{T-t}\right) \end{aligned}$$

by using equations (4.33) and (4.48).

Since

$$\Gamma_{\text{call}} = \Gamma_{\text{put}},$$

this means that the speed of a put option equals to that of a call option.

Vega

In all the risk parameters considered above, the implicit assumption we have made is that volatility σ is constant. In practice, volatility changes with time, which means that a derivative's value is liable to change due to movements in volatility and also due to changes in the price of the asset S and the passage of time t. The parameter *vega*, also known as *zeta* or *kappa* (see Wilmott [61]), is the sensitivity of the price of the option to volatility. It is the rate at which the price of the option changes with respect to the volatility of the underlying asset.

To compute vega for a call option, differentiate both sides of equation (4.32) with respect to σ to get

$$\begin{aligned}
\text{Vega}_{\text{call}} &= \tfrac{\partial c_t}{\partial \sigma} \\
&= \tfrac{\partial}{\partial \sigma}\left[SN(d_1) - Ke^{-r(T-t)}N(d_2)\right] \\
&= S\sqrt{T-t}N'(d_1)
\end{aligned}$$

by substituting equations (4.33) and (4.34) and then using equation (4.37).

For a put option we differentiate equation (4.36) with respect to σ as follows:

$$\begin{aligned}
\text{Vega}_{\text{put}} &= \frac{\partial p}{\partial \sigma} \\
&= \frac{\partial}{\partial \sigma}\left(Ke^{-r(T-t)}N(-d_2) - SN(-d_1)\right) \\
&= S\sqrt{T-t}N'(-d_1)
\end{aligned}$$

on substituting equations (4.33) and (4.34) and then using equation (4.43).

Rho

The sensitivity of the option value to the interest rate r used in the Black-Scholes formulae is called *rho*. To get the rho for a call option, differentiate both sides of equation (4.32) with respect to r to get

$$\begin{aligned}
\text{rho}_{\text{call}} &= \frac{\partial c_t}{\partial r} \\
&= \frac{\partial}{\partial r}\left[SN(d_1) - Ke^{-r(T-t)}N(d_2)\right] \\
&= K(T-t)e^{-r(T-t)}N(d_2)
\end{aligned}$$

on substituting equations (4.33) and (4.34) and then applying equation (4.37).

For a put option, rho is obtained by differentiating both sides of equation (4.36) with respect to r to get

$$\begin{aligned}\text{rho}_{\text{put}} &= \frac{\partial p}{\partial r} \\ &= \frac{\partial}{\partial r}\left[Ke^{-r(T-t)}N(-d_2) - SN(-d_1)\right] \\ &= -K(T-t)e^{-r(T-t)}N(-d_2)\end{aligned}$$

by using equations (4.33) and (4.34) and then applying equation (4.43).

4.2 Modified Option Valuation Theory

In the standard option valuation theory discussed in Section 4.1, we take the market (risk-neutral) dynamics as given and then calculate derivative prices.

However, when we have a feedback loop or market frictions it means that derivative hedging will lead to market dynamics. The feedback loop and market frictions resulting from hedging renders the use of the standard Black-Scholes model inappropriate in option valuation.

Nonlinearities in diffusion models can arise from insect dispersal, heat conduction and illiquid market effects. Our focus in this work is on the nonlinearity arising from illiquid market effects.

Two primary assumptions are used in formulating classical arbitrage pricing theory. These are the *frictionless* and *competitive markets* assumptions. Relaxing the competitive market assumption can completely change the standard theory. As such, manipulation of the market may become an issue and pricing of an option becomes market structure- and trader-dependent. Under market manipulation, the price process of a security can depend on the entire history of the investor's past trades up to the current trade. Eliminating this path-dependent condition rules out market manipulation and allows use of the classical arbitrage pricing theory.

The notion of *liquidity risk* (see Cetin *et al.* [11] for further details) is introduced on relaxing the two assumptions above. This risk, roughly speaking, is the additional risk resulting from timing and the size of a trade. Cash liquidity risk is of concern in a situation where a firm's involvement in derivatives is more pronounced [13].

The nonlinear Black-Scholes PDEs in illiquid markets have been derived in order to model the following.

1. *Transaction costs* arising in hedging of derivatives. The models under this category are called the *(quadratic) transaction-cost models.* The markets involved here are said to be markets with friction.

2. *Feedback effects* due to large traders. The models here are subdivided further into

   i. the *reduced-form SDE* models, and

   ii. the *equilibrium* or *reaction-function* models.

Two assets are used in all the models above. These are

1. a bond (or a risk-free *money market account*) with interest rate $r \geq 0$, and

2. a stock. This is an illiquid asset, i.e. its price is affected by trading.

The stock has no maturity date while a bond matures at the end of the term. The interest rate r is a *spot rate* of interest and is assumed to be zero in the illiquid market models above. The value in the money market account (or bond's value) is given by

$$\begin{aligned} B_t &= e^{\int_0^t r_s ds} \\ &= e^{\int_0^t r ds} \\ &= e^{rt} \end{aligned} \tag{4.49}$$

since $r_t = r$, i.e. the rate of interest r_t is assumed to be constant. For modelling purposes, the bond (or money market account) has been taken to be a *numéraire* whose value has been set at 1 [since $r = r_t = 0$ in equation (4.49)] so that the value of the bond (or money market account) throughout time t is $B_t \equiv 1$. This choice of the value of the bond simplifies the model (see Shreve [55]). It is assumed that the bond's market is liquid (or perfectly elastic). With this assumption, large amounts of the bond are traded without affecting its price. This is a reflection of the fact that money markets are more liquid compared to the stock markets [23].

There are two types of investors in all these nonlinear models. These are *fundamental investors* and *hedgers*.

4.2.1 Transaction-Cost Models

The market under the standard option valuation model discussed in Section 4.1 is assumed to be *frictionless.* Hence, there are no transaction costs incurred in trading either the stock or the option. This assumption is not realistic given the scale of hedging activities on many financial markets (see Bordag and Frey [6]). For instance we cannot avoid telephone charges in the process of placing an order.

The frictionless market assumption has been relaxed by Cetin *et al.* [11, 12] who have put forward the predominant model in the transaction cost model for illiquid markets. In the economy under consideration, the study is on the trading of a stock and a bond (or money market account).

In the transaction-cost model, a fundamental stock price process S_t^0 follows the dynamics

$$dS_t^0 = \mu S_t^0 dt + \sigma S_t^0 dW_t, \quad t \in [0, T]$$

for a Brownian motion W and constant parameters $\mu \in \mathbb{R}$ and $\sigma > 0$. The stock price S_t^0 is called the *bid* price.

In the classical theory discussed in Section 4.1, the trader receives the same price for any order size, i.e. the trader does not move the market. In the economy under transaction costs the price received depends on the hedger's order size, i.e. the trader moves the market.

Option prices increase due to transaction costs [12]. These costs make the price of a traded security dependent on the size of a trade (see Cetin *et al.* [11]). When trading $\alpha = \alpha_t$ shares, the *transaction price* to be paid by the investor at time t for his purchase/sale is

$$\overline{S}_t(\alpha) = e^{\rho\alpha} S_t^0, \quad \alpha \in \mathbb{R}, \quad 0 \leq \rho < 1. \tag{4.50}$$

The liquidity parameter ρ models the liquidity of the market. When $\rho = 0$ we have a perfectly liquid market that was described in Section 4.1. Large ρ means that trading has a substantial impact on the transaction price. The prices here are therefore a continuum of stochastic processes which are indexed by trade size. The transaction price $\overline{S}_t(\alpha)$ is called the *ask* price. Hence, a bid-ask-spread whose size depends on the amount α is modelled by the transaction price in the model (4.50). This leads to transaction costs which are proportional to quadratic variation of the stock trading strategy as shown in Cetin *et al.* [11].

To explain the statement above in more detail, consider a self-financing trading strategy $(\Phi_t, \beta_t)_{t\geq 0}$, where Φ_t and β_t are the stock and bond positions at time t respectively.

Definition 4.2.1 A self-financing portfolio is one where the change in its value only depends on the change of the prices of the asset(s).

From this definition, the change in the *marked-to-market (or paper) value* of the strategy at time t whose value is

$$V_t^M = \Phi_t S_t^0 + \beta_t \tag{4.51}$$

is given by

$$dV_t^M = \Phi_t dS_t^0 \tag{4.52}$$

since β_t is constant throughout time t. This marked-to-market value is the value of the position using current market prices. For a self-financing strategy $(\Phi_t, \beta_t)_{t\geq 0}$, the marked-to-market value in equation (4.51) represents the portfolio's value under the classical price-taking condition that the trader does not move the market.

The price process of the security in the transaction-cost models is independent of the past. This is the *Markov property* discussed in Subsection 3.1.3. A *Markovian trading strategy* is a trading strategy of the form $\Phi_t = \phi(t, S_t^0)$ where ϕ is a smooth function. The investor's trading strategy Φ_t has a temporary impact on the price process since the price process is independent of the past.

Since $\Phi_t = \phi = u_S(t, S_t^0)$, the quadratic variation for the stock trading strategy $(\Phi_t, \beta_t)_{t\geq 0}$ is obtained as follows.

Since the deterministic component of a SDE is not important in computing the quadratic variation of a given process, we will assume that the deterministic component is zero for simplicity. We now let

$$X_t = W_t$$

or

$$dX_t = dW_t, \tag{4.53}$$

where W_t is a Wiener process. By Theorem 28, pp. 17 in Protter [48], the quadratic variation of the process $\{X_t, t \geq 0\}$ satisfying the SDE (4.53) is given by $\langle X \rangle t = \langle W, W \rangle t = t$ (see Definition 3.1.5).

Suppose that the coefficient of dW_t in equation (4.53) is $\sigma(\neq 1)$. We can rewrite equation (4.53) as

$$dX_t = \sigma dW_t. \tag{4.54}$$

The quadratic variation of the process $\{X_t, t \geq 0\}$ satisfying the SDE (4.54) is given by

$$\langle X \rangle t = \langle \sigma W, \sigma W \rangle t = \sigma^2 t$$

or

$$\langle X \rangle t = \int_0^t \sigma^2 ds.$$

Suppose that instead of the arithmetic Brownian motion process described in equation (4.54) above we have a geometric Brownian motion process $\{S_t^0, t \geq 0\}$ such that

$$\frac{dS_t^0}{S_t^0} = \sigma dW_t$$

or

$$dS_t^0 = \sigma S_t^0 dW_t. \tag{4.55}$$

The quadratic variation of S_t^0 in equation (4.55) becomes

$$\langle S \rangle t = (\sigma S_t^0)^2 t$$

or

$$\langle S \rangle t = \int_0^t (\sigma S_s^0)^2 ds.$$

Hence,

$$d\langle S \rangle t = (\sigma S_t^0)^2 dt. \tag{4.56}$$

We now introduce a semimartingale $\Phi_t \in \mathbb{L}$ such that $\mathbb{E}\left\{\int_0^t \Phi_s^2 ds\right\} < \infty$ for all $t \geq 0$ where $\mathbb{L}$ is a left continuous process. To enable us derive the quadratic variation when semimartingales are involved, we let ϕ and S be continuous quadratic variation functions, where ϕ is smooth. Then

$$\begin{aligned}\langle \Phi \rangle t &= \langle \phi \circ S \rangle t \\ &= \int_0^t \phi'(S_s, s)^2 d\langle S \rangle s \\ &= \int_0^t \phi_S^2 d\langle S \rangle s.\end{aligned} \tag{4.57}$$

Hence, the change in quadratic variation of a semimartingale Φ_t in equation (4.57) becomes

$$d\langle \Phi \rangle t = \phi_S^2 d\langle S \rangle t. \tag{4.58}$$

Substitution of equation (4.56) into (4.58) gives

$$d\langle \Phi \rangle t = (\sigma \phi_S(t, S_t^0) S_t^0)^2 dt \tag{4.59}$$

or

$$\langle \Phi \rangle t = \int_0^t \left(\sigma \phi_S S_s^0\right)^2 ds.$$

We now apply Itô formula to the process $(u(t, S_t^0))_{t\geq 0}$ the way we did to obtain (4.3) to get the dynamics

$$du(t, S_t^0) = \left(u_t(t, S_t^0) + \tfrac{1}{2}\sigma^2 (S_t^0)^2 u_{SS}(t, S_t^0)\right) dt + u_S(t, S_t^0) dS_t^0. \tag{4.60}$$

The liquidity parameter ρ in equation (4.51) is zero since the equation is for standard option valuation. If $\rho > 0$, then the liquidation value of the portfolio is lower than its marked-to-market value (see Bordag and Frey [6]). This is because transaction costs incurred in the process of trading reduce the marked-to-market value V_t^M given in equation (4.51). The amount by which the change in the marked-to-market value reduces as trading goes on is an extra transaction cost which results from the limited market liquidity (see Frey and Polte [24]), i.e. when $\rho > 0$ a trader incurs an extra transaction cost. The amount of this extra transaction cost is $-\rho S_t^0 d\langle\Phi\rangle t$ (see for instance Bordag and Frey [6]). Adding this extra transaction cost into equation (4.52) gives the wealth dynamics of a self-financing strategy for a continuous semimartingale Φ_t with quadratic variation $\langle\Phi\rangle t$ as

$$dV_t^M = \Phi_t dS_t^0 - \rho S_t^0 d\langle\Phi\rangle t \quad \text{as} \quad t \to \infty. \tag{4.61}$$

A self-financing strategy by construction does not generate cash flows for all times $t \in [0, T)$, i.e. stock purchase/sale must be obtained through borrowing/investing in the money market account.

Substituting equation (4.59) into (4.61) yields the dynamics of V_t^M, i.e.

$$dV_t^M = \phi(t, S_t^0)dS_t^0 - \rho S_t^0 \left(\sigma\phi_S(t, S_t^0)S_t^0\right)^2 dt \quad \text{as} \quad t \to \infty, \tag{4.62}$$

where $\phi_S = \frac{\partial\phi}{\partial S}$ and $\Phi_t = \phi(t, S_t^0)$. Equations (4.61) and (4.62) are the results from Theorem A3 of Cetin *et al.* [11].

Since

$$V_t^M = u(t, S_t^0),$$

this means that

$$dV_t^M = du(t, S_t^0).$$

Equating the right hand side of equations (4.60) and (4.62) gives

$$\begin{aligned} u_S(t, S_t^0)dS_t^0 + \left(u_t(t, S_t^0) + \tfrac{1}{2}\sigma^2(S_t^0)^2 u_{SS}(t, S_t^0)\right) dt = \\ \phi(t, S_t^0)dS_t^0 - \rho S_t^0 \left(\sigma\phi_S(t, S_t^0)S_t^0\right)^2 dt. \end{aligned} \tag{4.63}$$

Using the relation $u_S = \phi$ means that the random terms in equation (4.63) are equivalent. We equate the deterministic terms and simplify to get

$$u_t + \tfrac{1}{2}\sigma^2 S^2 u_{SS} + \rho\sigma^2 S^3 (u_{SS})^2 = 0 \tag{4.64}$$

since $\phi_S = u_{SS}$. Rearranging equation (4.64) gives the nonlinear PDE for u as

$$u_t + \tfrac{1}{2}\sigma^2 S^2 u_{SS}(1 + 2\rho S u_{SS}) = 0, \quad u(S_T, T) = h(S_T), \quad S_T \geq 0. \tag{4.65}$$

The hedge-cost $u(S,t)$ of the terminal claim $h(S_T)$ is given by the solution of equation (4.65) which we intend to get.

If $\rho = 0$, then the asset's price S_t^0 in equation (4.65) follows the standard Black-Scholes model with constant volatility σ.

4.2.2 Feedback Models

Reduced-Form SDE Models

The assumption made in this modelling approach is that investors are *large traders* as their stock trading strategy affects equilibrium stock prices [6].

We now consider some Brownian motion W_t and two constants σ and $\rho \geq 0$ where σ is volatility and ρ is a liquidity parameter. The parameter ρ is a characteristic of the market. This parameter determines the strength of the price impact and it does not depend on the payoff of the hedged derivatives. The value of ρ is fixed in the process of trading. Use of the parameter ρ results into a model that can be viewed as a perturbation resulting from the standard Black-Scholes model (4.1). The parameter ρ controls the size of this perturbation.

If a semimartingale Φ_t represents a trader's stock trading strategy, the stock price process S_t is assumed to satisfy the SDE

$$dS_t = \rho S_t d\Phi_t + \sigma S_t dW_t. \tag{4.66}$$

The semimartingale Φ_t is a right-continuous process Φ^+ with $\Phi_t^+ = \lim_{s \underset{\to}{>} t} \Phi_s$ for Φ_t shares at time t. By choosing the strategy $(\Phi_t, \beta_t)_{t \geq 0}$, we are imposing exogenously rather than deriving the form and size of the hedger's trades' price-impact. This simplifies the analysis considerably. The property of the resulting stock-price dynamics is that if the large trader sells the stock, i.e. $\Delta\Phi_t < 0$, the stock price S_t falls (and if he buys, i.e. $\Delta\Phi_t > 0$, the price rises) by $\rho S_{t-} \Delta\Phi_t$ since the agent has increased (limited) the supply of shares. The notation S_{t-} stands for the left limit $\lim_{s \underset{\to}{<} t} S_t$. If the representative hedger does not trade, i.e. $\Phi_t \equiv 0$ and/or $\rho = 0$, the price of the asset follows the standard Black-Scholes model with constant volatility σ since the deterministic component in equation (4.66) vanishes when ρ (or $d\Phi_t$) $= 0$ leaving behind a linear Black-Scholes model.

The asset price process resulting when the liquidity parameter takes the value ρ and if the large trader uses a trading strategy Φ_t is denoted by

$S_t(\rho, \Phi_t)$. Suppose that the hedger uses a Markovian trading strategy $\Phi_t = \phi(t, S_t)$ for a smooth function ϕ and that ϕ satisfies the constraint

$$1 - \rho S \phi_S(t, S) > 0 \quad \text{for all} \quad (t, S).$$

Given a liquidity parameter ρ, the constraint above limits the permissible variations in the large trader's stock trading strategy.

Applying Itô formula to the stock price process $S_t = S_t(\rho, \Phi_t)$ in equation (4.66) gives the dynamics

$$dS_t = \nu(t, S_t) S_t dt + \upsilon(t, S_t) S_t dW_t, \tag{4.67}$$

where the function υ is the adjusted volatility and is given by

$$\upsilon(t, S) = \frac{\sigma}{1 - \rho S \phi_S(t, S)}, \tag{4.68}$$

while the function ν is given by

$$\nu(t, S_t) = \frac{\rho}{1 - \rho S \phi_S(t, S)} \left(\phi_t(t, S) + \frac{\sigma^2 S^2 \phi_{SS}(t, S)}{2(1 - \rho S \phi_S(t, S))^2} \right),$$

where $\phi_{SS} = \frac{\partial^2 \phi}{\partial S^2}$. See Frey [22] for a detailed derivation of equation (4.68).

We note that the adjusted volatility $v(t, S)$ increases (decreases) relative to the constant volatility σ if $\phi_S > 0$ ($\phi_S < 0$). This means that if a positive (negative) feedback strategy is used by the trader, when the stock price rises, this calls for additional buying (selling) and when it falls the hedger needs to sell (buy). Market movement will be accelerated (slowed) by this hedging demand. The dynamics of S are therefore affected by the form of the strategy Φ_t. This *feedback effect* (i.e. positive or negative ϕ_S) gives rise to the wealth dynamic's nonlinearity. The magnitude of the feedback effect of the strategy used by the hedger is determined by ρS in equation (4.68). Large ρ implies a big market-impact of hedging since this value of ρ leads to large ρS.

Using model (4.66) and Definition 4.2.1 we derive the nonlinear Black-Scholes PDE resulting from the SDE (4.67) as follows.

Suppose that $V_t^M = u(t, S_t)$ and the stock trading strategy used is $\Phi_t = \phi(t, S_t)$ for smooth functions u and ϕ. Applying Itô's formula to the process $(u(t, S_t(\rho, \Phi)))_{t \geq 0}$ gives $\phi = u_S$. We get the gains from the self-financing strategy Φ_t as follows.

Since

$$\int_0^t dV_s^M = V_t^M - V_0^M,$$

where V_0^M is an initial investment and V_t^M is the time t marked-to-market value, this means that the gains from the strategy are given by

$$\begin{aligned} G_t &= V_t^M - V_0^M \\ &= \int_0^t dV_s^M. \end{aligned} \tag{4.69}$$

Hence, from equations (4.52) and (4.69) we have

$$G_t = \int_0^t \Phi_s dS_s(\rho, \Phi).$$

The function $u(t, S_t)$ according to Bordag and Frey [6], and Frey [22] must satisfy

$$u_t + \tfrac{1}{2}v^2(t, S)S^2 u_{SS} = 0. \tag{4.70}$$

Since $\phi = u_S$, then $\phi_S = u_{SS}$. We substitute equation (4.68) into (4.70) and then use $\phi_S = u_{SS}$ to get the nonlinear PDE for $u(t, S)$ as

$$u_t + \tfrac{1}{2}\tfrac{\sigma^2}{(1-\rho S u_{SS})^2} S^2 u_{SS} = 0, \quad u(S_T, T) = h(S_T), \quad S_T \geq 0. \tag{4.71}$$

The *market depth* at time t is given by $\frac{1}{\rho S}$.

Equilibrium or Reaction-Function Models

There are two types of traders in the market modelled by *equilibrium models.* The first type of traders are *reference traders* who are ordinary investors. These traders are the majority in the market. Such traders are "price-takers". By this we mean that the traders cannot influence the asset price on the market. These traders trade in such a way that the equilibrium stock price S_t follows the dynamics

$$dS_t = a(S_t, t)dt + v(S_t; t)dW_t.$$

The price S_t is the proposed price of the stock [6]. We can consider all reference traders as a single aggregate reference trader who represents a total action of all reference traders [54]. We have two limitations for the single aggregate reference trader. These are

1. an aggregate stochastic income F_t, i.e. the total income of all the reference traders, which can be described by the equation

$$dF_t = \psi(F_t, t)dt + v(F_t, t)dW_t,$$

and

2. a demand function $\tilde{D}(S_t, F_t, t)$ which is a function of the income F_t and of the equilibrium price process S_t.

The second type of traders are *programme* or *large traders.* These traders trade the asset following a Black-Scholes dynamic hedging strategy. The main reason for trading of an asset is to hedge their position against the risk. Programme traders are large enough to change the price of the asset corresponding to their own trading strategy [54].

The large trader's demand function can be represented by

$$\phi(t, S_t) = \varsigma \Phi_t,$$

where ς is the volume of options hedged and Φ_t is a smooth function of time t representing the demand per security hedged.

The price process of the asset is determined by some fundamental value F and a market equilibrium. The assumption made in the model by Papanicolaou and Sircar [44] is that the supply of an asset s_0 is constant and that D is the demand of the reference traders relative to the constant supply s_0. This means that

$$\tilde{D}(S, F, t) = s_0 D(S, F, t),$$

where F is the value taken by the aggregate stochastic income F_t.

The relative demand function of both the reference and large traders at time t is

$$G(S, F, t) = D(S, F, t) + \rho\Phi(S, t),$$

where ρ is the liquidity parameter (i.e. ratio of the volume of options being hedged to the total supply of the stock) and $\rho\Phi(t, S) = \rho\phi(t, S) = \alpha$ is the stock position of a large trader for a smooth function ϕ.

We now set the overall demand and supply equal to one at each point in time. This means that market equilibrium will be equal to one, i.e.

$$\begin{aligned} G(S_t, F_t, t) &= D(S_t, F_t, t) + \rho\phi(t, S_t) \\ &\equiv 1. \end{aligned}$$

By determining the relationship between the stock price S_t and the income F_t in the function

$$G(S_t, F_t, t) \equiv 1,$$

we assume that the function G is smooth and satisfies the conditions of the Implicit Function Theorem [54] (see Theorem (§C.6), in Evans [18], pp. 633).

In this case we obtain

$$S_t = \psi(F_t, \rho\phi(t, S_t)),$$

where ψ is some smooth reaction function.

The smooth reaction function ψ is a function of F_t and defines the process S_t as a process that follows the same Brownian motion as the process F_t. The fundamental-value process F_t is assumed to follow a geometric Brownian motion with volatility σ. The reaction function ψ takes values in $\mathbb{R}_+$ and is also assumed to be of the form $\psi(f, \alpha) = f\lambda(\alpha)$ for some increasing function λ (see [46]). Hence,

$$S_t = \lambda(\rho\phi(t, S_t))F_t. \tag{4.72}$$

Assuming that the trading strategy of the large trader is of the form $\rho\phi(t, S_t)$ for a smooth function ϕ, Bordag and Frey [6] apply Itô's formula to equation (4.72) to get

$$dS_t = \lambda(\rho\phi(t, S_t))dF_t + \rho F_t \lambda_\alpha(\rho\phi(t, S_t))\phi_S(t, S_t)dS_t + b(t, S_t)dt, \tag{4.73}$$

where $\lambda_\alpha = \frac{\partial \lambda}{\partial \alpha}$. We now assume that

$$1 - \rho F_t \lambda_\alpha(\rho\phi(t, S_t))\phi_S(t, S_t) > 0 \qquad \text{a.s.} \tag{4.74}$$

We can view the constraint in (4.74) as an upper bound on permissible variations of the large trader's strategy [6].

Bordag and Frey [6] have shown that rearranging and integrating the inverse of the left hand side of inequality (4.74) over both sides of equation (4.73) gives the dynamics of S_t as

$$dS_t = \frac{1}{1-\rho\frac{\lambda_\alpha(\rho\phi(t,S_t))}{\lambda(\rho\phi(t,S_t))}S_t\phi_S(t,S_t)}\sigma S_t dW_t + \widetilde{b}(t,S_t)dt.$$

Reasoning the same way as in the reduced-form SDE models gives the PDE

$$u_t + \frac{1}{2}\frac{\sigma^2}{\left(1-\rho\frac{\lambda_\alpha(\rho u_S)}{\lambda(\rho u_S)}S u_{SS}\right)^2}S^2 u_{SS} = 0, \quad u(S_T,T) = h(S_T), \quad S_T \geq 0 \quad (4.75)$$

for the value function $u(t,S_t)$ of a self-financing strategy α. When $\lambda(\alpha) = \exp(\alpha)$ as in Platen and Schweizer [46] we have $\lambda = \lambda_\alpha$ and equation (4.75) reduces to (4.71). When the transformation $\lambda(\alpha) = \frac{1}{1-\alpha}$ is used in (4.75), the same PDE reduces to

$$u_t + \frac{1}{2}\sigma^2 S^2 u_{SS}\left(\frac{1-\rho u_S}{1-\rho u_S-\rho S u_{SS}}\right)^2 = 0, \quad u(S_T,T) = h(S_T), S \geq 0 \quad (4.76)$$

since $\alpha = \rho\phi = \rho u_S$.

4.2.3 Variants of Nonlinear Black-Scholes Equations

The nonlinear Black-Scholes PDEs (2.2), (4.65), (4.71), (4.75), and (4.76) are of the form

$$u_t + \frac{1}{2}\sigma^2 v_\rho^2(\rho u_S, \rho S u_{SS}) S^2 u_{SS} = 0, \quad u(S_T, T) = h(S_T), \quad S_T \geq 0, \tag{4.77}$$

where the "volatility" $\sigma v_\rho(\rho u_S, \rho S u_{SS})$ is an increasing function of the "gamma" u_{SS}.

The parameter ρ is often considered to be small. Therefore, replacing v_ρ^2 with its first order Taylor approximation around $\rho = 0$ using the linearization $v_\rho^2(\rho u_S, \rho S u_{SS}) \approx 1 + 2\rho S u_{SS}$ reduces equation (4.71) to the PDE (4.65). This linearization can be shown by taking

$$\begin{aligned} \tfrac{\partial}{\partial \rho} v_\rho^2(\rho u_S, \rho S u_{SS})|_{\rho=0} &= 2S u_{SS}(1 - \rho S u_{SS})^{-3}|_{\rho=0} \\ &= 2S u_{SS} \end{aligned}$$

so that

$$v_\rho^2(\rho u_S, \rho S u_{SS}) \approx 1 + 2\rho S u_{SS}.$$

Chapter 5

The Korteweg-de Vries Equation

Consider the Korteweg-de Vries equation (2.3) in Section 2.7. It is the simplest wave equation. We seek for a traveling wave solution which has the structure

$$u(x,t) = \nu(\xi), \quad \xi = x - Ct, \tag{5.1}$$

where $\xi, x \in \mathbb{R}$ and $t > 0$. This is a wave of permanent form and the wave translates to the right with speed $C > 0$. By the chain rule we have

$$u_t = -C\nu'(\xi), \quad u_x = \nu'(\xi), \quad \text{and} \quad u_{xxx} = \nu'''(\xi).$$

Substituting these expressions into the KdV equation (2.3), we conclude that u solves the KdV equation, provided that ν satisfies the third order ODE

$$-C\nu' + 6\nu\nu' + \nu''' = 0. \tag{5.2}$$

The ODE (5.2) can be solved in a closed-form since integrating it gives

$$-C\nu + 3\nu^2 + \nu'' = \delta_1, \tag{5.3}$$

where δ_1 denotes some constant. Multiplying this equation by ν' gives

$$-C\nu\nu' + 3\nu^2\nu' + \nu''\nu' = \delta_1\nu'. \tag{5.4}$$

Assume that the traveling wave is *localized*, that is, at large distances, the solution $u(t,x)$ together with its derivatives are small. This means that

$$\lim_{x\to\pm\infty} u(t,x) = \lim_{x\to\pm\infty} u_x(t,x) = \lim_{x\to\pm\infty} u_{xx}(t,x) = 0,$$

where $u_{xx} = \frac{\partial^2 u}{\partial x^2}$. We now impose the localizing boundary conditions

$$\lim_{\xi\to\pm\infty} \nu(\xi) = \lim_{\xi\to\pm\infty} \nu'(\xi) = \lim_{\xi\to\pm\infty} \nu''(\xi) = 0. \tag{5.5}$$

Integrating equation (5.4) gives

$$-\tfrac{C}{2}\nu^2 + \nu^3 + \tfrac{(\nu')^2}{2} = \delta_1\nu + \delta_2, \tag{5.6}$$

where δ_2 is also a constant of integration. With these localizing conditions, the function u with the form (5.1) is called a *solitary wave*. The localizing conditions imply that for equations (5.3), (5.4) and (5.6),

$$\delta_1 = \delta_2 = 0.$$

Hence, equation (5.6) simplifies to

$$\nu' = \pm\nu\sqrt{C - 2\nu}.$$

We take the negative sign for computational convenience. If we let

$$\xi = -\int_0^{\theta} \tfrac{dq}{q\sqrt{C-2q}} + \xi_0, \tag{5.7}$$

where ξ_0 is an integration constant and then substitute

$$q = \tfrac{C}{2}\mathrm{sech}^2 x,$$

it follows that

$$\frac{dq}{dx} = -C \text{ sech}^2 x \tanh x$$

and

$$q\sqrt{C-2q} = \frac{C^{3/2}}{2}\text{sech}^2 x \tanh x.$$

Then, equation (5.7) becomes

$$\xi = \frac{2}{\sqrt{C}}\theta + \xi_0, \tag{5.8}$$

where θ is implicitly given by the relation

$$\frac{C}{2}\text{sech}^2\theta = \nu(\xi). \tag{5.9}$$

Combining equations (5.8) and (5.9) we compute

$$\nu(\xi) = \frac{C}{2}\text{sech}^2\left(\frac{\sqrt{C}}{2}(\xi - \xi_0)\right) \quad (\xi \in \mathbb{R}).$$

We need to check routinely that ν as was defined actually solves the ODE (5.2). What we get is that

$$u(x,t) = \frac{C}{2}\text{sech}^2\left(\frac{\sqrt{C}}{2}(x - x_0 - Ct)\right) \quad (x \in \mathbb{R}, t \geq 0) \tag{5.10}$$

is a solution of the KdV equation for each $C > 0$ and $x_0 \in \mathbb{R}$. We call a solution of this form a *soliton* or *solitary wave solution* because equation (5.10) describes a localized travelling wave solution (see Figure 5.1). The localized wave travels with unchanged shape.

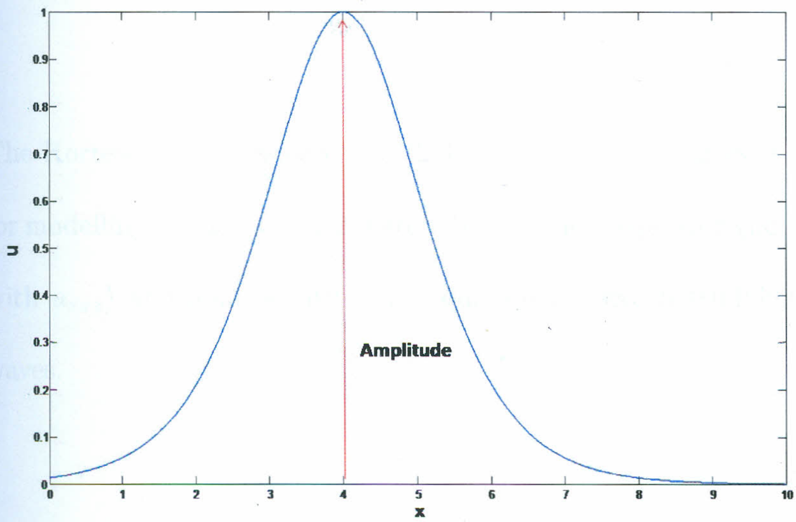

Figure 5.1: A solitary wave solution of KdV equation (5.10) for $C = 2$ and $x_0 = 2$ calculated at $t = 1$.

To compute the wave speed we rewrite equation (5.6) by assuming that

$$-\nu^3 + \tfrac{C}{2}\nu^2 = \nu^2(A - \nu), \quad 0 < A, \tag{5.11}$$

where A is the wave's amplitude (see Figure 5.1).

Equation (5.11) gives

$$C = 2A.$$

These solitons occur in nonlinear optics, fluid mechanics, and other nonlinear phenomena.

The Korteweg-de Vries equation (2.3) is a nonlinear dispersive equation for modelling surface waves in water. It combines dispersion (i.e. the term with u_{xxx}) and nonlinearity. The equation is used in studying solitary waves.

Chapter 6

Results and Discussion

The nonlinear Black-Scholes PDEs considered in Chapter 4 are all *singular perturbations* since they are modifications of the linear Black-Scholes PDE (4.9) through addition of a small multiple ρ times a higher order term u_{SS}.

The questions of *existence* and *uniqueness* of the solution to the nonlinear Black-Scholes equations discussed in Section 4.2 have been addressed in Theorem 3.1 of Frey and Polte [24]. General results in [24] have shown that a unique solution to equation (4.77) exists. These results are therefore applicable to the nonlinear equations (2.2), (4.65), (4.71), (4.75), and (4.76) since equation (4.77) is the general form of all the nonlinear Black-Scholes equations above.

Existence of the solution means that there is a solution to the problem satisfying all given conditions while *uniqueness* of the solution means that the problem has no more than one solution.

6.1 Solution to a Nonlinear Black-Scholes Equation

Now that a unique solution to the boundary-value problem (4.65) exists, we will solve it by direct integration. The primary solution methods we will apply to obtain its solution will be *transformations* and *traveling wave solution.*

To classify the nonlinear Black-Scholes equation (4.65) we differentiate it twice with respect to the spatial variable S and set $w = u_{SS}$ to get

$$w_t + \frac{\sigma^2 S^2}{2}(1+4\rho S w)w_{SS} + 2\rho\sigma^2 S^3 w_S^2 + 2\sigma^2 S(1+6\rho S w)w_S + \sigma^2(1+6\rho S w)w = 0, \tag{6.1}$$

where

$$w_t = \frac{\partial w}{\partial t}, \quad w_S = \frac{\partial w}{\partial S}, \quad \text{and} \quad w_{SS} = \frac{\partial^2 w}{\partial S^2}.$$

The general form of equation (6.1) is given by

$$F(S, t, w, w_S, w_t, w_{SS}, w_{St}, w_{tt}) = 0, \tag{6.2}$$

where

$$w_{St} = \frac{\partial^2 w}{\partial S \partial t}, \quad \text{and} \quad w_{tt} = \frac{\partial^2 w}{\partial t^2}.$$

We can rewrite equation (6.1) as

$$\frac{\sigma^2 S^2}{2}(1 + 4\rho S w) w_{SS} = f(S, t, w, w_t, w_S). \tag{6.3}$$

Equation (6.3) is a special case of equation (6.2) with

$$F(S, t, w, p, q, d) = \frac{\sigma^2 S^2}{2}(1 + 4\rho S w)p - f(S, t, w, w_t, w_S) = 0, \tag{6.4}$$

where

$$p = w_{SS}, \quad q = w_{St} = 0, \quad \text{and} \quad d = w_{tt} = 0.$$

Since from equation (6.4) we have

$$a = \frac{\partial F}{\partial p} = \frac{\sigma^2 S^2}{2}(1 + 4\rho S w), \quad b = \frac{1}{2}\frac{\partial F}{\partial q} = 0, \quad \text{and} \quad c = \frac{\partial F}{\partial d} = 0,$$

then, the discriminant of equation (6.4) is zero.

Hence, the nonlinear Black-Scholes equation (4.65) is a scalar parabolic equation.

If we let $\nu(\xi)$ be a twice continuously differentiable function, and x and t the spatial and time variables respectively, then there is a traveling wave solution of the form

$$V(x,t) = \nu(\xi), \quad \text{where} \quad \xi = x - Ct \tag{6.5}$$

for all $(x,t) \in \mathbb{R} \times (0,\infty)$. The function $V(x,t)$ in equation (6.5) is interpreted as the strength of the signal. Equation (6.5) is a bounded solution for the *signal* or *wave profile* at time t. When the conditions $\nu_1(\xi) > 0$ at $\xi \to +\infty$ and $\nu_2(\xi) > 0$ at $\xi \to -\infty$ are added to the equation that is solved to obtain $V(x,t)$, the traveling wave solution $V(x,t)$ is called a *wavefront solution.* The wavefront solution is termed as a *pulse* if $V(x,t)$ approaches the same constant values at both plus and minus infinity. Since the initial signal $V(x,0) = \nu(x)$, the profile at time t is represented by $\nu(x - Ct)$ which is an initial profile translated to the right

Ct spatial units [34]. The constant C represents the wave speed for a wave propagating undistorted along the characteristics $x - Ct =$ constant in spacetime [34]. We interpret the variable $\xi = x - Ct$ in equation (6.5) as a moving coordinate [34].

Proposition 6.1.1 *If $\nu(\xi)$ is a twice continuously differentiable function, and x and t are the spatial and time variables respectively, there exists a traveling wave solution to*

$$V_t + \left[D(V)\left(V_x + \tfrac{1}{2}V\right)\right]_x = 0 \quad in \quad \mathbb{R} \times (0, \infty) \tag{6.6}$$

of the form given in equation (6.5) for all $(x,t) \in \mathbb{R} \times (0,\infty)$ and $D(V) = \frac{\sigma^2}{2}V$ such that $V(x,t)$ is a traveling wave of permanent form which translates to the right with constant speed $C > 0$.

Proof 6.1.2 Applying the chain rule to equation (6.5) gives

$$V_t = -C\nu'(\xi), \quad V_x = \nu'(\xi), \quad \text{and} \quad V_{xx} = \nu''(\xi).$$

Substituting these expressions into equation (6.6) and since $D(\nu) = D'(\nu)\nu$ we conclude that $\nu(\xi)$ must satisfy the nonlinear second order ODE

$$-C\nu' + D\nu'' + D'(\nu')^2 + D\nu' = 0 \tag{6.7}$$

and hence V solves equation (6.6).

We also assume that the traveling wave is *localized.*

Equation (6.7) can now be solved in a closed-form. First write it as

$$\tfrac{d}{d\xi}(D\nu') + \tfrac{d}{d\xi}(\tfrac{1}{2}D\nu - C\nu) = 0 \tag{6.8}$$

since $D(\nu) = D'(\nu)\nu$. Integrating equation (6.8) we get the standard form [34]

$$\nu' = D^{-1}(C\nu - \tfrac{1}{2}D\nu + \kappa), \tag{6.9}$$

where κ is a constant of integration. Imposing the *localizing boundary conditions* (5.5) to equation (6.9) means that $\kappa = 0$. Rearranging equation (6.9) after applying the localizing boundary conditions (5.5) yields

$$\tfrac{\sigma^2}{2}\nu\nu' = C\nu - \tfrac{\sigma^2}{4}\nu^2 \tag{6.10}$$

since $D(\nu) = \frac{\sigma^2}{2}\nu$.

Introducing the two states of the signal at infinity (i.e. $\nu_1(\xi) > 0$ as

$\xi \to +\infty$ and $\nu_2(\xi) > 0$ as $\xi \to -\infty$) to the right hand side of (6.10) gives

$$C\nu_1 - \tfrac{\sigma^2}{4}\nu_1^2 = C\nu_2 - \tfrac{\sigma^2}{4}\nu_2^2.$$

Therefore, the wave speed is given by

$$C = \tfrac{\sigma^2}{4}(\nu_1 + \nu_2).$$

Since $\kappa = 0$ from the localizing boundary conditions (5.5) and $D(\nu) = \frac{\sigma^2}{2}\nu$, simplification of equation (6.9) further gives

$$\nu' = \tfrac{2}{\sigma^2}(C - \tfrac{\sigma^2}{4}\nu).$$

From this equation we conclude that $\nu(\xi)$ satisfies the first order linear autonomous ODE

$$-2\tfrac{d\nu}{d\xi} = \nu - \tfrac{4C}{\sigma^2}.$$

This equation is variable separable. Integrating it and simplifying gives

$$\nu(\xi) = e^{\frac{\xi_0 - \xi}{2}} + \tfrac{4C}{\sigma^2}, \quad \sigma > 0, \tag{6.11}$$

where ξ_0 is a constant of integration.

To get a twice continuously differentiable solution of equation (6.6) on $\mathbb{R}$ we use equations (6.5) and (6.11) to express it as

$$V(x,t) = e^{\frac{x_0-(x-Ct)}{2}} + \frac{4C}{\sigma^2}, \quad \sigma > 0, \quad t > 0, \tag{6.12}$$

where $\xi_0 = x_0 - Ct = x_0$ since the initial time $t = 0$.

In order to solve equation (4.65) analytically by direct integration we proceed as follows:

Theorem 6.1.3 *If $V(x,t)$ is any positive solution to the nonlinear groundwater equation $V_t + \left[D(V)\left(V_x + \frac{1}{2}V\right)\right]_x = 0$ in $\mathbb{R} \times (0,\infty)$, then,*

$$u(S,t) = \frac{1}{\rho}\left(-\sqrt{S}e^{\frac{Ct+s_0}{2}} + S(1 - lnS)(\tfrac{1}{4} - \tfrac{C}{\sigma^2}) + St\left(\tfrac{\sigma^2}{16} - \tfrac{C^2}{\sigma^2}\right) - \tfrac{\sigma^2}{16C}e^{Ct+s_0}\right)$$

solves the nonlinear Black-Scholes equation $u_t + \frac{1}{2}\sigma^2 S^2 u_{SS}(1 + 2\rho S u_{SS}) = 0$ for $S \in \mathbb{R}$, $x \in \mathbb{R}$, $t > 0$, $D(V) = \frac{\sigma^2}{2}V$ and for each $s_0 \in \mathbb{R}, C > 0$, $\sigma > 0$ and $1 > \rho > 0$.

Proof 6.1.4 Since the dynamical process (4.65) is first order in t, its solutions are expected to be uniquely prescribed by their initial values

$$u(S,0) = f(S), \quad -\infty < S < \infty.$$

We transform the reaction-advection-diffusion equation (6.1) by using two transformations. One of the transformations is

$$w = \frac{v}{\rho S}.$$

To understand the motivation behind this transformation we write it as follows:

$$\begin{aligned} v &= \frac{w}{(\rho S)^{-1}} \\ &= \frac{u_{SS}}{(\rho S)^{-1}} \end{aligned} \tag{6.13}$$

since $w = u_{SS}$. Equation (6.13) has the form of a dimensioned variable divided by a variable with the same dimension. We refer to the variable $(\rho S)^{-1}$ in the denominator as a *scale* [34]. This means that the parameter gamma (i.e u_{SS}) is being measured relative to the market depth $\frac{1}{\rho S}$.

The other transformation is

$$x = \ln S,$$

which is motivated by the fact that the stock price S follows a geometric Brownian motion, so that $\ln S$ describes a Brownian motion. Then, $\ln S$ should satisfy a diffusion equation.

Applying these two transformations to equation (6.1) and simplifying it gives

$$v_t + \tfrac{\sigma^2}{2}(1+4v)v_{xx} + 2\sigma^2 v_x^2 + \tfrac{\sigma^2}{2}(1+4v)v_x = 0, \tag{6.14}$$

where

$$v_t = \tfrac{\partial v}{\partial t}, \quad v_x = \tfrac{\partial v}{\partial x}, \quad \text{and} \quad v_{xx} = \tfrac{\partial^2 v}{\partial x^2}.$$

If we let

$$v = \tfrac{V-1}{4}$$

we get

$$v_t = \tfrac{V_t}{4}, \quad v_x = \tfrac{V_x}{4}, \quad \text{and} \quad v_{xx} = \tfrac{V_{xx}}{4},$$

where

$$V_t = \tfrac{\partial V}{\partial t}, \quad V_x = \tfrac{\partial V}{\partial x}, \quad \text{and} \quad V_{xx} = \tfrac{\partial^2 V}{\partial x^2}.$$

Substituting these expressions into equation (6.14) and simplifying the results gives

$$V_t + \frac{\sigma^2}{2}\left(VV_{xx} + V_x^2 + VV_x\right) = 0 \quad \text{in} \quad \mathbb{R} \times (0, \infty), \tag{6.15}$$

where VV_x represents a nonlinear advection or transport term. Equation (6.15) is a homogeneous second order nonlinear parabolic PDE of degree one.

Taking the diffusion coefficient to be $D(V)$, we can rewrite equation (6.15) as

$$V_t + D(V)V_{xx} + D'(V)V_x^2 + D(V)V_x = 0 \quad \text{in} \quad \mathbb{R} \times (0, \infty), \tag{6.16}$$

where $D(V)V_{xx}$ is a nonlinear *Fickian diffusion term* [34]. Comparing the terms in equations (6.15) and (6.16) we conclude that $D(V) = \frac{\sigma^2}{2}V$. Therefore, $D'(V) = \frac{\sigma^2}{2}$. The equation (6.16) can also be rewritten as

$$V_t + D(V)V_{xx} + \left[D'(V)V_x + D(V)\right]V_x = 0 \quad \text{in} \quad \mathbb{R} \times (0, \infty). \tag{6.17}$$

We recover from the *variable diffusion constant* in equation (6.17) a nonlinear advection term [34]

$$[D'(V)V_x + D(V)]\,V_x.$$

This implies a propagation signal of speed (see [34])

$$D'(V)V_x + D(V).$$

From the advection-diffusion equation (6.17), the Fick's law takes the form [34]

$$\phi(V) = D(V)\left(V_x + \tfrac{1}{2}V\right), \tag{6.18}$$

where $\phi(V)$ is the flux. We substitute equation (6.18) into (6.17) to get

$$V_t + \phi(V)_x = 0, \quad c(V) = \phi'(V), \tag{6.19}$$

where

$$\phi(V)_x = \tfrac{\partial}{\partial x}\phi(V)$$

and the *characteristic speed* $c(V)$ is a given smooth function of V. The nonlinear equation (6.19) is the same as equation (6.6).

Hence, from equation (6.6) we get the determined system

$$V_t + \tfrac{\sigma^2}{2}(VV_x + \tfrac{1}{2}V^2)_x = 0 \quad \text{in} \quad \mathbb{R} \times (0, \infty) \tag{6.20}$$

since $D(V) = \frac{\sigma^2}{2}V$.

Before assuming any form of a solution to the nonlinear PDE (6.6) we consider the *groundwater equation* (see equation (1.3.19) on pp. 31 of Logan [34]) given by

$$\kappa V_t - \tfrac{\delta}{\gamma}(Vf_x + gV^2)_x = 0, \tag{6.21}$$

where $f = f(x,t)$ is pressure, $f_x = \frac{\partial f}{\partial x}$, κ is the *porosity*, δ is *permeability* of the fluid, γ is the viscosity of the fluid, V is the density of the fluid with positive x measured downward, and g is the acceleration due to gravity. In this equation the parameters δ, γ and g are assumed to be positive constants. The porosity κ is also assumed to be constant. When $g = 0$, equation (6.21) becomes a *porous medium equation* used for modelling say a fluid (e.g., water) seeping downward through the soil. In a given volume of soil, a fraction κ of the total space (or volume) is available to the fluid while the remaining space is reserved for the soil itself [34] (see

Figure 6.1).

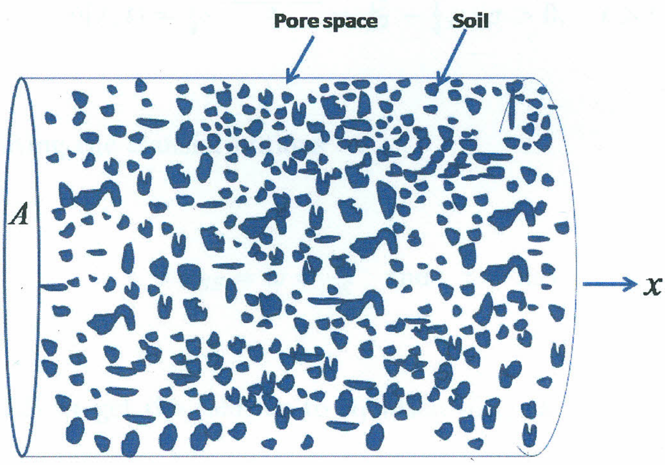

Figure 6.1: A chemical dissolved in a liquid contained in a porous medium of cross-sectional area A.

Comparing equations (6.20) and (6.21) we conclude that equation (6.20) is of the form (6.21) when $\kappa = -1$, $\frac{\delta}{\gamma} = \frac{\sigma^2}{2}$, $f_x = V_x$, and $g = \frac{1}{2}$.

Groundwater equations are applied in nonlinear problems of suspension transport in porous media e.g. a contaminant transport in groundwater (see [34, 47]).

Substituting

$$v = \frac{V-1}{4}$$

into (6.12) gives the solution to equation (6.14) as

$$v(x,t) = \tfrac{1}{4}e^{\frac{x_0-(x-Ct)}{2}} + \tfrac{C}{\sigma^2} - \tfrac{1}{4}, \quad \sigma > 0, \quad t > 0. \tag{6.22}$$

Substituting the transformations

$$u_{SS} = w = \frac{v}{\rho S} \quad \text{and} \quad x = \ln S$$

into (6.22) we get the solution to equation (6.1) as

$$u_{SS} = \frac{1}{\rho S}\left(\frac{1}{4\sqrt{S}}e^{\frac{Ct+s_0}{2}} + \frac{C}{\sigma^2} - \frac{1}{4}\right) \tag{6.23}$$

for all $\rho > 0$, $S > 0$, $\sigma > 0$, and $t > 0$, where

$$x_0 = s_0.$$

Hence, the parameter gamma given by u_{SS} in equation (6.23) is a solitary wave solution to equation (6.1).

Integrating u_{SS} in equation (6.23) twice with respect to the spatial variable S, we arrive at the (Black-Scholes formula) solution $u(S,t)$ of the nonlinear Black-Scholes PDE (4.65) (see Theorem 6.1.3).

REMARK 6.1.5 We call the solution u_{SS} in equation (6.23)

1. a *solitary wave solution* or simply a *soliton* as the solution u_{SS} decays to zero at large distances, i.e, $\lim_{S\to\infty} u_{SS} = 0$ [see equation (6.23)], and

2. *gamma* when taken as a risk parameter.

6.2 Applicability of the Solution and the Option Greeks

The stock price data from the Nairobi Stock Exchange will be used to plot curves of the solution of the nonlinear Black-Scholes equation and the risk parameters derived from the solution in order to test whether the solution is applicable in a real life situation. A subset of the data for KenGen and KPLC for the periods between 2nd January 2007 – 24th December 2007 and 3rd January 2003 – 2nd January 2004 respectively, i.e. one year for each company, will be used.

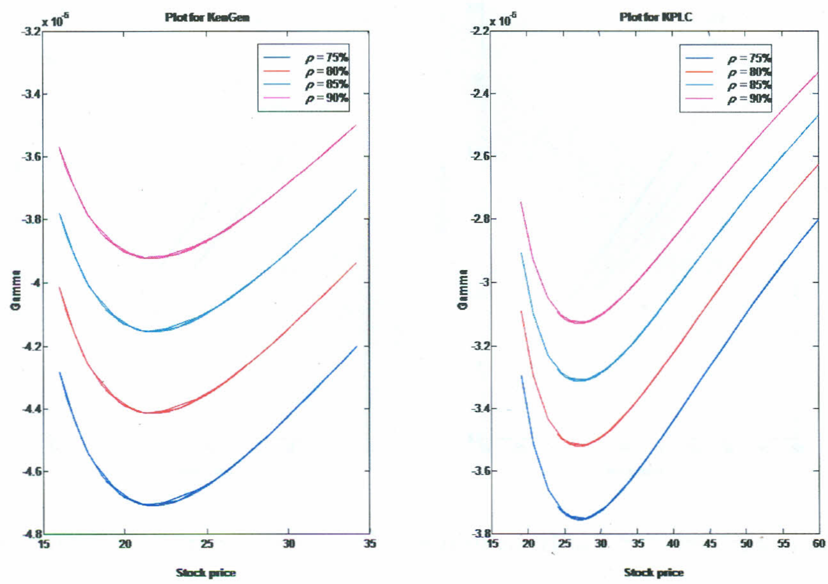

Figure 6.2: Variation of gamma u_{SS} with stock price S for a call option for $C = 2$ (KenGen), $C = 2.2$ (KPLC), $s_0 = 0.1$, $\sigma = 0.1$ and $t = 1$.

The curves in Figure 6.2 are going from up to down with decreasing value of the liquidity parameter ρ.

Figure 6.3 represents a plot of the solution of the nonlinear Black-Scholes equation (4.65) against stock prices for various values of the liquidity parameter ρ. The curves in this figure are going from down to up with

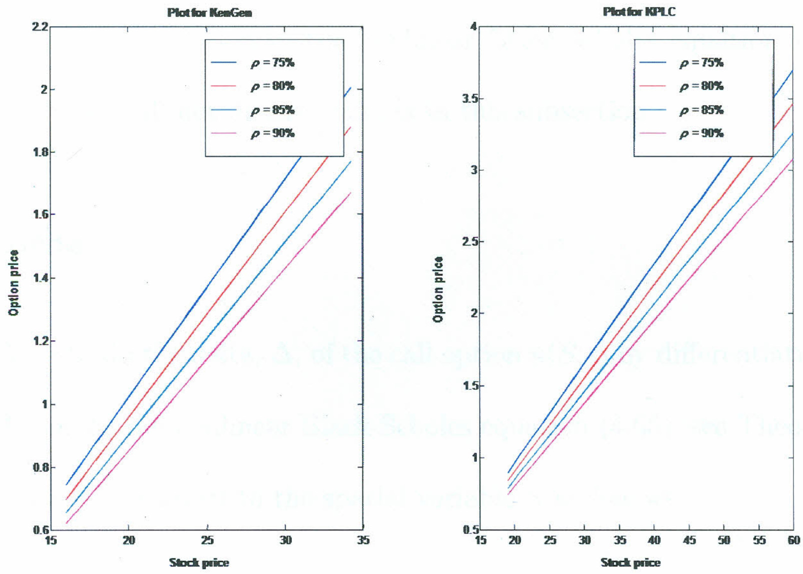

Figure 6.3: The solution of the nonlinear Black-Scholes equation (4.65) for $C = 2$ (KenGen), $C = 2.2$ (KPLC), $s_0 = 0.1$, $\sigma = 0.1$ and $t = 1$.

decreasing values of the liquidity parameter ρ. These rising curves with rising liquidity (i.e. reducing value of ρ) mean that the stock markets tend to become more liquid if derivative markets are introduced. This supports the empirical evidence by Mayhew [38]. It is clear from these curves that the gradient u_S is positive.

6.2.1 The Greeks

In this subsection we compute the risk parameters resulting from the solution of the nonlinear Black-Scholes equation. We then plot these risk parameters against the prices of the stock. The risk parameter gamma that is represented by u_{SS} was obtained in Section 6.1 in the process of finding the solution to the nonlinear Black-Scholes equation (4.65). As such, we will not discuss gamma in this subsection.

Delta

We obtain the delta, Δ, of the call option $u(S,t)$ by differentiating the solution to the nonlinear Black-Scholes equation (4.65) [see Theorem 6.1.3] once with respect to the spatial variable S as follows:

$$\begin{aligned}\Delta &= u_S \\ &= \frac{1}{\rho}\left(-\frac{1}{2\sqrt{S}}e^{\frac{Ct+s_0}{2}} - \ln S\left(\tfrac{1}{4} - \tfrac{C}{\sigma^2}\right) + t\left(\tfrac{\sigma^2}{16} - \tfrac{C^2}{\sigma^2}\right)\right)\end{aligned} \tag{6.24}$$

for all $\rho > 0$, $S > 0$, $\sigma > 0$ and $t > 0$.

The curves in Figure 6.4 are going from up to down with increasing value of the liquidity parameter ρ. These curves show that the gradient which

is given by u_{SS} is negative.

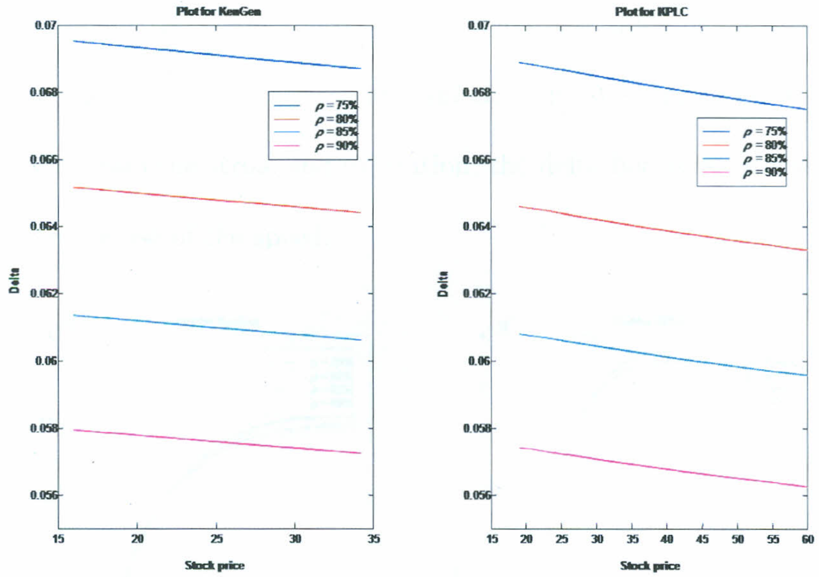

Figure 6.4: Variation of delta u_S with stock price S_t for a call option for $C = 2$ (KenGen), $C = 2.2$ (KPLC), $s_0 = 0.1$, $\sigma = 0.1$ and $t = 1$.

Speed

Differentiating gamma, u_{SS}, in equation (6.23) with respect to the spatial variable S, we get the option's speed as

$$\text{Speed} = \frac{1}{\rho S^2}\left(-\frac{3}{8\sqrt{S}}e^{\frac{Ct+s_0}{2}} - \frac{C}{\sigma^2} + \frac{1}{4}\right)$$

for all $\rho > 0$, $S > 0$, $\sigma > 0$, and $t > 0$.

Gamma is used by traders to estimate how much they will rehedge by if the stock price moves. An option delta may change by more or less the amount the traders have approximated the value of the stock price to change. If it is by a large amount that the stock price moves, or the option nears the strike and expiration, the delta becomes unreliable and hence the use of the speed.

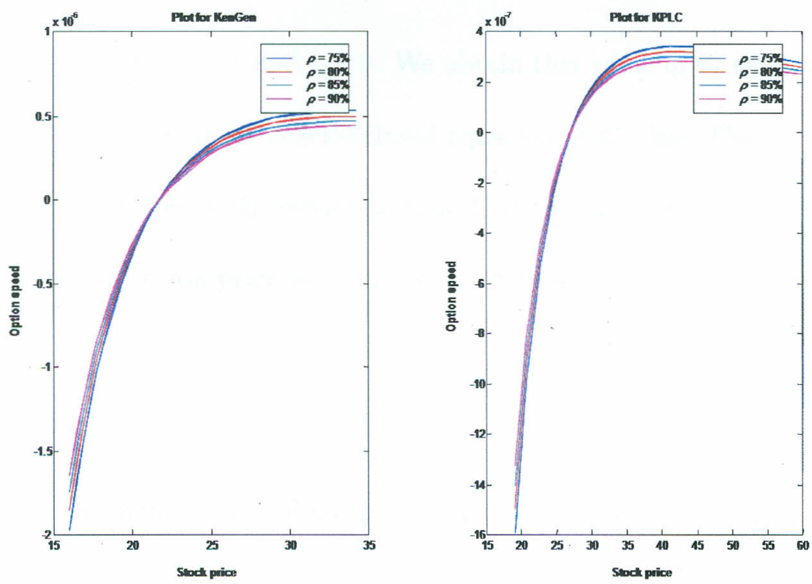

Figure 6.5: Variation of speed u_{SSS} with stock price S_t for a call option for $C = 2$ (KenGen), $C = 2.2$ (KPLC), $s_0 = 0.1$, $\sigma = 0.1$ and $t = 1$.

The curves in Figure 6.5 are going from down to up with increasing value of the liquidity parameter ρ when the option speed is negative. The curves converge when $u_{SSS} = 0$ and then go from up to down with increasing value of ρ when the speed is positive.

Theta

For a European call option resulting from the nonlinear Black-Scholes equation (4.65), the parameter theta, Θ, is computed as follows:

$$\begin{aligned}\Theta &= u_t \\ &= \frac{1}{\rho}\left(-\frac{C}{2}\sqrt{S}e^{\frac{Ct+s_0}{2}} + S\left(\frac{\sigma^2}{16} - \frac{C^2}{\sigma^2}\right) - \frac{\sigma^2}{16}e^{Ct+s_0}\right)\end{aligned} \tag{6.25}$$

for all $\rho > 0$, $\sigma > 0$, and $t > 0$. We obtain this value of u_t when the solution to the nonlinear Black-Scholes equation (4.65) [see Theorem (6.1.3)] is differentiated with respect to time t. If the price of the asset does not move, the option price will change by theta with time t.

The curves in Figure 6.6 are going from up to down with increasing value of the liquidity parameter ρ.

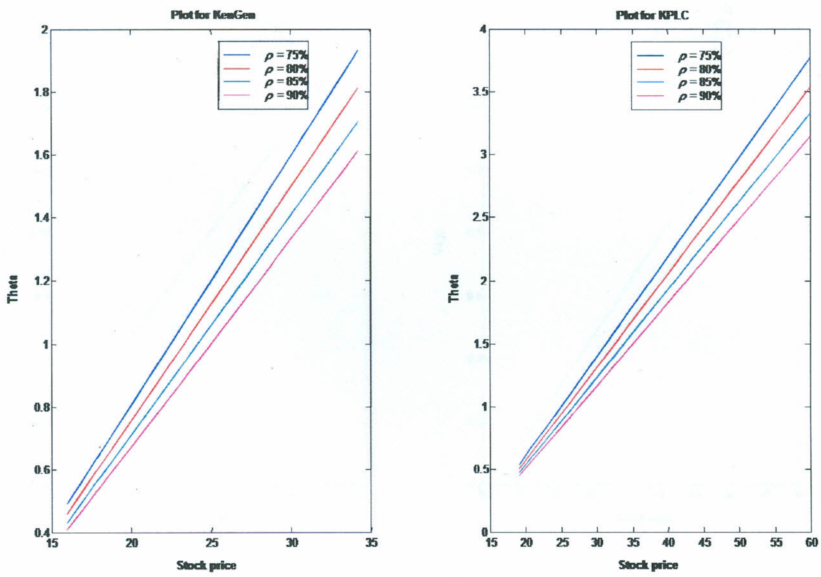

Figure 6.6: Variation of theta u_t with stock price S_t for a call option for $C = 2$ (KenGen), $C = 2.2$ (KPLC), $s_0 = 0.1$, $\sigma = 0.1$ and $t = 1$.

Vega

The vega, u_σ, of a call option $u(S, t)$ to the nonlinear Black-Scholes equation (4.65) is computed as follows:

$$\begin{aligned} u_\sigma &= \frac{\partial u}{\partial \sigma} \\ &= \frac{1}{\rho}\left(\frac{2C}{\sigma^3}S(1 - \ln S) + St(\frac{\sigma}{8} + \frac{2C^2}{\sigma^3}) - \frac{\sigma}{8C}e^{Ct+s_0}\right) \end{aligned} \tag{6.26}$$

for all $\rho > 0$, $S > 0$, $\sigma > 0$, $C > 0$, and $t > 0$. This value of vega u_σ is obtained on differentiating the solution to the nonlinear Black-Scholes equation (4.65) with respect to σ.

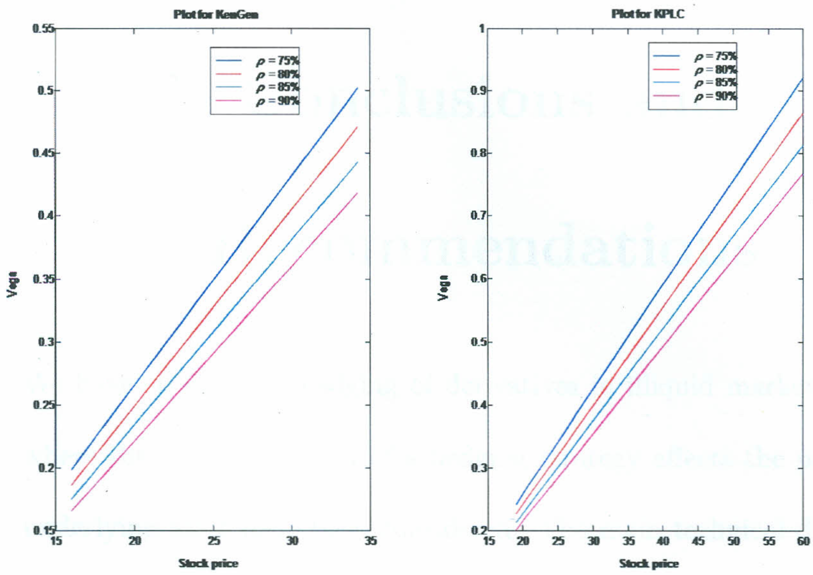

Figure 6.7: Variation of vega u_σ with stock price S_t for a call option for $C = 2$ (KenGen), $C = 2.2$ (KPLC), $s_0 = 0.1$, $\sigma = 0.1$ and $t = 1$.

The curves in Figure 6.7 are going from up to down with increasing value of the liquidity parameter ρ.

Conclusions and Recommendations

We have studied the hedging of derivatives in illiquid markets. Models where the implementation of a hedging strategy affects the price of the underlying asset have been considered. The main technical difficulty in the nonlinear Black-Scholes equation (4.65) came from the nonlinearity due to transaction costs. The difficulty was overcame by differentiating the nonlinear equation twice with respect to the spatial variable S. This led to the principal contribution in this work which is the reduction of the nonlinear Black-Scholes equation into a nonlinear groundwater equation that admits a solitary wave solution. Assuming the solution of a forward wave, a classical solution $u(S,t)$ of the nonlinear Black-Scholes equation was obtained from the the solitary wave solution u_{SS} by integrating u_{SS} twice with respect to the spatial variable S. The solution $u(S,t)$ can be applied in pricing a European call option at time $t > 0$. We have found out that the analytic solution to the nonlinear Black-Scholes equation is nontrivial when the liquidity parameter $\rho > 0$. We have further found out that transaction costs can be modelled by a parabolic nonlinear equation via a *soliton* u_{SS}.

The analytic solution to the nonlinear Black-Scholes equation can be used as a benchmark for numerical methods.

In conclusion, further research needs to be done to solve the nonlinear Black-Scholes equation using other boundary conditions. Future work will also involve evaluating the impact of Greek parameters.

References

[1] **L. Bachelier**, Théorie de la spéculation, *Annales Scientifiques de lÉcole Normale Supérieure,* **3** (1900), 21-86.

[2] **P. Bank, D. Baum**, Hedging and portfolio optimization in financial markets with a large trader, *Mathematical Finance,* **14** (2004), 1-18.

[3] **M. Baxter, A. Rennie**, *Financial Calculus, an Introduction to Derivative Pricing,* Ninth Edition, Cambridge University Press, UK (2003).

[4] **F. Black, M. Scholes**, The pricing of options and corporate liabilities, *The Journal of Political Economy,* **81**, 3 (1973), 637-654.

[5] **L.A. Bordag, A.Y. Chmakova**, Explicit solutions for a nonlinear model of financial derivatives, *Int. J. Theoret. Appl. Finance,* **10**, 1 (2007), 1-21.

[6] **L.A. Bordag, R. Frey**, *Pricing Options in Illiquid Markets: Symmetry Reductions and Exact Solutions,* Nova Science Publishers, Inc. (2008), 83-109.

[7] **W.E. Boyce, R.C. DiPrima**, *Elementary Differential Equations and Boundary Value Problems*, Seventh Edition, John Wiley and Sons, Inc., New York, USA (2001).

[8] **R. Brown**, A brief account of microscopical observations made in the months of June, July and August, 1827, on the particles contained in the pollen of plants; and on the general existence of active molecules in organic and inorganic bodies, *Philosophical Magazine, N.S. 4*, **17** (1828), 161-173.

[9] **J.R. Buchanan**, *An Undergraduate Introduction to Financial Mathematics*, World Scientific Publishing Co. Pte. Ltd, New Jersey, USA (2006).

[10] **J.M. Burgers**, *The Nonlinear Diffusion Equation, Asymptotic Solutions and Statistical Problems*, D. Reidel Publishing Company, Dordrecht-Holland (1974).

[11] **U. Cetin, R. Jarrow, P. Protter**, Liquidity risk and arbitrage pricing theory, *Finance and Stochastics*, **8** (2004), 311-341.

[12] **U. Cetin, R. Jarrow, P. Protter, M. Warachka**, Pricing options in an extended Black-Scholes economy with illiquidity: Theory and

empirical evidence, *Review of Financial Studies*, **19**, 2 (2006), 493-529.

[13] **D.N. Chorafas**, *Liabilities, Liquidity, and Cash Management, Balancing Financial Risk*, John Wiley and Sons, Inc., New York, USA (2002).

[14] **D. Cuoco, J. Cvitanic**, Optimal consumption choice for a large investor, *Journal of Economic Dynamics and Control*, **22** (1998), 401-436.

[15] **J. Cvitanić, F. Zapatero**, *Introduction to the Economics and Mathematics of Financial Markets*, Massachusetts Institute of Technology, London, England (2004).

[16] **A. Einstein**, *Investigations on the Theory of the Brownian Movement*, PhD Thesis, Dover Publications, Inc. (1956).

[17] **J. Esekon, S. Onyango, N.O. Ongati**, Analytic solution of a nonlinear Black-Scholes partial differential equation, *International Journal of Pure and Applied Mathematics*, **61**, 2 (2010), 219-230.

[18] **L.C. Evans**, *Partial Differential Equations*, Grad. Studies in Math., Volume 19, American Mathematical Society (1998).

[19] **W. Feller**, *An Introduction to Probability Theory and its Applications*, Volume II, Second Edition, John Wiley and Sons, Inc., New York, USA (1971).

[20] **E. Fermi, J. Pasta, S. Ulam**, Studies of nonlinear problems I., *Los Alamos Report LA-1940*, republished in *Fermi, E., Collected Papers, Volume 2*, University of Chicago Press, (1965), 978-988.

[21] **R. Frey**, Perfect option hedging for a large trader, *Finance and Stochastics*, **2** (1998), 115-148.

[22] **R. Frey**, *Market Illiquidity as a Source of Model Risk in Dynamic Hedging*, In R. Gibson (Editor), Model Risk, Risk Publications, London (2000), 125-136.

[23] **R. Frey, P. Patie**, Risk management for derivatives in illiquid markets: A simulation-study, *Advances in Finance and Stochastics*, Springer, Berlin (2002), 137-159.

[24] **R. Frey, U. Polte**, Nonlinear Black-Scholes equations in finance: associated control problems and properties of solutions, *SIAM J. on Control and Optimization*, **49**, 1 (2011), 185-204.

[25] **R. Frey, A. Stremme**, Market volatility and feedback effects from dynamic hedging, *Mathematical Finance*, **7**, 4 (1997), 351-374.

[26] **J. Harrison, S. Pliska**, Martingales and stochastic integrals in the theory of continuous trading, *Stochastic Processes and their Applications*, **11** (1981), 215-260.

[27] **D.C. Heath, R.A. Jarrow**, Arbitrage, continuous trading, and margin requirements, *The Journal of Finance*, **42**, 5 (1987), 1129-1142.

[28] **J.C. Hull**, *Options, Futures, and Other Derivatives*, Fifth Edition, Prentice Hall, New Jersey, USA (2005).

[29] **I. Karatzas, S.E. Shreve**, *Brownian Motion and Stochastic Calculus*, Springer-Verlag, New York, USA (1988).

[30] **D. Khoshnevisan**, *Probability*, Grad. Studies in Math., Volume 80, American Mathematical Society, Providence, Rhode Island (2007).

[31] **F.C. Klebaner**, *Introduction to Stochastic Calculus with Applications*, Second Edition, Imperial College Press, London (2005).

[32] **Knoll, Michael**, "The Ancient Roots of Modern Financial Innovation: The Early History of Regulatory Arbitrage" (2004), *Scholarship at Penn Law*, Paper 49.

[33] **H.-O. Kreiss, J. Lorenz**, *Initial-Boundary Value Problems and the Navier-Stokes Equations*, SIAM, Philadelphia, USA (2004).

[34] **J.D. Logan**, *An Introduction to Nonlinear Partial Differential Equations*, Second Edition, John Wiley and Sons, Inc., Hoboken, New Jersey, USA (2008).

[35] *MATLAB Reference Guide*, The MathWorks, Inc., Natick, MA (1992).

[36] *MATLAB: The Language of Technical Computing*, Student Edition, Version 5, The MathWorks, Inc. (1996).

[37] **B. Mawah**, *Option Pricing with Transaction Costs and a Non-linear Black-Scholes Equation*, U.U.D.M. Project Report 2007:18, Upssala University (2007).

[38] **S. Mayhew**, *The Impact of Derivatives on Cash Markets: What Have We Learned?*, Terry College of Business, University of Georgia (2000).

[39] **R.C. Merton**, Theory of rational option pricing, *The Bell Journal of Economics and Management Science*, 4, 1 (1973), 141-183.

[40] **L. Munteanu, S. Donescu**, *Introduction to Soliton Theory: Applications to Mechanics*, Kluwer Academic Publishers, New York, USA (2005).

[41] **N.O. Ongati**, *Stability of Lie Groups of Nonlinear Hyperbolic Equations*, PhD Thesis, pp. 3, University of Pretoria (1997).

[42] **S.N. Onyango**, *Extracting Stochastic Process Parameters from Market Price Data: a Pattern Recognition Approach*, PhD Thesis, University of Huddersfield (2003).

[43] **R.E.A.C. Paley, N. Wiener, A. Zygmund**, Notes on random functions, *Math. Zeitschrift*, **37** (1933), 647-668.

[44] **G. Papanicolaou, R. Sircar**, General Black-Scholes models accounting for increased market volatility from hedging strategies, *Applied Mathematical Finance*, **5** (1998), 45-82.

[45] **Y. Pinchover, J. Rubinstein**, *An Introduction to Partial Differential Equations*, Cambridge University Press, Cambridge, UK (2005).

[46] **E. Platen, M. Schweizer**, On feedback effects from hedging derivatives, *Mathematical Finance*, **8** (1998), 67-84.

[47] **A.D. Polyanin, A.V. Manzhirov**, *Handbook of Mathematics for Engineers and Scientists*, Chapman and Hall/CRC, Boca Raton, USA (2007).

[56] **D. Stirzaker**, *Stochastic Processes and Models*, Oxford University Press, New York, USA (2005).

[57] **G. Strang**, *Introduction to Applied Mathematics*, Wellesley Cambridge Press, Wellesley, Mass. (1986).

[58] **M.E. Taylor**, *Partial Differential Equations III, Nonlinear Equations*, Springer-Verlag, New York, Inc. (1996).

[59] **R.W. Ward**, *Options and Options Trading, a Simplified Course That Takes You From Coin Tosses to Black-Scholes*, McGraw-Hill, New York, USA (2004).

[60] **D. Williams**, *Probability with Martingales*, Statistical Laboratory, DPMMS, Cambridge University Press, Cambridge, Great Britain (1991).

[61] **P. Wilmott**, *Quantitative Finance*, Second Edition, John Wiley and Sons, Ltd., West Sussex, England (2006).

[62] **N.J. Zabusky, M.D. Kruskal**, Interaction of "solitons" in a collisionless plasma and the recurrence of initial states, *Physical Review Letters*, **15**, 6 (1965), 240-243.