STUDY OF NON-NORMAL OPERATORS IN A COMPLEX HILBERT SPACE

By

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ABSTRACT

A normal operator has been an object of much study in operator theory and possesses many nice properties and non-normal operators in the set of all bounded operators in a Hilbert space can be classified according to their satisfying some of such properties or generalizations of some of these properties.

We have classified some of the non-normal operators and investigated the relationship between these classes. In particular, we have shown that if a bounded operator is quasinormal then it is subnormal and hence hyponormal. If an operator is hyponormal, then it is paranormal and consequently its square is paranormal but not hyponormal. If a bounded operator is paranormal, then it is k-paranormal and hence is normaloid. We have also shown that the implications cannot, in general, be reversed.

We have also obtained a set of necessary and sufficient conditions for convexoidity and characterized those operators for which the real part of the spectrum equals the spectrum of the real part of the operator generally.

Using the result and the method of Lebow A., we have obtained results which indicate a connection between spectral sets, the numerical range and normal dilation of an operator.

Using the techniques employed by Paul Halmos in the course of studying reducible operators, we have investigated the class R_1 of operators and proved that this class includes normaloid, spectraloid, paranormal, hyponormal and T+k, where the operator T is isometric or has G_1-property, or hyponormal and k is compact.

CHAPTER ONE

INTRODUCTION

The theory of normal operators is so successful that much of the theory of non-normal operators is modeled after it.

If T is a normal operator in B(H), the set of all bounded operators on a Hilbert space H, then it has a number of interesting properties, among which are the following

N_1 $r(T) = \|T\|$ [14]

N_2 $r(T) = w(T)$, the numerical radius of T [2]

N_3 $\text{Conv}\,\sigma(T) = \overline{W(T)}$ [8]

N_4 $\sigma(T)$ and $\overline{W(T)}$ are spectral sets of T [8]

N_5 $\|Tx\| = \|T^*x\|$ for all $x \in H$ [5]

N_6 For all $\lambda \in \mathbb{C}$, $T + \lambda I$ is a normal operator [5]

N_7 If M is an invariant subspace of T, then the restriction $T|_M$ has property (i). [7]

N_8 If M is a reducing subspace of T, i.e, M is invariant under T and T*, then $T|_M$ is normal [7]

N_9 T satisfies the G_1-property, that is,

$$\|(T - \lambda I)^{-1}\| = \frac{1}{dist(\lambda; \sigma(T))}$$

for all $\lambda \in \rho(T)$ [8]

N_{10} $\text{Re}\,\sigma(T) = \sigma(\text{Re}\,T)$ [8]

Historically, the consideration of non-normal operators began with a famous paper of Wintner (1929) in an attempt to obtain a characterization of operators T for which $r(T) = \|T\|$. He asserted that for this property it is necessary and sufficient that $Conv\sigma(T) = \overline{W(T)}$.

An example of Paul R. Halmos (1967) [8]shows that this is not the case.

Thus properties (i) and (ii) are not equivalent.

The main aim was to consider generalized operators which satisfy some of the properties that normal operators fulfill or their generalized versions and find the interconnections, if any, between some of these classes of non-normal operators.

In chapter 2 of the thesis, some important prerequisites to the understanding of the rest of chapters are listed. For instance, discussions on some of the important results by Paul R. Halmos are done, that, if a bounded operator T is quasinormal, then it is subnormal. If T is subnormal, what about its adjoint, T*?

An example of hyponormal operators which are not subnormal is given. The works of Peter A. Fillmore on hyponormal operators and the Weierstrass approximation theorem and the S. K. Berberian's result on numerical range are discussed in this chapter.

Also some essential results on the characterization of non-normal operators are discussed in chapter 3 providing a proof to show that if $T \in B(H)$ is paranormal then it is k-paranormal. If $T \in B(H)$ is k-paranormal then it is normaloid.

A prove that if $T \in B(H)$ is k-hyperparanormal then it is k-paranormal is obtained. An important result on Von Neumann's theory of spectral sets which was first proved by Lebow is used to obtain some results which indicate a connection between spectral sets, the numerical range and the normal dilation. In this chapter the structure of operators with the G_1-property is also discussed. Lastly, the condition in the G_1-property is weakened to obtain loc-G_1 class and discuss the characterization of the class loc-G_1, spectral G_1, spectral loc-G_1 and the class R of operators.

In chapter 4, characterization of those operators $T \in B(H)$ for which $\mathrm{Re}\,\sigma(T) = \sigma(\mathrm{Re}\,T)$ holds generally is obtained.

Simple examples show that this remarkable relation does not hold for arbitrary operators. Hence in this chapter, we are motivated towards a natural problem to find classes of non-normal operators for which the relation holds.

In particular, we show that: If $T \in B(H)$ and one of the following holds, then T has the property $\mathrm{Re}\,\sigma(T) = \sigma(\mathrm{Re}\,T)$:

N_1 T is hyponormal.

N_2 T* is hyponormal

N_3 $\sigma(T)$ is a spectral set

N_4 T has G_1-property and $\sigma(T)$ is connected.

Finally, the class $\overline{R_1}$ of operators which was introduced by Paul R. Halmos [7] in the course of studying reducible operators is discussed. Halmos showed that $\overline{R_1}$ contains the normal and isometric operators.

Lastly we have proved that if $T \in B(H)$ is normaloid, spectraloid, paranormal, hyponormal, and T + k, where $T \in B(H)$ is isometric or has G_1-property, or hyponormal and k is compact; then T is in $\overline{R_1}$ class.

CHAPTER TWO

BACKGROUND MATERIAL

2.1 LITERATURE REVIEW

Some basic material about operator theory is given in Halmos [6] and Bonsall and Duncan's article [4] along with Functional analysis by Bachman and Narici [2].

The theory of normal operators is so successful that much of the theory of non-normal operators is modeled after it.

A natural way to extend a successful theory is to weaken some of its hypothesis slightly and hope that the results are weakened only slightly.

Historically the consideration of non-normal operators began with a famous paper of Wintner (1929) in an attempt to obtain a characterization of operators T for which

$r(T)\|T\|$.

He asserted that for this property, it is necessary and sufficient that conv $\sigma(T) = \overline{W(T)}$.

Later in 1967, Halmos [6] showed that this is not the case.

Halmos [6] gave a characterization of non-normal operators and proved that normal operators are quasinormal. He went on and proved that quasinormal operators are Fillmore [5] in 1970 extended Halmos' result and showed that if an operator T is hyponormal, then its normaloid.

Halmos however proved that the implications cannot, in general be reversed.

Halmos [6] extended Fillmore's [5] work on spectral sets and showed that for normal operators T, the spectrum and closure of the numerical range of T are spectral sets.

Lebow [11] gave a result on Von-Neumann's theory of spectral sets which we use to obtain results which indicate a connection between spectral sets, numerical range and the normal dilation.

Operators of class G_1, class loc-G_1, spectral loc-G_1 and their beautiful characterizations due to Halmos [6] and Fillmore [5] are discussed.

Halmos showed that $\overline{R}_1$ contains the normal and isometric operators.

It is assumed, unless otherwise mentioned, that H denotes a complex Hilbert space with the inner product function $\langle , \rangle : H \, x \, H \to \mathbb{C}$

2.2. CONCEPTS ASSOCIATED WITH NON-NORMAL OPERATORS

Definition 2.2.1.

We take the standard base $\{e_n : n = 0,1,2,...\}$ in ℓ^2, where $e_n = \{0,0,0...,1,0,...\}$ with 1 in the $(n+1)^{th}$ place. Let $x = (x_0, x_1, x_2,...) \in \ell^2, i.e; \sum_{n=0}^{\infty} |x_n|^2 < \infty$.

Define $U : \ell^2 \to \ell^2$ by

$$Ux = (0, x_0, x_1, x_2,...) \text{ for all } x \in \ell^2.$$

Thus $Ue_0 = e_1, Ue_1 = e_2,..., Ue_n = e_{n+1}$, for all $n = 0,1,2,...$ U is easily seen to be linear and is called the unilateral right shift operator on ℓ^2.

Let $x = (x_0, x_1, x_2,...), y = (y_0, y_1, y_2,...)$ both in ℓ^2.

Then

$$\langle Ux, y \rangle = \langle (0, x_0, x_1, x_2,...), (y_0, y_1, y_2,...) \rangle$$

$$= x_0 \overline{y}_1 + x_1 \overline{y}_2 + ...$$

$$= \langle (x_0, x_1, x_2,...), (y_1, y_2, y_3,...) \rangle$$

$$= \langle x, (y_1, y_2, y_3,...) \rangle$$

But $U \in B(\ell^2)$, so

$$\langle Ux, y \rangle = \langle x, U^* y \rangle \text{ for all } x \in \ell^2$$

Clearly U* is a contraction, i.e; $\|U^*\| \leq 1$.

U* is called the **unilateral left shift operator** on ℓ^2.

(Mile, Rao, Simiyu [13] pg.103)

Definition 2.2.2

Let $\{e_n : n = 0, \pm 1, \pm 2, ...\}$ be an orthonormal basis for ℓ^2. Order in orthonormal basis

$$\{...., e_{-3}, e_{-2}, e_{-1}, (e_o), e_1, e_2, e_3, ...\}.$$

If $x \in \ell^2$, we represent it by the Fourier series

$$x = \sum_{i=-\infty}^{\infty} x_i e_i$$

or coordinate wise as

$$(..., x_{-3}, x_{-2}, x_{-1}, (x_0), x_1, x_2, x_3, ...)$$

The **Bilateral shift** S is defined by

$$Sx = (..., x_{-3}, x_{-2,} (x_{-1}), x_0, x_1, x_2, ...)$$

where the shift is to the right by one place, so that x_{-1} has moved to the central position.

Clearly $\|Sx\| = \|x\|, \ \forall \ x \in \ell^2$.

Therefore S is 1-1 and onto.

Therefore S is unitary. (Mile, Rao, Simiyu [13] pg.103)

Definition 2.2.3

Let $\{e_n : n = 0, \pm 1, \pm 2, ...\} \ (or \{e_n : n = 0, 1, 2, ...\})$

be an orthonormal basis for ℓ^2.

The operator D defined by

$De_n = \alpha_n e_n$ for all n,

is called **a diagonal operator** with diagonal $\{\alpha_n\}$. D is bounded if and only if $\{\alpha_n\}$

is bounded, and it follows, then, that

$\|D\| = \sup\{|\alpha_n|\}$.   (Mile, Rao, Simiyu [13] pg.103)

Definition 2.2.4

An operator A = SD where S is a shift operator (Unilateral or bilateral) and D is a diagonal operator is called a **weighted shift** with weights $\{\alpha_n\}$, where $\{\alpha_n\}$ is the diagonal of D.

(Halmos[6] pg. 48)

Definition 2.2.5

An operator $T \in B(H)$ is said to be **quasinormal** if $T^*T \leftrightarrow T$, i.e. $(T^*T)T = T(T^*T)$.

It is obvious that if $T \in B(H)$ is normal, and then it is also quasinormal.

For T is normal $\Leftrightarrow T^*T = TT^*$

$$\therefore (T^*T)T = (TT^*)T = T(T^*T)$$

i.e. $T^*T \leftrightarrow T$, i.e. T is quasinormal.   (Halmos [6] pg. 75)

Definition 2.2.6

An operator $T \in B(H)$ is called **hyponormal** if $\|T^*x\| \leq \|Tx\|$ for all $x \in H$, or equivalently if $TT^* \leq T^*T$.   (Bachman[2])

Definition 2.2.7

An operator $T \in B(H)$ is called quasinilpotent if $r(T) = \lim_{n \to \infty} \|T^n\|^{1/n} = 0$ (Bachman[2])

Definition 2.2.8

An operator $T \in B(H)$ is said to be **subnormal** if there is a Hilbert space K of which H is a closed subspace and a normal operator $N \in B(K)$ such that

(i) H is invariant under N, i.e., $Nx \in H \quad \forall\, x \in H$.

and (ii) $N|_H = T$. (Halmos [6] pg. 105)

Theorem 2.2.9

If $T \in B(H)$ is quasinormal, then T is subnormal

Proof

First of all, for a $T \in B(H)$

$$Ker(T) = Ker(T^*T).$$

Indeed, let $x \in Ker(T)$. Then $Tx = \overline{0}$.

Hence $T*(Tx) = 0$, i.e., $x \in Ker(T^*T)$.

i.e. $Ker(T) \subseteq Ker(T^*T)$.

Conversely, let $x \in Ker(T^*T)$, i.e. $T*Tx = \overline{0}$.

Then

$$\langle T*Tx, x\rangle = 0, \text{ i.e, } \|Tx\|^2 = 0, \text{ i.e, } Tx = \overline{0},$$

i.e; $x \in Ker(T)$.

$\therefore Ker(T^*T) \subset Ker(T)$.

Thus

$$Ker(T^*T) = Ker(T).$$

Next, if T is quasinormal, then $T*T \leftrightarrow T$.

$\therefore Ker(T^*T)$ is invariant under T*.

Since T*T $\leftrightarrow$ T, so $Ker(T^*T)$ is invariant under T.

Thus

η_{T*T} is invariant under T and T*.

$\therefore Ker(TT^*)$ is a reducing subspace for T.

Since $Ker(T^*T) = Ker(T)$, we have,

$\ker(T)$ is a reducing subspace for T (when T is quasinormal). i.e., $Ker(T), \{Ker(T)\}^{\perp}$ are both invariant under T. We can therefore decompose T into two parts;

$T = T' \oplus T''$, where $T' : Ker(T) \to Ker(T)$

$T'' : \{Ker(T)\}^{\perp} \to \{Ker(T)\}^{\perp}$.

Replace T' by 0 and then T'' acts on the Hilbert space $\{Ker(T)\}^{\perp}$ and T'' has the trivial nullspace, i.e, $Ker(T'') = \{\bar{0}\}$, and since $T = 0 \oplus T''$ and T is quasinormal, it follows that T'' is quasinormal.

We can therefore, without loss of generality, consider a quasinormal $T \in B(H)$ with the trivial kernel.

Let T have the polar decomposition UP, where U is a partial isometry and $P \geq 0$.

Let E be the orthogonal projection UU*.

Then

$$(I-E)U = 0 = U^*(I-E)$$

Let $V, Q \in B(H \oplus H)$ defined by

$$V = \begin{pmatrix} U & (I-E) \\ 0 & U^* \end{pmatrix}, \quad Q = \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}.$$

The following facts are now verifiable:

(i) Q is positive

$$\begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Px \\ Py \end{pmatrix}$$

$$\therefore \left\langle \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} Px \\ Py \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle$$

$$= \langle Px, x \rangle + \langle Py, y \rangle \geq 0, \ for\ all \begin{pmatrix} x \\ y \end{pmatrix} \in H \oplus H.$$

i.e. $Q \geq 0.$

(ii) V is unitary.

$$V^* = \begin{pmatrix} U^* & 0 \\ (I-E)^* & (U^*)^* \end{pmatrix}$$

$$= \begin{pmatrix} U^* & 0 \\ I-E & U \end{pmatrix}$$

$$\therefore V^*V = \begin{pmatrix} U^* & 0 \\ I-E & U \end{pmatrix} \begin{pmatrix} U & I-E \\ 0 & U^* \end{pmatrix}$$

$$= \begin{pmatrix} U^*U & U^*(I-E) \\ (I-E)U & (I-E)^2 + UU^* \end{pmatrix}$$

$$= \begin{pmatrix} I & 0 \\ 0 & I-E+UU^* \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 1 \end{pmatrix}$$

Since E = UU*

$$VV^* = \begin{pmatrix} U & I-E \\ 0 & U^* \end{pmatrix} \begin{pmatrix} U^* & 0 \\ I-E & U \end{pmatrix}$$

$$= \begin{pmatrix} UU^* + (I-E)^2 & (I-E)U \\ U^*(I-E) & U^*U \end{pmatrix}$$

$$= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Also, $V \to Q$; indeed

$$VQ = \begin{bmatrix} U & I-E \\ 0 & U^* \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} = \begin{bmatrix} UP & (I-E)P \\ 0 & U^*P \end{bmatrix}$$

and

$$QV = \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} U & I-E \\ 0 & U^* \end{bmatrix} = \begin{bmatrix} PU & P(I-E) \\ 0 & PU^* \end{bmatrix}$$

Now

UP = PU and U*P = PU* and

P(I-E) = P-PUU*

= P- (PU)U*

= P- (UP)U*

= P-U(PU*)

= P -U(U*P)

= P – (UU*) P

= (I-E)P

The product

$$VQ = \begin{bmatrix} UP & (I-E)P \\ 0 & U^*P \end{bmatrix}$$

is the required normal extension of T = UP. (Halmos[6])

Theorem 2.2.10

If $T \in B(H)$ is subnormal then T is hyponormal.

Proof

Since T is subnormal, there is a Hilbert space $K \supset H$ and a normal operator $N \in B(K)$ and such that $N|_H = T$.

Therefore, for all $x, y \in H$

$$\langle T^*x, y\rangle = \langle x, Ty\rangle$$

$$= \langle x, Ny\rangle \text{ , since } N|_H = T.$$

$$= \langle N^*x, y\rangle$$

$$= \langle N^*x, Py\rangle,$$ where P is the orthogonal projection of K onto its subspace H.

$$\therefore \langle T^*x, y\rangle = \langle P^*N^*x, y\rangle$$

$$= \langle PN^*x, y\rangle.$$

Thus the operator $PN^* \in B(K)$ is invariant under H and T* is the restriction of PN* to H.

Thus

$$T^*x = PN^*x, \qquad \forall x \in H.$$

Also

$$\|T^*x\|^2 = \|PN^*x\|^2 \le \|N^*x\|^2$$ (since $\|P\| \le 1$ for P is an orthogonal projection).

Since N is normal

$$\|N^*x\| = \|Nx\|. \qquad \text{for all } x \in H$$

Thus $\|T^*x\|^2 \le \|Nx\|^2$ for all $x \in H$

i.e. $\|T^*x\|^2 \le \|Tx\|^2$ for all $x \in H$

i.e. $\langle T^*x, T^*x\rangle \le \langle Tx, Tx\rangle$ for all $x \in H$

i.e. $\langle TT^*x, x\rangle \le \langle T^*Tx, x\rangle$ .... (2.2.1)

Since TT*, T*T are positive operators for any $T \in B(H)$, (2.2.1) implies

$TT^* \le T^*T$.

Thus

T is subnormal $\Rightarrow TT^* \le T^*T$, i.e., T is hyponormal.

Lemma 2.2.11

Hyponormal operators are not necessarily subnormal.

Proof

There are hyponormal operators which are not subnormal. To see this, we deduce a necessary condition for subnormality.

Let $T \in B(H)$ be subnormal and N a normal extension of T. Let $\{x_0, x_1, x_2, \ldots, x_n\}$ be any finite set of vectors in H. Then

$$0 \leq \left\| \sum_i N^{*i} x_i \right\|^2 = \langle \sum_i N^{*i} x_i, \sum_j N^{*j} x_j \rangle$$

$$= \sum_i \sum_j \langle N^{*i} x_i, N^{*j} x_j \rangle$$

$$= \sum_i \sum_j \langle N^j N^{*i} x_i, x_j \rangle$$

$$= \sum_i \sum_j \langle N^j x_i, N^i x_j \rangle$$

$$\therefore 0 \leq \sum_i \sum_j \langle T^j x_i, T^i x_j \rangle, \text{ since } N|_H = T.$$

Let $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n$ be any complex numbers and replace x_i by $\lambda_i \tilde{x}_i$ where $x_i \in H$ $(i = 0,1,2,\ldots,n)$. Then we get for any finite set $\{\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_n\}$ of elements of H

$$\sum_i \sum_j \langle T^j \tilde{x}_i, T^i \tilde{x}_j \rangle \lambda_i \bar{\lambda}_j \geq 0.$$

In other words

$\sum_i \sum_j \langle T^j \tilde{x}_i, T^i \tilde{x}_j \rangle \lambda_i \bar{\lambda}_j$ is positive definite for all complex numbers $\lambda_0, \lambda_1, \ldots, \lambda_n$.

Hence the determinant of the finite square matrix $(i, j = 0,1,2,\ldots,n)$

$$[\langle T^j x_i, T^i x_j \rangle] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.2.2)$$

must be ≥ 0.

It can be shown that the condition (2.2.2), which is necessary, is also sufficient.

(Halmos,[6], pr.203, pg. 109).

Example (i)

An example of a hyponormal operator whose square is not hyponormal

(Halmos [6] pr.209, pg.111).

The following results on hyponormal operators are well-known. (Fillmore [5])

Proposition 2.2.12

Let $T \in B(H)$ be hyponormal. Then

N_1. $T - \lambda I$ and T^{-1} (if T is invertible) are hyponormal.

N_2. $Tx = \lambda x$ implies $T^* x = \lambda^* x$.

N_3. $Tx = \lambda x$, $Ty = \mu y$ and $\lambda \neq \mu$ imply that $x \perp y$.

N_4. If M is an invariant subspace of T, then the restriction $T|_M$ is hyponormal.

If $T|_M$ is normal, then M reduces T (Fillmore[5] pg.10).

Theorem 2.2.13

For any $T \in B(H)$, $\sigma(T) \subseteq \overline{W(T)}$ and if $d = dist(\lambda, \overline{W(T)})$ then $\lambda I - T$ has an inverse and

$\|(\lambda I - T)^{-1}\| < \frac{1}{d}$ or $\frac{1}{d} > \|R_\lambda(T)\|$.

Proof

If $\lambda \notin \overline{W(T)}$, then dist $(\lambda, \overline{W(T)}) > 0, i.e, d > 0$; and by definition of distance

$$d \leq |< Tx, x > - \lambda| \text{ for all } x \in H \text{ such that } \|x\| = 1.$$

This implies

$$d\|x\|^2 \leq |< (T - \lambda I)x, x >| \text{ for all } x \in H ;$$

using the Cauchy – Schwartz inequality, it is clear that

$$\|(T - \lambda I)x\| \geq d\|x\|.$$

Since $T - \lambda I$ is bounded from below, $(T - \lambda I)^{-1}$ exists on $R(T - \lambda I)$ and is bounded; moreover

$$\|(T-\lambda I)^{-1}y\| \leq d^{-1}\|y\| \text{..(2.2.3)}$$

for all $y \in R(T-\lambda I)$. Hence there are only two possibilities: either

$\lambda \in \rho(T)$ or $\lambda \in R_\sigma(T)$.

Suppose $\lambda \in R_\sigma(T)$. Since

$$\left\{R(\overline{T-\lambda I})\right\}^{\perp} = \left\{R(T-\lambda I)\right\}^{\perp} = Ker(T^*-\bar{\lambda} I)$$

if $\lambda \in R_\sigma(T)$, then

$$\left\{R(\overline{T-\lambda I})\right\}^{\perp} \neq \{\bar{0}\},$$

i.e. Ker $(T^*-\bar{\lambda} I) \neq \{\bar{0}\}$,

and hence $\bar{\lambda}$ is an eigenvalue of T*.

If $x \in H, \|x\| = 1$ and is such that $T^*x = \bar{\lambda}x$, then

$$< Tx, x >=< x, T^*x >=< x, \bar{\lambda}x >= \lambda,$$

which implies that $\lambda \in W(T)$, a contradiction.

Hence, if $\lambda \notin \overline{W(T)}$, then $\lambda \in \rho(T)$; this shows that $\sigma(T) \subseteq \overline{W(T)}$.

Now 2.2.3 implies that

$$\|(T-\lambda I)^{-1}\| \leq \frac{1}{d}.$$ (Bachman & Narici [2] pg 386, Theorem 21.11)

Theorem 2.2.14

Let $T \in B(H)$, then conv $\sigma(T) \subseteq \overline{W(T)}$.

Proof

See Halmos [8] pg. 170

Theorem 2.2.15 **(Spectral mapping theorem)**

Let H be a Hilbert space and $T \in B(H)$ and p be a complex polynomial

$a_0 I + a_1 z + a_2 z^2 + ... + a_n z^n$, where $a_0, a_1, a_2, ..., a_n \in \mathbb{C}$.

Let $p(T)$ denote the bounded operator $a_0 I + a_1 T + a_2 T^2 + ... + a_n T^n$.

Then

$$\sigma(p(T)) = p(\sigma(T))$$

where $p(\sigma(T)) = \{p(\lambda) : \lambda \in \sigma(T)\}$ (Bachman and Narici[2] pg .322)

Lemma 2.2.16

If $T \in B(H)$ is hyponormal, then

$$\|T^n\| = \|T\|^n, \qquad \text{for } n = 1,2,...,$$

and consequently

$$r(T) = \|T\|.$$

Proof

Fillmore [5] pg.10.

Theorem 2.2.17 (Extension by continuity)

Let X and Y be normed spaces, and suppose Y is complete. Every continuous linear operator u_0 from $\Omega \subset X$ into Y has a unique continuous linear extension u to the closure $\overline{\Omega}$ of Ω, and

$\|u\| = \|u_0\|$ (Bachman[2])

Definition 2.2.18

By analogy with the generalized integral, one can define a generalized limit,glim, for an arbitrary bounded sequence.

Consider the vector space l^∞ of bounded real sequences.

Let $a = (a; n_1, n_2, \ldots, n_k) = \overline{\lim_{n \to}} \frac{1}{k} a_n + n_j$.

$$p(a) = \inf \pi(a; n_1, n_2, \ldots, n_k),$$

where the infimum is taken over all sets of natural numbers $n_1, n_2, \ldots, n_k$.

P is a semi-additive positive homogeneous functional. Hence there exists a linear function f satisfying the condition

$$-p(-a) \le f(a) \le p(a)$$

If we set

$$\operatorname*{Lim}_{n \to \infty} a_n = f(a),$$

then this functional has the following properties:

N1: $\lim_{n \to \infty} [\alpha a'_n + \beta a''_n] = \alpha \lim_{n \to \infty} a'_n + \beta \lim_{n \to \infty} a''_n$

N1: $\lim_{n \to \infty} a_n \ge 0 \ \text{if} \ a_n \ge 0 \qquad (n = 1, 2, \ldots.,)$

N3: $\lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} a_n$

N4: $\lim_{n \to \infty} a_n^{(0)} = 1, \ \text{if} \ a_n^{(0)} = 1 \qquad (n = 1, 2, \ldots,)$

N5: $\underline{\lim}\, a_n \le \lim_{n \to \infty} a_n \le \overline{\lim_{n \to \infty}} a_n$

It follows from the last of these that, if $\lim a_n$ exists, then we must have $\lim_{n \to \infty} a_n = \lim_{n \to \infty} a_n$.

Thus the number $\lim_{n \to \infty} a_n$, is called the (Banach) generalized limit of the sequence (a_n).

Proposition 2.2.19

On the space m of all bounded sequences there exists a continuous linear functional denoted by "glim" having the following properties:

(i) If $a_n \geq 0$, then glim $(a_n) \geq 0$ for all n.

(ii) $\text{glim}(a_n) = g\lim\ (a_{n+1})$ (i.e; translation invariance)

(iii) If (a_n) is convergent, then glim $(a_n) = \lim\ (a_n)$.

(iv) If $a_n \in \mathbb{R}$, then $\underline{\lim}\ (a_n) \leq g\lim(a_n) \leq \overline{\lim}\ (a_n)$

Proof

We need to consider real sequences.

Denote the set of all bounded real sequences by M. Clearly M is a vector space.

Let $a \in M$ and let $n_1, n_2, ..., n_k$ be any set of real numbers. Write

$$\pi(a; n_1 n_2, ..., n_k) = \sup_{k \to \infty} \frac{1}{k} \sum_{j=1}^{k} (a_{n+n_j}).$$

Let

$$b_{n,k} = \frac{1}{k} \sum_{j=1}^{k} a_{n+n_j} \qquad , k = 1, 2, ...$$

and set

$$d_n = \lim_{k \to \infty} \sup b_{n,k}.$$

Define $p : M \to \mathbb{R}$ by

$$\begin{aligned} p(\{a_n\}) &= \inf_{n \in \mathbb{N}} d_n \\ &= \inf_{n \in \mathbb{N}} \left\{ \lim_{k \to \infty} \sup b_{n,k} \right\} \\ &= \inf_{n \in \mathbb{N}} \left\{ \lim_{k \to \infty} \sup \frac{1}{k} \sum_{j=1}^{k} a_{n+n_j} \right\} \end{aligned}$$

Clearly the map p is linear; since if $\alpha \in \mathbb{R}$, then

$$p(\alpha a_n) = \inf_{n \in \mathbb{N}} \left\{ \lim_{k \to \infty} \sup \frac{1}{k} \sum_{j=1}^{k} \alpha a_{n+n_j} \right\}$$

$$= \inf_{n \in \mathbb{N}} \left\{ \lim_{k \to \infty} \sup \frac{1}{k} \alpha \sum_{j=1}^{k} a_{n+n_j} \right\}$$

$$= \inf_{n \in \mathbb{N}} \left\{ \alpha \lim_{k \to \infty} \sup \frac{1}{k} \sum_{j=1}^{k} a_{n+n_j} \right\}$$

$$= \alpha \inf_{n \in \mathbb{N}} \left\{ \lim_{k \to \infty} \sup \frac{1}{k} \sum_{j=1}^{k} a_{n+n_j} \right\}$$

$$= \alpha d_n$$

$$= \alpha p(a_n).$$

Now set

$a = (a_n) \to p(a) = \inf \pi(a; n_1, n_2, ..., n_k),$

where the infimum is taken over all finite sets of numbers $n_1, n_2, ... n_k$.

Now

$p(sa) = sp(a)$ for all $s \geq 0$

and we show that

$$p(x+y) \leq p(x) + p(y)$$

i.e; p is semi-additive.

Indeed, for any real $\varepsilon > 0$, let $(n_1, n_2, ..., n_p)$ and $(n_1, n_2, ..., n_k)$ be sets of numbers such that

$\pi(a; n_1, n_2, ..., n_p) \leq p(a) + \varepsilon$ and

$\pi(b; n_1, n_2, ..., n_k) \leq p(b) + \varepsilon$.

Write

$n_{j,i} = n_j + n_i$.

Then we have, on one hand

$$p(a+b) \le \pi(a+b; n_{1,1}, n_{1,2}, \dots, n_{p,k}) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \text{(2.2.4)}$$

On the other hand

$$\pi(a+b; n_{1,1}, n_{1,2}, \dots, n_{p,k}) = \frac{1}{pk} \ \sup \sum_{j,1} \left(a_n + n_{j,1} + b_{n+n_{j,1}} \right)$$

$$\le \frac{1}{pk} \sup \sum_{j,1} \left(a_{n+n_{j,1}} \right) + \frac{1}{pk} \sup \sum \left(b_{n+n_{j,1}} \right)$$

$$\le \frac{1}{k} \sum_{j,1} \sup \frac{1}{m} \sum_{j=1}^{p} \left(a_{n+n_j+n_1} \right) + \frac{1}{p} \sum_{j=1}^{p} \ \frac{1}{k} \sum_{j=1}^{k} \left(b_{n+nj+n_1} \right)$$

$$= \pi(a; n_1, n_2, \dots, n_p) + \pi(b; n_1, n_2, \dots, n_k)$$

$$< p(a) + p(b) + 2\varepsilon .$$

Comparing this with (2.2.4) and bearing in mind that ε is arbitrary, we obtain

$$p(a+b) \le p(a) + p(b).$$

Since m_r contains the closed subspace c of all convergent sequences c and the functional

$$f(x) = \lim x_n \quad , \quad x = (x_n)$$

is continuous on c and

$$|f(x)| \le p(\tilde{x}) \ , \ \ \tilde{x} = (|x_n|), \ \ x = (x_n)$$

by the extension theorem, we find an extension of this functional to the entire space m_r; we denote this extension by g lim.

Now, this functional has the property

$$g \lim \, (a_n) = g \lim \, (a_{n+1}).$$

Since

$$-p(-a) = g \lim a \ \le p(a)$$

and taking the sequence $(a_{n+1} - a_n)$ we obtain the assertion.

The other properties stated in the proposition are obvious from the definition. □

Now there exist elements in B(H) for which the numerical range is not a closed set. There is a construction following S. Berberian [3] to show that for any operator $T \in B(H)$ we can define a new operator, say $\widetilde{T}$, such that

(a) $W(\widetilde{T}) = \overline{W(T)}$

(b) $P_\sigma(\widetilde{T}) = \pi(T)$.

The basic result due to S. Berberian [3] is the following:

Proposition 2.2.20

Let $T \in B(H)$. Then there exist a Hilbert space $\widetilde{H}$, and an application $T \mapsto \widetilde{T}$ such that

$W(\widetilde{T}) = \overline{W(T)}$ and $P_\sigma(\widetilde{T}) = \pi(T)$.

Proof

First we consider the space of all sequences $x = (x_n)$ such that $(\|x_n\|) \in \ell^\infty$ and we define a bilinear form,

$$\beta(x, y) = g\lim(< x_n, y_n >).$$

Since $|< x_n, y_n >| \le \|x_n\| \, \|y_n\|$, the function β is well – defined.

It is also clear that β is a positive symmetric functional on B(H), and by the Cauchy-Bunyakowskii-Schwarz inequality we see that the set

$$N = N(\beta) = \{x : \beta(x, x) = 0\}$$

is a linear subspace.

Now we can define the space S(H)|N with an inner product defined in a standard way: If

$\tilde{x} = x + N$, then

$$<\tilde{x}, \tilde{y}> = g\lim < x_n, y_n >$$

and if S(H) denotes the space of all sequences $x = (x_n)$, as above, we have an isometric

linear mapping of H into a closed linear subspace of S(H)/N.

Since S(H)/N is a linear subspace of a Hilbert space (its completion), we have the relation

$$H' \subset S(H)/N \subset K \text{ (the completion of S(H)/N)}$$

where H' denotes the image of H into S(H).

For any $T \in B(H)$ we can define an operator on K as follows:

If $x = (x_n) \in S(H)$, then we set

$$\tilde{T}(x) = (Tx_n).$$

It can be shown that $\tilde{T}$ is bounded and N is an invariant subspace for $\tilde{T}$ and this allows us to

define an operator on the space S (H)/N in a standard way. Denote this operator by $\tilde{T}_0$. In this

case we have $\|\tilde{T}_0\| \leq \|T\|$, and moreover, we have infact the equality of norms. From the density

of S (H)/N in the space K it follows that $\tilde{T}_0$ extends uniquely to a bounded operator on K. This

extension we denote by T^+. The following properties are then easily proved and we list them:

$$T \mapsto T^+$$

satisfies the relations

$$T + S \mapsto T^+ + S^+$$

$$\lambda T \mapsto \lambda T^+$$

$$ST \mapsto S^+ T^+$$

$$T^* \mapsto (T^+)^*$$

$$I \mapsto I$$

$$\|T\| = \|T^+\|$$

and

$$T \geq 0 \Rightarrow T^+ \geq 0.$$

From the properties of the mapping $T \mapsto T^+$ we obtain the assertion of the proposition in the following way:

If $\lambda \notin \pi(T)$ we find $\varepsilon > 0$ such that

$$(T - \lambda I)^*(T - \lambda I) > \varepsilon\ I$$

and this implies that the corresponding operator T^+ has the following property

$$(T^+ - \lambda I)^*(T^+ - \lambda I) > \varepsilon\ I$$

and this gives the second assertion of the proposition.

The first assertion of the proposition can be obtained by observing that the closure of the numerical range of T is the same as the closure of the numerical range of T^+.

But it is clear that the closure of W(T) is contained in $W(T^+)$, and thus the proposition is proved.

(Berberian and Orland[3])

Lemma 2.2.21

If $\{T_n\}$ is a sequence of invertible operators and if T is a non-invertible operator such that

$\|T_n - T\| \to 0$ as $n \to \infty$, then $0 \in \pi(T)$.

Proof

See Halmos [6]

Theorem 2.2.22

The boundary of the spectrum of an operator is included in the approximate point spectrum.

Proof

It is convenient to prove lemma 2.2.21. Since T is not invertible, either $0 \in \pi(T)$ or $0 \in \Gamma(T)$.

If $0 \in \pi(T)$, there is nothing to prove.

It is therefore sufficient to prove that T is not bounded from below (i.e; that $0 \in \pi(T)$) under the assumption that range (T) is not dense.

Suppose then that x is a non-zero vector orthogonal to range T, and write

$$x_n = \frac{T_n^{-1}x}{\|T_n^{-1}x\|}.$$

Since $\|x_n\| = 1$, it follows that

$$\|(T_n - T)x_n\| \le \|T_n - T\| \to 0.$$

Since, however, $Tx_n \in$ range T and $T_n x_n \perp$ range T, it follows that

$$\|T_n x_n - Tx_n\|^2 = \|T_n x_n\|^2 + \|Tx_n\|^2$$

$$\ge \|Tx_n\|^2,$$

and hence that

$$\|Tx_n\| \to 0.$$

To derive the original spectral assertion, suppose that λ is on the boundary of $\sigma(T)$, it follows that there exist numbers λ_n not in $\sigma(T)$ such that $\lambda_n \to \lambda$.

The operators $T - \lambda_n I$ are invertible and $T - \lambda I$ is not;

since

$$\|(T - \lambda_n I) - (T - \lambda I)\| = |\lambda_n - \lambda| \to 0,$$

it follows that $\lambda \in \pi(T)$.

Theorem: 2.2.23 (Banach Inverse Theorem)

Let X be a Banach space and $T \in B(X)$ which is one –to-one and onto. Then the set inverse $T^{-1} \in B(X)$; i.e; T is invertible.

Proof

See Bachman [2]

Definition 2.2.24

Let $T \in B(H)$. A point $\lambda \in \sigma(T)$ is in the continuous spectrum of T, $C_\sigma(T)$, if $T-\lambda I$ is one-to-one and the range of $T-\lambda I$ is dense in H but not equal to H.

Now it follows that λ is in the continuous spectrum of T if and only if there exists a sequence of elements $(x_n), \|x_n\| = 1$, such that

(i) $Tx_n - \lambda x_n \xrightarrow{S} \overline{0}$

(ii) $x_n \xrightarrow{w} \overline{0}$. ....(2.2.5)

Since $C_\sigma(T) = \pi(T) - \{\Gamma(T) \cup P_\sigma(T)\}$, it follows that $\lambda \in \pi(T)$ and hence there is a sequence (x_n) of unit vectors such that property (i) holds.

Since $T-\lambda I$ is one-to-one, Ker(T- λI) = $\{\overline{0}\}$ and hence the range of $T^*-\lambda^* I$ is dense in H.

Now, for each $y \in H$, we have

$$<x_n, (T^*-\lambda^* I)y> = <(T-\lambda I)x_n, y> \to 0 \text{ as } n \to \infty.$$

Hence $x_n \xrightarrow{w} \overline{0}$.

Conversely, if property (i) holds, we see that $\lambda \in \pi(T)$.

Now, property (i) implies that

$$<x_n, (T^*-\lambda^* I)y> \to 0 \;\; as \;\; n \to \infty, \text{ for all } y \in H.$$

Since $x_n \xrightarrow{w} \overline{0}$., we see that the range of the set ($T^*-\lambda^* I$) is dense in H, i.e, $T-\lambda I$ is one – to – one.

Hence $\lambda \notin P_\sigma(T)$. Also Range $(T - \lambda I) \neq H$ (for if Range $(T - \lambda I)$ was H, then $\lambda \in \rho(T)$ using the Banach inverse theorem).

Thus we have proved contention (2.2.5).

We know that if $T \in B(H)$ is normal, then

$$T \leftrightarrow T^*T \text{ and } T \leftrightarrow TT^*.$$

Using a result of Kleinecke and Shirokov (Halmos,[6]) which states;

Proposition 2.2.25

If P and Q are operators and if $C = PQ - QP$ and if $C \leftrightarrow P,$ then C is quasinilpotent.

Proof

See Halmos [6], pr. 232, pg. 130.

Proposition 2.2.26

If $T \in B(H)$ and satisfies

(i) $T \leftrightarrow T^*T$

(ii) $T \leftrightarrow TT^*$ or $T(T^*T - TT^*) = (T^*T - TT^*)T,$

then T is a normal operator.

Proof

Let Q = T*T - TT*.

The conditions of the proposition imply that Q is a quasinilpotent operator by proposition 2.2.25, i.e,

$$\lim_{n \to \infty} \|Q^n\|^{1/n} = 0.$$

Since Q is self-adjoint (and hence normal) we have $\|Q^n\| = \|Q\|^n$, and hence $Q = 0, i.e.,$

$T^*T = TT^*$, i.e., T is normal.

2.3. OBJECTIVES OF THE STUDY

The main objectives of this study were:

(i) To characterize some of the non-normal operators and investigate the relationship between these classes.

(ii) To obtain a set of necessary and sufficient conditions for convexoidity and characterize those operators $T \in B(H)$ for which $\mathrm{Re}\,\sigma(T) = \sigma(\mathrm{Re}\,T)$ generally.

(iii) To obtain results which indicate a connection between spectral sets, the numerical range and normal dilation of an operator T.

(iv) To investigate operators of class $\overline{R}_1$ and show that this class includes normaloid, spectraloid, paranormal, hyponormal and T+k, where $T \in B(H)$ is isometric or has G_1-property, or hyponormal and k is compact.

2.4. SIGNIFICANCE OF THE STUDY

It must be frankly said that the theory of normal operators is so successful that much of the theory of non-normal operators is modeled after it. Anybody working with particular realizations of Hilbert Spaces with concrete operators given in those particular spaces can apply these general abstract results.

In particular, the subnormal operators arise naturally in complex function theory, differential geometry, potential theory, and approximation theory, and their study has rich applications in many areas of applied sciences as well as in pure mathematics.

CHAPTER THREE

SOME ASPECTS OF NON-NORMAL OPERATORS IN A HILBERT SPACE

In this chapter, classification of non-normal operators which satisfy some properties as normal operators and their generalizations is obtained. Accordingly we have quasinormal, subnormal, hyponomal, paranormal, k-paranormal, normaloid, operators with G_1-property, operators with sequential G_1-property, etc. We also have generalizations such as operators with sequential G_1-property, convexoid operators, class loc-G_1, spectral loc-G_1, operators of class R etc. Classification of some of the non-normal operators and the relationship between these classes is investigated.

3.1. PARANORMAL, K-PARANORMAL, NORMALOID AND HYPONORMAL OPERATORS

Definition 3.1.1

$T \in B(H)$ is called **Paranormal** if for all $x \in H$ satisfying $\|x\| = 1$, we have $\|Tx\|^2 \leq \|T^2x\|$.

Definition 3.1.2

$T \in B(H)$ is called ***k-paranormal*** if for all $\|x\| = 1$ satisfying $\|x\| = 1$, we have

$$\|Tx\|^k \leq \|T^k x\|.$$

Of course, when $k = 2$, we simply use the phrase 'paranormal' instead of 2-paranormal.

Proposition 3.1.3

If $T \in B$ (H) is paranormal, so is T^2.

Proof

Let $x \in H$ and $\|x\| = 1$. Then Tx = $\bar{0}$ implies $\|T^n x\| = \bar{0}$ for all $n \in \mathbb{Z}$.

On the other hand, if $x \in H$, $\|x\| = 1$ and $Tx \neq \bar{0}$, then $\|Tx\| > 0$. Since T is paranormal, we have $\|T^2x\| > 0$.

Hence

$$\|T^4x\| = \|T^2(T^2x)\|$$

$$= \left\|T^2\left(\frac{T^2x}{\|T^2x\|}\right)\right\| \; \|T^2x\|$$

$$\geq \left\|T\left(\frac{T^2x}{\|T^2x\|}\right)\right\|^2 \; \|T^2x\|$$

$$= \frac{\|T^3x\|^2}{\|T^2x\|} \qquad \text{...(3.1.1)}$$

Now

$$\|T^3x\| = \|T^2(Tx)\| = \left\|T^2\left(\frac{Tx}{\|Tx\|}\right)\right\| \; \|Tx\|$$

$$\geq \left\|T\left(\frac{Tx}{\|Tx\|}\right)\right\|^2 \; \|Tx\|$$

$$= \frac{\|T^2x\|^2}{\|Tx\|} \qquad \text{...(3.1.2)}$$

From (3.1.1) and (3.1.2) we get

$$\|T^4x\| \geq \frac{\|T^2x\|^4}{\|Tx\|^2}\,\frac{1}{\|T^2x\|} = \frac{\|T^2x\|^3}{\|Tx\|^2}$$

$$= \|T^2x\|^2\,\frac{\|T^2x\|}{\|Tx\|^2}$$

Since T is paranormal,

$$\|T^2x\| \geq \|Tx\|^2, \text{ i.e } \frac{\|T^2x\|}{\|Tx\|^2} \geq 1.$$

Thus

$$\|T^4x\| \geq \|T^2x\|^2 \qquad (\|x\| = 1). \qquad \text{.......................................(3.1.3)}$$

which shows that T^2 is paranormal.

When $x \in H$, $\|x\| = 1$ and $\|T^2x\| = 0$ (in which case $\|Tx\| = 0$ since T is paranormal), the result (3.1.3) is trivially true.

Remark

$T \in B$ (H) is paranormal does not imply that T is hyponormal.

Now $T \in B$ (H) is paranormal $\Rightarrow$ T^2 is paranormal. It is enough to find a $T \in B(H)$ for which T^2 is not hyponormal, but T is hyponormal.

Thus we then have T is hyponormal $\Rightarrow$ T is paranormal, and hence T^2 is paranormal, but T^2 is not hyponormal.

Proposition 3.1.4

$T \in B(H)$ is paranormal $\Rightarrow$ T is k-paranormal.

Proof

To prove this assertion, we show that if $T \in B(H)$ is paranormal and k-paranormal, then it is (k+1)- paranormal.

Let $x \in H$ and $\|x\| = 1$ and $Tx \neq \overline{0}$. Now

$$\|T^{k+1}x\| = \left\|T^k\left(\frac{Tx}{\|Tx\|}\right)\right\| \|Tx\| \geq \left\|T\left(\frac{Tx}{\|Tx\|}\right)\right\|^k \|Tx\|$$

(where we use that T is k-paranormal)

$$= \|T^2x\|^k \ \|Tx\|^{1-k}$$

$$\geq \|Tx\|^{2k} \|Tx\|^{1-k} = \|Tx\|^{k+1}$$

In case $Tx = \bar{0}$, i.e; $\|Tx\| = \bar{0}$, the assertion is obvious.

Thus T is (k+1) – paranormal.

Now T is paranormal $\Rightarrow$ T is 3 – paranormal.

Indeed, let $x \in H$, $\|x\| = 1$ and $\|Tx\| \neq \bar{0}$.

Then

$$\|T^3x\| = \|T^2(Tx)\|$$

$$= \left\|T^2\left(\frac{Tx}{\|Tx\|}\right)\right\| \ \|Tx\|$$

$$\geq \left\|T(\frac{Tx}{\|Tx\|})\right\|^2 \ \|Tx\|$$

$$= \frac{\|T^2x\|^2}{\|Tx\|}$$

Now

$$\|T^2x\| \geq \|Tx\|^2 \quad (\|x\| = 1).$$

Here

$$\|T^3x\| \geq \frac{\|Tx\|^4}{\|Tx\|} = \|Tx\|^3$$

which is anyway trivially true if $\|x\| = 1$ and $Tx = \bar{0}$.

Thus T is 3-paranormal.

This completes the induction process and the proposition is proved.

Proposition 3.1.5

If T∈B (H) is k-paranormal, then it is normaloid.

Proof

We may assume, without loss of generality, that $\|T\| = 1$, and let (x_n), where $\|x_n\| = 1$, be such that $\|Tx_n\| \to 1$ as $n \to \infty$ (of course, then $\|Tx_n\| \le 1$, for all n∈ ℕ).

Since T is k-paranormal

$$\|T^k x_n\| \ge \|Tx_n\|^k \quad \text{for all } n \in \mathbb{N}.$$

Hence

$$\underline{\lim}_{n \to \infty} \|T^k x_n\| \ge 1 \quad \text{..........(3.1.4)}$$

Since

$$\|T^k x_n\| \le \|T^k\| \le \|T\|^k = 1$$

we get

$$\overline{\lim}_{n \to \infty} \|T^K x_n\| \le 1 \quad \text{..........(3.1.5)}$$

From (3.1.4) and (3.1.5), we obtain

$$\lim_{n \to \infty} \|T^k x_n\| = 1.$$

Now, for all $n \in \mathbb{N}$

$$\|T^k x_n\| \le \|T^{k-1} x_n\| \le \ldots \le \|Tx_n\|$$

and since

$$\lim_{n \to \infty} \|T^k x_n\| = 1 = \lim_{n \to \infty} \|Tx_n\|$$

we conclude that for all i ∈[2,k] (i an integer)

$$\lim_{n \to \infty} \|T^i x_n\| = 1.$$

Now

$$\|T^{K+1}x_n\| = \left\|T^k\left(\frac{Tx_n}{\|Tx_n\|}\right)\right\|\|Tx_n\| \geq \left\|T\left(\frac{Tx_n}{\|Tx_n\|}\right)\right\|^k \|Tx_n\|$$

since T is k-paranormal, i.e;

$$\|T^{K+1}x_n\| \geq \frac{\|T^2 x_n\|^k}{\|Tx_n\|^{k-1}} \to 1 \text{ as } n \to \infty;$$

and hence, using induction argument, we get

$$\lim_{n \to \infty} \|T^m x_n\| = 1, \text{ for all } m \in \mathbb{N} .$$

Thus

$$\|T^m\| = \sup\{\|T^m x\| : \|x\| = 1\} \geq 1$$

and $\|T^m\| \leq \|T\|^m = 1$. Consequently,

$$\|T^m\| = 1 \text{ for all } m \in \mathbb{N} .$$

Hence

$$r(T) = \lim_{m \to \infty} \|T^m\|^{\frac{1}{m}} = 1 = \|T\|.$$

which proves that T is normaloid.

Example 1

Consider a nilpotent operator T_1 on a Hilbert space H_1 with the property that $\|T_1\| = 1$.

Let T_2 be a normal operator on a Hilbert space H_2 such that $\|T_2\| = 1$.

Now, the operator $T = T_1 \oplus T_2$ on $H_1 \oplus H_2$ is normaloid and is not k- paranormal since T_1 is nilpotent.

Proposition 3.1.6

T∈B (H) is paranormal if and only if for all $\lambda \in \mathbb{R}$,

$$T^{*2}\,T^2 - 2\lambda T^*T + \lambda^2 I \geq 0.$$

Proof

If a, b are positive constants and λ is a positive real variable, then it is seen that

$$\sqrt{a}\sqrt{b} = \inf_{\lambda > 0} \frac{1}{2}\left(\lambda^{-1}a + \lambda b\right)$$

(through the differential calculus methods) . Hence, if T∈B (H) and x∈H, we have

$$\|T^2x\|\,\|x\| = \inf_{\lambda > 0} \frac{1}{2}\left(\lambda^{-1}\|T^2x\|^2 + \lambda\|x\|^2\right)$$

$$= \inf_{\lambda > 0} \frac{1}{2}\left\{< (\lambda^{-1}T^{*2}T^2 + \lambda I)\ x, x >\right\}$$

Taking x∈H with $\|x\| = 1$, and using the hypothesis

$$\|T^2x\| - \|Tx\|^2 \geq 0 \text{ (since T is paranormal)}$$

we obtain that T is paranormal if and only if

$$\frac{1}{2}\{< (\lambda^{-1}T^{*2}\,T^2 + \lambda I)x, x >\} - < T^*Tx, x > \geq 0$$

i.e; $T^{*2}\,T^2 - 2\lambda T^*T + \lambda^2 I \geq 0.$

The last result is obviously true when $\lambda \leq 0$.

Proposition 3.1.7

If $T \in B(H)$ is paranormal, then T^n is paranormal, for all $n \in \mathbb{N}$.

Proof

We have seen earlier (proposition 3.1.3) that T^2 is paranormal.

Now assuming that T^k is paranormal, we will prove that T^{K+1} is paranormal.

Now, for all $\lambda \in \mathbb{R}$,

$$T^{*(2k+1)}\ T^{(2k+1)} - 2T^{*(k+1)}T^{(k+1)} + \lambda T^*T$$
$$= T^*(T^{*2k}T^{2k} - 2T^{*k}T^k + \lambda I)T \geq 0 \qquad \text{...(3.1.6)}$$

Since, by proposition 3.1.6, the assumption T^k is paranormal amounts to, equivalently,

$$T^{*2k}T^{2k} - 2T^{*k}T^k + \lambda I \geq 0$$

and (3.1.6) implies that (using $\lambda = 1$)

$$\left\|T^{2k+1}x\right\|^2 - 2\left\|T^{k+1}x\right\|^2 + \left\|Tx\right\|^2 \geq 0 \qquad (\|x\| = 1)$$

and hence

$$\left\|T^{k+1}x\right\|^4 \leq \left\|T^{2k+1}x\right\|^2 \left\|T^2x\right\| \qquad (\|x\| = 1) \quad \text{...................................(3.1.7)}$$

since T is paranormal ($\left\|T^2x\right\| \geq \left\|Tx\right\|^2, \|x\| = 1$).

Now T is paranormal if and only if for all $n \in \mathbb{N}$

$$\left\|T^{n+1}x\right\|^2 \geq \left\|T^nx\right\|^2 \left\|T^2x\right\| \qquad (\|x\| = 1). \quad \text{..................................(3.1.8)}$$

Indeed, when $n = 1$, we have from paranormality of T

$$\left\|T^2x\right\| \geq \left\|Tx\right\|^2 \qquad (\|x\| = 1).$$

Multiplying both sides by $\left\|T^2x\right\|$, we get (3.1.8) for the case $n = 1$.

Assume the result (3.1.7) for $n+1$. Then if $Tx \neq o \qquad (\|x\| = 1)$

$$\left\|T^{n+2}x\right\|^2 = \left\|T^{n+1}\left(\frac{Tx}{\|Tx\|}\right)\right\|^2 \left\|Tx\right\|^2$$

$$\geq \left\| T^n \left(\frac{Tx}{\|Tx\|} \right) \right\|^2 \|Tx\|^2 \|T^2 x\|$$

i.e., $\|T^{n+2}x\|^2 \geq \|T^{n+1}x\|^2 \|T^2 x\|$

and by induction, the result is valid for all $n \in \mathbb{N}$.

Putting $n = 2k+1$ in (3.1.8), we obtain

$$\|T^{2k+2}x\|^2 \geq \|T^{2k+1}x\|^2 \|T^2 x\| \qquad \text{.......................................(3.1.9)}$$

From (3.1.7) and (3.1.9), we obtain

$$\|T^{k+1}x\|^2 \leq \|T^{2k+2}x\| \qquad (\|x\| = 1)$$

which is equivalent to paranormality of T^{k+1}. □

The next result gives a necessary condition for an operator T ∈ B (H) to be k-paranormal.

Proposition 3.1.8

If T ∈ B(H) satisfies

$$T^{*k}T^k - (T^*T)^k \geq 0 \qquad \text{..(3.1.10)}$$

then T is k-Paranormal.

Proof

Let $x \in H$ and $\|x\| = 1$. We then have from (3.1.10)

$$\langle (T^{*k}T^k)x, x\rangle - \langle (T^*T)^k x, x\rangle = \|T^k x\|^2 - \langle |T|^{2k} x, x\rangle \geq 0$$

where $|T|$ is the positive square root of the positive operator

T^*T. Now

$$\langle |T|^{2k} x, x\rangle = \langle |T|^k |T|^k x, x\rangle$$

$$= \langle |T^k| x, |T|^k x\rangle = \||T|^k x\|^2 .$$

Since $|T|$ is self –adjoint, it follows from the spectral theorem for self- adjoint operators that

$$\||T|^k x\|^2 = \||T|x\|^{2k} .$$

Now

$$\|Tx\|^2 = \langle T^*Tx, x\rangle = \langle |T|^2 x, x\rangle$$
$$= \langle |T|x, |T|x\rangle = \||T|x\|^2$$

and hence

$$\|Tx\|^{2k} = \||T|x\|^{2k} .$$

Hence $\langle T^{*k}T^k x, x\rangle - \langle (T^*T)^k x, x\rangle = \|T^k x\|^2 - \|Tx\|^{2k} \geq 0.$

Thus

$$\|Tx\|^k \leq \|T^k x\| \quad \text{for all } x \in H \text{ and } \|x\| = 1;$$

i.e.T is k-paranormal.

Definition 3.1.9

An operator $T \in B(H)$ which satisfies

$$T^{*k}T^k - (T^*T)^k \geq 0$$

is said to be **k-hyper paranormal** $(k \geq 2)$.

Thus if $T \in B(H)$ is k-hyper paranormal, then T is k-paranormal.

Proposition 3.1.10

The class of k-hyper paranormal operators $T \in B(H)$ is strictly smaller than the class of k-paranormal operators.

Proof

Let k be the direct sum of a countable number of copies of H. For positive operators A and B on H and for a positive integer $n > k$, we define the operator $T = T_{A,B,n}$ on k as follows:

$T(x_1, x_2,...) = (\overline{0}, Ax_1, Ax_2,..., Ax_n, Bx_{n+1}, Bx_{n+2},...)$.

A straight forward computation shows that $T^*(x_1, x_2,...) = (Ax_2, Ax_3,..., Ax_{n+1}, Bx_{n+2}, Bx_{n+3},...)$

Thus

$$\begin{aligned} T^*T(x_1, x_2,...) &= T^*(\overline{0}, Ax_1, Ax_2,..., Ax_n, Bx_{n+1}, Bx_{n+2},...) \\ &= (A^2x_1, A^2x_2,..., A^2x_n, B^2x_{n+1}, B^2x_{n+2},...) \end{aligned}$$

and $TT^*(x_1, x_2,....) = T(Ax_2, Ax_3,...Ax_{n+1}, Bx_{n+2}, Bx_{n+3....})$

$$= (\overline{o}, A^2x_2, A^2x_3,....., A^2x_{n+1}, B^2x_{n+2},....).$$

Hence $T^*T - TT^* \geq 0$ if and only if

$$B^2 - A^2 \geq 0.$$

Thus

(i) T is hyponormal if and only if $B^2 \geq A^2$.

(ii) It can be shown similarly that T^2 is k-hyper paranormal if and only if

$AB^{4K-2}A \geq (AB^2A)^k, A^{2k-m}B^{2m}A^{2K-m} \geq A^{4K}, \ m = 1,2...,2k-2.$

If we take H to be a 2-dimensional Hilbert space and A and B the operators which are the square roots of

$$C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

respectively, then $B^2 \geq A^2$ and thus $T_{A,B,n}$ is hyponormal, and it is known that in this case T is paranormal and k-paranormal for all k.

We now show that T^2 is not k-hyper paranormal. Suppose on the contrary that T^2 is k-hyperparanormal.

Then, since A is invertible, we have

$$B^{2m} \geq A^{2m} \text{ , m=1, 2,...,2k-2.}$$

But $B^4 - A^4 = D^2 - C^2$ is not positive and the contradiction proves the result.

For T^2 is paranormal (proposition 3.1.3) and hence k-paranormal (proposition 3.1.4).

But T^2 is not k-hyperparanormal.

Proposition 3.1.11

If $T \in B(H)$ and for all $\lambda \in \mathbb{R}$

$$T^{*k}T^k - 2\lambda\left(T^*T\right)^{k/2} + \lambda^2 I \geq 0, \quad \text{...(3.11)}$$

then T is k-paranormal.

Proof

Indeed (3.1.11) implies that

$$\sqrt{T^{*k}\, T^k} \geq (T^*T)^{k/2}$$

i.e. $\quad T^{*k}T^k \geq (T^*T)^k$,

and by proposition 3.1.8, it follows that T is k-paranormal.

Proposition 3.1.12

The strong closure of hyponormal operators is contained in the class of paranormal operators.

Proof

Let T be in the strong closure of hyponormal operators in B(H). Then there is a sequence (T_n) of hyponormal operators such that

$T_n x \xrightarrow{s} Tx$ for all $x \in H$.

Let $x \in H$ and $\|x\| = 1$. Then

$$\lim_{n \to \infty} \|T_n x\|^2 = \|Tx\|^2.$$

Now hyponormal operators are paranormal.

So

$$\|T_n x\|^2 \leq \|T_n^2 x\| \qquad (\text{for } x \in H \text{ and } \|x\| = 1).$$

Hence

$$\lim_{n \to \infty} \|T_n x\|^2 \leq \lim_{n \to \infty} \|T_n^2 x\| = \|T^2 x\|.$$

Thus T is paranormal.

3.2. SPECTRAL SETS, DILATIONS OF OPERATORS AND NUMERICAL RANGE

Definition 3.2.1

Let K be a Hilbert space and H a subspace of K. Let P be the orthogonal projection on K onto H.

If $T \in B(H)$ is an operator such that

$$Tx = PSx, \ \textit{for all} \ x \in H \text{ and for some } S \in B(K).$$

Then we say that S is a **dilation** of T to K and T is called the **compression** of S.

The space K is called the **dilation space.**

If S is a dilation of T and for all $n \in \mathbb{N}$

$$T^n x = PS^n x,$$

S is called a **power** or **strong dilation of T.** If K is the smallest space containing H and reducing for S, then S is also called **minimal dilation** of T.

Suppose now that X is a compact subset of $\mathbb{C}$ and R(X) represents the algebra of all complex-valued rational functions whose poles are not in X.

Then we say that S is an X-dilation of T if for all $f \in R(X)$, $f(T)$ is the compression of f(S) (of course, we have assumed that f(T) and f(S) exist). If the operator S is normal, we have X-normal dilation.

Definition 3.2.2

A closed subset X of the complex plane is called a **spectral set** for an operator $T \in B(H)$ if

(i) $\sigma(T) \subseteq X$ and

(ii) For all $f \in R(X)$

$$\|f(T)\| \leq \|f\|_X \left(= \sup\{|f(z)| : z \in X\}\right) \quad \text{(Fillmore[5],pg.62)}$$

For normal operators $T \in B(H)$, it is known that $\sigma(T)$ and $\overline{W(T)}$ are spectral sets. (Halmos[8],pg122)

The following result was proved by Lebow A, (1963).

Proposition 3.2.3

If X is a compact subset of $\mathbb{C}$ and is a spectral set for an operator $T \in B(H)$, then there exists a normal dilation N such that N^n is a dilation of T^n (all $n \in \mathbb{N}$) and $\sigma(N) \subset \partial X$.

We use this proposition to obtain some results which indicate a connection between spectral sets, the numerical range and the normal dilation. (Lebow A.[11] pg.64-90).

Definition 3.2.4

An operator $T \in B(H)$ is called **Translation-invariant –normaloid**

(in brief, we write T is T I N) if $T_\lambda = T + \lambda I$ satisfies

$$\|T_\lambda\| = r_{T_\lambda} \quad \text{for all } \lambda \in \mathbb{C}$$

Proposition 3.2.5

If $T \in B(H)$ is Translation- invariant-normaloid, then

$$\overline{W(T)} = \text{Conv}\,\sigma(T)$$

Proof

We know that $conv\sigma(T) \subseteq \overline{W(T)}$ (theorem 2.2.14)

To prove the reserve inclusion, let $D(\lambda,\alpha)$ represent the closed disc in $\mathbb{C}$ with centre λ and radius α and suppose $\sigma(T) \subset D(\lambda,\alpha)$.

Then

$\sigma(T-\lambda I) \subset D(0,\alpha)$ and here $r_{T_\lambda} \leq \alpha$.

But $T-\lambda I$ is Translation-invariant-normaloid, i.e; $r_{T_\lambda} = \|T-\lambda I\|$.

So $\|T-\lambda I\| \leq \alpha$. Since $w(T) \leq \|T\|$, we have $w(T-\lambda I) \leq \|T-\lambda I\|$ and so $W(T-\lambda I) \subseteq D(0,\alpha)$, and hence $W(T) \subseteq D(\lambda,\alpha)$.

Thus every closed disc containing $\sigma(T)$ contains $W(T)$, and therefore $\overline{W(T)} \subset conv\,\sigma(T)$.

Proposition 3.2.6

Let $T \in B(H)$ have $\overline{W(T)}$ as a spectral set, then

$$Conv\sigma(T) = \overline{W(T)} = \overline{W(N)}, \qquad \text{...(3.2.1)}$$

where N is a strong normal dilation of T.

Proof

As $\overline{W(T)}$ is a spectral set, $\sigma(T)$ being closed is also a spectral set and any

$f \in R(\overline{W(T)})$ also belongs to $R(\sigma(T))$.

For any $\lambda \in \mathbb{C}$,

$p(z) = z + \lambda \in R(\overline{W(T)})$ and take $f = p$.

Then since $\sigma(T)$ is a spectral set for T, we have

$$\|p(T)\| \leq \sup\{|p(z)| : z \in \sigma(T)\}$$

i.e. $\|T + \lambda I\| \leq \sup\{|z + \lambda| : z \in \sigma(T)\}$

$= \sup\{|z| : z \in \sigma(T + \lambda I)\}$ (by spectral mapping theorem)

$= r_{T_\lambda} \leq \|T + \lambda I\|$.

i.e $\quad r_{T_\lambda} = \|T + \lambda I\|$ for all $\lambda \in \mathbb{C}$.

Hence T is Translation-invariant-normaloid and by proposition 3.2.5 we get $\overline{W(T)} = Conv\,\sigma(T)$;

which proves the first assertion.

Using proposition 3.2.3 we assert that there is a strong normal dilation N such that

$\sigma(N) \subset \partial\ \overline{W(T)} = \partial\, conv\sigma(T)$.

However

$$conv\,\sigma(N) \subset conv\,\partial\sigma(T) = conv\,\sigma(T) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.2.2)$$

Since N is normal,

$$\overline{W(N)} = conv\sigma(N) \subseteq conv\sigma(T) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(3.2.3)$$

As N is a dilation of T, we have if $x \in H$ and $\|x\| = 1$, then (with P as the orthogonal projection on dilation space onto H)

$$\langle Tx, x\rangle = \langle PNx, x\rangle = \langle Nx, Px\rangle$$

$$=\langle Nx, x\rangle$$

and thus $W(T) \subseteq W(N)$;

accordingly, $\overline{W(T)} \subseteq \overline{W(N)}$.

Thus we get

$$conv\sigma(T) = \overline{W(T)} \subset \overline{W(N)} = conv\sigma(N)$$

$$\subseteq conv\sigma(T);$$

i.e.; set quality holds throughout, yielding (3.2.1)

The following proposition gives a characterization of spectral sets which are closures of numerical ranges.

Proposition 3.2.7

If $T \in B(H)$, then $\overline{W(T)}$ is a spectral set for T if and only if there exists a strong normal dilation N of T such that $\overline{W(T)} = \overline{W(N)}$.

Proof

From proposition 3.2.6 it follows that the condition is necessary sufficiency. For sufficiently large λ, the Neumann series.

$$\frac{-1}{\lambda}\left\{I + \frac{T}{\lambda} + \frac{T^2}{\lambda^2} + \ldots\right\}$$

converges for $(T - \lambda I)^{-1}$ and likewise

$$\frac{-1}{\lambda}\left\{I + \frac{N}{\lambda} + \frac{N^2}{\lambda^2} + \ldots\right\}$$

converges for $(N - \lambda I)^{-1}$.

Since T and N are related by

$$T^n x = PN^n x \quad \text{for all } x \in H$$

where P is the orthogonal projection of the dilation space (=Domain of N), we have for all

$x \in H$

$$(T-\lambda I)^{-1} x = P(N-\lambda I)^{-1} x.$$

Hence, for all $x, y \in H$, we obtain

$$\langle (T-\lambda I)^{-1} x, y\rangle = \langle P(N-\lambda I)^{-1} x, y\rangle$$

$$= \langle (N-\lambda I)^{-1} x, Py\rangle$$
$$= \langle (N-\lambda I)^{-1} x, y\rangle$$

and since $\overline{W(T)} = \overline{W(N)}$, for any $f \in R(\overline{W(T)})$, we have

$$\langle f(T)x, y\rangle = \langle Pf(N)x, y)\rangle$$

$$= \langle f(N)x, Py\rangle$$

$$= \langle f(N)x, y\rangle$$

(Note: f is a rational function with no poles in ($\overline{W(T)}$).

Thus $f(T)x = Pf(N)x$ for all $x \in H$.

Now, we show that $\overline{W(T)}$ is a spectral set for T.

Indeed, for any $f \in R(\overline{W(T)})$

$$\|f(T)\| = \sup\{\|f(T)x\| : x \in H \ and \ \|x\| = 1\}$$

$$= \sup\{\|Pf(N)x\| : x \in H \ and \ \|x\| = 1\}$$

$$\leq \sup\{\|f(N)x\| : x \in H \ and \ \|x\| = 1\}$$

$$\leq \|f(N)\| \qquad \text{..........(3.2.4)}$$

Since N is normal, $\overline{W(N)}$ is a spectral set for N, and thus

$$\|f(N)\| \leq \sup\{|f(z)| : z \in \overline{W(N)}\}$$

$$= \sup\{|f(z)| : z \in \overline{W(T)}\} \qquad \text{(for } \overline{W(N)} = \overline{W(T)}) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3.2.5)$$

Thus, for any $f \in R(\overline{W(T)})$, we obtain from (3.2.4) and (3.2.5):

$$\|f(T)\| \leq \sup\{|f(z)| : z \in \overline{W(T)}\},$$

which shows that $\overline{W(T)}$ is a spectral set for T.

Proposition 3.2.8

Let $T \in B(H)$ and $\lambda \in \rho(T)$. If $d = dist(\lambda, \sigma(T))$, then

$$r_{(T-\lambda I)^{-1}} = \frac{1}{d}.$$

Proof

This follows from the spectral mapping theorem.

Indeed, for $\lambda \in \rho(T)$, we have

$$\sigma\left((T-\lambda I)^{-1}\right) = \left\{\frac{1}{\alpha} : \alpha \in \sigma(T-\lambda I)\right\}$$

(Note $\alpha \neq 0$ since $T - \lambda I$ is invertible).

From the spectral mapping theorem, we have

$$\sigma\left((T-\lambda I)^{-1}\right) = \left\{\frac{1}{\theta - \lambda} : \theta \in \sigma(T)\right\}.$$

Now

$$r_{(T-\lambda I)^{-1}} = \sup\left\{\frac{1}{|\theta - \lambda|} : \theta \in \sigma(T)\right\}$$

$$= \frac{1}{\inf\{|\theta - \lambda| : \theta \in \sigma(T)\}} = \frac{1}{d}$$

Corollary 3.2.9

For any $T \in B(H)$ and $\lambda \in \rho(T)$,if $d = dist(\lambda, \sigma(T))$,

then $\left\|(T-\lambda I)^{-1}\right\| \geq \dfrac{1}{d}$.

Proof

This is immediate since for any $T \in B(H), r_T \leq \|T\|$.

Definition 3.2.10

An operator $T \in B(H)$ is said to have **G_1-property** if

$$\left\|(T-\lambda I)^{-1}\right\| = \frac{1}{dist(\lambda, \sigma(T))}$$

for all $\lambda \in \rho(T)$.

When $T \in B(H)$ is hyponormal, it is clear that $T - \lambda I$ is also hyponormal for all $\lambda \in \mathbb{C}$ and if, in addition, T is invertible, then T^{-1} is also hyponormal.
(Fillmore [5]).

Since $r_T = \|T\|$ for hyponormal $T \in B(H)$, we note that among the operators satisfying G_1-property, we have normal and hyponormal operators.

Proposition 3.2.11

If $T \in B(H)$ has G_1-property, then $conv\sigma(T) = \overline{W(T)}$.

Proof

As, for every $T \in B(H)$,we have $\sigma(T) \subseteq \overline{W(T)}$, we need to prove the reverse inclusion $\overline{W(T)} \subseteq \sigma(T)$ for T is satisfying the G_1-property.

It is sufficient to show that any closed half – plane which contains $\sigma(T)$ also contains $\overline{W(T)}$.

Then the intersection of all the closed half – planes that include the (compact) set $\sigma(T)$ is the convex hull, $conv\sigma(T)$ and this intersection obviously contains $\overline{W(T)}$.

By translation and rotation this reduces to showing that

$$\operatorname{Re}\sigma(T) \leq 0 \text{ implies } \operatorname{Re}\overline{W(T)} \leq 0.$$

Let $\|\tilde{x}\| = 1$ and $T\tilde{x} = (a+ib)\tilde{x} + \tilde{y}$, with $a, b \in \Re$ and $\tilde{x} \perp \tilde{y}$.

For all $c > 0$, we note that $c \in \rho(T)$ and hence $(T - cI)^{-1}$ exists.

Let $dist(c, \sigma(T)) = \tilde{c}$. The $\tilde{c} \geq c$ since $\operatorname{Re}\sigma(T) \leq 0$.

Since T satisfies property G_1, we have

$$\left\|(T-cI)^{-1}\right\| = \frac{1}{\tilde{c}} \leq \frac{1}{c}. \text{ Thus}$$

$$\left\|(T-cI)^{-1}z\right\|^2 \leq \frac{1}{c^2}\|z\|^2 \text{ for all } z \in H.$$

Replacing $(T-cI)^{-1}z$ by x and hence z by $(T-cI)x$, we get

$$c^2\|x\|^2 \leq \|(T-cI)x\|^2 \text{ for all } x \in H.$$

Put $x = \tilde{x}$. Then since $\|\tilde{x}\| = 1$ we get $c^2 \leq \|(T-cI)\tilde{x}\|^2$.

Now

$$\|(T-cI)\tilde{x}\|^2 = \|(a+ib)\tilde{x} + \tilde{y} - c\tilde{x}\|^2$$

$$= (a-c)^2 + b^2 + \|\tilde{y}\|^2.$$

Hence $c^2 \leq (a-c)^2 + b^2 + \|\tilde{y}\|^2$, i.e. $2ac \leq a^2 + b^2 + \|\tilde{y}\|^2$.

Since this holds for all $c > 0$,

$$\operatorname{Re}\langle T\tilde{x}, \tilde{x}\rangle = a \leq 0.$$

This completes the proof.

The next proposition tells us about the structure of operators with the G_1 property in the case when $\sigma(T)$ is a finite a set.

Proposition 3.2.12

Let $T \in B(H)$ have the following properties

(i) T has G_1 property

(ii) $\sigma(T)$ is a finite set.

Then T is normal.

Proof

Let $\sigma(T) = \{\lambda_1, \lambda_2, ... \lambda_n\}$ and for each λ_j let C_j be a circle with centre λ_j and radius ε. We assume that ε is so small that the intersection of the closed disk of boundary C_j with $\sigma(T)$ contains λ_j and no other points of $\sigma(T)$.

Define the projection P_j by

$$P_j = \frac{1}{2\pi i} \int_{C_j} (T - zI)^{-1} dz$$

which clearly commutes with T.

since

$$\|P_j\| = \frac{1}{2\pi} \left\| \int_{C_j} (T - zI)^{-1} dz \right\| \leq \frac{1}{2\pi} \int_{C_j} \|(T - zI)^{-1}\| \, |dz|$$

$$< \frac{1}{2\pi} (2\pi\varepsilon) \frac{1}{\varepsilon} = 1$$

for $\int_{C_j} |dz| = 2\pi\varepsilon$

and since $T \in G_1$, so

$$\left\|(T - zI)^{-1}\right\| = \frac{1}{dist(z, \sigma(T))}$$

$$= \frac{1}{dist(z, \lambda_j)} = \frac{1}{\varepsilon}$$

it follows that P_j is an orthogonal projection.

Let $x \in P_j(H)$. Now

$$(T - \lambda_j I)x = (T - \lambda_j I)P_j x \quad \text{for } P_j x = x .$$

We note that

$$(T - \lambda_j I)P_j = (T - \lambda_j I)\frac{1}{2\pi i}\int_{C_j}(T - zI)^{-1}\,dz$$

$$= \frac{1}{2\pi i}\int_{C_j}(T - zI + zI - \lambda_j I)(T - zI)^{-1}\,dz$$

$$= \frac{1}{2\pi i}\int_{C_j}\{I + (z - \lambda_j)(T - zI)^{-1}\}dz$$

$$= 0 + \frac{1}{2\pi i}\int_{C_j}(z - \lambda_j)(T - zI)^{-1}\,dz$$

Hence

$$\left\|(T - \lambda_j)x\right\| \le \left(\frac{1}{2\pi}\int_{C_j}\left|z - \lambda_j\right|\left\|(T - zI)^{-1}\right\|\left|dz\right|\right)\left\|x\right\| \qquad \text{..........................(3.2.6)}$$

$$\le \left(\frac{1}{2\pi}\varepsilon.\frac{1}{\varepsilon}.2\pi\varepsilon\right)\left\|x\right\| = \varepsilon\left\|x\right\| \to 0 \;\; as \;\; \varepsilon \to 0 .$$

We obtain $Tx = \lambda_j x$.

Since $T = \sum \lambda_j P_j$, it follows that T is Normal.

Corollary 3.2.13

On finite dimensional Hilbert spaces every operator with G_1-property is normal.

The following is an example of a bounded operator which has G_1- property but is not normal.

Example (ii)

Let T_1 be an operator on l_2^2 with the matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ which is nilpotent } \left(T_1^2 = 0\right).$$

Let T_2 be a bounded operator defined on an arbitrary Hilbert space with a countable basis

$\{e_i)_{i=1}^{\infty}$ by

$$T_2 e_i = \lambda_i e_i$$

where the λ_i are such that

$$\min_i |z - \lambda_i| < |z|^2$$

for all z, $|z| < 1$. The operator $T = T_1 \oplus T_2$ is of class G_1 and is clearly non-unitary (also non-normal).

This example also shows that the restriction of an operator in class G_1 fails to have the G_1-property on the invariant subspace.

Proposition 3.2.14

The set of all $T \in B(H)$ with G_1 property is strongly dense in B(H).

Proof

Let $T \in B(H)$ arbitrary and $\{x_1, x_2, \ldots, x_n\} \subset H$ and let M= subspace generated by $\{x_1, x_2, \ldots, x_n\}$.

We define the operator A on H as follows:

$$Ax_i = Tx_i \quad , \quad i = 1,2,\ldots, n,$$

and $Ax = \bar{o}$ if $x \in M^{\perp}$.

We can consider a normal operator N on $M^{\perp}$ such that

$$\sigma(N) = \{z : |z| \leq \|A\|\}.$$

The operator $A \oplus N$ satisfies the G_1 property and since it coincides with T on the x_i, the strong density is clear.

The following definition gives, in some sense, a generalization of the class of operators with G_1 property.

Definition 3.2.15

Let $T \in B(H)$. T satisfies the **sequential G_1- property** if for every $z \in \partial\sigma(T)$ there exists a sequence $(z_n) \subset \rho(T)$ such that

(i) $z_n \to z$ as $n \to \infty$

(ii) $\|(T - z_n I)^{-1}\| = \dfrac{1}{d(z_n, \sigma(T))}$ for all n.

Proposition 3.2.16

If $T \in B(H)$ is quasinilpotent and $0 \in \partial W(T)$, then T has sequential G_1- property.

Proof

Since W(T) is a convex set there is a line of support through 0. Hence we can assume, without loss of generality, that $W(T) \subset \{z : \operatorname{Re} z \geq 0\}$.

But $0 \in \partial W(T)$ and $\sigma(T) = \{0\}$. Since for any operator

$$\|(T - zI)^{-1}\| \leq \frac{1}{d(z, \overline{W(T)})}.$$

We can find that in our case, for any real negative number z,

$$d(z,W(T))=|z|.$$

So, taking $z=-\frac{1}{n}$, the proposition is proved.

An example of an operator with the sequential G_1 property we have the Voltera operator on L^2.

The later does not have the G_1 property.

Note that every Voltera operator is quasinilpotent.

Proposition 3.2.17

If $T\in B(H)$ and is with the sequential G_1- property and for some m, $T^m = 0$, then $T = 0$.

Proof

Since T has sequential G_1-property and is nilpotent, it follows that there exists a sequence (z_n) such that

$$\left\|(T-z_n)^{-1}\right\|=1 \qquad \text{for all n.}$$

Suppose now m > 1 and thus

$$(z_n I-T)^{-1}=\sum_{i=0}^{m-1}\frac{T^i}{z_n^{i+1}}.$$

This implies that

$$\frac{\left\|T^{m-1}\right\|}{|z_n|^m}=\sum_{i=0}^{m-2}\frac{\left\|T^i\right\|}{\left|z_n^{i+1}\right|}<\frac{1}{|z_n|}.$$

for all n.

Hence

$$\left\|T^{m-1}\right\| < \left|z_n^{\ m-1}\right| + \sum_{i=0}^{m-2} \frac{\left\|T^i\right\|}{\left|z_n\right|^{m-i-1}}$$

for all n. But

$$\left|z_n\right|^{m-1} + \sum_{i=0}^{m-2} \frac{\left\|T^i\right\|}{\left|z_n\right|^{m-i-1}} \to 0 \text{ as } n \to \infty$$

and this shows that $\left\|T^{m-1}\right\| = 0.$

An induction argument shows that T = 0.

Proposition 3.2.18

Let T∈B(H) and $z_o \in \sigma(T)$ have the following properties

(i) Tx = z_o x, $\|x\| = 1.$

(ii) There exists a subsequence (z_n) of elements of $\rho(T)$ such that $z_n \to z_o$ and

$$\frac{\left\|(T - z_n I)^{-1}\right\|}{\left|z_n - z_o\right|} \to 0.$$

Then $T^* x = \bar{z}_0 x$, i. e; $(T^* - \bar{z}_o I)x = \bar{o}.$

Proof

Without loss of generality, we can assume that $z_o = 0.$

Suppose that $T^* x = y.$ In this case

$$(T^* - \bar{z}_n I)x = y - \bar{z}_n x \quad \text{and thus}$$

$$\bar{z}_n (T^* - \bar{z}_n I)^{-1} x = -x + (T^* - \bar{z}_n I)^{-1} y.$$

But $x \perp (T^* - \bar{z}_n I)^{-1} y;$ thus we get

$$\|(T^*-\bar{z}_nI)^{-1}y\|^2+\|x\|^2=|\bar{z}_n|^2\|(T^*-\bar{z}_nI)^{-1}x\|^2$$

$$\begin{aligned}\|(T^*-\bar{z}_n)^{-1}y\| &\le \|(T^*-\bar{z}_nI)^{-1}\|\,\|y\| \\ &= \|((T-z_nI)^*)^{-1}\|\,\|y\|. \\ &= \|((T-z_nI)^{-1})^*\|\,\|y\| \\ &= \|(T-z_nI)^{-1}\|\,\|y\| \\ &= \frac{\|(T-z_nI)^{-1}\|}{|z_n|}|z_n|\,\|y\| \qquad \to 0 \ as \ n\to\infty\end{aligned}$$

i.e.

$$\lim_{n\to\infty}\|(T^*-\bar{z}_nI)^{-1}y\|=0.$$

Since

$$\|y\|=\lim_{n\to\infty}\|(T^*-\bar{z}_nI)\,(T^*-\bar{z}_nI)^{-1}y\|=0,$$

the proposition is proved.

Definition 3.2.19

An operator $T\in B(H)$ is called **algebraic** if there exists a polynomial P(z), not identically zero, such that P(T) = 0.

The next proposition gives the structure of a class of operators with the sequential G_1- property.

Proposition 3.2.20

Suppose $T\in B$ (H) and T has the following properties:

(i) T is ***algebraic***

(ii) T has the sequential G_1-property

Then T is normal.

Proof

Since T is algebraic it follows from the spectral mapping theorem that $\sigma(T) = \{\lambda_1, \lambda_2, ..., \lambda_m\}$ with multiplicities $\{n_i\}$. In this case,

$H = \sum_{i=1}^{m} H_i$, where

$$H_i = \{x \in H : (T - \lambda_i I)^{n_i} x = \overline{o}\}$$

and clearly $\sigma(T|_H) = \{\lambda_i\}$.

Since T has the sequential G_1- property, it follows that $T|_{H_i}$ also has the same property.

But $T - \lambda_i I$ is a nilpotent operator on H_i. From the above result (Proposition 3.2.17), it follows that $(T - \lambda_i I)|_{H_i} = 0$. By proposition 3.2.18, it follows that $(T^* - \overline{\lambda}_i I)|_{H_i} = 0$, which implies that for $i \neq j, H_i \perp H_j$. Thus we have $H = \sum H_i$ and $T = \sum \lambda_i p_i$, where P_i is the orthogonal projection of H onto H_i.

The concept of operators with G_1-property can be generalized.

Theorem 3.2.21

For any operator $T \in B$ (H) we have the following inequality

$$\frac{1}{dist(z, \overline{W(T)})} \geq \|R_z(T)\| \text{ if } z \notin \overline{W(T)} \quad \text{.........................(3.2.7)}$$

(Bachman and Narici,[2], pg. 386, Thm 21.11).

Now conv $\sigma(T) \subset \overline{W(T)}$ and therefore it does not necessarily follow that

$$\frac{1}{dist(z, conv\sigma(T))} \geq \|R_z(T)\| \text{ for all } z \notin conv\sigma(T).$$

We have therefore a natural generalization of operators of class G_1:

Definition 3.2.22

An operator $T \in B(H)$ is said to be in **spectral G_1-class** (or with **spectral G_1-property** or equivalently, a **convexoid operator)** if for all $z \notin conv\ \sigma(T)$

$$\|(T - zI)^{-1}\|, \text{ i.e., } \|R_z(T)\| \leq \frac{1}{dist(z, Conv\ \sigma(T))}.$$

First we prove

Lemma 3.2.23

An operator $T \in B(H)$ is convexoid (or is with spectral G_1– property) if and only if

$$\{dist(z, conv\ \sigma(T)\}\ \|y\| \leq \|Ty - zy\| \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.2.8)$$

for all $z \notin conv\ \sigma(T)$.

Proof

T is convexoid $\Leftrightarrow$ $\|(T - zI)^{-1}\| \leq \dfrac{1}{dist(z, conv\ \sigma(T))}$

for all $z \notin conv\sigma(T)$.

$$\Leftrightarrow \|(T - zI)^{-1} y\| \leq \frac{\|y\|}{dist(z, conv\sigma(T))} \quad \textit{for all } y \in H \text{ and}$$

for all $z \notin conv\sigma(T)$.

Putting (T-zI)y in place of y in the last line, we get

$$\|y\| \leq \frac{\|(T - zI)y\|}{dist(z, conv\ \sigma(T))}$$

i.e. relation (3.2.8) holds.

Lemma 3.2.24

Let $T \in B(H)$ with $\|(T-\lambda I)^{-1}\| \leq \frac{1}{dist(\lambda, conv\,\sigma(T))}$

for some $\lambda \notin conv \;\; \sigma(T)$.

Then for any complex numbers $a \neq 0$ and b, the transformation $z \mapsto az+b$ maps the complement of $conv\sigma(T)$ on to the complement of conv $\sigma(aT+bI)$ and

$$\|R_{a\lambda+b}(aT+bI)\| \leq \frac{1}{dist(a\lambda+b, conv\sigma(aT+bI))}$$

Proof

Since $a \neq 0,$ the transformation $z \mapsto az+b$ maps $\mathbb{C}$ onto $\mathbb{C}$ and is one-to-one and by the spectral mapping theorem $\sigma(T)$ is mapped onto $\sigma(aT+bI)$.

From this it follows that conv $\sigma(T)$ is mapped onto conv $\sigma(aT+bI)$. Taking the complements in $\mathbb{C}$ the first assertion of the lemma follows.

For the second assertion, we use lemma 3.2.23, by which for all y$\in$ H, we have

$\{dist(\lambda,\; conv\;\; \sigma(T))\}\; \|y\| \leq \|(T-\lambda I)y\|,$

and from this, we have

$$\begin{aligned}|a|\{dist(\lambda, conv\sigma(T))\}\|y\| &= \{\text{dist } (a\lambda, a\; conv\sigma(T))\}\|y\|. \\ &= \{dist(a\lambda, conv\sigma(aT))\}\|y\| \\ &= dist(a\lambda+b, conv\sigma(aT+bI))\|y\| \\ &\leq |a|\|Ty-\lambda y\| \\ &= \|(aT+bI)y-(a\lambda+b)y\|\end{aligned}$$

and the assertion of the lemma is proved.

Lemma 3.2.25

If $T \in B(H)$, then $\|R_\lambda(T)\| \le \frac{1}{\lambda}$ for all $\lambda > 0$ if and only if Re W(T) is in $\{x : x \le 0\}$.

Proof

Let y∈ H and $\lambda > 0$. Then $\lambda\|y\| \le \|(T - \lambda I)y\|$ if and only if

$$\lambda^2\|y\|^2 \le \|Ty\|^2 + \lambda^2\|y\|^2 - 2\lambda \operatorname{Re}\langle Ty, y\rangle$$

and this implies that

$$2\lambda \operatorname{Re}\langle Ty, y\rangle \le \|Ty\|^2.$$

This holds for all $\lambda > 0$ if and only if $\operatorname{Re}\langle Ty, y\rangle \le 0$.

With the help of these results, we now prove

Proposition 3.2.26

An operator $T \in B(H)$ is convexoid if and only if $\operatorname{conv}\sigma(T) = \overline{W(T)}$.

Proof

First, let $T \in B(H)$ be convexoid. We know that, for any operator $T \in B(H)$

$$\operatorname{conv}\sigma(T) \subseteq \overline{W(T)}.$$

To prove the assertion, we show the reverse inclusion.

Let L be a line of support of $\operatorname{conv}\sigma(T)$ and the closed half plane which contains $\operatorname{conv}\sigma(T)$; we show that it contains $\overline{W(T)}$

If L touches conv $\sigma(T)$ at a point, say λ_o, we find the complex numbers a and b, $|a| = 1$, such that $z \mapsto az + b$ sends λ_o to the origin, L into the y-axis, and conv $\sigma(T)$ into a part of the left closed half plane.

From Lemma 3.2.24 we obtain

$$\|R_{az+b}(aT + bI)\| \leq \frac{1}{dist(az + b, conv\sigma(aT + bI))}$$

for all $az + b \notin conv\sigma(aT + bI)$.

We can choose a and b such that az + b is a strictly positive number and apply Lemma 3.2.25 above, and thus

$$\operatorname{Re} W(aT + bI) \leq 0$$

and this implies that $W(T)$ is on the same side of L as conv $\sigma(T)$.

Thus W(T)$\subseteq conv\sigma(T)$ and consequently $\overline{W(T)} \subseteq conv\sigma(T)$.

Conversely, suppose that conv $\sigma(T)=\overline{W(T)}$; we prove that T is convexoid.

Let z be any point in $\overline{W(T)}=conv\sigma(T)$. We find a unique point $\lambda_o \in \overline{W(T)}$ such that

$$|\lambda_o - z| = dist(z, \overline{W(T)}).$$

The line through λ_o and perpendicular to the line joining λ_0 and z is a line of support of $W(T)$ we denote this line as L. Hence we can find complex numbers a and b such that $z \mapsto az + b$ carries λ_o into the origin, L into the y-axis, and z into $|z - \lambda_o|$.

Since $\overline{W(aT + b)} =$ conv σ (aT + b), we have, by lemma 3.2.25

$$\|R_{|z-\lambda o|}(aT + b)\| \leq \frac{1}{|z - \lambda_o|} = \frac{1}{dist(|z - \lambda_o|, conv\sigma(aT + bI))}.$$

Since the inverse of the map $z \mapsto az + b$ has an inverse mapping of the same form, it follows from lemma 3.2.24 that

$$\|R_z(T)\| \leq \frac{1}{dist(z, conv\sigma(T))}$$, i.e; T is convexoid.

The converse part has a simpler alternative solution:

Suppose that $conv\sigma(T) = \overline{W(T)}$ for a $T \in B(H)$. ....(3.2.9)

Since for all $z \notin \overline{W(T)}$, we have

$$\|R_z(T)\| \leq \frac{1}{dist(z, \overline{W(T)})}.$$

It follows that

$$\|R_z(T)\| \leq \frac{1}{dist(z, conv\sigma(T))} \text{ for } z \notin conv\sigma(T)$$

since (3.2.8) holds. But this means that T is convexoid.

We can weaken the condition in the G_1 property by, supposing that z is in an open neighborhood of $\sigma(T)$.

Definition 3.2.27

An operator $T \in B(H)$ is said to be of **class loc-G_1** if there exists an open neighborhood U of $\sigma(T)$ such that for all $z \in U / \sigma(T)$

$$\|R_z(T)\| \leq \frac{1}{d(z, \sigma(T))}.$$

This class loc-G_1 is strictly larger than the class of operators with G_1- property. To see this, we have

Proposition 3.2.28

Let H and K be two Hilbert spaces and $H \oplus K$ their orthogonal sum, A be an arbitrary operator in B (H) and N be a normal operator in B (K). Then the operator $T = A \oplus N$ is in loc-G_1 class when $\sigma(A) \subset U \subset \sigma(N)$, where U is open.

Proof

We have $\sigma(T) = \sigma(A \oplus N) = \sigma(A) \cup \sigma(N) = \sigma(N)$.

If $z \notin \sigma(A \oplus N)$, then

$$\|R_z(T)\| = \max\{\|R_z(A)\|, \|R_z(N)\|\}$$

$$= \max\left\{\|R_z(A)\|, \frac{1}{d(z, \sigma(N))}\right\}$$

since N is a normal operator.

Now since there exists an open set U containing $\sigma(A)$ and contained in $\sigma(N)$, we can find V open such that

$$V \supset \sigma(N)(= \sigma(T))$$

and thus for all $z \in V|\sigma(T)$

$$\|R_z(T)\| \leq \frac{1}{d(z, \sigma(T))}$$

Definition 3.2.29

A complex number λ is called a **normal eigenvalue** for $T \in B(H)$ if

$$\ker(T^* - \overline{\lambda} I) = \ker(T - \lambda I).$$

Proposition 3.2.30

If $T \in loc - G_1$ and λ_0 is an isolated point in $\sigma(T)$, then λ_0 is a normal eigenvalue for T.

Proof

Let $\varepsilon > 0$ and C_ε be a circle with centre λ_0 and radius ε such that the intersection of the closed disc with boundary C_ε with $\sigma(T)$ is just $\{\lambda_0\}$.

Let

$$P_{\lambda_0} = \frac{1}{2\pi i} \int_{C_\varepsilon} (T - \lambda I)^{-1} d\lambda$$

which is an orthogonal projection commuting with T (see proposition 3.2.12).

The argument given there also shows that if $x \in P_{\lambda_0}(H)$, then $Tx = \lambda_0 x$ and since

$$\|R_\lambda(T)\| = \|R_{\bar{\lambda}}(T^*)\|$$

we obtain $T^* x = \bar{\lambda}_0 x$.

Indeed, since $\sigma(T^*) = \{\bar{z} : z \in \sigma(T)\}$, if λ_0 is an isolated point of $\sigma(T), \bar{\lambda}_0$ is an isolated point of $\sigma(T^*)$. Let C'_ε be the circle of radius ε centered at $\bar{\lambda}_0$ (with the usual orientation). So the closed disc with centre $\bar{\lambda}_0$ and radius ε intersects $\sigma(T^*)$ at the point $\bar{\lambda}_0$ only.

Consider the orthogonal projection

$$P_{\bar{\lambda}_0} = \frac{1}{2\pi i} \int_{C'_\varepsilon} (T^* - \lambda I)^{-1} d\lambda .$$

Then it is easily seen that

$$P^*_{\bar{\lambda}_0} = \frac{1}{2\pi i} \int_{C_\varepsilon} (T - \lambda I)^{-1} d\lambda \qquad \text{.......................................(3.2.10)}$$

But $P^*_{\bar{\lambda}_0} = P_{\bar{\lambda}_0}$, whereas the right side of (3.2.10) is P_{λ_0} and hence $P_{\bar{\lambda}_0} = P_{\lambda_0}$.

Now for $x \in P_{\lambda_0}(H) = P_{\bar{\lambda}_0}(H)$, we have (see equation (3.2.6), proposition 3.2.12).

$$\|(T^* - \bar{\lambda}_0)x\| \le \left(\frac{1}{2\pi} \int_{C'_\varepsilon} |z - \bar{\lambda}_0| \, \|(T^* - zI)^{-1}\| \, |dz| \right) \|x\|$$

$$= \left(\frac{1}{2\pi} \int_{C_\varepsilon} |z - \lambda_0| \, \|(T - zI)^{-1}\| \, |dz| \right) \|x\|$$

$$\leq \varepsilon \|x\|$$

and $\varepsilon\|x\| \to 0$ as $\varepsilon \to 0$, to give $T^* x = \overline{\lambda}_0 x$.

As a corollary we have

Corollary 3.2.31

On a finite dimensional Hilbert space the class of normal operators and the class loc-G_1 are the same.

We give an illustration. Let A be the operator with the matrix

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

and it is seen that

$$\overline{W(A)} = \{z : |z| \leq 1\}.$$

Consider a normal operator N on a Hilbert space such that

$$\sigma(N) = \left\{ z : |z| \leq \frac{1}{2} \right\}.$$

Then the operator $T = A \oplus N$ is in class loc-G_1 and is not with G_1 property. Note that

$conv\sigma(T) \neq W(T)$.

We can consider a new class of operators which is related to the class of operators with spectral G_1 property and loc-G_1.

Definition 3.2.32

An operator $T \in B(H)$ is said to be in the class **spectral loc-G$_1$** if there exists an open neighborhood U of $conv\sigma(T)$ such that

$$\|R_z(T)\| \leq \frac{1}{d(z, conv\sigma(T))}$$

for all $z \in U \setminus conv\sigma(T)$.

We define another class as follows:

Definition 3.2.33

An operator $T \in B(H)$ is said to be of **class R** if for all $z \notin \sigma(T)$

$$\|R_z(T)\| = \frac{1}{d(z,(\overline{W(T)})}.$$

In the next proposition, we give a characterization of this class of operators.

Proposition 3.2.34

An operator $T \in B(H)$ is of **class R** if and only if $\partial W(T) \subset \sigma(T)$.

Proof

First let $\partial W(T) \subset \sigma(T)$ and $z \notin W(T)$. In this case we have $dist(z, W(T) = dist(z, \sigma(T))$

and

$$\begin{aligned} \frac{1}{d(z,\overline{W(T)})} &= \frac{1}{d(z,\sigma(T))} \\ &\leq \|R_z(T)\| \\ &\leq \frac{1}{d(z,\overline{W(T)})} \end{aligned}$$

and thus T is of class R.

Suppose now that T is of class R and $z_0 \in \partial W(T)$.

Then we can find a sequence (z_n) such that $|z_n - z_0| \to 0$, $d(z_n, W(T)) > 0$ for all $n \in \mathbb{N}$. But we have

$$\|R_{z_n}(T)\| = \frac{1}{d\left(z_n, \overline{W(T)}\right)}$$

and thus $z_0 \in \sigma(T)$.

Proposition 3.2.35

If $T \in B(H)$ is of class R, then the following assertions hold:

(i) $Conv\sigma(T) = \overline{W(T)}$

(ii) if $\sigma(T)$ is a finite set, then T is a multiple of I

Proof

(i) The first assertion is a simple consequence of the characterization of operators of class R.

(ii) For the second, from the convexity of W(T), it follows that W(T) contains only one point, which clearly implies the rest of the proposition.

CHAPTER FOUR

ADDITIONAL RESULTS ON NON-NORMAL OPERATORS

This chapter continues with additional results on non-normal operators.

We obtain a set of necessary and sufficient conditions for convexoidity and characterize those operators $T \in B(H)$ for which $\operatorname{Re}\sigma(T) = \sigma(\operatorname{Re} T)$ generally.

We also obtain results which indicate a connection between spectral sets, the numerical range and normal dilation of an operator T.

We finally study operators of class $\overline{R}_1$ and show that this class includes normaloid, spectraloid, paranormal, hyponormal and T + k, where $T \in B(H)$ is isometric or has G_1-property, or has sequential G_1-property, or hyponormal and k is compact.

4.1. RESULTS AND CONSEQUENCES

If X is a complex Banach space and $T \in B(X)$ for a polynomial $p(\lambda) = a_0 + a_1\lambda + \ldots + a_n\lambda^n$ with complex coefficients, we represent by p(T) the element $a_0 I + a_1 T + \ldots + a_n T^n \in B(X)$.

By the spectral mapping theorem we have

$$\sigma(p(T)) = \{p(\lambda) : \lambda \in \sigma(T)\} = p(\sigma(T)).$$

Now if H is a complex Hilbert space and $T \in B(H)$ is self-adjoint we have

$$\|T\| = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

Then $\sigma(T)$ is a compact subset of $\Re$ and the set $C(\sigma(T))$ of all complex- valued continuous functions on $\sigma(T)$ is a Banach space with respect to the uniform norm

$$\|f\| = \sup\{|f(\lambda)| : \lambda \in \sigma(T)\}, \qquad (f \in C(\sigma(T))).$$

When $f, g \in C(\sigma(T))$, the product fg and the complex conjugate function $\bar{f}$ (both defined pointwise) are also in $C(\sigma(T))$. By the classical Weierstrass approximation theorem the set p of all polynomials (considered as functions on $\sigma(T)$) is an everywhere dense subspace of $C(\sigma(T))$.

Proposition 4.1.1

Let $T \in B(H)$ be self-adjoint. Then there is a unique linear mapping $f \mapsto f(T)$ from the algebra $C(\sigma(T))$ into B(H) such that

(i) f(T) has its elementary meaning when f is a polynomial;

(ii) $\|f(T)\| = \|f\| \quad (f \in c(\sigma(T)))$

Furthermore, for each f and g in $C(\sigma(T))$,

(iii) (fg)(T)=f(T)g(T).

(iv) f(T) = f(T*).

(v) f(T) is normal.

(vi) f(T)S = Sf(T), whenever $S \in B(H)$ and TS = ST.

(vii) If $\lambda \in P_\sigma(T), x \in H$ and Tx = λx, then $f(T)x = f(\lambda)x$.

Proof

If $p(\lambda) = a_0 + a_1\lambda + ... + a_n\lambda^n$, then $\overline{p}(\lambda) = \overline{a}_0 + \overline{a}_1\lambda + ... + \overline{a}_n\lambda^n$ when $\lambda \in \sigma(T)$.

Thus

$$p(T^*) = (a_0 I + a_1 T + ... + a_n T^n)^*$$

$$= \overline{a}_0 I + \overline{a}_1 T + ... + \overline{a}_n T^n = \overline{p}(T).$$

It follows that p(T) is normal (for p(T)p(T*)=p(T*)p(T)).

Thus

$$\|P(T)\| = r(P(T)) = \sup\{|\mu| : \mu \in \sigma(p(T))\}$$

$$= \sup\{|p(\lambda)| : \lambda \in \sigma(T)\} = \|p\|$$

(since $\sigma(p(T)) = p(\sigma(T))$),

where $\|p\|$ denotes the norm of p as an element of $C(\sigma(T))$.

This shows that the linear mapping $p \mapsto p(T)$ from p into B(H) is isometric.

Since p is dense in $C(\sigma(T))$ and B(H) is complete, this mapping extends uniquely by continuity, to an isometric linear mapping $f \mapsto f(T)$ from $C(\sigma(T))$ into B(H).

We have thus proved the existence and uniqueness of the linear mapping $f \mapsto f(T))$ from $C(\sigma(T))$ into B(H), satisfying parts (i) and (ii) of the proposition. It is evident that parts (iii), (iv), (v), (vi) and (vii) are satisfied when f, g are polynomials; by continuity, they remain valid whenever f, g$\in C(\sigma(T))$.

The mapping $f \mapsto f(T))$ described in this proposition is called the functional calculus for the self-adjoint operator T.

It is thus clear that the linear mapping $f \mapsto f(T))$ of $C(\sigma(T))$ into B(H) is an isomorphism from the algebra $C(\sigma(T))$ onto a sub algebra of B(H).

Proposition 4.1.2

If $T \in B(H)$ is a normal operator and if $o \in \sigma(T)$, then for any real $\varepsilon > 0$ there exists a closed subspace $N \neq \{\overline{0}\}$ such that

1) N is invariant for any operator commuting with T*T.

2) $\|T|_N\| \leq \varepsilon$

Proof

Since $o \in \sigma(T)$, we can find a sequence (x_n), $\|x_n\| = 1$ for all n$\in \mathbb{N}$, such that

s-lim $Tx_n = 0$, and thus lim $T^*Tx_n = 0$.

Consequently, $o \in \sigma(T^*T)$. Let $\varepsilon > 0$; since T*T is self-adjoint, we can define the operator $f(T^*T)$, where

$$f(t)=\begin{cases}1 & , for\ |t|\le\frac{\varepsilon}{2}\\ 2\left(1-\left|\frac{t}{\varepsilon}\right|\right), & for\ \frac{\varepsilon}{2}\le|t|\le\varepsilon\\ 0 & , for\ |t|\ge\varepsilon\end{cases}$$

is continuous on $[-\|T\|,\|T\|]$, we can define

$$N=\{x;: f(A)x=x\}, \qquad T^*T=A.$$

Clearly N is a closed invariant subspace for all operators commuting with T*T. It remains to show that $N\neq\{0\}$ and that $\|T|_N\|\le\varepsilon.$

Let $x\in N,$ with $\|x\|=1.$ In this case we have

$$\begin{aligned}\|Ax\| &= \|Af(T^*T)x\| = \|f(T^*T)Ax\|\\ &\le\|f(T^*T)A\|\\ &=\sup\{|sf(s)|: s\in\sigma(A)\}\\ &\le\varepsilon\end{aligned}$$

and since $\|Tx\|^2=\langle T^*Tx,x\rangle=\langle Ax,x\rangle,$ it follows that $\|T|_N\|\le\varepsilon^{1/2}.$

Next we show that N is nonzero. Since

$$\|(I-f(A))f(2A)\|=\sup|(1-f(t))f(2t)|=0.$$

We see that $f(2A)x$ is in N for all $x\in H;$ furthermore this is not zero because

$$\|f(2A)\|=\sup\{|f(2t)|: t\in\sigma(A)\}\ge f(0)=1.$$

We have thus proved proposition 4.1.2

Proposition 4.1.3.

If $T\in B(H)$ is a normal operator and $p(t,s)$ is a polynomial, then

$$\sigma(P(T^*,T))=\{p(z^*,z): z\in\sigma(T)\}.$$

Proof

Let $p(s,t) = \sum a_{n,m} s^n t^m$ and thus the operator $p(T^*,T)$ is equal to

$$\sum a_{n,m} T^{*n} T^m.$$

Let $S \in \sigma(T)$; then there exists a sequence $(x_n), \|x_n\| = 1$, such that

$$(T - S)x_n \to 0, \quad (T^* - S^*)x_n \to 0.$$

and this implies that

$$p(T^*,T)x_n - p(S^*,S)x_n \to 0.$$

Thus for any $S \in \sigma(T)$, $p(S^*,S)$ is in $\sigma(p(T^*,T))$.

Let $r \in \sigma(p(T^*,T))$ and since the operator

$A = p(T^*,T) - r$ is normal and $0 \in \sigma(A)$ by the above proposition 4.1.2, for $\varepsilon = \frac{1}{n}$ we can find

a closed invariant subspace N_n such that

$$\left\| A|_N \right\| < \frac{1}{n}.$$

Let $S_n \in \sigma(T|_{N_n})$ and since $T \leftrightarrow A$, N_n reduces T, and we can find $y_n, \|y_n\| = 1$ such that

$$\|(S_n I - T)y_n\| < \frac{1}{n}.$$

We can suppose, without loss of generality, that $S_n \to S \in \sigma(T)$ and also

$$\|(T^* - S^*_n)y_n\| \le \frac{1}{n}.$$

Thus we have that

and since $p(T^*,T)y_n - p(S^*,S)y_n \to 0, r = p(S^*,S)$ and the proposition is proved.

4.2: OPERATORS WITH THE PROPERTY Reσ(T) = σ(Re T).

By proposition 4.1.3, if $T \in B(H)$ is a normal operator, then for any polynomial $p(z,z^*)$, we have

$$\sigma(p(T,T^*)) = \{p(z,z^*) : z \in \sigma(T)\},$$

and if $p(z,z^*) = \frac{1}{2}(z + z^*)$, then we get (since $p(T,T^*) = \frac{1}{2}(T + T^*)$)

$$\sigma(\mathrm{Re}\,T) = \mathrm{Re}\,\sigma(T).$$

Simple examples show that this remarkable relation does not hold for arbitrary operators. It is clear that for nilpotent operators or more generally for quasinilpotent operators this fails to hold. Hence we are motivated towards a natural problem to find classes of non-normal operators for which the relation holds.

We first prove a few results:

Proposition 4.2.1

If $T \in B(H)$ is a hyponormal operator, then

$$\mathrm{Re}\,\pi(T) \subset \sigma(\mathrm{Re}\,T).$$

Proof

Let $\lambda = x + iy \in \pi(T)$. Then there exists a sequence (x_n) of elements of H such that $\|x_n\| = 1$ and

$$(T - \lambda I)x_n \xrightarrow{s} \overline{0}.$$

Since $T - \lambda I$ is hyponormal

$$(T^* - \overline{\lambda} I)x_n \xrightarrow{s} \overline{0}.$$

Hence

$$(\mathrm{Re}\,T)x_n - \mathrm{Re}\,\lambda x_n = \frac{1}{2}\left[T + T^* - (\lambda + \overline{\lambda})I\right]x_n \xrightarrow{s} \overline{0}$$

and this proves that

$$\operatorname{Re}\lambda \in \pi\left(\frac{T+T^*}{2}\right) = \sigma(\operatorname{Re}T) \text{ for Re T is self-adjoint. Thus the result.}$$

Proposition 4.2.2

If $T \in B(H)$ is hyponormal, then $\operatorname{Re}\sigma(T) \subset \sigma(\operatorname{Re}T)$.

Proof

Let $\lambda_o \in \sigma(T)$ and the line $\operatorname{Re}\lambda = \operatorname{Re}\lambda_o$ meet the spectrum of T in a boundary point λ_0'. Since every boundary point of $\sigma(T)$ is in the approximate point spectrum $\pi(T)$, it follows that $\operatorname{Re}\lambda_o'$ is in $\operatorname{Re}\pi(T)$ and hence in $\sigma(\operatorname{Re}T)$ (by Proposition 4.2.1).

But $\operatorname{Re}\lambda_o = \operatorname{Re}\lambda_0'$, and we have thus proved the assertion of the proposition.

Remark: Clearly, the assertion of proposition 4.2.2 also holds when T* is hyponormal.

Proposition 4.2.3

If $T \in B(H)$ and if either T or T* is hyponormal, then $\operatorname{Re}\sigma(T) = \sigma(\operatorname{Re}T)$.

Proof

We need only consider the case when T is hyponormal. We use the result that the approximate point spectrum of a self-adjoint operator is exactly its spectrum. Using proposition 2.2.20, we can assume, without loss of generality, that if $T = A + iB$, then $\sigma(A) = P_\sigma(A)$.

Let $\lambda = a + ib$ and $a \in \sigma(A)$. Consider the subspace

$$M = \ker(A - aI).$$

Since a is an eigenvalue, $M \neq \{\overline{0}\}$. We show that M is an invariant subspace for B. Let $x \in M$ and thus $Ax = ax$.

Since we have

$$(A - aI)B - B(A - aI) = \frac{1}{2} iD$$

where $D = T^*T - TT^*$, it follows that

$$-\frac{1}{2} i\langle Dx, x\rangle = \langle (A - aI)Bx, x\rangle - \langle B(A - aI)x, x\rangle = 0$$

and thus $\langle Dx, x\rangle = 0$.

But $D > 0$ and thus $D^{\frac{1}{2}}$ exists and is self-adjoint. Hence we have

$$0 = \langle Dx, x\rangle = \langle D^{\frac{1}{2}}x, D^{\frac{1}{2}}x\rangle = \left\| D^{\frac{1}{2}}x \right\|^2 .$$

which implies that $D^{\frac{1}{2}}x = \overline{0}$.

From this we have $Dx = \overline{0}$ and hence

$$0 = A(Bx) - B(Ax) = (A - aI)Bx = \overline{0}$$

that is, Bx $\in$M.

Clearly M is an invariant subspace for T and $T|_M$ is of the form

$$\widetilde{T} = aI + \widetilde{B}$$

and since $\widetilde{B}$ is hermitian (self-adjoint), $\widetilde{T}$ is a normal operator.

But T is a hyponormal operator whose restriction to an invariant subspace is normal.

Then M is also invariant for T*. Clearly,

$$T = (T|_M) \oplus (T|_{M^\perp}) = T_1 \oplus T_2 .$$

Since $\sigma(T) = \sigma(T_1) \cup \sigma(T_2)$, we obtain that the same relation holds for the real parts,

$$\operatorname{Re}\sigma(T) = \operatorname{Re}\sigma(T_1) \cup \operatorname{Re}\sigma(T_2)$$

and since $\operatorname{Re}\sigma(T_1) = \{a\}$, the assertion of the proposition follows.

Earlier we had defined an operator $T \in B(H)$ to be **convexoid** if for all $\lambda \notin conv\sigma(T)$

$$\|(T-\lambda I)^{-1}\| \leq \frac{1}{dist(\lambda, conv\sigma(T))}$$

and showed that $T \in B(H)$ is convexoid if and only if $conv\sigma(T) = \overline{W(T)}$.

Proposition 4.2.4

If $T \in B(H)$ is a convexoid operator and $[a,b]$ is the smallest segment containing $\operatorname{Re}\sigma(T)$, then a and b are in $\sigma(\operatorname{Re}T)$.

Proof

As we know, we can assume without loss of generality that W(T) is a closed convex set and $\pi(T) = P_\sigma(T)$ (Proposition 2.2.20).

Let $\lambda_o \in \sigma(T)$ such that $\operatorname{Re}\lambda_o = a$ and λ_o is a point on $\delta\sigma(T)$ such that $\lambda_o \in W(T)$.

Since T is convexoid, W(T) is closed and $\operatorname{Re}\sigma(T) \geq a$, it follows that

$$\operatorname{Re}\overline{W(T)} = \operatorname{Re}(conv\sigma(T)) \geq a$$

and thus λ_o is a boundary point of W(T). ie, $\lambda_o \in \delta W(T)$.

Hence, as is easily seen, λ_o is a normal eigenvalue.

As in proposition 4.2.1, we can show that a is in $\sigma(\operatorname{Re}T)$, and similarly for b.

Proposition 4.2.5

If $T \in B(H)$ is a convexoid operator and [c,d] is the smallest segment containing $\sigma(\operatorname{Re}T)$, then

$$\sigma(\operatorname{Re}T) \subset [c,d] \subset \operatorname{Re}\sigma(T)$$

if $\sigma(T)$ is a connected set.

Proof

Since $\sigma(T)$ is connected, it follows that $\operatorname{Re}\sigma(T)$ is a segment on $\mathbb{R}$.

To prove the proposition it suffices to show that $c, d \in \operatorname{Re}\sigma(T)$. Let us suppose the contrary for c first, i.e $m \in \mathbb{N}$., $c \notin \operatorname{Re}\sigma(T)$.

Let l be the straight line $\operatorname{Re}\lambda = c$. l is disjoint from $\sigma(T)$ since otherwise $c \in \operatorname{Re}\sigma(T)$.

But $\sigma(T)$ is a connected set and thus it is strictly on one side of l. Suppose that it is on the right side of l. Then we can find an $\varepsilon > 0$ such that $\operatorname{Re}\sigma(T) \geq c + \varepsilon$.

Since T is convexoid, we have $conv\sigma(T) = \overline{W(T)}$ .So $W(T) \subset conv\sigma(T)$ and hence

$\operatorname{Re}W(T) \geq c + \varepsilon$, i.e,

$\operatorname{Re}\langle Tx, x\rangle \geq (c + \varepsilon)\|x\|^2$ for all $x \in H$.

Since $\operatorname{Re}\langle Tx, x\rangle = \langle(\operatorname{Re}T)x, x\rangle$ for all $x \in H$, we obtain

$\langle(\operatorname{Re}T)x, x\rangle \geq (c + \varepsilon)\|x\|^2$ for all $x \in H$,

i.e. $\operatorname{Re}T \geq (c + \varepsilon)I$, which implies

$\|(\operatorname{Re}T)x - cIx\| \geq \varepsilon\|x\|$ for all $x \in H$.

This shows that

$c \notin \pi(\operatorname{Re}T) = \sigma(\operatorname{Re}T)$ (for $\operatorname{Re}T$ is self–adjoint)

and this contradicts the hypothesis that [c, d] is the smallest segment containing $\sigma(\operatorname{Re}T)$.

Likewise, dealing with d, we can show that $d \notin \sigma(\operatorname{Re}T)$ and this gives again a contradiction of the hypothesis.

Hence $c, d \in \operatorname{Re}\sigma(T)$.

Note

[c,d] is the smallest segment containing $\sigma(\mathrm{Re}\,T)$ implies that

$$c = \inf\{\langle(\mathrm{Re}\,T)x, x\rangle : x \in H \text{ and } \|x\| = 1\}$$

$$d = \sup\{\langle(\mathrm{Re}\,T)x, x\rangle : x \in H \text{ and } \|x\| = 1\}$$

Moreover $c, d \in \sigma(\mathrm{Re}\,T)$.

Proposition 4.2.6

If $T \in B(H)$ and $\sigma(T)$ is a spectral set for T, then $\sigma(\mathrm{Re}\,T) \subseteq \mathrm{Re}(\sigma(T))$.

Proof

Since $\sigma(T)$ is a spectral set for T, we have

$$\|f(T)\| \le \sup\{|f(\lambda)| : \lambda \in \sigma(T)\}$$

for all rational functions f without poles in $\sigma(T)$.

Now

$r_{f(T)} = \sup\{|\lambda| : \lambda \in \sigma(f(T))\}$.

By the spectral mapping theorem

$$\sigma(f(T)) = f(\sigma(T)).$$

Hence

$$\begin{aligned} r_{f(T)} &= \sup\{|\lambda| : \lambda \in f(\sigma(T))\} \\ &= \sup\{|f(\lambda)| : \lambda \in \sigma(T)\} \\ &\ge \|f(T)\|. \end{aligned}$$

However $\|f(T)\| \ge r_{f(T)}$ always and hence

$$r_{f(T)} = \|f(T)\|.$$

This shows that f(T) is normaloid.

Taking $f(\lambda)=\lambda-z$ (where z is a constant), we see that $T-zI$ is normaloid , and this says that T is transaloid. Therefore (by proposition 3.2.5) we have $conv\sigma(T)=\overline{W(T)}$.

Let $a\in\sigma(\operatorname{Re}T)$ and suppose that $a\notin\operatorname{Re}\sigma(T)$. Let l be the straight line $\operatorname{Re}\lambda=a$ which is disjoint from $\sigma(T)$. Suppose that $\sigma(T)$ is on the left side of l. Then there exists an $\varepsilon>0$ such that $\operatorname{Re}\sigma(T)\leq a-\varepsilon$. But T is convexoid and hence we obtain that

$\operatorname{Re}W(T)\leq a-\varepsilon$, i.e., $W(\operatorname{Re}T)\leq a-\varepsilon$ .Thus $\sigma(\operatorname{Re}T)\leq a-\varepsilon$, which is a contradiction since $a\leq a-\varepsilon$ is not possible.

Incase $\sigma(T)$ is on the right side of l, we can proceed in the same manner.

If $\sigma(T)=A_1\cup A_2$,where A_1 is on the left of l and A_2 on the right side, then we can decompose $T=T_1\oplus T_2,\sigma(T_i)=A_i,i=1,2$ and we can apply the above result.

Proposition 4.2.7

If $T\in B(H)$ has the G_1-property, i.e.,

$$\|(T-\lambda I)^{-1}\|=\frac{1}{dist(\lambda,\sigma(T))}\quad \text{for all } \lambda\in\rho(T)$$

then $\quad \operatorname{Re}(\sigma(T))\subset\sigma(\operatorname{Re}T)$,

Proof

Assume, without loss of generality that $\pi(T)=P_\sigma(T)$ (see Berberian's result).

This implies that

$$\partial\sigma(T)\subset\pi(T)=P_\sigma(T).$$

Let $a\in\operatorname{Re}\sigma(T)$ and l be the line $\operatorname{Re}\lambda=a$.

Let λ_0 be the point where l exits the spectrum of T; it is clear that it is a boundary point.

Obviously $\operatorname{Re}\lambda_0 = a$.

We can construct a sequence of unit vectors(x_n) such that

(i) $(T - \lambda_0 I)x_n \to \overline{0}$

(ii) $(T^* - \lambda_0^* I)x_n \to \overline{0}$

and this implies that $a \in \sigma(\operatorname{Re} T)$.

For $n = 1,2,3,\ldots$, let $D_n = \left\{\lambda : |\lambda - \lambda_0| \leq \frac{1}{n}\right\}$.

Since $\lambda_0 \in \sigma(T)$, the set D_n contains a point $a_n \in \rho(T)$ such that $|a_n - \lambda_0| \leq \frac{1}{2n}$.

Let b_n be such that

$$dist(a_n, \sigma(T)) = |a_n - b_n|.$$

In this case b_n is in $\sigma(T)$, and since T has G_1-property, we obtain

$$\eta_{T-b_n I} = \eta_{T^* - b_n^* I}$$

Now let x_n be such that

$$(T - b_n I)x_n = (T^* - b_n^* I)x_n = \overline{0}$$

and from the definition of b_n we obtain that

$$(T - \lambda_o I)x_n = (b_n - \lambda_o)x_n$$

and

$$(T^* - \lambda_o^* I)x_n = (b_n^* - \lambda_0^* I)x_n.$$

Thus since (x_n) is a sequence satisfying the requirements and the proposition is proved.

Proposition 4.2.8

If $T \in B(H)$ has one of the following properties

(i) $\sigma(T)$ is a spectral set for T

(ii) T has the G_1-property and $\sigma(T)$ is connected,

then T has the property $\mathrm{Re}\,\sigma(T) = \sigma(\mathrm{Re}\,T)$.

Proof

First, if $\sigma(T)$ is spectral set, then by proposition 4.2.6

$$\sigma(\mathrm{Re}\,T) \subset \mathrm{Re}\,\sigma(T).$$

Since T satisfies trivially the G_1- property (by the fact that $\sigma(T)$ is a spectral set), the opposite inclusion follows from proposition 4.2.7.

Now suppose that T has G_1-property and $\sigma(T)$ is connected. From proposition 4.2.7 we have one inclusion.

For the opposite inclusion, we proceed as follows:

Let [a,b] be the smallest segment containing $\sigma(\mathrm{Re}\,T)$ and since T is convexoid, by proposition 4.2.4, it follows that $[a,b] \subset \mathrm{Re}\,\sigma(T)$.

Thus we have

$$\sigma(\mathrm{Re}\,T) \subset [a,b] \subset \mathrm{Re}\,\sigma(T) \subset \sigma(\mathrm{Re}\,T)$$

and the second part follows.

Combining proposition 4.2.8 with proposition 4.2.3, we summarize:

Proposition 4.2.9

If $T \in B(H)$ and one of the following conditions holds, then T has the property

$$\mathrm{Re}\,\sigma(T) = \sigma(\mathrm{Re}\,T)$$

S_1. T is hyponormal

S_2. T^* is hyponormal

S_3. $\sigma(T)$ is a spectral set for T

S_4. T has the G_1-property and $\sigma(T)$ is connected.

Example (i)

Now part (S_4) of proposition 4.2.9 does not generalize to arbitrary convexoid operators.

Let A be an operator with matrix

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

and B with matrix

$$\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$$

where the eigenvalues lie off the imaginary axis and are vertices of a triangle containing the disc

$$D_{\frac{1}{2}} = \left\{ z : |z| \leq \frac{1}{2} \right\}.$$

Since $W(A) = D_{\frac{1}{2}}$ and $W(B) = conv\{a_1, a_2, a_3\} \supset D_{\frac{1}{2}}$, it follows that $T = A \oplus B$ is convexoid and

since the spectrum of $A \oplus B$ is $\{0, a_1, a_2, a_3\}$, it follows that

$$0 \in \operatorname{Re}\sigma(A \oplus B).$$

Since $\operatorname{Re}(A \oplus B) = \operatorname{Re} A \oplus \operatorname{Re} B$,

$$\sigma(\operatorname{Re}(A \oplus B)) = \sigma(\operatorname{Re} A) \cup \sigma(\operatorname{Re} B)$$

$$= \left\{ -\frac{1}{2}, \frac{1}{2} \right\} \cup \operatorname{Re}\sigma(B)$$

and thus

$$0 \notin \sigma(\operatorname{Re}(A \oplus B)).$$

4.3. FURTHER PROPERTIES OF NON-NORMAL OPERATORS

The following class of operators was introduced by Paul Halmos in the course of studying reducible operators (Halmos, [7]).

Definition 4.3.1

An operator $T \in B(H)$ is in $\overline{R}_1$ if T has an approximate reducing eigenvector, i.e., there exists λ_0 such that for any real $\varepsilon < 0$ there exists $x_\varepsilon, \|x_\varepsilon\| = 1$,

and

$$\|(T - \lambda_0 I)x_\varepsilon\| < \varepsilon, \quad \|(T^* - \lambda_0^* I)x_\varepsilon\| < \varepsilon .$$

From the definition it is clear that $\overline{R}_1$ represents the closure of operators with one-dimensional reducing spaces.

Halmos showed that $\overline{R}_1$ contains the normal and isometric operators.

Proposition 4.3.2

If $T \in B(H)$ and $\lambda \in \sigma(T)$ with $|\lambda| = \|T\|$, then

(i) If $(T - \lambda I)x = \overline{0}$ then $(T^* - \lambda^* I)x = \overline{0}$.

(ii) If $\lim_{n \to \infty} (T - \lambda I)x_n = \overline{0}$, then $(T^* - \lambda^* I)x_n \to \overline{0}$.

Proof

We prove the first assertion only.

Let x be such that $(T - \lambda I)x = \overline{0}$. Then

$$\|(T^* - \lambda^* I)x\|^2 = \|T^* x\|^2 - \langle T^* x, \lambda^* x\rangle - \langle \lambda^* x, T^* x\rangle + |\lambda|^2 \|x\|^2$$

$$= \|T^* x\|^2 - |\lambda|^2 \|x\|^2$$

$$\leq \|T\|^2 \|x\|^2 - \|T\|^2 \|x\|^2 = 0.$$

Hence $(T^* - \lambda^* I)x = \overline{0}$.

For the second part we observe that $\lambda \in \partial\sigma(T)$ and hence belongs to the approximate point spectrum of T (Halmos,[8], pr.63, pg.228).Also since $\lambda^* \in \sigma(T^*)$ and $|\lambda^*| = \|T^*\|$, we note that $\lambda^* \in \pi(T^*)$, the approximate point spectrum of T^* since $\lambda^* \in \partial\sigma(T^*)$.

Corollary 4.3.3

If $T \in B(H)$ is normaloid, then $T \in \overline{R}_1$.

Proof

If T is normaloid, then $r(T) = \|T\|$. So there exists a $\lambda \in \sigma(T)$ (which is closed in $\mathbb{C}$)such that $|\lambda| = \|T\|$. So $\lambda \in \pi(T)$ and $\lambda^* \in \pi(T^*)$ and there exists for each $\varepsilon > 0$ an $x_\varepsilon \in H$ with $\|x_\varepsilon\| = 1$ that $\|(T - \lambda I)x_\varepsilon\| < \varepsilon$ and $\|(T^* - \lambda^* I)x_\varepsilon\| < \varepsilon$.

Definition 4.3.4

An operator $T \in B(H)$ is said to be **spectraloid** if its numerical radius is equal to its spectral radius, i.e, w(T) = r(T). Note that every normaloid operator is spectraloid.

Then the corollary 4.3.3 is a consequence of the following result:

Proposition 4.3.5

If $T \in B(H)$ is spectraloid, then $T \in \overline{R}_1$.

Proof

Let λ be such that $|\lambda| = r(T) = w(T)$.

It suffices to show that if λ is with the properties

(i) $Tx = \lambda x$

(ii) $\lambda \in \partial W(T)$

then $T^*x = \lambda^* x$.

By transformation of the form $\lambda \to a\lambda + b$, the point spectrum and the numerical range are transformed in an obvious way and we can assume without loss of generality that $\lambda = 0$ and $\operatorname{Re} T < 0$.

Since $\operatorname{Re}\langle Tx, x\rangle = \langle(\operatorname{Re} T)x, x\rangle$

and since $\operatorname{Re} T$ is a self-adjoint operator, we obtain that $(\operatorname{Re} T)x = \overline{0}$ and thus $Tx = \overline{0} = (\operatorname{Re} T)x + i(\operatorname{Im} T)x,$ which gives $\operatorname{Im} Tx = \overline{0}$.

Now $T^*x = (\operatorname{Re} T)x - i(\operatorname{Im} T)x = \overline{0}$. The case of a sequence of elements in H can be treated in a similar way.

Since $\lambda \in \sigma(T)$, we can find a sequence of elements such that

$$Tx_n - \lambda I x_n \xrightarrow{s} \overline{0}$$

and from the above remark

$$T^* x_n - \lambda^* I x_n \xrightarrow{s} \overline{0}$$

which proves the proposition.

Corollary 4.3.6

If $T \in B(H)$ is paranormal, then $T \in \overline{R_1}$.

Proof

For paranormal operators are normaloid and thus spectraloid.

Corollary 4.3.7

If $T \in B(H)$ is a hyponormal operator, then $T \in \overline{R}_1$.

Proof

Every hyponormal operator is paranormal.

{Note

We know that

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\|.$$

(Halmos,[9],pg.33).

Since $\sigma(A) \subseteq \overline{W(A)}$, we get $r(T) \leq w(T)$. Suppose T is normaloid, so $r(T) = \|T\|$. Hence

$w(T) = \|T\|$ and consequently $w(T) = r(T)$, i.e. T is spectraloid}.

In the case of operators in the above classes, we prove a more general result. We recall

Definition 4.3.8

Let $T \in B(H)$. A point $\lambda \in \sigma(T)$ is in the **continuous spectrum of** T if $T - \lambda I$ is one-to- one and the range of $T - \lambda I$ is dense in H but not equal to H.

Now it follows that λ is in the continuous spectrum of T if and only if there exists a sequence of elements $(x_n), \|x_n\| = 1$, such that

(i) $Tx_n - \lambda x_n \xrightarrow{s} \overline{0}$

(ii) $x_n \xrightarrow{w} \overline{0}$ ....(4.3.1)

Since the continuous spectrum of T is $\pi(T)-\{\Gamma(T)\cup P_\sigma(T)\}$ (Halmos,[6], pg.41).

Here $P_\sigma(T)$ is the point spectrum of T and $\Gamma(T)$ is the residual spectrum of T), it follows that

$\lambda\in\pi(T)$ and hence there is a sequence (x_n) of unit vectors such that property (i) holds.

Since $(T-\lambda I)$ is one- to- one, $Ker(T-\lambda I)=\{\overline{0}\}$ and hence the range of $T^*-\lambda^* I$ is dense in H.

Now, for each $y\in H$, we have

$$\langle x_n,(T^*-\lambda^* I)y\rangle=\langle (T-\lambda I)x_n,y\rangle\to 0 \text{ as } n\to\infty .$$

Hence by (Bachman and Narici,[2], pg.235, theorem 14.3)

$$x_n \xrightarrow{w} \overline{0} .$$

Conversely, if property (i) holds, we see that $\lambda\in\pi(T)$.

Now, property (i) implies that

$$\langle x_n,(T^*-\lambda^* I)y\rangle\to 0 \; as \; n\to\infty \text{ , for all } y\in H.$$

Since $x_n \xrightarrow{w} \overline{0}$, we see that the range of the set ($T^*-\lambda^* I$) is dense in H, i.e; $T-\lambda I$ is one- to-

one.

Hence $\lambda\notin P_\sigma(T)$. Also Range $(T-\lambda I)\neq H$ (for if Range $(T-\lambda I)$ was H, then $\lambda\in\rho(T)$ using

the Banach inverse theorem).

Thus we have proved contention (4.3.1).

Proposition 4.3.9

If $T\in B(H)$ is a hyponormal operator on H, then

$$\left\{x\in H:\left\|T^k x\right\|=\left\|T^{*k}x\right\|,\; for\; all\, k=1,2,...\right\}$$

$$=\bigcap_{k=1}^{\infty}\left\{x\in H:T^{*k}T^k x=T^k T^{*k}x\right\}$$

and is a subspace which reduces T.

Proof

It is clear that

$$\{x \in H : \|T^k x\| = \|T^{*k} x\|,\ for\ all\, k = 1,2,\ldots\}$$

$$\supseteq \bigcap_{k=1}^{\infty} \{x \in H : T^{*k} T^k x = T^k T^{*k} x\}$$

and we note that

$$\bigcap_{k=1}^{\infty} \{x \in H : T^{*k} T^k x = T^k T^{*k} x\}$$

is a subspace.

Now, for any $x \in \{x \in H : \|T^k x\| = \|T^{*k} x\|,\ for\ all\, k = 1,2,\ldots\}, \|Tx\| = \|T^* x\|$ implies

$$T^* Tx = TT^* x.$$

Since T is hyponormal

$$\|T^2 x\| = \|T(Tx)\| \geq \|T^*(Tx)\|$$

$$= \|TT^* x\| \geq \|T^{*2} x\|,$$

and by the equation

$\|T^2 x\| = \|T^{*2} x\|$, we have

$$\|T(Tx)\| = \|T^*(Tx)\|,$$

$$= \|T(T^* x)\|$$

$$= \|T^*(T^* x)\|$$

and hence

$$T^* T(Tx) = TT^*(Tx) \text{ and } T^* T(T^* x) = TT^*(T^* x)$$

(we use: For a hyponormal T on H, $\|Ty\| = \|T^* y\|$ if and only if $T^* Ty = TT^* y$.

Since $T^* T - TT^* \geq 0$,

$$\left\|(T^*T - TT^*)^{\frac{1}{2}}\right\|^2 = \langle (T^*T - TT^*)y, y\rangle$$

$$= \|Ty\|^2 - \|T^*y\|^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.3.2)$$

Since $T^*Tx = TT^*x$,

$$T^*T^2x = TT^*Tx = T^2T^*x \text{ and}$$

$$T^{*2}Tx = T^*TT^*x = TT^{*2}x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.3.3)$$

By (4.3.3) and the fact that T is hyponormal

$$\|T^3x\| \geq \|T^*T^2x\|$$

$$= \|TT^*Tx\| = \|T^2T^*x\|$$

$$\geq \|T^*TT^*x\|$$

$$= \|T^{*2}Tx\| = \|TT^{*2}x\| \geq \|T^{*3}x\|$$

and, by the equation $\|T^3x\| = \|T^{*3}x\|$, we have

$$\|T(T^2x)\| = \|T^*(T^2x)\|$$

$$= \|T(T^*Tx)\| = \|T^*(T^*Tx)\|$$

$$= \|T(TT^*x)\|$$

$$= \|T^*(TT^*x)\|$$

$$= \|T(T^{*2}x)\|$$

$$= \|T^*(T^{*2}x)\|$$

and hence using (4.3.2) again

$$T^*T(T^2x) = TT^*(T^2x), \quad T^*T(T^*Tx) = TT^*(T^*Tx)$$

$$T^*T(TT^*x) = TT^*(TT^*x)\, and\ T^*T(T^{*2}x) = TT^*(T^{*2}x).$$

Hence by (4.3.2) and (4.3.3), we have

$$T^*T^3x = TT^*T^2x = T^2T^*Tx = T^3T^*x,$$

$$TT^{*3}x = T^*TT^{*2}x = T^{*2}TT^*x = T^{*3}Tx$$

and

$$T^{*2}T^2x = T^*TT^*Tx = T^*T^2T^*x = TT^*TT^*x$$

$$= TT^{*2}Tx = T^2T^{*2}x.$$

By repeating the same argument as above, we have

$$\|T^k(Tx)\| = \|T^{*k}(Tx)\| = \|T^k(T^*x)\| = \|T^{*k}(T^*x)\|$$

and $\quad T^{*k}T^kx = T^kT^{*k}x$ for all k = 1,2,….

Therefore

$$\{x \in H : \|T^kx\| = \|T^{*k}x\|, \text{ for all K} = 1,2,\ldots\}$$

reduces T and

$$\{x \in H : \|T^kx\| = \|T^{*k}x\|, \text{ for all } k = 1,2,\ldots\} \subseteq \bigcap_{k=1}^{\infty}\{x \in H : T^{*k}T^kx = T^kT^{*k}x\}$$

Proposition 4.3.10

If T is a hyponormal operator on H, then

$$H_T^{(n)} \underline{\underline{defn}} \{x \in H : \|T^kx\| = \|T^{*k}x\|, for\ all\ k = 1,2,\ldots\}$$

$$= \bigcap_{k=1}^{\infty}\{x \in H : T^{*k}T^kx = T^kT^{*k}x\}$$

is the maximal reducing subspace on which its restriction is normal.

If $P^{(n)}$ is the orthogonal projection on H onto $H_T^{(n)}$, then $P^{(n)}$ commutes with any operator which commutes with T and T^*.

Proof

Since $H_T^{(n)}$ is a subspace which reduces T,

$$P^{(n)} \leftrightarrow T$$

and the restriction $T|_{H_T^{(n)}}$ of T is normal.

If M is a subspace which contains $H_T^{(n)}$ and reduces T and if $T|_M$ is normal, then, for any $x \in M$,

$$\|T^k x\| = \|(T|_M)^k x\| = \|(T|_M)^{*k} x\|$$

$$= \|(T^*|_M)^k x\| = \|T^{*k} x\|$$

and $x \in H_T^{(n)}$ and hence

$$M = H_T^{(n)}.$$

Corollary 4.3.11

If $T \in B(H)$ is a hyponormal operator and if $H_T^{(n)} = \{\overline{0}\}$, *then* $P_\sigma(T) = \phi$.

Proof

If $Tx = \lambda x$, then $T^* x = \overline{\lambda} x$ and for each $k = 1,2,...,$ we have

$$\begin{aligned} T^{*k}T^k x &= \lambda^k T^{*k} x \\ &= \lambda^k \lambda^{*k} x \\ &= \lambda^{*k} T^k x = T^k T^{*k} x \end{aligned}$$

and, by proposition 4.3.10, $x \in H_T^{(n)} = \{\overline{0}\}$ and hence $x = \overline{0}$.

Therefore, $P_\sigma(T) = \phi$.

Definition 4.3.12

Let $T \in B(H)$ be hyponormal. The normal operator $T|_{H_T^{(n)}}$ is called the **normal part** of T.

Proposition 4.3.13

If $T \in B(H)$ is hyponormal, then for any compact operator $K \in B(H), T + K \in \overline{R}_1$.

Proof

Since T is hyponormal, we can (using proposition 4.3.10) write $T = T_1 \oplus T_2$ *on* $H_1 \oplus H_2$ (here $H_1 = H_T^{(n)}$ and $H_2 = H\Theta H_T^{(n)}$ or $H_2 = H \cap H_T^{(n)}$), where T_1 is normal and $P_\sigma(T_2) = \phi$. The assertion of the proposition is clear when H_1 is infinite – dimensional since we can choose an infinite sequence of eigenvectors which are mutually orthogonal.

In the case $\dim H_1 < \infty$, we choose a point $\lambda \in \sigma(T_2)$ and find a sequence $(x_n), \|x_n\| = 1$ such that

$$(T_2 - \lambda I)x_n \xrightarrow{s} \overline{0}$$

and also

$$x_n \xrightarrow{w} \overline{0}.$$

From the hyponormality it follows that

$$(T_2^* - \lambda^* I)x_n \xrightarrow{s} \overline{0}$$

and the weak convergence of (x_n) *to* $\overline{0}$.

The fact that K is a compact operator implies that for a subsequence of x_n's (and we can suppose that this is just x_n),

$$(T_2 - \lambda I + K)x_n \xrightarrow{s} \overline{0}$$

and

$$(T_2^* - \lambda^* I + K^*)x_n \xrightarrow{s} \overline{0}.$$

Now we use the corollary 4.3.3 of proposition 4.3.2 to complete the proof.

Lemma 4.3.14

Let $T \in B(H)$ and (x_n) be a sequence converging weakly to x_0 and $\lim_{n\to\infty} \|Tx_n\| = 0$.

Then $Tx_0 = 0$.

Proof

Since $x_n \xrightarrow{w} x_0$ as $n \to \infty$, so $\langle x_n, y\rangle \to \langle x_0, y\rangle \quad \forall y \in H$.

Now

$$\langle Tx_n, y\rangle = \langle x_n, T^* y\rangle \to \langle x_0, T^* y\rangle = \langle Tx_0, y\rangle \ .$$

Now

$$|\langle Tx_n, y\rangle| \le \|Tx_n\| \, \|y\| \to 0 \text{ as } n \to 0 \text{ since } \lim_{n\to\infty} \|Tx_n\| = 0.$$

Hence

$$\langle Tx_n, y\rangle = 0 \text{ for all } y \in H.$$

Consequently,

$$\langle Tx_0, y\rangle = 0 \text{ for all } y \in H,$$

i.e; $\quad Tx_0 = \overline{0}$.

Proposition 4.3.15

If for some sequence $(x_n), \|x_n\| = 1$, converging weakly to $\overline{0}$ and

$$(T - \lambda I)x_n \to \overline{0}, \qquad (T^* - \lambda^* I)x_n \to \overline{0}$$

then for any compact operator K, $T+K \in \overline{R}_1$.

Proof

Follows immediately from the definition of the class $\overline{R}_1$.

Definition 4.3.16

For any $T \in B(H)$, let

$\gamma_T = \{\lambda \in \sigma(T):$ there exists an $x \in H$ such that $x \neq 0$ and $(T-\lambda I)x = \overline{0}, (T^* - \lambda^* I)x = \overline{0}\}$.

It is clear that if γ_T is a nonvoid set, then $T \in \overline{R}_1$.

Also, if γ_T is an infinite set, then $T \in \overline{R}_1$.

Proposition 4.3.17

If $\lambda \notin \gamma_T$ and there exists a sequence of unit vectors (x_n) such that

$$(T-\lambda I)x_n \xrightarrow{s} \overline{0}, \quad (T^* - \lambda^* I)x_n \xrightarrow{s} \overline{0},$$

then $T+K \in \overline{R}_1$ for any compact operator K.

Proof

We can assume, without loss of generality, that $x_n \xrightarrow{w} x_0$.

Then, by the Lemma 4.3.14 above

$$(T-\lambda I)x_0 = \overline{0}, \text{ and } (T^* - \lambda^* I)x_0 = \overline{0}.$$

Since $\lambda \notin \gamma_T$, it follows that $x_0 = \overline{0}$. Thus, by proposition 4.3.15, it follows that $T+K \in \overline{R}_1$ for any compact operator K.

Proposition 4.3.18

If $T \in B(H)$ is with the sequential G_1-property and $\gamma_T = \sigma(T)$, then for any compact operator $K \in B(H)$,

$$T + K \in \overline{R}_1.$$

Proof

Let $\sigma(T) = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ and since $\sigma(T) = \gamma_T$, each λ_i is an eigenvalue for T, each λ_i is a normal eigenvalue.

Moreover, if for some i, the nullspace of $T - \lambda_i I$ is infinite - dimensional, then clearly $T + K \in \overline{R}_1$.

In the contrary case $H = H_1 \oplus H_2$, where $H_i = \sum N(T - \lambda_i I)$. Since H_1 reduces T and $\sigma(T)$ is a finite set, the restriction of T on H_2 is an operator with the sequential G_1-property and γ_{T_2} is void.

In this case $T_2 + K \in \overline{R}_1$.

Proposition 4.3.19

If $T \in B(H)$ is with the sequential G_1-property, then for any compact operator $K \in B(H), T + K \in \overline{R}_1$

Proof

We can consider the case γ_T is a finite set and different from $\sigma(T)$.

Since T is with the sequential G_1-property for any point λ_0 in $\sigma(T) - \gamma_T$ there exists a sequence $(a_n) \subset \sigma(T)$ such that

$$\left\|(T - a_n I)^{-1}\right\| = \frac{1}{dist(a_n, \sigma(T))} \quad a_n \to \lambda_0.$$

Since $a_n - \lambda_0 \notin \gamma_T$ for any $a_{m_0} \in \sigma(T) - \gamma_T$ such that

$$|a_{m_0} - \lambda_0| < \min\{|\lambda_0 - \lambda| : \lambda \in \gamma_T\}, \text{ we have}$$

$$dist(a_{m_0}, \sigma(T)) = |a_{m_0} - \lambda_{m_0}|$$

for some $\lambda_{m_0} \in \sigma(T) - \gamma_T$.

Thus we obtain that

$$\left\|(T - a_{m_0})^{-1}\right\| = \frac{1}{dist(a_{m_0}, \sigma(T))} = \frac{1}{|a_{m_0} - \lambda_{m_0}|}$$

and $$\frac{1}{\lambda_{m_0} - a_{m_0}} \in \sigma((T - a_{m_0} I)^{-1}).$$

Then by proposition 4.3.2, there exists a sequence (x_n) of unit vectors such that

$$\left[(T - a_{m_0} I)^{-1} - \frac{1}{(\lambda_{m_0} - a_{m_0})^{-1}}\right] x_n \to \overline{0}$$

$$\left[(T^* - a^*_{m_0})^{-1} - \frac{1}{(\lambda^*_{m_0} - a^*_{m_0})^{-1}}\right] x_n \to \overline{0}$$

and from this we see that a_{m_0} is not in γ_T.

The assertion of the proposition follows then from proposition 4.3.17.

Corollary 4.3.20

If $T \in B(H)$ has the G_1-property, then for any compact operator $K, T + K \in \overline{R}_1$.

Corollary 4.3.21

If $T \in B(H)$ is an isometric operator, then for any compact operator $K, T + K \in \overline{R}_1$.

4.4. CONCLUSIONS

From the study we can make the following conclusions:

We could consider the class C of operators $T \in B(H)$ which satisfy the property:

For any $x \in H$ which is not $\overline{0}$ the sequence

$$\left(\frac{\|T^n x\|}{\|T^{n-1} x\|} \right)$$

is monotone.

So every paranormal operator has this property. It is clear that any power of T is also in the class C. It would be an interesting problem to investigate if operators which are k-paranormal are also of class C?

If $T \in B(H)$ is hyponormal then it is paranormal and consequently T^2 is paranormal but T^2 is not hypornormal.

A necessary and sufficient condition for $\overline{W(T)}$ to be a spectral set for T is if there exists a strong normal dilation N of T such that $\overline{W(T)} = \overline{W(N)}$.

If $T \in B(H)$ and one of the following conditions holds, then T has the property $\operatorname{Re}\sigma(T) = \sigma(\operatorname{Re} T)$:

- T is hyponormal
- T^* is hyponormal
- $\sigma(T)$ is a spectral set of T
- T has the G_1-property and $\sigma(T)$ is connected.

Finally, we can say that if $T \in B(H)$ is normaloid, spectralod, paranormal, hyponormal and T +K, where $T \in B(H)$ is isometric,or has the G_1-property and K is compact, then $T + K \in \overline{R}_1$.

4.5. RECOMMENDATIONS

As we have seen, a normal operator satisfies quite a number of conditions weaker than normality, for example, hyponormality, convexoidity, transloidity.
Results in the converse direction can be obtained by supplementing or strengthening such conditions.

Quite possibly, the supplementary conditions may refer to the spectrum, finite- dimensionality, compactness, etc. The work on non-normal operators may be pursued in this direction. To mention a few cases, an interesting characterization of normality was obtained by Stampfli G.J using the class of hyponormal operators:

- If $T \in B(H)$ is a hyponormal operator and for some positive integer p, T^p is normal, then T is normal.

Ando, T. [1] extended this result to paranormal operators:

- A necessary and sufficient condition for an operator $T \in B(H)$ to be normal is that for some integer $p > 1$ the operator T^p is paranormal.

The work of this thesis can be moved in the reverse direction and we would suggest some problems deserving attention, but the means through which the answers may be obtained may be far reaching and too involved:

(1). If T is paranormal and for some p, $(T^*)^p$ is paranormal, does it follow that T is normal?

(2). If T is hyponormal (or paranormal) and for some integers p and q, $(T^*)^p(T^q)$ is hyponormal, is T normal?

(3). If $T \in B(H)$ has the property that for all $\lambda \in \mathbb{C}, T + \lambda I$ is also paranormal, and then is T is hyponormal?

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