# COMPACT OPERATORS ON SEQUENCE AND FUNCTION SPACES: CHARACTERISATIONS AND DUALITY RESULTS

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# Abstract

Since the early seventies numerous papers appeared in which the authors considered the question "For which Banach spaces $X$ and $Y$ is the space $K(X,Y)$ of compact linear operators uncomplemented in the space $L(X,Y)$ of bounded linear operators?". The most general answer to this question has close connection with the question whether the scalar sequence space $c_0$ of null sequences embeds isomorphically into $K(X,Y)$. In one of the early papers ([28] (1979)) J. Johnson followed a path through dual spaces of spaces of operators when he proved that there is a projection on $L(X,Y)^*$ with range isomorphic to $K(X,Y)^*$ and kernel the annihilator of $K(X,Y)$ when $Y$ has the bounded approximation property. Johnson then applied his result to consider $L(X,Y)$ as an isomorphic subspace of $K(X,Y)^{**}$ and could then derive necessary and sufficient conditions for $K(X,Y)$ to be reflexive, provided that either $X$ or $Y$ has the bounded approximation property. It turns out that the spaces $X$ and $Y$ have to be reflexive and that the property $K(X,Y) = L(X,Y)$ necessarily has to hold for $K(X,Y)$ to be reflexive in this case.

The questions described in the previous paragraph led to research into different directions. Much work went into the study of $K(X,Y)$ as a subspace of $L(X,Y)$ and the question when (i.e. for which Banach spaces) is the equality $K(X,Y) = L(X,Y)$ true? For instance an extensive investigation on when $K(X,Y)$ is an $M$-ideal in $L(X,Y)$ was done. And some researchers (for example in the papers [2] and [6]) considered the question on the equality of $K(X,Y)$ and $L(X,Y)$ in the context of scalar sequence spaces and Banach function spaces - i.e. where either $X$ or both $X$ and $Y$ are such spaces. Also, especially in recent papers (for example in [10], [22], [25], [26] and [27]), the same questions and the question about projections from $L(X,Y)^*$ onto $K(X,Y)^*$ were considered in the setting of Banach spaces which fail the approximation property. These studies also extended to similar research activities in the setting of locally convex spaces (for example in the papers [8], [19] and [20]).

The objective in the present thesis is to contribute to the above mentioned study, in the following ways:

(a) In line with recent developments we want to show the existence of a suitable projection onto the space $K(X,Y)^*$, which will allow us to find necessary and sufficient conditions for the reflexivity of $K(X,Y)$ without relying on the presence of the bounded approximation property on $X$ or $Y$. The idea is to put recent work of others in connection with continuous dual spaces of spaces of bounded linear operators in a suitable framework and to improve the present known results and techniques.

(b) Use techniques from the theory of vector sequence spaces to simplify proofs of existing results in connection with the equality $K(X,Y) = L(X,Y)$ when $X$ is a Banach scalar sequence space and then to extend the existing results to include more general sequence spaces $X$.

(c) Exposing that recent studies in connection with "absolutely summing multipliers" and " sequences in the range of a vector measure" are intertwined, we intend to extend the concept of "absolutely summing multiplier" to more general types of "summing mul-

tipliers" and to apply our work to consider properties of Banach space valued operators on scalar sequence spaces.

**What are our contributions in this thesis?**

∗ Firstly, we introduce an operator ideal approach which seems to provide a natural setting in which to consider the existence of projections from $L(X,Y)^*$ onto $K(X,Y)^*$ and derive necessary and sufficient conditions for the reflexivity of $K(X,Y)$ in the absence of the approximation property. Thus we simplify the proofs of existing results in the literature and also generalise these results to such an extend that at least the well known spaces without the approximation property are included.

∗ Secondly, in line with modern trend to provide proofs for theorems about operators on Banach spaces which do not rely on the approximation properties, we expand the concept of *conjugate ideal* to introduce the *operator dual space* of spaces of bounded linear operators. It turns out that the operator dual space is a handy tool to study inclusion theorems for spaces of operators. Also, using operator dual techniques, we are able to prove existing characterisations of continuous dual spaces of important classes of operators without relying on the continuity of the trace functional with respect to the nuclear norm – thus the proofs do not depend on the approximation property.

∗ Thirdly, we provide a direct and easy proof of a known result which provides necessary and sufficient condition for all weak $p$-summable sequences in a Banach space to be norm null. Our proof uses sequence space arguments only, thereby allowing us to extend the proof to more general sequence spaces, including certain Orlicz sequence spaces. Applying these results, together with some known characterisations of operators on sequence spaces in terms of vector sequence spaces, we succeed on the one hand to provide easier proofs for existing results about necessary and sufficient conditions for the equalities $K(\ell^p, X) = L(\ell^p, X)$ and $K(c_0, X) = L(c_0, X)$ and on the other hand to obtain further improvements.

∗ An absolutely summing multiplier of a Banach space $X$ is a scalar sequence $(\alpha_i)$ such that $(\alpha_i x_i)$ is absolutely summable in $X$ for all weakly absolutely summable sequences $(x_i)$ in $X$. Recently there were several papers by Spanish mathematicians about sequences in the range of a vector measure. We expose the fact that these concepts are intertwined and thereby show that various results in one of the papers about sequences in the range of a vector measure can easily be obtained, using the absolutely summing multiplier concept. In the last chapter of the thesis we generalise the idea of *absolutely summing multiplier* to that of *p-summing multiplier,* $\Lambda$- *summing multiplier* and even more general, $(\Lambda, \Sigma)$-*summing multiplier* and use these concepts to obtain results about Banach space valued bounded linear operators on $\Lambda$.

**Keyterms:** Compact linear operator, Banach space, sequence space, dual space,

$p$-summing multiplier, range of an $X$-valued measure, sectional convergence, normed operator ideal.

# Introduction

There is a long standing conjecture that the space $K(X,Y)$ of compact linear operators (on Banach spaces) either equals the space $L(X,Y)$ of all bounded linear operators or is uncomplemented in $L(X,Y)$. Since the early seventies this conjecture attracted many Functional Analysts and also led to a study of the continuous dual spaces of the spaces of compact and bounded linear operators – in particular, the question of the complementation of $K(X,Y)^*$ in $L(X,Y)^*$ received a lot of attention. A result by J. Johnson (in 1979) (cf. [28]) for Banach spaces with the approximation property more or less settled the last question, although there was still some activity around this in the context of locally convex spaces. Although many examples of Banach spaces which fail the approximation property – even among spaces with nice properties – became known after Enflo's example in 1973, the fact that the classical spaces have the approximation property made Johnson's result a "final" one. However, after Pisier's paper in 1983 (cf. [38]) which introduced counterexamples to a conjecture of Grothendieck in connection with the strong and weak norms on tensor products of Banach spaces, a new interest in the study of operators on Banach spaces without the approximtion property was evoked.

J. Johnson proved in [28] that if $Y$ is a Banach space having the bounded approximation property then the annihilator $K(X,Y)^{\perp}$ in the (continuous) dual space $L(X,Y)^*$ (of the space of bounded linear operators from the Banach space $X$ into $Y$) is the kernel of a projection $p$ on $L(X,Y)^*$. Here $K(X,Y)$ denotes the space of compact linear operators $T : X \to Y$. The range space of the projection $p$ is isomorphic to the dual space $K(X,Y)^*$. K. John's observation in [25] that the same statement is true in case of any separable Pisier space $X = P$ and its dual $Y = P^*$ - they are both spaces which do not have the approximation property - motivated his more general results in [27] where it is for instance proved that Johnson's result holds for couples of Banach spaces $X$, $Y$ such that each $T : X \to Y$ factors through a Banach space $Z$, its dual $Z^*$ being a separable Banach space which has the bounded approximation property. Moreover, a substantially generalised version of the result is proved there.

Following Kalton [29] we denote by $w'$ the *dual weak operator topology* on $L(X,Y)$ which is defined by the linear functionals

$$T \mapsto e^{**}(T^*f^*), \qquad f^* \in Y^*, e^{**} \in X^{**}.$$

Although the weak topology of $L(X,Y)$ is in general stronger than $w'$, it is shown by Kalton in [29] that $w'$-compact subsets of $K(X,Y)$ are weakly compact. In particular, it is important to notice that

**Property (K).** If $(T_n) \subset K(X,Y)$ is a $w'$- convergent sequence which converges to a $T \in K(X,Y)$, then $T_n \to T$ in the weak topology of $L(X,Y)$.

Using Property (K) and some fundamental properties of complete normed spaces, K. John recently proved (in [27]) that if for each $T \in L(X,Y)$ there exists a sequence $(T_n) \subset K(X,Y)$ such that $T_n \to T$ in the dual weak operator topology, then the annihilator

$K(X,Y)^{\perp}$ in $L(X,Y)^*$ is the kernel of a projection on $L(X,Y)^*$. The conditions of this theorem are for instance satisfied if each $T \in L(X,Y)$ factors through a Banach space (depending on $T$) which has a separable dual with the approximation property. For example, each $T \in L(P, P^*)$ factors through a Hilbert space. It should be noted that John's theorem provides information on spaces which are excluded by the conditions (bounded approximation property) in Johnson's result, but it does not provide a strict generalisation of that result. In Chapter 1 we set out to find an operator ideal approach to John's results; in this way we are not only able to state his results in a (in our view) more proper setting, but also to prove extended versions of the same. Also, extended versions of results by Kalton and J. Johnson about reflexivity of the space $K(X,Y)$ of compact operators are proved by means of our appraoch. Our results in Chapter 1 are contributions in the recent and present research activities around the problem of finding proofs for results in operator theory which do not depend on the approximation property of the underlying spaces.

In the paper [15], the author (JH Fourie) considers generalisations of John's results in the setting of locally convex spaces. In order to follow a similar (to John's work) approach in the general context, the author in [15] had to consider a generalisation of Kalton's result on the weak convergence of $w'$-convergent sequences of compact operators in the setting of operators on locally convex spaces (cf. Theorem 1.3.5). Then the proof of the final result about the existence of projections on dual spaces (cf. Proposition 1.3.6) depends on difficult arguments which involve the application of the Riesz representation theorem (for $C(K)$-spaces) and the Lebesgue Dominated Convergence Theorem. We consider an alternative approach which results in both simplification and extension of Fourie's result and its proof (cf. Theorem 1.3.9 and Theorem 1.3.10). The results of this chapter will appear in the manuscript [4], which is now in preparation.

Our study of the continuous dual spaces of spaces of bounded linear operators, leads us to the question: "When can the continuous dual space of a (quasi-) normed space of bounded linear operators be characterised as another space of bounded linear operators?" There are of course many examples of such characterisations of dual spaces of classical spaces of bounded linear operators on Banach spaces with suitable properties. Mostly, the proofs of these characterisations depend on the continuity of the trace functional (tr) with respect to the nuclear norm – thus asking for the presence of the (bounded) approximation property on the underlying Banach spaces or their dual spaces. In Chapter 2 we consider possible answers to this question, by extending the known concept of "conjugate ideal" of a quasi-normed operator ideal to the concept of "operator dual space" of a topological space $(\mathcal{A}(X,Y), \mu)$ of bounded linear operators on Banach spaces. We restrict our discussion to spaces $\mathcal{A}(X,Y)$ which are either components of quasi-normed operator ideals (with quasi norm $\mu$) or which contain the space $\mathcal{F}(X,Y)$ of bounded linear operators of finite rank. An operator $T \in L(Y,X)$ belongs to the operator dual space of $\mathcal{A}(X,Y)$ if the linear functional $\mathcal{F}(X,Y) \mapsto \mathrm{tr}(ST)$ is $\mu$-continuous. In Theorem 2.1.2 and Theorem 2.1.3 we prove general conditions so that the operator dual space will be isomorphic to either a subspace of the dual space or to the dual space itself. We also discuss some examples of operator dual spaces of important Banach ideals of operators. In particular, we try to indicate how a study of dual spaces of spaces of bounded linear operators in the context of the operator dual space, normally do not depend on the approximation property of the underlying Banach spaces. In fact, in some instances we are also able to arrive at known characterisations of the continuous dual spaces of certain well known spaces of bounded linear operators, without having to rely on the presence of the approximation property on the given Banach spaces - as is the case for the known characterisations in the literature. In the last section of Chapter 2 we restrict our discussion to operator spaces $(\mathcal{A}(X,Y),\mu)$ which are continuously embedded into $L(X,Y)$ with the uniform operator norm. It is then showed how the operator dual space can be used to find inclusion theorems for spaces of bounded linear operators.

In Chapter 3 we continue our study of compact operators, but in this case for compact operators whose domains are scalar sequence spaces. Our work in this chapter makes extensive use of the characterisations of compact operators on sequence spaces in terms of vector sequence spaces which appeared in [13], [16], [17] and [18] (refer to Chapter 0). Therefore, we have to consider some results about vector sequence spaces first. Most of the material of Chapter 3 can also be found in the joint paper [3]. Thoerem 3.0.1 is in a sense the main result of the chapter. This result, which gives a necessary and sufficient condition for each weakly $p$-summable sequence in a Banach space to be norm null, is well known in the literature. Its proof can for instance be found in the book [8] on tensor products. However, we discuss a much simplified proof which does not depend on deep results as in the case of the existing proof. In fact, our proof leads to extension of the result to include necessary and sufficient conditions for weakly $\Lambda$-summable sequences ($\Lambda$ more general scalar sequence spaces than $\ell^p$) to be norm null. These results are then used to prove necessary and sufficent conditions for all bounded linear operators from $\Lambda$ into a Banach space $X$ to be compact. In this way we improve on results in the papers [2] and [6].

In the paper [14], Fourie introduced the concept of *absolutely summing multiplier* of a Banach space $E$ as follows: A sequence $(\xi_i) \in \omega$ is called an *absolutely summing multiplier of* $E$ if $(\xi_i x_i)$ is absolutely summable in $E$ whenever $(x_i)$ is weakly absolutely summable in $E$; hence, in the notation of Chapter 0, $(\xi_i x_i) \in \ell^1_s(E)$ for all $(x_i) \in \ell^1_w(E)$. The perfect scalar sequence space of all absolutely summing multipliers of $E$ is denoted by $m(E)$. It is a vector subspace of $\ell^\infty$ such that $\ell^1 \subseteq m(E)$ and is normed in an obvious way. Using the well known Dvoretzky-Rogers Theorem, it is easily verified (cf. Theorem 4.1.3) that $m(E) \subseteq \ell^2$ if $E$ is infinite dimensional. It is interesting to note that for an infinite dimensional Banach space $E$, $m(E) = \ell^2$ if and only if $E$ has the well known Orlicz property (cf. Theorem 4.1.5). For instance, all Hilbert spaces have this property. We discuss (in Chapter 4) the absolutely summing multiplier spaces of the Banach spaces $\ell^p$ and $L^p$. In a recent paper (cf. [32]) Marchena and Piñeiro introduced the space $\lambda_X$ of all scalar sequences $(\alpha_i)$ such that for each null sequence $(x_i)$ in $X$, the sequence $(\alpha_i x_i)$ lies in the range of a vector measure. Although the authors of the paper [32] do not mention the space $m(X)$, it follows from their main result that $m(X^*) = \lambda_X$ for any Banach space $X$. Because of this connection, it is possible to derive several results in the paper [32] from our results in section 4.1 – and in fact, mostly the proofs of our results are straightforward. On the other hand, one easily verifies from the proof of $m(X^*) = \lambda_X$ (the result is differently stated in [32]) given by the authors in [32], that $(\alpha_i) \in m(X^*)$ if and only if the operator

$$\sum_{i=1}^{\infty} \alpha_i e_i^* \otimes x_i : \ell^1 \to X$$

is nuclear for all norm null sequences $(x_i)$ in $X$. Benefiting from their arguments in the proof of $m(X^*) = \lambda_X$, we prove (using some duality arguments) that $m(X) = m(X^{**})$. In doing so we also show that $(\alpha_i) \in m(X)$ if and only if

$$\sum_{i=1}^{\infty} \alpha_i e_i^* \otimes x_i^* : \ell^1 \to X^*$$

is nuclear for all norm null sequences $(x_i^*)$ in $X^*$.

In Chapter 5 we extend the notion of absolutely summing multiplier to "general families of summing multipliers". We consider (in §5.1) the scalar sequence space $m_p(X)$ of $p$-summing multipliers of the Banach space $X$. An inclusion theorem, $m_p(X) \subseteq m_q(X)$ if $1 < p \leq q < \infty$, is proved and for $1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1$ it is proved that $(\alpha_i) \in m_p(X^*)$ if and only if

$$\ell^p \to X : (\beta_i) \mapsto \sum_{i=1}^{\infty} \beta_i \alpha_i x_i$$

is integral for all sequences $(x_i) \in \ell_s^q(X)$. It is also shown that $m_p(X) = m_p(X^{**})$ – this follows after the proof of an important result (Lemma 5.1.7) in which it is demonstrated that the adjoint $P^*$ of the operator

$$P : K(\ell^q, X) \to \ell_s^p(X) : \sum_{n=1}^{\infty} e_n^* \otimes x_n \mapsto (\alpha_n x_n)$$

maps $\ell_s^q(X^*)$ into $\mathcal{N}(X, \ell^q)$. In §5.2 the results of the previous section are extended to the case where $\ell^p$ is replaced by a $BK$ sequence space with the $AK$ property. Finally, we find some conditions for the sequence space of $(\Lambda, \Sigma)$-summing multipliers of a Banach space to have the $AK$ property. It turns out that the equality $L(\Lambda, X) = K(\Lambda, X)$ for reflexive spaces $\Lambda$ and $X$ implies the $AK$ property on $m_{(\Lambda,\Sigma)}(X)$. Hence we see the connection with the work in Chapter 3. The results of Chapter 4 and Chapter 5 will appear in a joint paper (cf. [5]), which is in preparation.

# Chapter 0

# Notation and basic facts

## 0.1 About sequence spaces

Let $E, F$ be Banach spaces. $L(E,F)$ denotes the space of all bounded linear operators between $E$ and $F$, whereas $K(E,F)$ denotes the space of all compact linear operators between $E$ and $F$. The closed unit ball in $E$ will be denoted $B_E$. Sequences in $E$ will be denoted $(x_i), (y_i)$ etc. Let

$$\begin{aligned}(x_i)(\leq n) &:= (x_1, x_2, \ldots x_n, 0, 0, \ldots)\\ (x_i)(\geq n) &:= (0, 0, \ldots 0, x_n, x_{n+1}, \ldots)\end{aligned}$$

Let $w$ denote the vector space (with respect to coordinate wise addition and scalar multiplication) of all (scalar) sequences of complex numbers. A vector space $\Lambda$ whose elements are sequences $(\alpha_n)$ of numbers (real or complex), is called a sequence space. $\Lambda$ is said to be normal if whenever it contains $(\alpha_n)$, it also contains all sequences $(\beta_n)$ with $|\beta_n| \leq |\alpha_n|$ for all $n \in \mathbb{N}$. To each sequence space $\Lambda$ we assign another sequence space $\Lambda^\times$, its Köthe-dual. $\Lambda^\times$ is defined to be the set of all sequences $(\beta_n)$ for which the scalar products $\sum_{n=1}^{\infty} \alpha_n \beta_n$ converge absolutely for all $(\alpha_n) \in \Lambda$.

$$\Lambda^\times = \left\{ (\alpha_n) \in w : \sum_{n=1}^{\infty} |\alpha_n \beta_n| < \infty,\ \forall (\beta_n) \in \Lambda \right\}.$$

A sequence space $\Lambda$ is said to be **perfect** if $\Lambda^{\times\times} = \Lambda$. $\Lambda$ is said to be **symmetric** if $(\alpha_i) \in \Lambda$ if and only if $(\alpha_{\pi(i)}) \in \Lambda$ for all permutations $\pi$ of the positive integers.

A Banach sequence space $\Lambda$ is said to be a ***BK*-space** if each coordinate projection mapping $(\alpha_n) \mapsto \alpha_i$ is continuous.

Let $e_n = (\delta_{i,n})_i$, with $\delta_{i,n} = 1$ if $i = n$ and $\delta_{i,n} = 0$ if $i \neq n$. A normed sequence space is said to have the ***AK*-property** if all its elements can be approximated by their sections. That is, if each element $(\beta_i)$ in the sequence space satisfies $(\beta_i) = \lim_{n\to\infty} (\beta_i)(\leq n)$, where $(\beta_i)(\leq n) = \sum_{i=1}^{n} \beta_i e_i$.

A $BK$-space $\Lambda$ has the $AK$-property if and only if $\{e_n : n = 1, 2, \ldots\}$ is a Schauder basis for $\Lambda$ if and only if $\lim_{n\to\infty} \|(\mu_i)(\geq n))\|_\Lambda = 0$. If $\Lambda$ is a normal $BK$-space with $AK$, then $\{e_n : n = 1, 2, \dots\}$ is an unconditional basis for $\Lambda$, called the *standard coordinate basis* or the unit vector basis of $\Lambda$. In this case a standard argument shows that $\Lambda^\times$ is algebraically isomorphic to the continuous dual space $\Lambda^*$ with respect to the obvious duality. We call $\Lambda$ a $DAK$ space if both $\Lambda$ and $\Lambda^\times$ have the $AK$ property.

If not otherwise stated, all scalar sequence spaces $\Lambda \neq \ell^\infty$ will throughout be assumed to be normal $BK$-spaces with the $AK$-property.

The following standard sequence spaces will be referred to:

$w$, the vector space (with respect to coordinate wise vector operations) of all (complex and real) scalar sequences;

$\phi \subset w$, the space of all sequences with only finite number of non-zero terms;

$\ell^\infty$, the space of all bounded sequences;

$c_0$, the space of all zero sequences;

$\ell^p$, $1 \leq p < \infty$, the space of all absolutely $p$-summable sequences.

The vector sequence space $\Lambda_s(E) := \{(x_i) \subset E : (\|x_i\|) \in \Lambda\}$ is a complete normed space with respect to the norm

$$\pi_\Lambda((x_i)) := \|(\|x_i\|)\|_\Lambda.$$

We put $\pi_\Lambda((x_i)) = \pi_p((x_i))$ when $\Lambda = \ell^p$, the Banach space of $p$-absolutely summable scalar sequences (with $1 \leq p < \infty$).

The vector sequence space $\Lambda_w(E) := \{(x_i) \subset E : (\langle a, x_i \rangle) \in \Lambda, \ \forall a \in E^*\}$ is a complete normed space with respect to the norm

$$\epsilon_\Lambda((x_i)) := \sup_{\|a\| \leq 1} \|(\langle a, x_i \rangle)\|_\Lambda.$$

We put $\epsilon_p = \epsilon_\Lambda$ when $\Lambda = \ell^p$, (with $1 \leq p < \infty$).

The vector sequence space

$$\begin{aligned} \Lambda_c(E) &= \{(x_i) \in \Lambda_w(E) : (x_i) = \epsilon_\Lambda - \lim_{n\to\infty} (x_1, \dots, x_n, 0, \dots)\} \\ &= \{(x_i) \in \Lambda_w(E) : \epsilon_\Lambda((x_i)(\geq n)) \to 0 \text{ if } n \to \infty\} \end{aligned}$$

is a closed subspace of $\Lambda_w(E)$. On $\Lambda_c(E)$ we will consider the induced subspace norm, inherited from $\Lambda_w(E)$.

It follows from Proposition 2 in the paper [20], that the continuous dual space $\Lambda_c(E)^*$ can be identified with the vector space of all sequences $(x_i^*)$ in $E^*$ such that $\sum_{i=1}^\infty |\langle x_i, x_i^* \rangle| < \infty$ for all $(x_i) \in \Lambda_w(E)$.

It is proved in [13] that $(x_i) \in \Lambda_w^\times(E)$ if and only if $\sum_{i=1}^{\infty} \lambda_i x_i$ converges in $E$ for every $(\lambda_i) \in \Lambda$ and that

$$\epsilon_{\Lambda^\times}((x_i)) = \sup_{(\lambda_i)\in B_\Lambda} \|\sum_{i=1}^{\infty} \lambda_i x_i\|.$$

Moreover, the following characterisations of $E$-valued operators on $\Lambda$ can also be found in [13] and in the paper [18]:

**Theorem 0.1.1** *Consider a Banach space $E$.*

*a) Let $\Lambda$ be a Banach sequence space with the AK-property. Then $\Lambda_w^\times(E)$ is isometrically isomorphic to $L(\Lambda, E)$. The isometry is given by $(x_n) \mapsto T_{(x_n)}$, where $T_{(x_n)}((\xi_i)) = \sum_{i=1}^{\infty} \xi_i x_i$.*

*b) Let $\Lambda$ be a Banach sequence space with the AK-property such that $\Lambda^\times$ has AK. Then $\Lambda_c^\times(E)$ is isometrically isomorphic to $K(\Lambda, E)$. This isometry is given by $(x_n) \mapsto T_{(x_n)}$, where $T_{(x_n)}((\xi_i)) = \sum_{i=1}^{\infty} \xi_i x_i$.*

## 0.2 About operator ideals

**Definition 0.2.1** *An ideal of operators on the family of all Banach spaces is an assignment $\mathcal{A}$ which associates with each pair $(X, Y)$ of Banach spaces a subset $\mathcal{A}(X, Y)$ of $L(X, Y)$ such that the following are satisfied:*

*(i) $a \otimes y \in \mathcal{A}(X, Y)$ for all $a \in X^*$ for all $y \in Y$, where $a \otimes y : X \to Y : x \to \langle x, a\rangle y$.*

*(ii) If $S_1, S_2 \in \mathcal{A}(X, Y)$ then $S_1 + S_2 \in \mathcal{A}(X, Y)$.*

*(iii) If $T \in L(X, X_0)$, $S \in \mathcal{A}(X_0, Y_0)$, $R \in L(Y_0, Y)$, then $R \circ S \circ T \in \mathcal{A}(X, Y)$.*

We note also that $\alpha T \in \mathcal{A}(X, Y)$ if $T \in \mathcal{A}(X, Y)$ and $\alpha \in \mathbb{K}$ because of property (iii) in the definition. Thus $\mathcal{A}(X, Y)$ is a subspace of $L(X, Y)$.
There is an equivalent definition by Pietsch [34].

**Definition 0.2.1′** (cf Pietch [34], pp 45) *Let $L$ denote the class of all operators between arbitrary Banach spaces. An operator ideal $\mathcal{U}$ is a subclass of $L$ such that the components*

$$\mathcal{U}(E, F) := \mathcal{U} \cap \mathcal{L}(E, F)$$

*satisfy the following conditions:*

($OI_0$) *$I_K \in \mathcal{U}$, where $K$ denotes the 1-dimensional Banach space.*

($OI_1$) *It follows from $S_1, S_2 \in \mathcal{U}(E, F)$ that $S_1 + S_2 \in \mathcal{U}(E, F)$.*

($OI_2$) *If $T \in L(E_0, E)$, $S \in \mathcal{U}(E, F)$, and $R \in L(F, F_0)$, then $RST \in \mathcal{U}(E_0, F_0)$.*

**Definition 0.2.2** *Let $\mathcal{A}$ be an operator ideal. Let $\alpha$ be an assignment which associates with each $S$ belonging to some component $\mathcal{A}(X,Y)$ of $\mathcal{A}$ a non-negative real number $\alpha(S)$. We call $\alpha$ an ideal-quasi norm if for arbitrary Banach spaces $X, X_0, Y, Y_0$, the following are satisfied:*

*(i)* $\alpha(a \otimes y) = \|a\|\|y\| \quad \forall a \in X^*, \quad \forall y \in Y.$

*(ii) There exists a constant $k \geq 1$ such that $k$ is independent of the choices of Banach spaces $X, Y$ such that $\alpha(S_1 + S_2) \leq k[\alpha(S_1) + \alpha(S_2)]$ for all $S_1, S_2 \in \mathcal{A}(X,Y)$.*

*(iii)* $\alpha(R \circ S \circ T) \leq \|R\|\alpha(S)\|T\|$ *for all* $R \in L(Y_0, Y)$, $T \in L(X, X_0)$, $S \in \mathcal{A}(X_0, Y_0)$.

The couple $(\mathcal{A}, \alpha)$ is called a quasi-normed ideal of operators. It is easy to check that each of the components $\mathcal{A}(X,Y)$ becomes a metrizable Hausdorff topological vector space. If each component is complete, then the quasi-normed ideal is called a complete metrizable operator ideal. Although one sometimes meets topological ideals which do not admit a reasonable ideal quasi-norm, we assume throughout that the ideal topologies are defined by ideal quasi-norms. It is a well known fact that for any pair $(X,Y)$ of Banach spaces and every $T \in \mathcal{A}(X,Y)$, we have $\|T\| \leq \alpha(T)$. This shows in particular that a non-trivial $\mathcal{A}(X,Y)$ has non-trivial dual space $\mathcal{A}(X,Y)^*$. As a consequence of the closed graph theorem we mention that if $(\mathcal{A}_1, \alpha_1)$ and $(\mathcal{A}_2, \alpha_2)$ are complete metrizable operator ideals such that $\mathcal{A}_1 \subset \mathcal{A}_2$, then for every pair $(X,Y)$ of Banach spaces, the canonical injection of $\mathcal{A}_1(X,Y)$ into $\mathcal{A}_2(X,Y)$ is continuous.

If $k = 1$ in the above definition of an ideal quasi-norm, we speak of an *ideal norm* and correspondingly, the couple $(\mathcal{A}, \alpha)$ is called a normed ideal of operators. In this case the components $\mathcal{A}(X,Y)$ are normed spaces. The ideal is called a Banach ideal if each of these is complete.

The following well known examples of quasi-normed (some are normed) operator ideals will be considered in this work:

$*$ $(L, \|\cdot\|)$, where $T \in L(X,Y)$ if and only if $T$ is a bounded linear operator and $\|\cdot\|$ is the usual uniform operator norm. Recall that a linear operator $T : X \to Y$ is called bounded if there exists $k > 0$ such that $\|Tx\| \leq k\|x\|$ for all $x \in X$ and that

$$\|T\| = \sup\{\|Tx\| : \|x\| \leq 1\}.$$

$*$ $(K, \|\cdot\|)$, where $T \in K(X,Y)$ if and only if $T$ is a compact bounded linear operator and $\|\cdot\|$ is the usual uniform operator norm. Recall that a linear operator $T : X \to Y$ is called compact if $T(B_X)$ is relatively compact in $Y$.

$*$ $(\mathcal{F}, \|\cdot\|)$, where $T \in \mathcal{F}(X,Y)$ if and only if $T$ is a finite rank bounded linear operator and $\|\cdot\|$ is the usual uniform operator norm. Recall that $T \in \mathcal{F}(X,Y)$ if and only if $T$ has a representation of the form $T = \sum_{i=1}^{n} a_i \otimes x_i$ where $a_i \in X^*$ and $x_i \in Y$. Hence $Tx = \sum_{i=1}^{n} a_i(x)x_i$. Also, recall that the **trace** of $T$ is the number

$$tr(T) = \sum_{i=1}^{n} a_i(x_i).$$

* $(\mathcal{N}, \nu_1)$, where $T \in \mathcal{N}(X,Y)$ if and only if $T$ is a nuclear operator, i.e. $T$ has a representation

$$Tx = \sum_{i=1}^{\infty} \lambda_i \langle x, x_i^* \rangle y_i$$

where $(\lambda_i) \in \ell^1$, $(x_i^*)$ is bounded in $X^*$ and $(y_i)$ is bounded in $Y$. Here

$$\nu_1(T) := \inf \sum_{i=1}^{\infty} |\lambda_i|,$$

where the infimum is extended over all such representations for which $\|x_i^*\| \leq 1$ and $\|y_i\| \leq 1$ for all $i$.

* $(\mathcal{N}_p, \nu_p)$, where $T \in \mathcal{N}_p(X,Y)$ if and only if $T$ is a $p$-nuclear operator, i.e. $T$ has a representation

$$Tx = \sum_{i=1}^{\infty} \langle x, x_i^* \rangle y_i$$

where $(x_i^*) \in \ell_s^p(X^*)$ and $(y_i) \in \ell_w^q(Y)$ and where $\frac{1}{p} + \frac{1}{q} = 1$. Here

$$\nu_p(T) := \inf \pi_p((x_i^*)) \epsilon_q((y_i)),$$

where the infimum is extended over all such representations of $T$.

* $(\mathcal{I}_1, i_1)$, where $T \in \mathcal{I}_1(X,Y)$ if and only if $T$ is an integral operator, i.e. if and only there exists $\rho \geq 0$ such that

$$|tr(TS)| \leq \rho \|S\|, \qquad \forall S \in \mathcal{F}(Y,X).$$

The integral norm $i_1(T)$ equals the smallest of all numbers $\rho \geq 0$ admissible in these inequalities.

* $(\mathcal{P}_p, \pi_p)$, where $T \in \mathcal{P}_p(X,Y)$ if and only if $T$ is a $p$-absolutely summing operator, i.e. if and only if $(Tx_i) \in \ell_s^p(Y)$ for all $(x_i) \in \ell_w^p(X)$. The $p$-summing norm $\pi_p(T)$ of $T$ equals the operator norm of the bounded linear operator

$$\ell_w^p(X) \to \ell_s^p(Y) :: (x_i) \mapsto (Tx_i),$$

i.e.

$$\pi_p(T) = \sup\{ \left( \sum_{i=1}^{\infty} \|Tx_i\|^p \right)^{1/p} : \epsilon_p((x_i)) \leq 1 \}.$$

* $(\mathcal{S}_p, \sigma_p)$, where $T \in \mathcal{S}_p(X,Y)$ if and only if $T$ is a $p$-approximable operator, i.e. if and only if $(s_n(T)) \in \ell^p$, where

$$s_n(T) := \inf\{\|T - S\| : S \in \mathcal{F}(X,Y), \dim S(X) \leq n\}$$

is the $n$-th approximation number of $T$. Here

$$\sigma_p(T) := \|(s_n(T))\|_p.$$

**Theorem 0.2.3** (cf. Pietsch [34] 6.2.3, pp 91) *Let $\mathcal{U}$ be a subclass of $L$ with an $\mathbb{R}^+$-valued function $\alpha$ such that the following conditions are satisfied:*

*(i) If $X, Y$ are Banach spaces, then $a \otimes y \in \mathcal{U}(X,Y)$ for all $a \in X^*$, $y \in Y$ and $\alpha(a \otimes y) = \|a\|\|y\|$.*

*(ii) If $S_1, S_2, \cdots \in \mathcal{U}(X,Y)$ and $\sum_{i=1}^{\infty} \alpha(S_i) < \infty$. Then $S = \sum_{i=1}^{\infty} S_i = \|\cdot\| - \lim_n \sum_{i=1}^{n} S_i \in \mathcal{U}(X,Y)$. And $\alpha\left(\sum_{i=1}^{\infty} S_i\right) \leq \sum_{i=1}^{\infty} \alpha(S_i)$.*

*(iii) $RST \in \mathcal{U}(X,Y)$ and $\alpha(RST) \leq \|R\|\alpha(S)\|T\|$ whenever $T \in L(X, X_0)$, $S \in \mathcal{U}(X_0, Y_0)$ and $R \in L(Y_0, Y)$.*

*Then $(\mathcal{U}, \alpha)$ is a complete normed operator ideal.*

By a **trace** $\tau$ defined on an operator ideal $\mathcal{U}$ we mean a function which assigns to every operator $T \in \mathcal{U}(X)$ and each Banach space $X$, a complex number $\tau(T)$ such that the following conditions are satisfied:

(T1) $\tau(a \otimes x) = \langle x, a \rangle$ for $a \in X^*$ and $x \in X$.

(T2) $\tau(ST) = \tau(TS)$ for $T \in \mathcal{U}(X,Y)$ and $S \in L(Y,X)$.

(T3) $\tau(S+T) = \tau(S) + \tau(T)$ for $S, T \in \mathcal{U}(X)$.

(T4) $\tau(\lambda T) = \lambda \tau(T)$ for $T \in \mathcal{U}(X)$ and $\lambda \in \mathbb{C}$.

On $\mathcal{F}$ there exists a unique trace, namely for each Banach space $X$ and $T \in \mathcal{F}(X)$ we put

$$\tau(T) = \mathrm{tr}(T) = \sum_{i=1}^{n} \langle x_i, x_i^* \rangle,$$

where $Tx = \sum_{i=1}^{n} \langle x, x_i^* \rangle x_i$ is any representation of $T$.

A trace $\tau$ defined on a quasi-Banach operator ideal $(\mathcal{U}, \mu)$ is said to be continuous if the function $T \to \tau(T)$ has this property on all components $\mathcal{U}(X)$. Then there exists a constant $c \geq 1$ such that

$$|\tau(T)| \leq c\mu(T).$$

Although there does not exist any continuous trace on the Banach operator ideal $(\mathcal{I}_1, i_1)$ itself, it is a well known result by A. Pietsch that $\mathcal{U} \subseteq \mathcal{I}_1$ for each quasi-Banach ideal $\mathcal{U}$ with a continuous trace. For more information on traces on operator ideals and examples of operator ideals with continuous traces, we refer to the book of Pietsch (cf. [35]).

## 0.3 About vector measures

**Definition 0.3.1** *A function $F$ from a field $\Sigma$ of subsets of a set $\Omega$ to a Banach space $X$ is called a finitely additive vector measure if whenever $E_1$ and $E_2$ are disjoint members* *of $\Sigma$ then*

$$F(E_1 \cup E_2) = F(E_1) + F(E_2).$$

*If in addition*

$$F\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} F(E_n)$$

*in the norm topology of $X$ for all sequences $(E_n)$ of pairwise disjoint members of $\Sigma$ such that $\bigcup_{n=1}^{\infty} E_n \in \Sigma$, then $F$ is termed a countably additive vector measure.*

**Definition 0.3.2** *Let $F : \Sigma \to X$ be a vector measure. The variation of $F$ is the extended non-negative function $|F|$ whose value on a set $E \in \Sigma$ is given by*

$$|F|(E) = \sup_{\pi} \sum_{A \in \pi} \|F(A)\|$$

*where the supremum is taken over all partitions $\pi$ of $E$ into a finite number of pairwise disjoint members of $\Sigma$. If the total variation of $F$ i.e. $tv(F) = |F|(\Omega) < \infty$, then $F$ will be called a measure of bounded variation.*

**Definition 0.3.3** *The range of a vector-valued measure is defined to be the set of the form*

$$rg(F) := \{F(E) : E \in \Sigma\}$$

*where $\Sigma$ is a $\sigma$-field of sets and $F$ is a countably additive measure on $\Sigma$ with values in an appropriate Banach space $X$ with dual $X^*$.*

We state the following facts:

**Theorem 0.3.4** ([10] The Liapunoff convexity Theorem): The range of a nonatomic vector measure with values in a finite dimensional space is compact and convex.

**Theorem 0.3.5** ([10] Bartle, Dunford and Schwartz) *The range of a vector-valued measure is always relatively weakly compact.*

**Theorem 0.3.6** ([10] Diestel and Seifert, Anantharaman) *Any sequence in the range of a vector-valued measure admits a subsequence with norm convergent arithmetic means.*

The closed convex hull and closed absolutely convex hull of the range of a vector-valued measure are each, themselves, the range of a vector-valued measure.

**Theorem 0.3.7** ([9] Diestel & Uhl; Chap IX, I Kluvanek and G Knowles) *A closed convex set that is the range of a vector-valued measure has extreme points, denting points, exposed points and strongly exposed points, similar to all weakly compact convex sets. However, any closed convex range of a vector-valued measure has extreme points, strongly exposed points, similar to norm compact convex sets but not similar to all weakly compact convex sets.*

**Theorem 0.3.8** ([10] Diestel & Uhl, Kluvanek & Knowles) *Any extreme point of the closed convex hull of the range of a vector-valued measure lies inside the range of the measure.*

**Theorem 0.3.9** ([1] Kluvanek & Knowles) *The closed unit ball of an infinite dimensional Banach space* **can** *be the range of a vector-valued measure.*

**Theorem 0.3.10** ([1] Anantharaman & Garg) *The closed unit ball $B_X$ of $X$ is the range of a measure iff $X^*$ (with the dual norm) is isometrically isomorphic to a reflexive subspace of $L^1(\mu)$ for some probability measure $\mu$.*

The *Pietch integral operators* are defined as follows:

* $T \in \mathcal{P}I(X, Y)$ if and only if there exists a $Y$-valued countably additive vector measure $G$ of bounded variation defined on the Borel (for the weak*-topology) sets of the closed unit ball $B_{X^*}$ of $X^*$ such that for each $x \in X$ we have

$$T(x) := \int_{B_{X^*}} x^*(x) dG(x^*).$$

The space $\mathcal{P}I(X, Y)$ becomes a Banach space under the norm

$$\|T\|_{pint} = \inf\{|G|(B_{X^*})\}$$

where the infimum is taken over all measures $G$ that satisfy the above definition.

# Chapter 1

# Spaces of compact operators and their dual spaces

## 1.1 A summary of some existing results.

Recall the following definitions:

**Definition 1.1.1** *The* **weak-operator topology** $w$ *on* $L(E,F)$ *is defined by the linear functionals* $T \mapsto f^*(Te)$ *for* $f^* \in F^*$ *and* $e \in E$.

*The* **dual weak-operator topology** $w'$ *on* $L(E,F)$ *is defined by the linear functionals* $T \mapsto e^{**}(T^*f^*)$ *for* $e^{**} \in E^{**}$ *and* $f^* \in F^*$. *Clearly* $w' \geq w$.

Let $B_{E^{**}}$ denote the unit ball of $E^{**}$ with the weak*-topology, $\sigma(E^{**}, E^*)$. Let $B_{F^*}$ be the unit ball of $F^*$ with the weak*-topology, $\sigma(F^*, F)$. Then $B_{E^{**}}$ and $B_{F^*}$ are compact Hausdorff topological spaces. For $T \in L(E,F)$ we define $\psi_T$ a function on $B_{E^{**}} \times B_{F^*}$ by $\psi_T(u,v) = u(T^*v)$. For the following results we refer to Kalton [29].

**Theorem 1.1.2** ([29], pp 268) $T \to \psi_T$ *defines a linear isometry of* $K(E,F)$ *onto a closed linear subspace of* $C(B_{E^{**}} \times B_{F^*})$.

**Theorem 1.1.3** ([29], pp 268) *Let* $A$ *be a subset of* $K(E,F)$. *Then* $A$ *is weakly compact if and only if* $A$ *is* $w'$*-compact.*

**Theorem 1.1.4** ([29], pp 269) *Let* $(T_n)$ *be a sequence of compact operators such that* $T_n \to T$ *in* $w'$ *where* $T$ *is compact. Then* $T_n \to T$ *weakly and there is a sequence* $(S_n)$ *of convex combinations of* $\{T_n : n = 1, 2, \dots\}$ *with* $\|T - S_n\| \to 0$.

**Remark** ([29], pp 269) A Banach space $E$ is called a Grothendieck space if every weak*-convergent sequence in $E^*$ converges weakly in $E^*$. $E$ is a Grothendieck space if and only if for any Banach space $F$, if $T_n \to T$ in the weak-operator topology $w$ on $K(E,F)$, then $T_n \to T$ weakly.

We recall the following well known definition.

**Definition 1.1.5** ([34], pp 130–131) *$E$ has the approximation property if there is a sequence, say $(T_n)$ of bounded finite rank operators $T_n : E \to E$ such that $T_n \to \mathrm{Id}_E$ uniformly on compact subsets of $E$. If there is a $\lambda > 0$ such that $\|T_n\| \leq \lambda$ for all $n$, then we say that $E$ has the $\lambda$-bounded approximation property. If $E$ has the $\lambda$-bounded approximation property for some $\lambda > 0$,then we say $E$ has the bounded approximation property. If $E$ has the 1-bounded approximation property, then we say $E$ has the metric approximation property.*

In his paper [28] J. Johnson proved the following results on projections and imbeddings on dual spaces of $L(E, F)$.

**Theorem 1.1.6** ([28], pp 305) *Let $E$ and $F$ be Banach spaces and suppose $F$ has the $\lambda$-bounded approximation property. Then there is a projection $P$ on $L(E, F)^*$ such that $\|P\| \leq \lambda$, the range of $P$ is isomorphic to $K(E, F)^*$ (isometric if $\lambda = 1$) and the kernel of $P$ is the annihilator of $K(E, F)$.*

**Theorem 1.1.7** ([28], pp 307) *If $E$ and $F$ are Banach spaces and $F$ has the $\lambda$-bounded approximation property, then there is an isomorphism (isometry if $\lambda = 1$) of $L(E, F)$ into $K(E, F)^{**}$ whose restriction to $K(E, F)$ is the canonical imbedding.*

Several papers have appeared dealing with the question "For which spaces $E$ and $F$ is $K(E, F)$ (respectively $L(E, F)$) reflexive?" The early papers used deep results from Grothendieck's theory of topological tensor products to attempt answering this question. From the paper of Johnson [28] we have the following result.

**Theorem 1.1.8** ([28], pp 307) *Let $E$ and $F$ be Banach spaces, one of which has the approximation property. The following are equivalent:*

*1) $K(E, F)$ is reflexive.*

*2) $L(E, F)$ is reflexive.*

*3) $E$, $F$ are reflexive and $K(E, F) = L(E, F)$.*

**Proof** If $K(E, F)$ is reflexive then $F$ is reflexive. Hence $F$ has the bounded (in fact metric) approximation property. Thus Theorem 1.1.7 applies and yields (2) and (3). If $E$ has the approximation property and $K(E, F)$ is reflexive, then $E$ is reflexive and so $E^*$ has the bounded approximation property. We now apply the previous argument to $L(F^*, E^*)$ and $K(F^*, E^*)$ which are canonically identifiable with $L(E, F)$ and $K(E, F)$ respectively. □

**Remark** Recall that by JP Kahane (cf [9], pp 141) a Banach space $E$ has cotype $p$ ($p \geq 2$) provided $\sum_{i=1}^{\infty} \|x_i\|^p$ converges whenever $\sum_{i=1}^{\infty} \sigma_i x_i$ is convergent in $E$ for almost all sequences $(\sigma_i)$ of signs $\sigma_i = \pm 1$ in $\{-1, 1\}^{\mathbb{N}}$, where the product space $\{-1, 1\}^{\mathbb{N}}$ is endowed with the natural product measure whose coordinate measures assign each singleton the probability of $1/2$.

**Definition 1.1.9** ([27], pp 69) *By Pisier space we will mean an infinite dimensional Banach space $P$ such that*

*(1) on $P \otimes P$ the extremal $\epsilon$ and $\pi$-tensor norms are equivalent.*

*(2) $P$ and $P^*$ are both cotype 2 spaces.*

Theorem 1.1.6 cannot be obtained for the space $E = P$ and $F = P^*$ using Johnson's technique, since both $P$ and $P^*$ lack the approximation property. However, the same result (1.1.6) has been proved by K John for this case, using the fact that $T : P \to P^*$ is factorable through a Hilbert space. Later, in his paper [27], John succeeded in proving a much more general result which includes the case when $E = P$ and $F = P^*$.

K John considered some extensions of Johnson's result (Theorem 1.1.6) about projections on the dual space of $L(X, Y)$, namely:

**Theorem 1.1.10** (cf. John, [25]) Let $P$ be a separable Pisier space. Then the annihilator $K(P, P^*)^\perp$ in the continuous dual space $L(P, P^*)^*$ is the kernel of a projection $P$ on $L(P, P^*)^*$. Also the range space of the projection $P$ is isomorphic to the dual space $K(P, P^*)^*$.

**Theorem 1.1.11** (cf. John, [27]) Johnson's result (Theorem 1.1.6 ) holds for couples of Banach spaces $X, Y$ such that each $T : X \to Y$ factors through a Banach space $Z$, its dual $Z^*$ being a separable Banach space which has the bounded approximation property.

**Theorem 1.1.12** (cf. John, [27]) If for each $T \in L(X, Y)$ there exists a sequence $(T_n) \subset K(X, Y)$ such that $T_n \to T$ in the dual weak operator topology, then the annihilator $K(X, Y)^\perp$ in $L(X, Y)^*$ is the kernel of a projection in $L(X, Y)^*$.

The conditions of Theorem 1.1.12 are for instance satisfied if each $T \in L(X, Y)$ factors through a Banach space (depending on $T$) which has a separable dual with the approximation property. For example, each $T \in L(P, P^*)$ factors through a Hilbert space. It should be said that Theorem 1.1.12 provides information on spaces which are excluded by the conditions (bounded approximation property) in Johnson's result, but it does not provide a strict generalisation of that result. One crucial observation in the proof of Theorem 1.1.12 is

**Theorem 1.1.13** (cf. John, [27]) Suppose the Banach spaces $X$ and $Y$ are such that for every $T \in L(X, Y)$ there is a sequence $(T_n) \subset K(X, Y)$ such that $T_n \to T$ in the dual weak operator topology, then there exists a $c > 0$ such that for each $T \in L(X, Y)$ the sequence $(T_n)$ can be chosen to satisfy $\|T_n\| \leq c\|T\|$, for all $n$.

John provides an elegant proof (in [27]) of the fact that with the hypothesis of Theorem 1.1.12 the norm $|||T||| := \inf\{\sup_n \|T_n\| : T_n \in K(X, Y), T_n \stackrel{w'}{\to} T\}$ on $L(X, Y)$ is equivalent to the uniform norm. This of course implies Theorem 1.1.13.

## 1.2 An alternative approach

In this section we present an alternative approach to John's paper [27] which is based on his techniques but which, in our opinion, is more natural. The main idea in the paper [27] was to present an alternative version to Theorem 1.1.6 which includes spaces which do not have the approximation property. This study was initiated by John's interest in results on compact operators on Pisier spaces. It seems that examples of (separable) reflexive Pisier spaces are not known yet (or they may not exist at all). This may be the reason why John was not interested in considering Theorem 1.1.8 in the context of Banach spaces without the approximation property. However, by now we know that Enflo's example (in 1973) paved the way to showing the existence of many separable reflexive Banach spaces which do not have the approximation property (cf. [23], pp 414).

Our aim in the present section is to consider John's results (in more general form) in the environment of operator ideals and then to demonstrate how these results lead to extended versions of results of Kalton and J Johnson about reflexivity of $K(X,Y)$. We believe that our presentation does not only provide extensions of existing results but also provides more insight into the structures of the proofs. The reader is referred to Chapter 0 for the information about operator ideals that will be needed here.

**Definition 1.2.1** *Let $T \in L(X,Y)$. $T$ is said to have the $w'$-compact approximation property ($w'$-cap) if there is a sequence $(T_n) \subset K(X,Y)$ such that $T_n \stackrel{w'}{\to} T$. Let $\mathcal{L}^{w'}(X,Y)$ be the family of all $T \in L(X,Y)$ which have the $w'$-compact approximation property.*

**Lemma 1.2.2** *If $T_n \to T$ in the $w'$-topology of $L(X,Y)$ then $(T_n)$ is norm bounded.*

**Proof** Given $x^{**}(T_n^* y^*) \to x^{**}(T^* y^*)$ for all $x^{**} \in X^{**}$ and for all $y^* \in Y^*$. Then we have $\langle x, T_n^* y^*\rangle \to \langle x, T^* y^*\rangle$ $\forall x \in X$, $\forall y^* \in Y^*$. Hence

$$\langle T_n x, y^*\rangle \to \langle Tx, y^*\rangle \ \forall x \in X, \ \forall y^* \in Y^*.$$

Thus

$$T_n x \to Tx \text{ weakly } \forall x \in X.$$

That is, the set $\{T_n x | n \in \mathbb{N}\}$ is weakly bounded, hence norm bounded in $Y$. Hence $(T_n)$ is pointwise bounded in $L(X,Y)$. By the uniform boundedness theorem $(T_n)$ is also norm bounded in $L(X,Y)$. □

Let $X, Y$ be fixed Banach spaces. For $T \in \mathcal{L}^{w'}(X,Y)$ we put

$$(*) \qquad |||T||| = \inf\left\{\sup_n \|T_n\| : T_n \in K(X,Y), T_n \stackrel{w'}{\to} T\right\}.$$

Clearly, if $T \in K(X,Y)$, then $|||T||| = \|T\|$.

Although our following results differ from John's results in [27], we make extensive use of the ideas and techniques developed in that paper.

**Theorem 1.2.3** *Let $\mathcal{L}^{w'}$ denote the assignment which associates with each pair of Banach spaces $X, Y$ the vector space $\mathcal{L}^{w'}(X,Y)$. And let $|||\cdot|||$ be the assignment that associates with every pair of Banach spaces $X, Y$ and with every operator $S$ belonging to $\mathcal{L}^{w'}(X,Y)$ the real number $|||S|||$ in $(*)$. Then $(\mathcal{L}^{w'}, |||\cdot|||)$ is a Banach operator ideal.*

**Proof** We observe that $\|\cdot\| \leq |||\cdot|||$ on $\mathcal{L}^{w'}(X,Y)$ where $\|\cdot\|$ is the uniform operator norm on $\mathcal{L}(X,Y)$. In fact for any $\epsilon > 0$, let $\|x\| \leq 1$, $\|y^*\| \leq 1$ such that $\|T\| - \epsilon \leq |y^*(Tx)| = \lim_n |y^*(T_n x)| \leq \sup_n \|T_n\|$ where $(T_n) \subset K(X,Y)$ such that $T_n \xrightarrow{w'} T$ is arbitrary chosen. Clearly $\|T\| \leq |||T||| + \epsilon$. To prove that $(\mathcal{L}^{w'}, |||\cdot|||)$ is a complete normed ideal we make use of Theorem 0.2.3:

(i) $|||I_K||| = 1$ where $I_K \in \mathcal{L}^{w'}(K)$ is the identity map on the 1-dimensional Banach space $K$.

(iii) Let $T \in \mathcal{L}(X, X_0)$, $S \in \mathcal{L}^{w'}(X_0, Y_0)$ and $R \in \mathcal{L}(Y_0, Y)$. Then if $S_n \xrightarrow{w'} S$, $S_n \in K(X,Y)$ arbitrary, then $RS_nT \xrightarrow{w'} RST$. Hence

$$|||RST||| \leq \sup_n \|RS_nT\| \leq \|R\|(\sup_n \|S_n\|)\|T\|.$$

Since $(S_n)$ was arbitrarily chosen, it is clear that $|||RST||| \leq \|R\|\ |||S|||\ \|T\|$.

(ii) Now suppose that $(T_n) \subset \mathcal{L}^{w'}(X,Y)$ with $\sum_{n=1}^{\infty} |||T_n||| < \infty$. We have to show that $\sum_{i=1}^{\infty} T_i = \|\cdot\| - \lim_n \sum_{i=1}^{n} T_i$ exists and is in $\mathcal{L}^{w'}(X,Y)$ with $|||\sum_{i=1}^{\infty} T_i||| \leq \sum_{i=1}^{\infty} |||T_i|||$: Let $T_{n,i} \in K(X,Y)$ such that $T_{n,i} \underset{n}{\xrightarrow{w'}} T_i$, $\sup_n \|T_{n,i}\| \leq |||T_i||| + \epsilon/2^i$. For arbitrary $\|x^{**}\| \leq 1$, $\|y^*\| \leq 1$ we have $|x^{**}(T^*_{n,i}y^*)| \leq |||T_i||| + \epsilon/2^i$, $\forall i$ and $\forall n$. Hence $\sum_{i=1}^{\infty} x^{**}(T^*_{n,i}y^*)$ converges uniformly in $n \in \mathbb{N}$, thus showing that

$$(*) \qquad \sum_{i=1}^{\infty} x^{**}(T_i^* y^*) = \lim_n \sum_{i=1}^{\infty} x^{**}(T^*_{n,i}y^*)\,.$$

It follows from the completeness of $(L(X,Y), \|\cdot\|)$ and $(K(X,Y), \|\cdot\|)$ and the inequalities $\|T_i\| \leq |||T_i|||$ for all $i$ and $\|T_{n,i}\| \leq |||T_i||| + \epsilon/2^i$ for all $i$, that $\sum_{i=1}^{\infty} T_i \in L(X,Y)$ and $\sum_{i=1}^{\infty} T_{n,i} \in K(X,Y)$ for all $n$. Since $(*)$ holds for arbitrary $x^{**} \in B_{X^{**}}$ and $y^* \in B_{Y^*}$, it follows that $\sum_{i=1}^{\infty} T_{n,i} \xrightarrow{w'} \sum_{i=1}^{\infty} T_i$. Hence $\sum_{i=1}^{\infty} T_i$ is in $\mathcal{L}^{w'}(X,Y)$ and

$$|||\sum_i T_i||| \leq \sup_n \|\sum_{i=1}^{\infty} T_{n,i}\| \leq \sup_n \sum_{i=1}^{\infty} \|T_{n,i}\| \leq \epsilon + \sum_i |||T_i|||\,.$$

This shows that $|||\sum_{i=1}^{\infty} T_i||| \leq \sum_{i=1}^{\infty} |||T_i|||$. By Theorem 0.2.3, $(\mathcal{L}^{w'}(X,Y), |||\cdot|||)$ is a Banach ideal of operators.

□

**Theorem 1.2.4** *Fix the Banach spaces $X$ and $Y$. Let $\mathcal{L}^{w'} = \mathcal{L}^{w'}(X,Y), K = K(X,Y)$ and $\mathcal{L}^* = \mathcal{L}^{w'}(X,Y)^*$. There exists a continuous bilinear form $J : \mathcal{L}^* \times \mathcal{L}^{w'} \to \mathbb{K}$ such that*

(a) *$J(\phi, T) = \phi(T)$ for all $(\phi, T) \in \mathcal{L}^* \times K$.*

(b) *$|J(\phi, T)| \leq \|\phi\| |||T|||$ for all $T \in \mathcal{L}^{w'}$ and $\phi \in \mathcal{L}^*$.*

(c) *$J(\phi, T) = \lim_n \phi(T_n)$, where $(T_n)$ is any sequence of compact operators $T_n \in K(X, Y)$ tending to $T$ in $w'$-topology.*

**Proof** (c) First we observe that if $T \in \mathcal{L}^{w'}(X, Y)$, $T_n \xrightarrow{w'} T$, $T_n \in K(X, Y)$ for all $n$, then $\lim_n \phi(T_n)$ exists for all $\phi \in \mathcal{L}^*$ : Indeed since $\{T_n\}_{n \in \mathbb{N}}$ is norm bounded in $K(X, Y)$ (by Lemma 1.2.2), it is also weakly bounded. Hence $\{\phi(T_n)\}_{n \in \mathbb{N}}$ is bounded. Hence, in case of $\mathbb{K} = \mathbb{R}$, there are subsequences $(T_{n_k})$ and $(T_{m_k})$ such that $\lim_n \sup \phi(T_n) = \lim_k \phi(T_{n_k})$ and $\lim_n \inf \phi(T_n) = \lim_k \phi(T_{m_k})$.

Thus $\lim_n \sup \phi(T_n) - \lim_n \inf \phi(T_n) = \lim_k \phi(T_{n_k} - T_{m_k}) = 0$ because $T_{n_k} - T_{m_k} \to 0$ weakly by Theorem 1.1.4. For the case $\mathbb{K}$ is complex, we write $\phi(T_n) = Re(\phi(T_n)) + Im(\phi(T_n))$ and proceed as in the real case. Thus $\lim_n \phi(T_n)$ exists. Similarly, it follows that if $T_n \xrightarrow{w'} T$ and $S_n \xrightarrow{w'} T$ with $S_n, T_n \in K(X, Y)$ for all $n$, then $\lim_n \phi(T_n) = \lim_n \phi(S_n)$ for all $\phi \in \mathcal{L}^*$. Thus $J(\phi, T)$ in (c) is well defined.

(a) $J$ is evidently bilinear and if $T \in K(X, Y)$ then $J(\phi, T) = \phi(T)$.

(b) To prove (b) let $\phi \in \mathcal{L}^*$ and $T \in \mathcal{L}^{w'}(X, Y)$ be given. Remember that $\phi$ is continuous on $K(X, Y)$ with respect to the uniform operator norm. For any $\epsilon > 0$ there is a sequence $(T_n) \subset K(X, Y)$ such that $\sup_n \|T_n\| < (1 + \epsilon)|||T|||$ and $T_n \xrightarrow{w'} T$. Hence,

$$\begin{aligned} |J(\phi, T)| &= |\lim_n \phi(T_n)| \leq \|\phi\| \sup_n \|T_n\| \\ &\leq (1 + \epsilon)\|\phi\| |||T|||. \end{aligned}$$

Since this holds for all $\epsilon > 0$, it follows that $|J(\phi, T)| \leq \|\phi\| |||T|||$.

□

John's result now follows from this theorem:

**Theorem 1.2.5** (cf John [27]) *Let $X, Y$ be Banach spaces such that for every $T \in L(X, Y)$ there is a sequence $(T_n) \subset K(X, Y)$ such that $T_n \xrightarrow{w'} T$. Then there exists a continuous bilinear form $J : K(X, Y)^* \times L(X, Y) \to \mathbb{K}$ and a number $c > 0$ such that:*

(a) *If $T \in K(X, Y)$ and $\phi \in K(X, Y)^*$, then $J(\phi, T) = \phi(T)$.*

(b) *$|J(\phi, T)| \leq c\|\phi\| \|T\|$ for all $T \in L(X, Y)$ and $\phi \in K(X, Y)^*$.*

(c) *$J(\phi, T) = \lim_n \phi(T_n)$, where $(T_n)$ is any sequence of operators, $T_n \in K(X, Y)$ tending $w'$ to $T$.*

**Proof** In this case $L(X,Y) = \mathcal{L}^{w'}(X,Y)$. Since they are both Banach spaces, the Open Mapping Theorem tells us that the norms $\|\cdot\|$ and $|||\cdot|||$ are equivalent. Hence there is a $c > 0$ such that $\|T\| \leq |||T||| \leq c\|T\|$ for all $T \in L(X,Y)$. □

**Corollary 1.2.6** *Let $X, Y$ be Banach spaces. There is a projection $P : \mathcal{L}^{w'}(X,Y)^* \to \mathcal{L}^{w'}(X,Y)^*$ such that $Ker(P) = K(X,Y)^{\perp} = \{\phi \in \mathcal{L}^{w'}(X,Y)^* : \phi|_{K(X,Y)} = 0\}$ and the range of $P$ is isomorphic to $K(X,Y)^*$.*

**Proof** Let $P : \mathcal{L}^{w'}(X,Y)^* \to \mathcal{L}^{w'}(X,Y)^*$ be defined by $P\phi(T) = \lim_n \phi(T_n) = J(\phi, T)$ for all $\phi \in \mathcal{L}^{w'}(X,Y)^*$ and $T \in \mathcal{L}^{w'}(X,Y)$. Then $P$ is well-defined by Theorem 1.2.4. By the same theorem we have

$$|(P\phi)(T)| = |J(\phi,T)| \leq \|\phi\| |||T|||, \ \forall T \in \mathcal{L}^{w'}(X,Y).$$

Therefore

$$\|P\phi\| \leq \|\phi\|.$$

Hence $P$ is continuous with $\|P\| \leq 1$.

To show $P$ is a projection:

$$\begin{aligned} P^2\phi(T) &= (P(P(\phi)))(T) = J(P\phi, T) \\ &= \lim_n P\phi(T_n) \\ &= \lim_n J(\phi, T_n) = \lim_n \phi(T_n), \ T_n \in K(X,Y) \\ &= P\phi(T), \ \forall\, T \in \mathcal{L}^{w'}(X,Y) \text{ and} \\ &\quad\ \phi \in \mathcal{L}^{w'}(X,Y)^*. \end{aligned}$$

Hence $P^2 = P$, that is $P$ is a projection and the range of $P$ is topologically complemented. If $\phi \in K(X,Y)^{\perp}$ then $P\phi(T) = \lim_n \phi(T_n) = 0, \ \forall\ T \in \mathcal{L}^{w'}(X,Y)$. Thus $P\phi = 0$. Therefore

$$K(X,Y)^{\perp} \subseteq \text{Ker}P.$$

Conversely, if $\phi \in \text{Ker}\,(P)$. Then, for each $T \in K(X,Y)$ we have $0 = P\phi(T) = J(\phi,T) = \phi(T)$ by Theorem 1.2.4(a). Hence $\phi(T) = 0 \,\forall\, T \in K(X,Y)$, i.e. $\phi \in K(X,Y)^{\perp}$. Thus $\text{Ker}\,(P) \subseteq K(X,Y)^{\perp}$. We conclude that $\text{Ker}\,(P) = K(X,Y)^{\perp}$.

FACT ( cf [7], Conway pp 132, Theorem 2.3) $(K(X,Y), |||\cdot|||)^* \simeq \mathcal{L}^{w'}(X,Y)^*/K(X,Y)^{\perp}$.

Hence

$$\begin{aligned} \mathcal{L}^{w'}(X,Y)^* &\approxeq K(X,Y)^{\perp} \oplus \mathcal{L}^{w'}(X,Y)^*/K(X,Y)^{\perp} \\ &\approxeq K(X,Y)^{\perp} \oplus \text{Range of } P \end{aligned}$$

Thus Range of $P \approxeq \mathcal{L}^{w'}(X,Y)^*/K(X,Y)^{\perp} \approxeq K(X,Y)^*$. □

Since the norms $\|\cdot\|$ and $|||\cdot|||$ are equivalent when $L(X,Y) = \mathcal{L}^{w'}(X,Y)$, it follows from Corollary 1.2.6 that

**Corollary 1.2.7** *Let $X, Y$ be Banach spaces such that for each $T \in L(X,Y)$ there is a sequence $(T_n) \subset K(X,Y)$ such that $T_n \xrightarrow{w'} T$. Then there exists a projection*

$$P : L(X,Y)^* \to L(X,Y)^* \text{ such that}$$

$$Ker(P) = K(X,Y)^{\perp} = \{\phi \in L(X,Y)^* : \phi|_{K(X,Y)} = 0\}.$$

*and the range of $P$ is isomorphic to $K(X,Y)^*$.*

The bilinear form $J$ in Theorem 1.2.4 gives rise to two embeddings, which are discussed below.

**Theorem 1.2.8** *(i) $J_K : K(X,Y)^* \to \mathcal{L}^{w'}(X,Y)^*$ is defined by $(J_K\phi)(T) = J(\tilde{\phi}, T)$ where $\tilde{\phi} \in \mathcal{L}^*$ is a continuous linear extension of $\phi$.*

*(ii) $J_{\mathcal{L}} : \mathcal{L}^{w'}(X,Y) \to K(X,Y)^{**}$ is defined by $(J_{\mathcal{L}}T)(\phi) = J(\tilde{\phi}, T)$, where $\tilde{\phi} \in \mathcal{L}^*$ is any continuous linear extension of $\phi$.*

*Then $J_K$ is an isometry into $\mathcal{L}^{w'}(X,Y)^*$. Also $J_{\mathcal{L}}$ satisfies the following*

*a) $J_{\mathcal{L}}$ is a bounded injective linear operator, with $||J_{\mathcal{L}}|| \le 1$.*

*b) $J_{\mathcal{L}} = J_K^*|_{\mathcal{L}^{w'}(X,Y)}$*

*c) $J_{\mathcal{L}}|_{K(X,Y)}$ is the canonical injection of $K(X,Y)$ into $K(X,Y)^{**}$.*

**Proof** Let $T \in K(X,Y)$ and suppose $\tilde{\phi}_1, \tilde{\phi}_2 \in \mathcal{L}^*$ are any two continuous linear extensions of $\phi \in K(X,Y)^*$. Then $J(\tilde{\phi}_1, T) = \phi(T) = J(\tilde{\phi}_2, T)$ by Theorem 1.2.4(a). Thus $J_K$ and $J_{\mathcal{L}}$ are well defined. Furthermore, if we let $||\tilde{\phi}|| = ||\phi||$ (by Hahn-Banach), then

$$\begin{aligned} |(J_K\phi)(T)| &= |J(\tilde{\phi}, T)| \le ||\tilde{\phi}|| |||T||| \\ &= ||\phi|| |||T|||. \end{aligned}$$

Hence $J_K\phi \in \mathcal{L}^{w'}(X,Y)^*$ and $||J_K|| \le 1$.

On the other hand, since $||T|| = |||T|||$ for all $T \in K(X,Y)$ we also have

$$\begin{aligned} ||\phi|| &= \sup\{|J_K\phi(T)| : \quad |||T||| \le 1, T \in K(X,Y)\} \\ &\le \sup\{|J_K\phi(T)| : \quad |||T||| \le 1, T \in \mathcal{L}^{w'}(X,Y)\} \\ &= ||J_K\phi|| \end{aligned}$$

Hence $||J_K\phi|| = ||\phi||$ for all $\phi \in K(X,Y)^*$. This proves that $J_K$ is an isometry.

$J_{\mathcal{L}}$ is linear because of the bilinearity of $J$. It is also bounded with norm $\le 1$ because, let $\phi \in K(X,Y)^*$ and $\tilde{\phi}$ an extension of $\phi$ such that $||\tilde{\phi}|| = ||\phi||$, then

$$|(J_{\mathcal{L}}T)(\phi)| = |J(\tilde{\phi}, T)| \le ||\tilde{\phi}|| |||T||| = ||\phi|| |||T|||.$$

Let $T \in K(X,Y)$. Then we have

$$\langle J_{\mathcal{L}}T, \phi\rangle = J(\tilde{\phi}, T) = \phi(T) = \langle T, \phi\rangle, \ \forall\, \phi \in K(X,Y)^*.$$

Hence $J_{\mathcal{L}}|_{K(X,Y)}$ is the canonical embedding into $K(X,Y)^{**}$. □

**Remark** Let $T \in \mathcal{L}^{w'}(X,Y)$. For $\epsilon > 0$ there exists $x \in X$, $\|x\| \leq 1$ and $a \in Y^*$, $\|a\| \leq 1$ such that

$$\begin{aligned} \|T\| &< |\langle Tx, a\rangle| + \epsilon \\ &= \lim_n |\langle T_n x, a\rangle| + \epsilon \end{aligned}$$

for any sequence $(T_n) \subset K(X,Y)$ such that $T_n \overset{w'}{\to} T$.

Let $\phi(S) = a(Sx)$ for all $S \in K(X,Y)$. Then

$$|\phi(S)| = |a(Sx)| \leq \|a\|\|x\|\|S\| \leq \|S\|.$$

Hence $\phi \in K(X,Y)^*$ and $\|\phi\| \leq 1$. Let $\tilde{\phi}$ be any bounded linear extension of $\phi$ to $\mathcal{L}^{w'}(X,Y)^*$ such that $\|\tilde{\phi}\| \leq 1$. Then

$$\begin{aligned} \|T\| &< \lim_n |\phi(T_n)| + \epsilon \\ &= |\lim_n \phi(T_n)| + \epsilon \\ &= |J(\tilde{\phi}, T)| + \epsilon \\ &= |J_{\mathcal{L}}(T)(\phi)| + \epsilon. \end{aligned}$$

In particular $\|T\| < \|J_{\mathcal{L}}(T)\| + \epsilon$. This holds for all $\epsilon > 0$. Therefore $\|T\| \leq \|J_{\mathcal{L}}(T)\|$ for all $T \in \mathcal{L}^{w'}(X,Y)$. However, we also have $\|J_{\mathcal{L}}(T)\| \leq |||T|||$ for all $T \in \mathcal{L}^{w'}(X,Y)$.

Now, if $X$ and $Y$ are such that $\mathcal{L}^{w'}(X,Y) = L(X,Y)$, then $\|\cdot\|$ and $|||\cdot|||$ are equivalent norms. Hence there is a $c > 0$ such that $\|T\| \leq \|J_{\mathcal{L}}(T)\| \leq c\|T\|$. This shows that in this case the linear operator $J_{\mathcal{L}}$ is also an isomorphism.

**Corollary 1.2.9** (cf K. John [27], pp 237) *Let $X, Y$ be Banach spaces such that for each $T \in L(X,Y)$ there is a sequence $(T_n) \subset K(X,Y)$ such that $T_n \overset{w'}{\to} T$. Then the bilinear form $J$ in theorem 1.2.4 gives rise to the following two isomorphic embeddings:*

*i)* $J_K : K(X,Y)^* \to L(X,Y)^* : (J_K\phi)(T) = J(\tilde{\phi}, T)$

*ii)* $J_L : L(X,Y) \to K(X,Y)^{**} : (J_LT)(\phi) = J(\tilde{\phi}, T)$ *where* $\tilde{\phi} \in L(X,Y)^*$ *is any continuous linear extension of* $\phi \in K(X,Y)^*$.

**Lemma 1.2.10** *Let $X, Y$ be Banach spaces, with $Y$ reflexive. Then $\mathcal{L}^{w'}(X,Y)$ and $\mathcal{L}^{w'}(Y^*, X^*)$ are isometrically identifiable. In particular, $T \in \mathcal{L}^{w'}(X,Y)$ if and only if $T^* \in \mathcal{L}^{w'}(Y^*, X^*)$ and $|||T||| = |||T^*|||$.*

**Proof** Suppose $T \in \mathcal{L}^{w'}(X,Y)$ and $(T_n) \subset K(X,Y)$ such that $T_n \xrightarrow{w'} T$. Then, for $y^* \in Y^{***} = Y^*$ and $x^{**} \in X^{**}$ it is clear that

$$\langle T_n^{**}x^{**}, y^*\rangle = \langle x^{**}, T_n^* y^*\rangle \underset{\infty}{\overset{n}{\rightarrow}} \langle x^{**}, T^*y^*\rangle = \langle T^{**}x^{**}, y^*\rangle.$$

Thus $T_n^* \xrightarrow{w'} T^*$ and by Schauder's Theorem $(T_n^*) \subset K(Y^*, X^*)$. Hence $T^* \in \mathcal{L}^{w'}(Y^*, X^*)$ and $|||T^*||| \leq \sup_n \|T_n^*\| = \sup_n \|T_n\|$. Since the sequence $(T_n)$ was arbitrarily chosen, it is clear that

$$|||T^*||| \leq |||T|||. \tag{A}$$

Now let $S \in \mathcal{L}^{w'}(Y^*, X^*)$ and suppose $S_n \xrightarrow{w'} S$ where $(S_n) \subset K(Y^*, X^*)$. Then each $T_n = S_n^*|_X$ is compact (from $X$ into $Y$) and $T_n^* = S_n$. Similarly if $T = S^*|_X$, then $T^* = S$. And

$$\begin{aligned}\langle T_n^* y^*, x^{**}\rangle &= \langle y^*, T_n^{**}x^{**}\rangle = \langle y^*, S_n^* x^{**}\rangle \rightarrow \langle y^*, S^*x^{**}\rangle \\ &= \langle T^*y^*, x^{**}\rangle \text{ for all } x^{**} \in X^{**} \text{ and } y^* \in Y^*.\end{aligned}$$

Thus $T_n \xrightarrow{w'} T$. Hence $T \in \mathcal{L}^{w'}(X,Y)$ and $T^* = S$. Moreover $|||T||| \leq \sup_n \|T_n\| = \sup_n \|T_n^*\| = \sup_n \|S_n\|$. The sequence $(S_n)$ was arbitrarily chosen. Hence

$$|||T||| \leq |||S||| = |||T^*|||. \tag{B}$$

From (A) and (B) we have $|||T||| = |||T^*|||$.
We have thus established that $T \rightarrow T^*$ defines an isometric isomorphism. □

We are now ready to consider some results (extending existing ones) on reflexivity of $K(X,Y)$.

**Theorem 1.2.11** *Let $X$ and $Y$ be Banach spaces. Consider the following statements:*

*(1) $K(X,Y)$ is reflexive.*

*(2) $\mathcal{L}^{w'}(X,Y)$ is reflexive.*

*(3) $X$ and $Y$ are reflexive and $K(X,Y) = \mathcal{L}^{w'}(X,Y)$.*

*(4) $X$ and $Y$ are reflexive and $K(X,Y)$ is $w$-closed in $L(X,Y)$.*

*We have* (1) $\iff$ (2) $\implies$ (3), (4) $\implies$ (3) *and* (4) $\implies$ (1).

**Proof** (2) $\implies$ (1) is clear because $K(X,Y)$ is a closed subspace of $\mathcal{L}^{w'}(X,Y)$.
(1) $\implies$ (2) In this case $J_{\mathcal{L}} : \mathcal{L}^{w'}(X,Y) \rightarrow K(X,Y)^{**} = K(X,Y)$ (cf. Theorem 1.2.8) is surjective. By the Open Mapping Theorem $J_{\mathcal{L}}$ defines an isomorphism. Hence $\mathcal{L}^{w'}(X,Y)$ is reflexive.

(2) $\Longrightarrow$ (3) If $\mathcal{L}^{w'}(X,Y)$ is reflexive, then $Y$ is reflexive (because it is isomorphic to a closed subspace of $K(X,Y)$). We have to show that $X$ is reflexive. Since $Y$ is reflexive, the spaces $\mathcal{L}^{w'}(X,Y)$, and $\mathcal{L}^{w'}(Y^*,X^*)$ are isometrically identifiable (by Lemma 1.2.10). Thus $\mathcal{L}^{w'}(Y^*,X^*)$ is reflexive. As before this implies that $X^*$ is reflexive. Hence $X$ is reflexive. Also, $J_{\mathcal{L}} = J_K^*$ is surjective and an isometry in this case.

(4) $\Longrightarrow$ (3) This is clear, since $w = w'$ in this case.

(4) $\Longrightarrow$ (1) Suppose $X$ and $Y$ are reflexive and $\overline{K(X,Y)}^w = K(X,Y)$. Recall that $w = w'$ since $X$ is reflexive. Thus by Kalton (cf. Theorem 1.1.3) each $w$-compact subset of $K(X,Y)$ is weakly compact. We prove that $B_{K(X,Y)}$ is $w$-compact, hence also weakly compact: Let $(T_\delta) \subset B_{K(X,Y)}$ be any net. There exists a subnet $(S_\delta)$ of $(T_\delta)$ and $f \in B_{K(X,Y)^{**}}$ such that $S_\delta \to f$ in $wk^*$-topology. For each $x \in X$, $y^* \in Y^*$ consider $y^* \otimes x : K(X,Y) \to \mathbb{K} : S \to \langle Sx, y^* \rangle$, which is a bounded linear functional. We have

$$\begin{aligned} f(y^* \otimes x) &= \lim_\delta \langle S_\delta, y^* \otimes x \rangle \\ &= \lim_\delta \langle S_\delta x, y^* \rangle \quad \forall\, x \in X, y^* \in Y^*. \end{aligned}$$

Let $T : X \to Y^{**} = Y$ be defined by $\langle Tx, y^* \rangle = f(y^* \otimes x)$, $\forall x \in X\ \forall y^* \in Y^*$. Then $T : X \to Y$ is a bounded linear operator with

$$|\langle Tx, y^* \rangle| = |f(y^* \otimes x)| \leq \|f\| \|y^*\| \|x\|.$$

Clearly, $\|T\| \leq \|f\| \leq 1$ and $T = w - \lim_\delta S_\delta$.

Hence, we see that $T \in \overline{K(X,Y)}^w = K(X,Y)$ with $\|T\| \leq 1$. Thus any net in $B_{K(X,Y)}$ has a subnet which converges in $B_{K(X,Y)}$ with the $w$-topology, that is $B_{K(X,Y)}$ is weakly compact. This proves that $K(X,Y)$ is reflexive. □

**Corollary 1.2.12** *If $\mathcal{L}^{w'}(X,Y) = \overline{K(X,Y)}^{w'}$ then the following are equivalent:*

*(1) $K(X,Y)$ is reflexive.*

*(2) $\overline{K(X,Y)}^{w'}$ is reflexive.*

*(3) $X$, $Y$ are reflexive and $K(X,Y) = \overline{K(X,Y)}^{w'}$.*

**Corollary 1.2.13** *If $\mathcal{L}^{w'}(X,Y) = L(X,Y)$. Then the following are equivalent:*

*(1) $K(X,Y)$ is reflexive.*

*(2) $L(X,Y)$ is reflexive.*

*(3) $X$, $Y$ are reflexive and $K(X,Y) = L(X,Y)$.*

**Remark:** Although (3) $\Longrightarrow$ (2) in Theorem 1.1.8 was proved by Kalton (cf. Kalton [29]) and independently by Kheinrich (cf. [27]) without assuming the presence of the approximation property, it was mentioned in Johnson's paper [28] that it is apparently still open whether (2) $\Longrightarrow$ (3), in the same theorem, is true without assuming the presence of the approximation property. Our result now shows that the implication is still true if we have the assumption $L(X,Y) = \mathcal{L}^{w'}(X,Y)$, where it may be that neither $X$ nor $Y$ has the approximation property.

**Lemma 1.2.14** *Suppose $X$ and $Y$ are Banach spaces such that $X^{**}$ and $Y^*$ are separable. Then the closed unit ball $B_{L(X,Y)}$ is $w'$-metrisable.*

**Proof** Let $\{e_n^{**} : n \in \mathbb{N}\}$ and $\{f_m^* : m \in \mathbb{N}\}$ be countable dense subsets of $X^{**}$ and $Y^*$ respectively. We show that the countable family of continuous seminorms

$$T \mapsto |e_n^{**}(T^* f_m^*)|, \forall m, n \in \mathbb{N}$$

generates the $w'$ toplogy on $B_{L(X,Y)}$: Let $(T_\delta)$ be a net in $B_{L(X,Y)}$ such that $|e_n^{**}(T_\delta^* f_m^*)| \xrightarrow{\delta} 0$ for each pair $(e_n^{**}, f_m^*)$. Consider arbitrary $e^{**} \in X^{**}$ and $f^* \in Y^*$. For $\epsilon > 0$ given, let $e_{n_0}^{**} \in X^{**}$ such that $\|e^{**} - e_{n_0}^{**}\| < \epsilon/(3\|f^*\|)$. Then let $f_{m_0}^* \in Y^*$ such that $\|f^* - f_{m_0}^*\| < \epsilon/(3\|e_{n_0}^{**}\|)$. Let $\delta_0$ be an index such that $|e_{n_0}^{**}(T_\delta^* f_{m_0}^*)| < \epsilon/3$ for all $\delta \geq \delta_0$. Then we have

$$\begin{aligned}
|e^{**}(T_\delta^* f^*)| &\leq |e^{**}(T_\delta^* f^*) - e_{n_0}^{**}(T_\delta^* f^*)| + |e_{n_0}^{**}(T_\delta^* f^*) - e_{n_0}^{**}(T_\delta^* f_{m_0}^*)| + |e_{n_0}^{**}(T_\delta^* f_{m_0}^*)| \\
&\leq \|e^{**} - e_{n_0}^{**}\|(\sup_\delta \|T_\delta^* f^*\|) + \|e_{n_0}^{**}\|\|T_\delta^* f^* - T_\delta^* f_{m_0}^*\| + |e_{n_0}^{**}(T_\delta^* f_{m_0}^*)| \\
&< \epsilon, \ \forall \delta \geq \delta_0.
\end{aligned}$$

Having a countable neighbourhood basis of the origin, it follows that the Hausdorff topology $w'$ on $B_{L(X,Y)}$ is metrisable (cf. [39], Theorem 4, pp 16).

□

In the setting of separable Banach spaces we have the following generalisation of Theorem 1.1.8:

**Corollary 1.2.15** *Let $X$ and $Y$ be separable Banach spaces. The following are equivalent:*

*(1) $K(X,Y)$ is reflexive.*

*(2) $\mathcal{L}^{w'}(X,Y)$ is reflexive.*

*(3) $X, Y$ are reflexive and $K(X,Y) = \mathcal{L}^{w'}(X,Y)$.*

**Proof** We only have to check that (3) $\Longrightarrow$ (1) holds. Refer to the proof of (4) $\Longrightarrow$ (1) in (1.2.11). Following the same arguments it follows for a given net $(T_\delta) \subset B_{K(X,Y)}$ that a subnet $(S_\gamma) \subset B_{K(X,Y)}$ converges to a $T \in \overline{K(X,Y)}^{w'}$ with $\|T\| \leq 1$. Thus $T$ is in the closure of $B_{K(X,Y)}$ in the metrisable space $(B_{L(X,Y)}, w')$ (refer to Lemma 1.2.14). Hence there is a sequence $(T_n) \subset B_{K(X,Y)}$ such that $T_n \xrightarrow{w'} T$. Thus $T \in \mathcal{L}^{w'}(X,Y) = K(X,Y)$. Therefore, $B_{K(X,Y)}$ is $w'$-compact and hence weakly compact. □

## 1.3 Extensions to locally convex spaces

Let $E$ and $F$ denote Hausdorff locally convex spaces. The algebraic dual space of a linear space $E$ is denoted by $E'$. $E^*_\beta$ will denote the continuous dual space $E^*$ of $E$ when it carries the strong topology $\beta(E^*, E)$. However, if not otherwise specified, the topology on $E^*$ will throughout the section be assumed to be the strong topology. The bidual is defined by $E^{**} = (E^*_\beta)^*$. The space $L(E,F)$ of continuous linear operators is a Hausdorff locally convex space when it is endowed with the topology of uniform convergence in the bounded sets in $E$, in which case it is denoted by $L_b(E,F)$. The topology is generated by zero neighbourhoods of the form

$$[A, V] := \{T \in L(E,F) : T(A) \subseteq V\}$$

where $V$ and $A$ run through a zero neighbourhood basis $\mathcal{U}(F)$ of closed absolutely convex sets in $F$ and a fundamental system $\mathcal{B}(E)$ of absolutely convex closed bounded sets in $E$ respectively. The gauge function of $[A, V]$ is denoted by $P_{A,V}$.

**Definition 1.3.1** *An operator $T \in L(E,F)$ is said to be quasi compact if $T(A)$ is precompact in $F$ for each bounded set $A$ in $E$. The vector space of quasi compact operators in $L(E,F)$ is denoted by $K(E,F)$. We use the notation $K_b(E,F)$ when it carries the subspace topology of uniform convergence on the bounded sets in $E$.*

The existence of a (continuous with respect to the strong topology) projection on $L_b(E,F)^*$ with kernel the annihilator $K(E,F)^\perp$ and its applications to topological decomposition of the bidual space $K_b(E,F)^{**}$ and results on semi-reflexivity of $L_b(E,F)$ are considered in the papers [12] and [19]. The results in the paper [19] are generalisations of those in [12] and [28]. Especially in [19] the following generalisations of Johnson's result are proved.

**Theorem 1.3.2** (cf. [19], Theorem 2.2) There exists a (continuous with respect to the strong topology) projection $P : L_b(E,F)^* \to L_b(E,F)^*$ with kernel $K(E,F)^\perp$ if either

(a) $F$ has the quasi compact approximation property, that is there exists an equicontinuous net $(T_\delta) \subset K_b(F,F)$ such that $T_\delta x \to x$ in $F$ for each $x \in F$, or

(b) $E$ has the shrinking quasi compact approximation property, that is there exists an equicontinuous net $(T_\delta) \subset K_b(E,E)$ such that $T_\delta x \to x$ for each $x \in E$ and $T^*_\delta x^* \to x^*$ strongly for each $x^* \in E^*$.

We recall two important results from the theory of locally convex spaces:

**Theorem 1.3.3** ([23], pp 191) *If $E$ is a Fréchet space, then (relative) $\sigma(E, E^*)$-compactness and (relative) sequential $\sigma(E, E^*)$-compactness are equivalent.*

**Theorem 1.3.4** ([23], pp 191) *Let $K$ be a compact topological space. A bounded subset $M$ of the Banach space $C(K)$ is weakly compact if and only if it is compact for the topology of pointwise convergence.*

Motivated by John's paper [27], Fourie (in [15]) studied projections on dual spaces of spaces of operators in the locally convex space setting. Before we take a closer look at the results obtained in [15], we consider a locally convex version of Kalton's theorem (Theorem 1.1.4).

Define $w'$ on $L(E, F)$ as in the Banach space case. Hence, a net $(T_\delta)$ converges $w'$ to $T$ in $L(E, F)$ if

$$x^{**}(T_\delta^* y^*) \to x^{**}(T^* y^*)$$

for all $x^{**} \in E^{**}$ and $y^* \in F^*$.

For the sake of completeness we present a complete proof of the following generalisation of Kalton's result to the locally convex setting:

**Theorem 1.3.5** ([15], Proposition 1) *Suppose that $(T_n)$ is a bounded sequence in $K_b(E, F)$ which converges in the dual weak operator topology to $T \in K(E, F)$. Then $\phi(T_n) \to \phi(T)$ for all $\phi \in L_b(E, F)^*$.*

**Proof** Let $A \in \mathcal{B}(E)$ and $V \in \mathcal{U}(F)$ such that $|\phi(S)| \leq 1$ for all $S \in [A, V]$. Denote the bipolar set of $A$ in $E^{**}$ by $A^{\circ\circ}$ and the polar set of $V$ in $F^*$ by $V^\circ$. Consider $A^{\circ\circ} \times V^\circ$, which is compact with respect to the product topology defined by $\sigma(E^{**}, E^*)$ and $\sigma(F^*, F)$. For each $S \in K(E, F)$ define on $A^{\circ\circ} \times V^\circ$ a mapping $\psi(S)$ by $\psi(S)(x^{**}, y^*) = x^{**}(S^* y^*)$.

We show that $\psi(S) \in C(A^{\circ\circ} \times V^\circ)$ : Take $(x_0^{**}, y_0^*) \in A^{\circ\circ} \times V^\circ$. For $\epsilon > 0$ we let $W_1$ be a neighbourhood of $x_0^{**}$ such that $|x^{**}(S^* y_0^*) - x_0^{**}(S^* y_0^*)| < \epsilon/3$ for all $x^{**} \in W_1$ (by the continuity of $S^* y_0^*$ with respect to the $\sigma(E^{**}, E^*)$-topology). Since $S(A)$ is precompact, there is a finite set $\mathcal{F} \subset A$ such that $S(A) \subseteq S(\mathcal{F}) + (\epsilon/6)V$. Now let $W_2 = (y_0^* + (\epsilon/3)S(\mathcal{F})^\circ) \cap V^\circ$. Then $W_2$ is a neighbourhood of $y_0^*$ in $V^\circ$ and $|y^*(Sx_0) - y_0^*(Sx_0)| < \epsilon/3$ for all $y^* \in W_2$ and for all $x_0 \in \mathcal{F}$. ((We note that $y^* \in y_0^* + (\epsilon/3)S(\mathcal{F})^\circ$, $y^* - y_0^* \in (\epsilon/3)S(\mathcal{F})^\circ$, $|\langle y^* - y_0^*, Sx_0\rangle| < \epsilon/3$)). So, for $x \in A$ there are $x_0 \in \mathcal{F}$, $v \in V$ such that $|(S^* y^* - S^* y_0^*)(x)| = |(y^* - y_0^*)(Sx_0 + (\epsilon/6)v)|$. That is

$$\begin{aligned}
|(S^* y^* - S^* y_0^*)(x)| &= |\langle y^* - y_0^*, Sx\rangle| \\
&= |\langle y^* - y_0^*, Sx_0 + (\epsilon/6)v\rangle| \\
&= |(y^* - y_0^*)(Sx_0 + (\epsilon/6)v)| \\
&\leq |y^*(Sx_0) - y_0^*(Sx_0)| + |(y^* - y_0^*)((\epsilon/6)v)| \\
&\leq \epsilon/3 + (\epsilon/6)|y^*(v)| + (\epsilon/6)|y_0^*(v)| \\
&\leq \epsilon/3 + \epsilon/3 \\
&= 2\epsilon/3 \quad \forall\, y^* \in W_2.
\end{aligned}$$

Hence $S^*y^* - S^*y_0^* \in (2\epsilon/3)A^\circ$, $\forall\, y^* \in W_2$. By the triangle inequality

$$\begin{aligned}|\psi(S)(x^{**},y^*) - \psi(S)(x_0^{**},y_0^*)| &= |x^{**}(S^*y^*) - x_0^{**}(S^*y_0^*)| \\ &\leq |x^{**}(S^*y^*) - x^{**}(S^*y_0^*)| + |x^{**}(S^*y_0^*) - x_0^{**}(S^*y_0^*)| \\ &< 2\epsilon/3 + \epsilon/3 = \epsilon \qquad \forall\, x^{**} \in W_1 \cap A^{\circ\circ}, \forall y^* \in W_2.\end{aligned}$$

Hence $|\psi(S)(x^{**},y^*) - \psi(S)(x_0^{**},y_0^*)| < \epsilon$, $\forall\, (x^{**},y^*) \in (W_1 \cap A^{\circ\circ}) \times W_2 := W$, where $W$ is a neighbourhood of $(x_0^{**},y_0^*)$ in $A^{\circ\circ} \times V^\circ$. Therefore $\psi(S)$ is continuous at $(x_0^{**},y_0^*)$.

Thus we have that $\psi(S) \in C(A^{\circ\circ} \times V^\circ)$ for all $S \in K(E,F)$. Since $(T_n)$ is bounded, there is $\lambda_0 > 0$ such that $A \subset \lambda_0(\cap_n T_n^{-1}(V))$. Hence for $y^* \in V^\circ$ and $x \in A$ we have

$$|\langle \frac{1}{\lambda_0}x, T_n^*y^*\rangle| = |\langle \frac{1}{\lambda_0}T_n x, y^*\rangle| \leq 1.$$

Thus $T_n^*y^* \in \left(\frac{1}{\lambda_0}A\right)^\circ = \lambda_0 A^\circ$, $\forall\, n \in \mathbb{N}$, $\forall\, y^* \in V^\circ$.

On $C(A^{\circ\circ} \times V^\circ)$ we have the norm defined by $\|f\|_\infty = \max\limits_{\substack{x^{**}\in A^{\circ\circ}\\ y^*\in V^\circ}} |f(x^{**},y^*)|$.

Hence we see that

$$\begin{aligned}\|\psi(T_n)\|_\infty &= \max_{\substack{x^{**}\in A^{\circ\circ}\\ y^*\in V^\circ}} |\psi(T_n)(x^{**},y^*)| \\ &= \max_{\substack{x^{**}\in A^{\circ\circ}\\ y^*\in V^\circ}} |x^{**}(T_n^*y^*)| \leq \lambda_0, \ \forall n\end{aligned}$$

Thus $(\psi(T_n))$ is norm bounded in $C(A^{\circ\circ} \times V^\circ)$. Hence $\{\psi(T_n) : n \in \mathbb{N}\} \cup \{\psi(T)\}$ is norm bounded. It is however, also pointwise compact in $C(A^{\circ\circ} \times V^\circ)$, because

$$\psi(T_n)(x^{**},y^*) = x^{**}(T_n^*y^*) \xrightarrow[\infty]{n} x^{**}(T^*y^*) = \psi(T)(x^{**},y^*).$$

By the Grothendieck result (Theorem 1.3.4), every pointwise compact norm bounded set in $C(A^{\circ\circ} \times V^\circ)$ is weakly compact. Hence $\{\psi(T_n) : n \in \mathbb{N}\} \cup \{\psi(T)\}$ is weakly compact in $C(A^{\circ\circ} \times V^\circ)$.

Define $\psi : K_b(E,F) \to C(A^{\circ\circ} \times V^\circ)$ by $\psi : S \to \psi(S)$.

We show that $\psi$ is continuous:

$$\begin{aligned}P_{A,V}(S) &= \inf\{\lambda > 0 : S \in \lambda[A,V]\} \\ &= \inf\{\lambda > 0 : S(A) \subseteq \lambda V\}\end{aligned}$$

and

$$S(A) \subseteq \lambda V \iff S(x) \in \lambda V, \forall x \in A \iff Sx \in \lambda V^{\circ\circ}, \forall x \in A.$$

Thus

$$\begin{aligned}
P_{A,V}(S) \leq 1 &\iff S \in [A,V] \iff S(A) \subseteq V = V^{\circ\circ} \\
&\iff |\langle Sx, y^* \rangle| \leq 1 \ \forall\, x \in A,\ \forall\, y^* \in V^{\circ} \\
&\iff |\langle x, S^* y^* \rangle| \leq 1,\ \forall\, x \in A, \forall y^* \in V^{\circ} \\
&\iff S^* y^* \in A^{\circ}, \forall\, y^* \in V^{\circ} \\
&\iff |\langle S^* y^*, x^{**} \rangle| \leq 1,\ \forall\, y^* \in V^{\circ},\ \forall\, x^{**} \in A^{\circ\circ} \\
&\iff \max_{\substack{y^* \in V^{\circ} \\ x^{**} \in A^{\circ\circ}}} |\langle S^* y^*, x^{**} \rangle| \leq 1 \\
&\iff \max_{\substack{y^* \in V^{\circ} \\ x^{**} \in A^{\circ\circ}}} |\psi(S)(x^{**}, y^*)| \leq 1 \\
&\iff \|\psi(S)\|_{\infty} \leq 1.
\end{aligned}$$

Thus it follows that

$$P_{A,V}(S) = \|\psi(S)\|,\ \forall\, S \in K(E,F).$$

Hence, if $S_\delta \to S$ in $K_b(E,F)$ then,

$$\|\psi(S) - \psi(S_\delta)\| = \|\psi(S - S_\delta)\| = P_{A,V}(S - S_\delta) \xrightarrow{\delta} 0.$$

Thus the map $\psi$ is continuous. Also,

$$|\phi(S)| \leq \sup_{\theta \in [A,V]^{\circ}} |\theta(S)| = P_{A,V}(S) = \|\psi(S)\|,\ \forall\, S \in K(E,F).$$

Hence Ker $\psi \subseteq$ Ker $\phi$ and $\psi(K(E,F)) \subseteq C(A^{\circ\circ} \times V^{\circ})$ is a linear space. Let $\tilde{\theta} : \psi(K(E,F)) \to \mathbb{K}$ be defined by $\tilde{\theta}(\psi(S)) = \phi(S)$. Suppose $\psi(S_1) = \psi(S_2)$. Then $S_1 - S_2 \in$ Ker $\psi \subseteq$ Ker $\phi$. Hence $\phi(S_1 - S_2) = 0$ or equivalently $\phi(S_1) = \phi(S_2)$. Therefore $\tilde{\theta} : \psi(K(E,F)) \to \mathbb{K}$ is well defined. Also $\tilde{\theta}$ is continuous. This is because $|\tilde{\theta}(\psi(S))| = |\phi(S)| \leq \|\psi(S)\|$. By the Hahn-Banach Theorem there is a continuous linear extension $\theta \in C(A^{\circ\circ} \times V^{\circ})^*$ such that $\theta|_{\psi(K(E,F))} = \tilde{\theta}$ and $\|\theta\| = \|\tilde{\theta}\|$. Clearly $\theta \circ \psi = \tilde{\theta} \circ \psi = \phi$. Since we have that $\{\psi(T_n) : n \in \mathbb{N}\} \cup \{\psi(T)\}$ is weakly compact, it follows that there is a subsequence $(T_{n_k})$ such that $\theta(\psi(T_{n_k}))$ converges to $\theta(\psi(T))$. Suppose $\theta(\psi(T_n)) \nrightarrow \theta(\psi(T))$. Then we can construct a subsequence such that $|\theta(\psi(T_{n_k})) - \theta(\psi(T))| \geq \epsilon > 0$ for some $\epsilon > 0$ and all $k \in \mathbb{N}$. Now $\{\psi(T_{n_k}) | k \in \mathbb{N}\}$ is relatively weakly compact. Thus there is a subsequence $\{\psi(T_{n_{kl}})\}$ which converges weakly. This weak limit should be $\psi(T)$, because we know that $\psi(T_{n_{kl}}) \to \psi(T)$ pointwise and the pointwise convergence topology is Hausdorff. Hence $\theta(\psi(T_{n_{k\ell}}) \to \theta(\psi(T))$ if $l \to \infty$. This is a contradiction!

Thus we have established that $\theta(\psi(T_n)) \to \theta(\psi(T))$. □

This generalisation of Kalton's result will later on make it possible for us to follow our alternative approach in section 1.2 in order to extend our results in the previous section to the locally convex space setting.

Once we have the above "generalised version" of Theorem 1.0.4, it is a matter of mimicking John's argument in [27] (also refer to the proof of Theorem 1.2.4) to show that if a bounded sequence $(T_n) \subset K(E,F)$ converges to $T \in L(E,F)$ with respect to the dual weak operator topology, then $\lim_n \phi(T_n)$ exists for each $\phi \in L_b(E,F)^*$. The idea is to take subsequences $(\phi(T_{n_k}))$ and $(\phi(T_{mk}))$ such that

$$\begin{aligned} \alpha_0 &= \limsup_n \phi(T_n) = \lim_k \phi(T_{n_k}) \\ \beta_0 &= \liminf_n \phi(T_n) = \lim_k \phi(T_{m_k}) \end{aligned}$$

and then to notice that since the sequence $(T_{n_k} - T_{m_k})$ is bounded and converges to 0 in the dual weak operator topology, we have from Theorem 1.3.5 that

$$\alpha_0 - \beta_0 = \lim_k \phi(T_{n_k} - T_{m_k}) = 0.$$

A similar argument shows that the limit $\lim_n \phi(T_n)$ is independent of the choice of the bounded sequence $(T_n)$. Thus if each $T \in L(E,F)$ is the $w'$-limit of a bounded sequence $(T_n)$ in $K_b(E,F)$, then the operator

$$\begin{aligned} P : L_b(E,F)^* &\to L_b(E,F)' \\ \phi &\to P\phi : P\phi(T) = \lim_n \phi(T_n) \end{aligned}$$

into the algebraic dual space $L_b(E,F)'$ is well defined.

We have to decide whether $P$ maps into the continuous dual space $L_b(E,F)^*$. Using the Riesz representation theorem we show that if each $T \in L_b(E,F)$ is the $w'$-limit of a bounded sequence $(T_n) \subset K_b(E,F)$, then this is true.

**Proposition 1.3.6** ([15], Proposition 2) *Suppose that each $T \in L_b(E,F)$ is the $w'$-limit of a bounded sequence $(T_n) \subset K_b(E,F)$. Then the linear operator*

$$P : L_b(E,F)^* \to L_b(E,F)' : P\phi(T) = \lim_n \phi(T_n)$$

*maps into $L_b(E,F)^*$ and is a closed graph projection with kernel $K_b(E,F)^{\perp}$. In particular, $K_b(E,F)^*$ and $K_b(E,F)^{\perp}$ are isomorphic to closed (in the strong topology) subspaces of $L_b(E,F)^*$.*

**Proof** Let $\phi \in L_b(E,F)^*$. Suppose $\phi$ is bounded on $[A,V]$ with $|\phi(T)| \leq 1$ for all $T \in [A,V]$. We refer to the proof of Proposition 1.3.5. Let $\theta \in C(A^{\circ\circ} \times V^{\circ})^*$ be such that $\phi = \theta \circ \psi$. By the Royden version of the Riesz representation theorem there exist a positive Borel measure $\mu$ and a measurable function $h$, $|h| = 1$ a.e., such that $\|\mu\| = \|\theta\|$ and

$$\phi(S) = \theta(\psi(S)) = \int_{A^{\circ\circ} \times V^{\circ}} h(x^{**}, y^*)\psi(S)(x^{**}, y^*)d\mu\ , \quad \forall\ S \in K_b(E,F).$$

Let $f(x^{**}, y^*) = x^{**}(T^*y^*)$, $\forall\ (x^{**}, y^*) \in A^{\circ\circ} \times V^{\circ}$. Then $f$ is well defined and bounded. Also,

$$(\psi(T_n))(x^{**}, y^*) = x^{**}(T_n^* y^*) \xrightarrow[\infty]{n} x^{**}(T^*y^*) = f(x^{**}, y^*).$$

Hence $\psi(T_n) \xrightarrow[\infty]{n} f$ pointwise. Moreover, $(\psi(T_n))$ is norm bounded in $C(A^{\circ\circ} \times V^\circ)$. By the Lebesgue Dominated Convergence Theorem, $f$ is integrable and

$$\int_{A^{\circ\circ}\times V^\circ} f(x^{**}, y^*)d\mu = \lim_n \int_{A^{\circ\circ}\times V^\circ} \psi(T_n)d\mu.$$

Thus

$$\begin{aligned}
|P\phi(T)| &= |\lim_n \phi(T_n)| \\
&= \left|\lim_n \int_{A^{\circ\circ}\times V^\circ} h(x^{**}, y^*)\psi(T_n)(x^{**}, y^*)d\mu\right| \\
&= \left|\int_{A^{\circ\circ}\times V^\circ} h(x^{**}, y^*)f(x^{**}, y^*)d\mu\right| \\
&\le \int_{A^{\circ\circ}\times V^\circ} |f(x^{**}, y^*)|d\mu = \int_{A^{\circ\circ}\times V^\circ} |x^{**}(T^*y^*)|d\mu \\
&\le \mu(A^{\circ\circ} \times V^\circ) = \|\mu\|, \quad \forall\, T \in [A, V].
\end{aligned}$$

Hence $P\phi \in L_b(E, F)^*$. And

$$\begin{aligned}
(P^2\phi)(T) &= P(P\phi(T)) = \lim_n (P\phi)(T_n) \\
&= \lim_n \phi(T_n) = (P\phi)(T).
\end{aligned}$$

We show $P$ has closed graph: Let $(\phi, \theta) \in \overline{G(P)} \subseteq L_b(E, F)^* \times L_b(E, F)^*$ with the product topology defined by the strong topology. Hence $\phi_\delta \to \phi$ and $P\phi_\delta \to \theta$ in the strong topology of $L_b(E, F)^*$. Take any $T \in L_b(E, F)$. Let $\epsilon > 0$ be given. We show that $|P\phi(T) - \theta(T)| < \epsilon$. Let $\delta_0$ be an index such that $|P\phi_\delta(T) - \theta(T)| < \epsilon/2,\ \forall\, \delta \ge \delta_0$. Then let $\delta_1 \ge \delta_0$ such that

$$|\phi(T_n) - \phi_\delta(T_n)| < \epsilon/2, \ \forall\, \delta \ge \delta_1 \text{ and } \forall\, n.$$

((The set $\{T_n | n \in \mathbb{N}\}$ is bounded in $L_b(E, F)$, so that $\phi_\delta \to \phi$ uniformly on $\{T_n\}$.)) Hence

$$|\phi(T_n) - \theta(T)| \le |\phi(T_n) - \phi_{\delta_1}(T_n)| + |\phi_{\delta_1}(T_n) - \theta(T)| < \epsilon/2 + |\phi_{\delta_1}(T_n) - \theta(T)|.$$

And

$$\begin{aligned}
|P\phi(T) - \theta(T)| &= \lim_n |\phi(T_n) - \theta(T)| \\
&\le \epsilon/2 + \lim_n |\phi_{\delta_1}(T_n) - \theta(T)| \\
&\le \epsilon/2 + |P\phi_{\delta_1}(T) - \theta(T)| \\
&< \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

This holds for all $\epsilon > 0$. Thus

$$P\phi(T) = \theta(T) \quad \forall\, T \in L_b(E, F),$$

showing that $P\phi = \theta$. □

**Corollary 1.3.7** *Let $X$ and $Y$ be normed spaces. If for each $T \in L(X,Y)$ there exists a bounded sequence $(T_n) \subset K(X,Y)$ such that $T_n \to T$ in the dual weak operator topology, then the annihilator $K(X,Y)^\perp$ in $L(X,Y)^*$ is the kernel of a projection on $L(X,Y)^*$. In particular if $X$ is $\aleph_0$-barrelled, then the hypothesis that the sequence $(T_n)$ should be norm bounded can be ignored (cf [23], pp 252).*

Moreover, the different versions of the closed graph theorem for locally convex spaces and Theorem 1.3.5 provide more general versions of Proposition 1.3.6 such that the projection mapping $P$ is indeed continuous. In particular we have

**Corollary 1.3.8** *If for instance $L(E,F)$ is a df-space, then $L(E,F)^*$ with the strong topology is a Fréchet space (cf. [23], p 257) so that with the hypotheses of Proposition 1.3.6, the mapping $P$ is continuous.*

We intend to show that by following our "alternative approach", as was outlined in the previous section in the Banach space setting, one can easily obtain and extend the above result (Proposition 1.3.6), which is also proved in Fourie's paper [15]. In the above proof of Proposition 1.3.6 one has to use the Riesz representation theorem and the Lebesgue Dominated Convergence Theorem in order to show that the projection $P$ maps into $L_b(E,F)^*$. Following is an alternative approach which results in both simplification and extention of Proposition 1.3.6 and its proof.

Denote by $\mathcal{L}^{w'}(E,F)$ the vector space of all $T \in L(E,F)$ such that $T_n \stackrel{w'}{\to} T$ for some bounded sequence $(T_n) \subset K_b(E,F)$. On $\mathcal{L}^{w'}(E,F)$ we define a locally convex topology by means of the seminorms

$$\pi_{A,V}(T) = \inf\{\sup_n P_{A,V}(T_n) : T_n \in K(E,F), T_n \stackrel{w'}{\to} T\},$$

where $A \in \mathcal{B}(E)$ and $V \in \mathcal{U}(F)$. Put

$$W_{A,V} = \{T \in \mathcal{L}^{w'}(E,F) : \pi_{A,V}(T) \leq 1\}$$

and let

$$q_{A,V}(\phi) = \sup\{|\phi(S)| : S \in W_{A,V}\}$$

for every (continuous) linear functional $\phi$ on $\mathcal{L}^{w'}(E,F)$. Clearly $\phi$ is bounded on $W_{A,V}$ if and only if $q_{A,V}(\phi) < \infty$.

**Theorem 1.3.9** *Fix the locally convex spaces $E$ and $F$. Let $\mathcal{L}^{w'} = \mathcal{L}^{w'}(E,F), K = K(E,F)$ and $\mathcal{L}^* = \mathcal{L}^{w'}(E,F)^*$. There exists a bilinear form $J : \mathcal{L}^* \times \mathcal{L}^{w'} \to \mathbb{K}$ such that*

*(a) $J(\phi,T) = \phi(T)$ for all $(\phi,T) \in \mathcal{L}^* \times K$.*

*(b) For all $T \in \mathcal{L}^{w'}$, $\phi \in \mathcal{L}^*$ and for all $A \in \mathcal{B}(E)$, $V \in \mathcal{U}(F)$, we have*

$$|J(\phi,T)| \leq q_{A,V}(\phi)\pi_{A,V}(T).$$

*(c) $J(\phi, T) = \lim_n \phi(T_n)$, where $(T_n)$ is any bounded sequence of operators $T_n \in K_b(E,F)$ tending to $T$ in $w'$-topology.*

**Proof** (c) The proof is similar to the proof of (c) in Theorem 1.2.4, this time using that each $T \in \mathcal{L}^{w'}$ is the $w'$-limit of a bounded sequence $(T_n)$ in $K_b(E,F)$ and the fact that the same sequence converges weakly to $T$ (as is proved in Theorem 1.3.5).

(b) Let $\phi \in \mathcal{L}^*$ and $T \in \mathcal{L}^{w'}(E,F)$ be given. Suppose $\phi$ is bounded on $W_{A,V}$. For any $\epsilon > 0$ there is a bounded sequence $(T_n) \subset K_b(E,F)$ such that $\sup_n P_{A,V}(T_n) < (1+\epsilon)\pi_{A,V}(T)$ and $T_n \xrightarrow{w'} T$. Hence,

$$\begin{aligned} |J(\phi,T)| &= |\lim_n \phi(T_n)| \leq q_{A,V}(\phi) \sup_n P_{A,V}(T_n) \\ &\leq (1+\epsilon) q_{A,V}(\phi)\pi_{A,V}(T). \end{aligned}$$

Since this holds for all $\epsilon > 0$ it follows that $|J(\phi,T)| \leq q_{A,V}(\phi)\pi_{A,V}(T)$. □

We are now ready to discuss an alternative result for Proposition 1.3.6, which has easier proof and which is formulated in more general context.

**Proposition 1.3.10** *Let $E$ and $F$ be locally convex spaces. The linear operator*

$$P : \mathcal{L}^{w'}(E,F)^* \to \mathcal{L}^{w'}(E,F)' : P\phi(T) = \lim_n \phi(T_n),$$

*where $T \in \mathcal{L}^{w'}(E,F)$ is the $w'$-limit of the bounded sequence $(T_n) \subset K_b(E,F)$, maps into $\mathcal{L}^{w'}(E,F)^*$ and is a closed graph projection with kernel $K_b(E,F)^\perp$. In particular, $K_b(E,F)^*$ and $K_b(E,F)^\perp$ are isomorphic to closed (in the strong topology) subspaces of $\mathcal{L}^{w'}(E,F)^*$.*

**Proof** Let $\phi \in \mathcal{L}^{w'}(E,F)^*$. Suppose $\phi$ is bounded on $W_{A,V}$ with $|\phi(S)| \leq 1$ for all $S \in W_{A,V}$. From the proof of Proposition 1.3.9 (b) it follows that

$$|J(\phi,T)| \leq \pi_{A,V}(T)$$

for all $T \in \mathcal{L}^{w'}(E,F)$. This shows (cf. also (1.3.9)(c)) that $P\phi$ is continous, hence that $P$ maps into $\mathcal{L}^{w'}(E,F)^*$. Clearly, $P^2 = P$.

The argument to show that $P$ has closed graph is similar to that used in the proof of Proposition 1.3.6. For the sake of completeness, we discuss the proof of this fact: Let $(\phi,\theta) \in \overline{G(P)} \subseteq \mathcal{L}^{w'}(E,F)^* \times \mathcal{L}^{w'}(E,F)^*$ with the product topology defined by the strong topology. Hence $\phi_\delta \to \phi$ and $P\phi_\delta \to \theta$ in the strong topology of $\mathcal{L}^{w'}(E,F)^*$. Take any $T \in \mathcal{L}^{w'}(E,F)$. Let $\epsilon > 0$ be given. We show that $|P\phi(T) - \theta(T)| < \epsilon$. Let $\delta_0$ be an index such that $|P\phi_\delta(T) - \theta(T)| < \epsilon/2$, $\forall\, \delta \geq \delta_0$. Then let $\delta_1 \geq \delta_0$ such that

$$|\phi(T_n) - \phi_\delta(T_n)| < \epsilon/2, \ \forall\, \delta \geq \delta_1 \text{ and } \forall\, n.$$

((The set $\{T_n | n \in \mathbb{N}\}$ is bounded in $K_b(E,F)$, hence also in $\mathcal{L}^{w'}(E,F)$. This is so because for all $A \in \mathcal{B}(E)$ and $V \in \mathcal{U}(F)$ it is clear that $[A,V] \cap K_b(E,F) \subseteq W_{A,V} \cap K_b(E,F)$. We have $\phi_\delta \to \phi$ uniformly on $\{T_n\}$.)) Hence

$$|\phi(T_n) - \theta(T)| \leq |\phi(T_n) - \phi_{\delta_1}(T_n)| + |\phi_{\delta_1}(T_n) - \theta(T)| < \epsilon/2 + |\phi_{\delta_1}(T_n) - \theta(T)|.$$

And

$$\begin{aligned}
|P\phi(T) - \theta(T)| &= \lim_n |\phi(T_n) - \theta(T)| \\
&\leq \epsilon/2 + \lim_n |\phi_{\delta_1}(T_n) - \theta(T)| \\
&\leq \epsilon/2 + |P\phi_{\delta_1}(T) - \theta(T)| \\
&< \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

This holds for all $\epsilon > 0$. Thus

$$P\phi(T) = \theta(T) \quad \forall\, T \in \mathcal{L}^{w'}(E, F),$$

showing that $P\phi = \theta$. □

# Chapter 2

# Operator dual space and applications

## 2.1 Introducing the Operator dual space and other notation

Characterising the continuous dual space of a normed subspace $(\mathcal{A}(X,Y),\mu)$ of the Banach space $(L(X,Y),\|\cdot\|)$, is often an important exercise. Mostly, if a characterisation of $(\mathcal{A}(X,Y),\mu)^*$ is possible, then this dual space turns out to be a space of bounded linear operators from $Y$ into $X^{**}$. Such a characterisation normally depends on the presence of the approximation property on either $X^*$ or $Y$. In this chapter we intend to study the so called *operator dual space* of a space of bounded linear operators and to discuss the operator dual spaces of some classical spaces of bounded operators. It is also our intention to show the relationship between the continuous dual space and the operator dual space.

From Chapter 0 we recall that $\mathcal{F}$ denotes the operator ideal of finite rank bounded linear operators on Banach spaces and that we use $\operatorname{tr}(S)$ to denote the trace $(=\sum_{i=1}^{n}\langle x_i, a_i\rangle)$ of $S := \sum_{i=1}^{n} a_i \otimes x_i \in \mathcal{F}(X)$. The concept *conjugate ideal* $\mathcal{A}^{\triangle}$ of a (complete) quasi-normed ideal $(\mathcal{A},\alpha)$ has already been studied and significantly applied in the literature (cf. for instance the papers [33], [22] and [24]). Recall that $T \in \mathcal{A}^{\triangle}(Y,X)$ if there is a $\rho > 0$ such that

$$|\operatorname{tr}(LT)| \leq \rho\alpha(L)$$

for any $L \in \mathcal{F}(X,Y)$. The ideal $\mathcal{A}^{\triangle}$ is normed by the (complete) ideal norm $\alpha^{\triangle}$ which is defined by

$$\alpha^{\triangle}(T) := \inf\{\rho > 0 : |\operatorname{tr}(LT)| \leq \rho\alpha(L),\ \forall L \in \mathcal{F}(X,Y)\}.$$

Also recall that an operator $T \in L(Y,X)$ belongs to the *adjoint ideal* $(\mathcal{A}^*,\alpha^*)$ if there exists $\rho > 0$ such that for all finite dimensional Banach spaces $X_0$, $Y_0$ and for all $V \in$

$L(Y_0, Y)$, $U \in \mathcal{A}(X_0, Y_0)$ and $W \in L(X, X_0)$ we have

$$|\operatorname{tr}(WTVU)| \leq \rho \|W\| \|V\| \alpha(U).$$

It follows from a result of Pietsch ([33], lemma 3) that

$$(\mathcal{A}^{\triangle}(Y, X), \alpha^{\triangle}) = (\mathcal{A}^{*}(Y, X), \alpha^{*})$$

if both $X$ and $Y$ have the metric approximation property.

In this chapter we consider a generalisation of the concept "conjugate ideal". The background is the following: Suppose $\mathcal{A}(X, Y)$ is a fixed component of a quasi-normed operator ideal $(\mathcal{A}, \mu)$ on the family of all Banach spaces. We denote by $\mathcal{A}_{\mu}^{\triangle}(Y, X)$ (where $\mathcal{A}^{\triangle}$ is the conjugate ideal) the *operator dual space* of $(\mathcal{A}(X, Y), \mu)$; hence $T \in \mathcal{A}_{\mu}^{\triangle}(Y, X)$ if and only if the mapping

$$\mathcal{F}(X, Y) \to \mathbb{K} : S \mapsto \operatorname{tr}(TS)$$

is a $\mu$-continuous linear functional. If no confusion can arise, we write $\mathcal{A}^{\triangle}(Y, X)$ for the operator dual space.

Moreover, if $\mu$ is a linear topology on a vector space $\mathcal{A}(X, Y)$ of bounded linear operators which contains $\mathcal{F}(X, Y)$, then for each $\mu$- neighbourhood $U$ of the origin we let

$$U_{\mathcal{L}} := \{T \in L(Y, X) : |\operatorname{tr}(TS)| \leq 1, \forall S \in U \cap \mathcal{F}(X, Y)\}$$

and then define the vector space

$$\mathcal{A}_{\mu}^{\triangle}(Y, X) := \cup\{U_{\mathcal{L}} : U \in \mathcal{U}\},$$

where $\mathcal{U}$ is a zero neighbourhood basis for the linear topology $\mu$. It is clear that $T \in \mathcal{A}_{\mu}^{\triangle}(Y, X)$ if and only if the mapping $\mathcal{F}(X, Y) \to \mathbb{K} : S \mapsto \operatorname{tr}(TS)$ is a $\mu$-continuous linear functional. Thus we define the *operator dual space* for general topological vector spaces of bounded linear operators between Banach spaces, when they contain the bounded linear operators of finite rank.

**Remark.** If $\mathcal{G}$ and $\mathcal{A}$ are complete metrizable operator ideals such that $\mathcal{G} \subseteq \mathcal{A}$, then the embedding $\mathcal{G}(X, Y) \hookrightarrow \mathcal{A}(X, Y)$ is continuous with respect to the corresponding ideal topologies. It is thus easily verified that $\mathcal{G}^{\triangle}(Y, X) \supseteq \mathcal{A}^{\triangle}(Y, X)$. By the closed graph theorem, if $\mathcal{G}(Y, X)$ is closed in $\mathcal{A}(Y, X)$, then $\mathcal{G}^{\triangle}(Y, X) = \mathcal{A}^{\triangle}(Y, X)$.

We agree that in general, if either $\mathcal{A}$ is a topological ideal of operators or else, if a linear topology on a subspace $\mathcal{A}(X, Y)$ of $L(X, Y)$ is given, then $\mathcal{A}^{\triangle}(Y, X)$ will denote the operator dual space of $\mathcal{A}(X, Y)$ with respect to the ideal topology or else, with respect to the given linear topology.

The operator dual space may be regarded as the "operator version" of the so called *functional dual* of a sequence space, which is especially considered and applied in the context of $FK$-spaces (cf. [40] for more information). We somehow demonstrate this statement later in the chapter when some inclusion theorems for $FH$-spaces of bounded linear operators are considered, where in this case $H$ is the space $L(H_1, H_2)$ of bounded linear operators on Hilbert spaces which is endowed with the uniform operator norm topology. Recall the definition of an $FH$ space:

**Definition 2.1.1** *Assume given a fixed vector space $H$ which has a (not necessarily vector) Hausdorff topology. An $FH$ space is a vector subspace $X$ of $H$ which is a Fréchet space (hence a complete metrizable locally convex space) and is continuously embedded in $H$, that is, the topology of $X$ is larger than the relative topology of $H$.*

The closed graph theorem is again the reason why one may conclude that the inclusion of one $FH$ space into another is always continuous, the topological space $H$ being fixed of course. In particular, the topology of an $FH$ space is unique, so that there is at most one way to make a vector subspace of $H$ into an $FH$ space.

If $\mathcal{F}(X,Y)$ is dense in $(\mathcal{A}(X,Y),\mu)$, then with each $T \in \mathcal{A}^{\triangle}(Y,X)$ we associate a continuous linear functional $\tilde{\phi}_T$ on $\mathcal{A}(X,Y)$ as follows:

First let

$$\phi_T(S) := \operatorname{tr}(TS)$$

for all $S \in \mathcal{F}(X,Y)$; the linear functional $\phi_T$ is continuous on $\mathcal{F}(X,Y)$ with respect to the induced $\mu$-topology. Then let $\tilde{\phi}_T$ be its unique continuous linear extension to $\mathcal{A}(X,Y)$. The mapping $T \mapsto \tilde{\phi}_T$ defines a linear isomorphism from $\mathcal{A}^{\triangle}(Y,X)$ onto a subspace of $\mathcal{A}(X,Y)^*$. Thus we have

**Theorem 2.1.2** *Suppose $\mathcal{F}(X,Y)$ is $\mu$-dense in $\mathcal{A}(X,Y)$. Then $\mathcal{A}^{\triangle}(Y,X)$ is linear isomorphic to a subspace of $\mathcal{A}(X,Y)^*$. In case of $\mathcal{A}$ being a normed operator ideal, the embedding is an isometry.*

If $X$ is norm one complemented in $X^{**}$ with norm one projection $P: X^{**} \to X$, then for each $\phi \in \mathcal{A}(X,Y)^*$ let $R_\phi : Y \to X^{**}$ be the linear operator defined by

$$\langle R_\phi(y), a\rangle = \phi(a \otimes y), \quad \forall a \in X^*.$$

Put $T_\phi = P \circ R_\phi$. $P^*$ being an injection, it follows that $\phi(S) = \operatorname{tr}(T_\phi S)$ for all $S \in \mathcal{F}(X,Y)$. Thus $T_\phi \in \mathcal{A}^{\triangle}(Y,X)$ (and $\mu^{\triangle}(T_\phi) \leq \|\phi\|$ in case of $\mu$ being an ideal quasi-norm). If moreover, in this case $\mathcal{F}(X,Y)$ is also dense in $\mathcal{A}(X,Y)$, then the linear isomorphism $T \mapsto \tilde{\phi}_T$ in the proof of the previous theorem is surjective. Thus we have the following

**Theorem 2.1.3** *Suppose $\mathcal{F}(X,Y)$ is dense in $\mathcal{A}(X,Y)$. If $X$ is norm one complemented in $X^{**}$, then $\mathcal{A}^{\triangle}(Y,X)$ is linearly isomorphic to $\mathcal{A}(X,Y)^*$. The linear isomorphism is an isometry in case of $\mathcal{A}$ being a normed operator ideal.*

Omitting that $\mathcal{F}(X,Y)$ is dense in $\mathcal{A}(X,Y)$ in the previous two results, the same arguments show that if $X$ is reflexive, then

$$\mathcal{A}^{\triangle}(Y,X) = \{S \in \mathcal{L}(Y,X) : \exists \phi \in \mathcal{A}(X,Y)^*, \langle Sy, a\rangle = \phi(a \otimes y), \forall a \in X^*, \forall y \in Y\}.$$

The mapping $\mathcal{A}(X,Y)^* \to \mathcal{A}^{\triangle}(Y,X) : \phi \mapsto R_\phi$ is surjective in this case; if moreover the $\mu$-topology is locally convex, then $\phi \to R_\phi$ defines an isomorphism if and only if $\mathcal{F}(X,Y)$ is dense in $\mathcal{A}(X,Y)$.

**Proposition 2.1.4** *Let $(\mathcal{A}(X,Y),\mu)$ be either a component of a quasi-normed operator ideal or a metrizable topological vector space of bounded linear operators which contains $\mathcal{F}(X,Y)$. We have the following inclusions:*

*(a) If $X$ is reflexive, then $\overline{\mathcal{F}(X,Y)}^{\mu} \subset \mathcal{A}^{\triangle\triangle}(X,Y) = (\mathcal{A}^{\triangle})^{\triangle}_{\pi(\beta)}(X,Y)$, where $(\mathcal{A}^{\triangle}(Y,X),\pi(\beta))$ denotes the quotient space of $\mathcal{A}(X,Y)^*$ (with respect to the mapping $\phi \mapsto R_\phi$) with the strong topology;*

*(b) if $\mathcal{F}(X,Y)$ is dense in $\mathcal{A}(X,Y)$, then (2.1.2) applies and $\mathcal{A}(X,Y) \subset \mathcal{A}^{\triangle\triangle}(X,Y)$, where the strong topology of $\mathcal{A}(X,Y)^*$ is restricted to $\mathcal{A}^{\triangle}(Y,X)$.*

**Proof** We prove (b) and omit the (similar) proof of (a): Choose an arbitrary $T \in \mathcal{A}(X,Y)$ and let $\lim_n T_n = T$ (with respect to $\mu$), for some sequence $(T_n) \subset \mathcal{F}(X,Y)$. Consider any net $\{S_\delta : \delta \in \mathcal{I}\}$ in $\mathcal{F}(Y,X)$ which converges with respect to the induced $\beta$-topology (strong topology) of $\mathcal{A}(X,Y)^*$ to $S \in \mathcal{F}(Y,X)$. Let $\epsilon > 0$ be given. Since the polar set $B^\circ$ of the bounded set $B := \{T_n : n \in \mathbb{N}\}$ is a zero-neighbourhood in the $\beta$-topology, there exists an index $\delta_0$ such that

$$|\operatorname{tr}(T_n S_\delta) - \operatorname{tr}(T_n S)| \leq \frac{\epsilon}{3} \text{ for all } n = 1,2,\ldots \text{ and all } \delta \geq \delta_0.$$

Each $S_\delta$ and also $S$ are in $\mathcal{A}^{\triangle}(Y,X)$, so that the mappings $R \mapsto \operatorname{tr}(RS_\delta)$ and $R \mapsto \operatorname{tr}(RS)$ are continuous on $\mathcal{F}(X,Y)$ with respect to the $\mu$-topology. Fix any $\delta \geq \delta_0$. There exists $n_0 = n_0(\delta, S) \in \mathbb{N}$ such that

$$|\operatorname{tr}(T_n S_\delta) - \operatorname{tr}(T S_\delta)| < \frac{\epsilon}{3} \quad \text{and} \quad |\operatorname{tr}(T_n S) - \operatorname{tr}(TS)| < \frac{\epsilon}{3}$$

for all $n \geq n_0$. Hence from the triangle inequality we have $|\operatorname{tr}(TS_\delta) - \operatorname{tr}(TS)| < \epsilon$. Since this is true for all $\delta \geq \delta_0$, it follows that the mapping

$$\mathcal{F}(Y,X) \to \mathbb{K} : R \mapsto \operatorname{tr}(TR)$$

is continuous with respect to the induced $\beta$-topolgy. Hence $T \in \mathcal{A}^{\triangle\triangle}(X,Y)$. □

That $\mathcal{F}(X,Y)$ being dense in $\mathcal{A}(X,Y)$, is not a necessary condition for the inclusion $\mathcal{A}(X,Y) \subseteq \mathcal{A}^{\triangle\triangle}(X,Y)$. This is illustrated by the following example:

Let $(\mathcal{N}_1, \nu_1)$ and $(K, \|\cdot\|)$ be the Banach ideals of nuclear and compact operators, respectively. The components $(\mathcal{N}_1(Y,X), \nu_1)$ and $(K(X,Y), \|.\|)$ are Banach spaces in this case. If, for instance, $X$ and $Y$ are Hilbert spaces, then it is well known that $\mathcal{N}_1(Y,X) = \mathcal{K}(X,Y)^*$ (cf. [23], 20.2.6 and 20.2.5). From (2.2) and the remark, we have $\mathcal{N}_1(Y,X) = L^{\triangle}(Y,X)$. Hence

$$L^{\triangle\triangle}(X,Y) = \mathcal{N}_1^{\triangle}(X,Y) = \mathcal{N}_1(Y,X)' = L(X,Y).$$

## 2.2 Operator dual spaces of some important classes of operators

The discussion in [22] regarding conjugate ideals, concentrates on Banach ideals $(\mathcal{A}, \alpha)$ of operators on Banach spaces. It is clear from the same paper and others in literature that the conjugate ideal has important applications; for instance, although some of the ideas of Gordon, Lewis and Retherford which are used in [22] go back to the theory of tensor products as developed by Schatten and Grothendieck, their theory of conjugate ideals allows the authors to prove many results without the hypothesis of the (metric) approximation property. Unfortunately some characterisations in [22] of the components of conjugate duals of several classical operator ideals still rely on the metric approximation property on the underlying Banach spaces. This is because the continuity of the trace functional with respect to the nuclear norm $\nu_1$ plays important role in establishing the characterisations.

In recent papers (cf. for instance [25], and [27]) there was a new interest in proving results on spaces of operators between Banach spaces and their duality, from an infinite dimensional point of view. This was of course also the case in our discussion in Chapter 1 about the projections on dual spaces of spaces of operators. The effect of this is that some known results of Grothendieck, J. Johnson and others in which the (metric) approximation property on the underlying Banach spaces is critical, are generalised to spaces of operators on Banach spaces without the approximation property. In some instances weaker kinds of approximations are needed – for instance, by considering the space $\mathcal{L}^{w'}(X,Y)$ in stead of $L(X,Y)$ in Chapter 1, we actually established the situation where the weaker kind of approximation in the $w'$-topology is present. The existence of various examples of Banach spaces without the metric approximation property, in particular the counterexamples by Pisier to a conjecture of Grothendieck, motivates the study in these references.

In this section we recall some classical examples of conjugate ideals and prove two propositions ((2.2.1) and (2.2.3) below) in which we remove the metric approximation property from two results in [22].

**Example 1.** Let $B_{\mathcal{L}}$ denote the closed unit ball in $L(X,Y)$. $T \in L(Y,X)$ is integral if and only if there exists a $\rho > 0$ such that

$$|\operatorname{tr}(TS)| \leq \rho \|S\|, \quad \forall S \in \mathcal{F}(X,Y).$$

Hence,

$$\mathcal{I}_1(Y,X) = L^{\triangle}(Y,X) = K^{\triangle}(Y,X), \text{ isometrically.}$$

If $X$ is reflexive, then $\mathcal{I}_1(Y,X)$ (in other words, $L^{\triangle}(Y,X)$) and the space $\mathcal{N}_1(Y,X)$ of nuclear operators are isometrically isomorphic (cf. [23], 17.4.5; 17.6.4; 17.6.5). If moreover, $X$ has the approximation property then it follows from (2.1.3) that $\mathcal{N}_1(Y,X) = \mathcal{K}(X,Y)^*$, which is a well known result of Persson-Pietsch and Grothendieck (cf. [23], p.449). It is also well known that $\mathcal{I}_1(Y,X) = \mathcal{N}_1(Y,X)$ if any one of the following properties holds:

(a) $X$ is seperable and representable as the dual of some Banach space ([23], 17.6.6);

(b) $X$ has the Radon-Nikodym property and is complemented in $X^{**}$ by a norm one projection ([10], Cor. 10, p.235 and Th. 8, p.175);

(c) $Y^*$ has the Radon-Nikodym property and the approximation property ([10], Th. 6, p. 248).

Hence in each case $K^{\triangle}(Y,X) = \mathcal{N}_1(Y,X)$ holds.

**Example 2.** Let $(\mathcal{A}, \alpha)$ be a (quasi-) Banach ideal of operators which admits a continuous trace $\tau$. In this case, since $(\mathcal{F}(X,Y),\alpha) \to (\mathcal{F}(X),\alpha) : S \mapsto TS$ is continuous with $\alpha(TS) \leq \|T\|\alpha(S)$ for each $T \in L(Y,X)$, it follows that

$$(\mathcal{F}(X,Y),\alpha) \to \mathbb{K} : S \mapsto \tau(TS) = \mathrm{tr}(TS)$$

is continuous. Hence $\mathcal{A}^{\triangle}(Y,X) = L(Y,X)$. In fact it is clear from the definitions that a quasi-normed ideal $\mathcal{A}$ is traceable if and only if $\mathcal{A}^{\triangle} = L$. The ideal $\mathcal{S}_1$ of *1-approximable operators* is for instance traceable (cf. [23], p.442). Therefore $\mathcal{S}_1^{\triangle}(Y,X) = L(Y,X)$. Hence, if $X$ is reflexive (or norm one complemented in $X^{**}$), then $\mathcal{S}_1(X,Y)^* = L(Y,X)$.

Let $X$ have the approximation property. Since in this case the linear functional $\mathcal{F}(X) \to \mathbb{K} : R \mapsto \mathrm{tr}(R)$ is continuous with respect to the nuclear norm on $\mathcal{F}(X)$ (cf. [23], p.406), it follows that $\mathcal{N}_1^{\triangle}(Y,X) = L(Y,X)$. Furthermore, if $X$ is also reflexive (or norm one complemented in $X^{**}$), then $L(Y,X) = \mathcal{N}_1(X,Y)^*$.

In [24] (p.20) it is mentioned that a Banach space $X$ has

(i) the approximation property if and only if $\mathcal{N}_1^{\triangle}(X) = L(X)$;

(ii) the bounded approximation property if and only if $\mathcal{I}_1^{\triangle}(X) = L(X)$;

(iii) the metric approximation property if and only if $\mathcal{I}_1^{\triangle}(X)$ and $L(X)$ are isometrically isomorphic.

For a finite set $\{x_1, \dots x_N\}$ in a Banach space $X$ and for a finite set $\{a_1, \dots, a_N\}$ in the dual space $X^*$ of a Banach space $X$ (or for denumerable sets in $X$ and $X^*$, respectively) and for $1 \leq p < \infty$ the following quantities are well known (and important!):

(i) $\epsilon_p((x_i)) := \sup\{(\sum_i |\langle x_i, a\rangle|^p)^{\frac{1}{p}} : a \in X', \|a\| \leq 1\}$;

(ii) $\epsilon_p((a_i)) := \sup\{(\sum_i |\langle x, a_i\rangle|^p)^{\frac{1}{p}} : x \in X, \|x\| \leq 1\}$;

(iii) $\pi_p((x_i)) := (\sum_i \|x_i\|^p)^{\frac{1}{p}}$;

(iv) $\kappa_p((x_i)) := \sup\{|\sum_i \langle x_i, a_i\rangle| : a_i \in X', \epsilon_q((a_i)) \leq 1\}$, $\frac{1}{p} + \frac{1}{q} = 1$.

**Example 3.** Let $(\mathcal{N}_p(X,Y), \nu_p)$ and $(\mathcal{P}_q(Y,X), \pi_q)$ denote the Banach spaces of $p$-nuclear and $q$-absolutely summing operators on the underlying Banach spaces, respectively. In [22] it is proved (cf. [22], Theorem 2.5(b)) that $\mathcal{N}_p^{\triangle}(Y,X) = \mathcal{P}_q(Y,X)$ isometrically (for $1 < p,q < \infty$, with $\frac{1}{p} + \frac{1}{q} = 1$) if either $X$ or $Y$ has the metric approximation property, using both the continuity of the trace functional on $\mathcal{F}(X)$ with respect to the nuclear norm and the equality of the nuclear norm and the integral norm in this case.

We discuss the same example without the restriction (metric approximation property) on the underlying spaces.

**Proposition 2.2.1** *Let $X$ and $Y$ be Banach spaces. The normed spaces $\mathcal{N}_p^{\triangle}(Y,X)$ and $\mathcal{P}_q(Y,X)$ are isometrically isomorphic for $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.*

**Proof** Let $T \in \mathcal{P}_q(Y,X)$. For $S \in \mathcal{F}(X,Y)$ and each representation $S = \sum_{i=1}^k a_i \otimes y_i$ we have

$$|\operatorname{tr}(TS)| \leq \sum_{i=1}^k \|a_i\|\|Ty_i\| \leq \|(\|a_i\|)(\leq k)\|_p \|(\|Ty_i\|)(\leq k)\|_q.$$

Hence

$$|\operatorname{tr}(TS)| \leq \inf\{\pi_p((a_i)(\leq k))\epsilon_q((y_i)(\leq k)) : S = \sum_{i=1}^k a_i \otimes y_i\}\pi_q(T) = \nu_p(S)\pi_q(T).$$

Thus $T \in \mathcal{N}_p^{\triangle}(Y,X)$. It is also clear that $\nu_p^{\triangle}(T) \leq \pi_q(T)$.
Conversely, let $T \in \mathcal{N}_p^{\triangle}(Y,X)$; then $\phi_T(S) = \operatorname{tr}(TS)$ defines a $\nu_p$-continuous linear functional on $\mathcal{F}(X,Y)$. Fix

$$(y_i) \in \ell_w^q(Y) := \{(y_i) \in Y^{\mathbb{N}} : (\langle y_i, a\rangle) \in \ell^q, \forall a \in Y'\}.$$

For each $n \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$ there exists $a_i \in X'$ with $\|a_i\| = 1$ and $\langle Ty_i, a_i\rangle = \|Ty_i\|$. Put $\lambda_i := \|Ty_i\|^{q-1}$ for $1 \leq i \leq n$ and let $S := \sum_{i=1}^n \lambda_i a_i \otimes y_i$. Then we have

$$\sum_{i=1}^n \lambda_i \|Ty_i\| = \operatorname{tr}(TS) \leq \|\phi_T\|\nu_p(S) \leq \|\phi_T\|(\sum_{i=1}^n |\lambda_i|^p)^{\frac{1}{p}}\epsilon_q((y_i)).$$

Hence

$$(\sum_{i=1}^n \|Ty_i\|^q)^{\frac{1}{q}} \leq \|\phi_T\|\epsilon_q((y_i))$$

for all $n \in \mathbb{N}$; i.e. $T \in \mathcal{P}_q(Y,X)$ and $\pi_q(T) \leq \|\phi_T\| = \nu_p^{\triangle}(T)$. □

If $X$ is norm one complemented in $X^{**}$, then it follows from (2.1.3) that

$$\mathcal{N}_p(X,Y)^* = \mathcal{P}_q(Y,X)$$

isometrically. Here the approximation property on $X^*$ is not needed in the proof as is the case in the characterisation $\mathcal{N}_p(X,Y)^* = \mathcal{P}_q(Y,X^{**})$ in Theorem 6 of ([23], p. 448).

**Example 4** Let $\mathcal{K}_p$ denote the operator ideal of $p$-compact operators (with $1 \leq p < \infty$), i.e.

$$T \in \mathcal{K}_p(X,Y) \iff T = Q \circ P \text{ with } P \in K(X,\ell^p), Q \in K(\ell^p,Y).$$

This operator ideal is extensively studied in [16] and [17]. It is normed by the ideal norm

$$c_p(T) := \inf \|Q\|\|P\|$$

where the infimum is taken over all such factorizations. Let $(\mathcal{J}_q, j_q)$ be the ideal of Cohen $q$-nuclear operators, i.e.

$$T \in \mathcal{J}_q(X,Y) \iff \exists \rho > 0 \text{ such that } \kappa_q((Tx_i)) \leq \rho \epsilon_q((x_i))$$

for all finite sets $\{x_1, \ldots, x_N\}$ in $X$. Here $j_q(T) := \inf \rho$. In [22] it is proved (cf. [22], theorem 2.5(d)) that $\mathcal{K}_p^{\triangle}(Y,X) = \mathcal{J}_q(Y,X)$ isometrically (for $1 < p, q < \infty$, with $\frac{1}{p} + \frac{1}{q} = 1$) if either $X$ or $Y$ has the metric approximation property, again using among other things the continuity of the trace functional on $\mathcal{F}(X)$ with respect to the nuclear norm.

We discuss the same example without the restriction (metric approximation property) on the underlying spaces. In doing so we make use of the characterisation of $p$-compact operators in ([16], theorem 2.5).

Let $\ell_w^p(X^*) := \{(a_i) \in (X^*)^{\mathbb{N}} : (\langle x, a_i \rangle) \in \ell^p, \forall x \in X\}$. It is proved in [16] (also, see Theorem 0.1.1) that $T \in \mathcal{K}_p(X,Y) \iff T = \sum_{i=1}^{\infty} a_i \otimes y_i$, where

$$(a_i) \in \ell_c^p(X^*) := \{(a_i) \in \ell_w^p(X^*) : \epsilon_p((a_i)(\geq k)) \to 0 \text{ as } k \to \infty\}$$

and

$$(y_i) \in \ell_c^q(Y) := \{(y_i) \in \ell_w^q(Y) : \epsilon_q((y_i)(\geq k)) \to 0 \text{ as } k \to \infty\}.$$

In this case we have

$$c_p(T) = \inf \epsilon_p((a_i))\epsilon_q((y_i)),$$

where the infimum is taken over all representations of $T$.

**Proposition 2.2.2** *Let $X$ and $Y$ be Banach spaces. Then $\mathcal{K}_p^{\triangle}(Y,X)$ and $\mathcal{J}_q(Y,X)$ are isometrically isomorphic (for $1 < p, q < \infty$, and with $\frac{1}{p} + \frac{1}{q} = 1$).*

**Proof** Let $T \in \mathcal{J}_q(Y,X)$. For $S \in \mathcal{F}(X,Y)$ and each representation of $S$ in the form $S = \sum_{i=1}^{k} a_i \otimes y_i$ we have $|\operatorname{tr}(TS)| \leq \epsilon_p((a_i)(\leq k))\epsilon_q((y_i)(\leq k)) j_q(T)$. Hence

$$|\operatorname{tr}(TS)| \leq \inf\{\epsilon_p((a_i)(\leq k))\epsilon_q((y_i)(\leq k)) : S = \sum_{i=1}^{k} a_i \otimes y_i\} j_q(T) = c_p(S) j_q(T).$$

Thus, $T \in \mathcal{K}_p^{\triangle}(Y,X)$. It is also clear that $c_p^{\triangle}(T) \leq j_q(T)$.

Conversely, let $T \in \mathcal{K}_p^{\triangle}(Y,X)$; then $\phi_T(S) = \operatorname{tr}(TS)$ defines a $c_p$-continuous linear functional on $\mathcal{F}(X,Y)$ with $\|\phi_T\| = c_p^{\triangle}(T)$. Fix $(a_i) \in \ell_w^p(X^*)$. For any finite set $\{y_1, y_2, \ldots, y_N\} \subset Y$, let $S = \sum_{i=1}^{N} a_i \otimes y_i$. We have

$$|\sum_{i=1}^{N} \langle Ty_i, a_i \rangle| = |\operatorname{tr}(TS)| \leq c_p^{\triangle}(T)\epsilon_p((a_i)(\leq N))\epsilon_q((y_i)(\leq N)).$$

Hence, $\kappa_q((Ty_i)(\leq N)) \leq c_p^{\triangle}(T)\epsilon_q((y_i)(\leq N))$. This shows that $T \in \mathcal{J}_q(Y,X)$ and also that $j_q(T) \leq c_p^{\triangle}(T)$. □

## 2.3 Quotients of operator ideals.

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be operator ideals on the family of all Banach spaces. An operator $S \in L(X,Y)$ belongs to the *lefthand quotient* $\mathcal{A}_1^{-1} \circ \mathcal{A}_2$ if for all Banach spaces $Z$ and for all $R \in \mathcal{A}_1(Y,Z)$ we have $RS \in \mathcal{A}_2(X,Z)$. Similarly, an operator $S \in L(X,Y)$ belongs to the *righthand quotient* $\mathcal{A}_2 \circ \mathcal{A}_1^{-1}$ if for all Banach spaces $Z$ and for all $R \in \mathcal{A}_1(Z,X)$ we have $SR \in \mathcal{A}_2(Z,Y)$. Both the left- and righthand quotients are operator ideals.

If $(\mathcal{A}_1, \alpha_1)$ and $(\mathcal{A}_2, \alpha_2)$ are quasi-normed operator ideals, then the left- and righthand quotients are also quasi-normed ideals with respect to the quasi-norms

$$(\alpha_1^{-1} \circ \alpha_2)(S) := \sup\{\alpha_2(RS) : \alpha_1(R) \le 1\}$$

and

$$(\alpha_2 \circ \alpha_1^{-1})(S) := \sup\{\alpha_2(SR) : \alpha_1(R) \le 1\},$$

respectively.

Let $X$ be a fixed Banach space. Throughout the section $(\mathcal{A}_0, \alpha)$ is a complete quasi-normed operator ideal. We define linear topologies on an ideal $\mathcal{A}(X)$ of bounded linear operators (which contains $\mathcal{F}(X)$) as follows:

(a) The *left weak* $(\mathcal{A}, \mathcal{A}^{-1} \circ \mathcal{A}_0)$*- topology* (which is denoted by $\sigma_l$) has a subbase for the neighbourhoods of 0 consisting of the sets $U_T := \{S \in \mathcal{A}(X) : \alpha(ST) \le 1\}$, where $T$ runs through $(\mathcal{A}^{-1} \circ \mathcal{A}_0)(X)$.

(b) The *right weak* $(\mathcal{A}, \mathcal{A}_0 \circ \mathcal{A}^{-1})$*- topology* (which is denoted by $\sigma_r$) has a subbase for the neighbourhoods of 0 consisting of the sets $W_T := \{S \in \mathcal{A}(X) : \alpha(TS) \le 1\}$, where $T$ runs through $(\mathcal{A}_0 \circ \mathcal{A}^{-1})(X)$.

If $(\mathcal{A}_0, \alpha)$ is an operator ideal which admits a continuous trace (hence, in this case $\operatorname{tr} : \mathcal{F}(X) \to \mathbb{K}$ is continuous with respect to the induced $\alpha$-topology) then convergence of a net in $\mathcal{A}(X)$ with respect to the $\sigma_l$-topology (respectively, $\sigma_r$-topology) implies convergence to the same limit with respect to the weak operator topology. For instance, if $S_\delta \to S$ in $(\mathcal{A}(X), \sigma_l)$ and $x \in X$, $a \in X^*$ are given, then for $\epsilon > 0$ there is an index $\delta_0$ such that $\alpha((S_\delta - S) \circ (a \otimes x)) \le \epsilon$ for all $\delta \ge \delta_0$; hence it follows that

$$|a(Sx) - a(S_\delta x)| = |\operatorname{tr}((a \otimes x) \circ S - (a \otimes x) \circ S_\delta)| = |\operatorname{tr}(S \circ (a \otimes x) - S_\delta \circ (a \otimes x))| \to 0.$$

Also, if $\operatorname{tr} : \mathcal{F}(X) \to \mathbb{K}$ is continuous with respect to the $\alpha$-topology, then for each $T \in (\mathcal{A}^{-1} \circ \mathcal{A}_0)(X)$ the linear functional $S \mapsto \operatorname{tr}(ST)$ is bounded on $\{S \in \mathcal{F}(X) : \alpha(ST) \le 1\}$, hence $T \in \mathcal{A}^{\triangle}_{\sigma_l}(X)$; thus showing that

**Proposition 2.3.1** *If* $(\mathcal{A}_0(X), \alpha)$ *is given such that* $\operatorname{tr} : (\mathcal{F}(X), \alpha) \to \mathbb{K}$ *is continuous, then*

$$(\mathcal{A}^{-1} \circ \mathcal{A}_0)(X) \subseteq \mathcal{A}^{\triangle}_{\sigma_l}(X).$$

*Similarly, it follows in this case that*

$$(\mathcal{A}_0 \circ \mathcal{A}^{-1})(X) \subseteq \mathcal{A}^{\triangle}_{\sigma_r}(X).$$

It is proved in [24] that if $\mathcal{A}$ is an injective (resp. surjective) operator ideal, then $\mathcal{A}^{\triangle} = \mathcal{A}^{-1} \circ \mathcal{I}_1$ (resp. $\mathcal{A}^{\triangle} = \mathcal{I}_1 \circ \mathcal{A}^{-1}$). Sometimes the inclusions in the last proposition are equalities, as is for instance demonstrated in

**Proposition 2.3.2** *Let $(\mathcal{A}_0, \alpha)$ be a complete quasi-normed operator ideal and $(\mathcal{A}, \mu)$ a quasi-normed operator ideal. Suppose $X$ is a Banach space such that the following conditions are satisfied:*

*(a)* $\mathcal{F}(X)$ *is dense in* $(\mathcal{A}(X), \mu)$.

*(b)* *The mapping* $\mathrm{tr} : \mathcal{F}(X) \to \mathbb{K}$ *is* $\alpha$*-continuous;*

*(c)* *There exists* $k > 0$ *such that*

$$\alpha(S) \leq k \sup\{|\,\mathrm{tr}(QS)| : Q \in \mathcal{F}(X), \|Q\| \leq 1\},$$

*for all* $S \in \mathcal{F}(X)$.

*Then*

$$(\mathcal{A}^{-1} \circ \mathcal{A}_0)(X) = \mathcal{A}^{\triangle}_{\sigma_l}(X) \text{ and } (\mathcal{A}_0 \circ \mathcal{A}^{-1})(X) = \mathcal{A}^{\triangle}_{\sigma_r}(X).$$

*If $X$ is reflexive, then*

$$(\mathcal{A}(X), \sigma_l)^* = (\mathcal{A}^{-1} \circ \mathcal{A}_0)(X) \text{ and } (\mathcal{A}(X), \sigma_r)^* = (\mathcal{A}_0 \circ \mathcal{A}^{-1})(X).$$

**Proof** $(\mathcal{A}^{-1} \circ \mathcal{A}_0)(X) \subseteq \mathcal{A}^{\triangle}_{\sigma_l}(X)$ follows from (2.3.1). Conversely, let $T_0 \in \mathcal{A}^{\triangle}_{\sigma_l}(X)$; then $T_0 \in (U_R)_{\mathcal{L}}$ (see §2, the definition of the operator dual space) for some $R \in (\mathcal{A}^{-1} \circ \mathcal{A}_0)(X)$. We show that $\mathcal{A}(X)$ is contained in the $\sigma_l$- closure of $\mathcal{F}(X)$. Let $\beta := \mu^{-1} \circ \alpha$ be the quasi-norm on the operator ideal $\mathcal{A}^{-1} \circ \mathcal{A}_0$. Consider arbitrary $S \in \mathcal{A}(X)$. For any $0 \neq T \in (\mathcal{A}^{-1} \circ \mathcal{A}_0)(X)$ we have

$$0 \neq \beta(T) := \sup\{\alpha(PT) : P \in \mathcal{A}(X), \mu(P) \leq 1\} < \infty.$$

If $Q \in \mathcal{F}(X)$ such that $\mu(S - Q) < \frac{1}{\beta(T)}$ (thus using condition (a)), then

$$\alpha(ST - QT) \leq \mu(S - Q)\beta(T) < 1.$$

Thus it follows that $\overline{\mathcal{F}(X)}^{\sigma_l} = \mathcal{A}(X)$. Hence for $S \in \mathcal{A}(X)$ there is a net $(S_\delta) \subset \mathcal{F}(X)$ which converges to $S$ in $(\mathcal{A}(X), \sigma_l)$. For $\epsilon > 0$ there exists an index $\gamma_0$ such that $(S_\delta - S_\gamma) \in \epsilon U_R$ for all $\gamma, \delta \geq \gamma_0$; hence $\alpha((QS_\delta - QS_\gamma)R) < \epsilon$ for all $\delta, \gamma \geq \gamma_0$ and for all $Q \in \mathcal{F}(X)$ with $\|Q\| \leq 1$. Since $T_0 \in (U_R)_{\mathcal{L}}$, this implies that

$$(*) \qquad |\,\mathrm{tr}((QS_\delta - QS_\gamma)T_0)| \leq \epsilon, \quad \forall \gamma, \delta \geq \gamma_0, \forall\ \|Q\| \leq 1,\ Q \in \mathcal{F}(X).$$

Thus it follows from (c) that $\alpha((S_\delta - S_\gamma)T_0) \leq k\epsilon \quad \forall \gamma, \delta \geq \gamma_0$. Because of the completeness of $(\mathcal{A}_0(X), \alpha)$, this implies that the net $(S_\delta T_0)$ converges with respect to the $\alpha$-topology in $\mathcal{A}_0(X)$. Since the same net converges to $ST_0$ in the (weaker) weak operator topology, it follows that $ST_0 \in \mathcal{A}_0(X)$.

The proof of $(\mathcal{A}_0 \circ \mathcal{A}^{-1})(X) = \mathcal{A}_{\sigma_r}^{\triangle}(X)$ is similar. □

**Remark.** If $X$ is a Banach space with the approximation property, then the conditions (b) and (c) of Proposition 2.3.2 are satisfied if we replace $(\mathcal{A}_0, \alpha)$ by the Banach operator ideal $(\mathcal{N}_1, \nu_1)$ of nuclear operators. We refer to [23] (18.3.4) and the remark following Lemma 3 in [23] (§17.5) for this information. See also [34] (§6.8).

Let $(\mathcal{S}_1, \sigma_1)$ be the quasi-normed ideal of 1-approximable operators (cf. Chapter 0), which is often called the trace class. This ideal admits a continuous trace (cf. [23], 19.8.7); in particular, the mapping $\mathrm{tr} : \mathcal{F}(X) \to \mathbb{K}$ is $\sigma_1$-continuous. Although in general no complete ideal-norm on $\mathcal{S}_1$ exists, it is true that $(\mathcal{S}_1, \sigma_1)$ is a Banach ideal of operators on the family of Hilbert spaces. Moreover, in this case we have $(\mathcal{S}_1, \sigma_1) = (\mathcal{N}_1, \nu_1)$ (cf. [23], 20.2.5). In [21] the concept $\alpha$-dual $\mathcal{A}^\times(H)$ of an ideal $\mathcal{A}(H)$ of bounded linear operators on a Hilbert space $H$ is introduced. It is namely defined by

$$\mathcal{A}^\times(H) := \{T \in \mathcal{L}(H) : TS \in \mathcal{S}_1(H), \forall\, S \in \mathcal{A}(H)\}.$$

It is proved in ([21], Proposition 6, p.122) that

$$\mathcal{A}^\times(H) := \{T \in \mathcal{L}(H) : ST \in \mathcal{S}_1(H), \forall\, S \in \mathcal{A}(H)\}.$$

From (2.3.2) follows that

$$(\mathcal{A}^{-1} \circ \mathcal{S}_1)(H) = (\mathcal{S}_1 \circ \mathcal{A}^{-1})(H) = \mathcal{A}^\times(H)$$

for any quasi-normed ideal $(\mathcal{A}, \mu)$ such that $\mathcal{F}(H)$ is dense in $(\mathcal{A}(H), \mu)$.

## 2.4 Inclusions in $FH$-spaces

Throughout this section, if $X$, $Y$ are given Banach spaces, then we let $H = (L(X,Y), \|.\|)$. The components $(\mathcal{A}(X,Y), \mu)$ of Banach operator ideals $(\mathcal{A}, \mu)$ are $FH$ spaces of operators containing $\mathcal{F}(X,Y)$. All $FH$ spaces of operators, hence complete metrizable locally convex subspaces $\mathcal{A}(X,Y)$ of $L(X,Y)$ which are continuously embedded into $H$, are from now on assumed to contain $\mathcal{F}(X,Y)$. As in the case of $FK$-spaces, it follows from the properties of $FH$ spaces of operators (or components of Banach operator ideals) and the closed graph theorem that

**Proposition 2.4.1** *Let $(\mathcal{A}_1(X,Y), \mu_1)$ and $(\mathcal{A}_2(X,Y), \mu_2)$ be $FH$-spaces of operators. Suppose a subspace $\mathcal{A}(X,Y)$ of $L(X,Y)$ satisfies the following conditions:*

*(a)* $\mathcal{A}(X,Y) \subseteq \mathcal{A}_1(X,Y) \cap \mathcal{A}_2(X,Y)$.

*(b)* $\overline{\mathcal{A}(X,Y)}^{\mu_2} = \mathcal{A}_2(X,Y)$.

*(c)* $\forall \phi \in \mathcal{A}_1(X,Y)^*$ *there exists* $\theta \in \mathcal{A}_2(X,Y)^*$ *such that* $\theta/_{\mathcal{A}(X,Y)} = \phi/_{\mathcal{A}(X,Y)}$ *holds.*

*Then the inclusion* $\mathcal{A}_2(X,Y) \subseteq \mathcal{A}_1(X,Y)$ *holds.*

**Proposition 2.4.2** *Let* $(\mathcal{A}_1(X,Y),\mu_1)$ *and* $(\mathcal{A}_2(X,Y),\mu_2)$ *be* $FH$*-spaces of operators. If* $X$ *is reflexive and* $\mathcal{F}(X,Y)$ *is dense in* $\mathcal{A}_2(X,Y)$*, then*

$$\mathcal{A}_1(X,Y) \supseteq \mathcal{A}_2(X,Y) \iff \mathcal{A}_2^{\triangle}(Y,X) \supseteq \mathcal{A}_1^{\triangle}(Y,X).$$

**Proof** If $\mathcal{A}_1(X,Y) \supseteq \mathcal{A}_2(X,Y)$, then $\mathcal{A}_2^{\triangle}(Y,X) \supseteq \mathcal{A}_1^{\triangle}(Y,X)$ by an earlier remark. To prove the converse, we need only verify that for each $\theta \in \mathcal{A}_1(X,Y)^*$, the bounded linear functional $\tilde{\phi}_{R_\theta}$ on $\mathcal{A}_2(X,Y)$ satisfies $\tilde{\phi}_{R_\theta}(S) = \theta(S)$ for all $S \in \mathcal{F}(X,Y)$. Then it follows from (2.4.1) that $\mathcal{A}_1(X,Y) \supseteq \mathcal{A}_2(X,Y)$. □

Let the Hilbert spaces $H_1$ and $H_2$ be fixed and put $H := L(H_1,H_2)$. In the following examples we demonstrate how to apply (2.4.2) to find necessary and sufficient conditions for an $FH$-space $\mathcal{A}(H_1,H_2)$ of operators on the given Hilbert spaces to contain some important classes of operators. Recall the definition of "Schatten class of index $p$", for $1 \le p \le \infty$. It is the restriction of the ideal of $p$-approximable operators to the family of Hilbert spaces. A bounded linear operator $T$ belongs to $\mathcal{S}_p(H_1,H_2)$ if and only if it can be represented in the form $Tx = \sum_{i=1}^{\infty} \alpha_i \langle x, e_i\rangle g_i$ for $(\alpha_i) \in \ell^p$ ($c_0$ if $p=\infty$) and orthonormal sequences $(e_i)$ and $(g_i)$ in $H_1$ and $H_2$, respectively. In this case the norm on $\mathcal{S}_p(H_1,H_2)$ is given by $\sigma_p(T) = \|(\alpha_i)\|_p$. It is well known that a bounded linear operator $T$ from $H_1$ into $H_2$ belongs to $\mathcal{S}_p(H_1,H_2)$ if and only if the scalar sequence $(\langle Te_i, g_i\rangle)$ belongs to $\ell^p$ (respectively, $c_0$ if $p=\infty$) for all orthonormal sequences $(e_i)$ and $(g_i)$ in $H_1$ and $H_2$, respectively (cf. [23], p.453–454). In the following examples we make use of Theorem 20.2.6 in [23], which states that $\mathcal{S}_p(H_1,H_2)^* = \mathcal{S}_q(H_2,H_1)$ for $1 < p < \infty$, $\frac{1}{p}+\frac{1}{q} = 1$ and $\mathcal{S}_1(H_1,H_2)^* = L(H_2,H_1)$.

We demonstrate the application of the operator dual to find necessary and sufficient conditions for $FH$ spaces of operators on Hilbert spaces to contain the Schatten classes of index $p$. In the following examples $\mathcal{A}(H_1,H_2)$ denotes an $FH$ space of operators and for all $\phi \in \mathcal{A}(H_1,H_2)^*$, $R_\phi$ always refers to the operator defined in section 1 (proof of Theorem 2.1.3) – from a Hilbert space $H_2$ into a Hilbert space $H_1$ in this case and where of course now we use the Riesz Theorem to represent the bounded linear functionals on $H_1$. Thus $\langle R_\phi(y), x\rangle = \phi(x \otimes y)$ for all $y \in H_2$ and $x \in H_1$.

It follows from (2.1.3) that $K^{\triangle}(H_2,H_1) = K(H_1,H_2)^* = \mathcal{S}_1(H_2,H_1)$ (cf. [23], p.456). Hence by (2.4.2) we have

$$\mathcal{A}(H_1,H_2) \supseteq K(H_1,H_2) \iff \mathcal{A}^{\triangle}(H_2,H_1) \subseteq K^{\triangle}(H_2,H_1) = \mathcal{S}_1(H_2,H_1).$$

This shows that

**Proposition 2.4.3** *For all Hilbert spaces $H_1$ and $H_2$ we have*

$$\begin{aligned} \mathcal{A}(H_1,H_2) \supseteq K(H_1,H_2) &\iff R_\phi \in \mathcal{S}_1(H_2,H_1), \quad \forall \phi \in \mathcal{A}(H_1,H_2)^* \\ \iff (\phi(e_i \otimes g_i)) \in \ell^1, &\quad \forall \phi \in \mathcal{A}(H_1,H_2)^*, \forall \textit{ orthonormal sequences } (e_i) \subset H_1, (g_i) \subset H_2. \end{aligned}$$

Let $1 < p, q < \infty$, with $\frac{1}{p} + \frac{1}{q} = 1$. As before, we have

$$\mathcal{S}_p^{\triangle}(H_2,H_1) = \mathcal{S}_p(H_1,H_2)^* = \mathcal{S}_q(H_2,H_1)$$

and hence that

**Proposition 2.4.4** $\mathcal{A}(H_1,H_2) \supseteq \mathcal{S}_p(H_1,H_2) \iff (\phi(e_i \otimes g_i)) \in \ell^q$, $\forall \phi \in \mathcal{A}(H_1,H_2)^*$, *and for all orthonormal sequences $(e_i) \subset H_1$ and $(g_i) \subset H_2$.*

The Banach ideal of nuclear operators is the smallest complete normed ideal of operators. On the family of Hilbert spaces it coincides with the Schatten class of index 1. Hence $\mathcal{A}(H_1,H_2) \supseteq \mathcal{S}_1(H_1,H_2)$ holds for all Banach ideals $\mathcal{A}$. In fact, for every $FH$ space of operators $\mathcal{A}(H_1,H_2)$ containing $\mathcal{F}(H_1,H_2)$ (in particular when $\mathcal{A}$ is a Banach ideal), it follows as before that

$$\mathcal{S}_1^{\triangle}(H_2,H_1) = \mathcal{S}_1(H_1,H_2)^* = L(H_2,H_1)$$

and hence by (2.4.2) that

$$\mathcal{A}(H_1,H_2) \supseteq \mathcal{S}_1(H_1,H_2) \iff R_\phi \in L(H_2,H_1), \quad \forall \phi \in \mathcal{A}(H_1,H_2)^*.$$

However, it is easily verified that $R_\phi$ is indeed a bounded linear operator for each $\phi \in \mathcal{A}(H_1,H_2)^*$; hence the inclusion $\mathcal{S}_1(H_1,H_2) \subseteq \mathcal{A}(H_1,H_2)$ holds for all $FH$ spaces of operators.

# Chapter 3

# Convergence of sections of weak $\Lambda^{\times}$-sequences in Banach spaces

We continue our study of compact operators, but in this case for compact operators whose domains are scalar sequence spaces. Due to the characterisations of compact operators on sequence spaces in terms of vector sequence spaces (refer to Chapter 0), we have to consider some results about vector sequence spaces first. Most of the material of the present chapter are contained in the joint paper [3].

Consider an arbitrary Banach space $X$. The following two results (both depending on the Bessaga-Pelczyński selection principle, but with different proofs) are briefly discussed in ([8] p 104-105):

a) $\ell^1_w(X) \subset c_0\{X\}$ if and only if $\ell^1_w(X) = \ell^1_c(X)$, hence if and only if each weak absolutely summable sequence (also called weakly unconditionally Cauchy sequence) is unconditionally summable. This is proved by using the fact that the second condition is equivalent to $X$ not containing $c_0$ isomorphically (cf. [9], p 45) - a result which follows from utilizing the Bessaga-Pelczyński selection principle.

b) Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then $\ell^p_w(X) \subset c_0(X)$ if and only if $\ell^p_w(X) = \ell^p_c(X)$. In the proof of this fact (due to Castillo) the Bessaga-Pelczyński selection principle as well as the isometric characterisation of $\ell^p_c(X)$ as the space $K(\ell^{p'}, X)$ of compact operators and the fact that $T : \ell^{p'} \to X$ is compact if and only if $\|T((\lambda_{n,i})_i)\| \to 0$ if $n \to \infty$ whenever $((\lambda_{n,i})_i)_n \subset \ell^{p'}$ converges weakly to zero if $n \to \infty$, are important. Hence, the proof makes use of properties of weak null sequences in and compact operators on the reflexive Banach space $\ell^{p'}$.

Our main result of this chapter includes both the above mentioned results and its proof makes use of elementary sequence space properties only.

**Theorem 3.0.1** $\ell^p_w(X) = \ell^p_c(X)$ *(for* $1 \leq p < \infty$*) if and only if* $\ell^p_w(X) \subset c_0\{X\}$.

**Proof** Let $\ell^p_w(X) \subset c_0\{X\}$. Suppose, to the contrary, there is $(x_i) \in \ell^p_w(X)$ for which

$$\epsilon_p((x_i)(\geq n)) \nrightarrow 0.$$

That is there is an $\epsilon > 0$ such that $\epsilon_p((x_i)(\geq n)) \geq 2\epsilon$ for all $n$. We construct an increasing sequence $n_0 < n_1 < n_2 < \ldots$ of natural numbers and a sequence $(a_n) \subset B_{X^*}$ as follows: If $n_k$ is given, take $a_k \in B_{X^*}$ such that

$$\|(a_k(x_i))(\geq n_k)\|_p \geq 3/2\epsilon.$$

Then by the $AK$-property, let $n_{k+1} > n_k$ be such that $\|(a_k(x_i))(\geq n_{k+1})\|_p < \epsilon/2$, hence

$$\|(a_k(x_i))(n_k < i < n_{k+1})\|_p \geq \epsilon.$$

Thus for each $k = 1, 2, 3, \ldots$ there exists a norm one sequence $(\lambda_i^k) \subset B_{\ell^{p'}}$, with $\frac{1}{p} + \frac{1}{p'} = 1$ (or $(\lambda_i^k) \subset B_{c_0}$ if $p = 1$) such that

$$\sum_{i=n_k+1}^{n_{k+1}} |\lambda_i^k||a_k(x_i)| \geq \epsilon.$$

It is easy to see that we may assume that each $(\lambda_i^k)$ is a finite "block" sequence $(\lambda_i^k)(n_k + 1 \leq i \leq n_{k+1})$ (with norm one). Put

$$z_k := \sum_{i=n_k+1}^{n_{k+1}} sgn(\lambda_i^k a_k(x_i))\lambda_i^k x_i.$$

Then

$$\|z_k\| \geq |a_k(z_k)| = \sum_{i=n_k+1}^{n_{k+1}} |\lambda_i^k a_k(x_i)| \geq \epsilon$$

for all $k = 1, 2, \ldots$. We have to show that $(z_i) \in \ell_w^p(X)$. To do so, take arbitrary $a \in X^*$ and $(\alpha_i) \in \ell^{p'}$ (resp. $(\alpha_i) \in c_0$ if $p = 1$). Direct calculation, for instance

$$(*) \qquad |\alpha_1\lambda_{n_1+1}^1|^{p'} + \cdots + |\alpha_1\lambda_{n_2}^1|^{p'} + |\alpha_2\lambda_{n_2+1}^2|^{p'} + \ldots$$
$$+|\alpha_2\lambda_{n_3}^2|^{p'} + \cdots \leq \|(\alpha_i)\|_{p'}^{p'},$$

shows that the sequence

$$\sum_k \alpha_k \left(\sum_{i=n_k+1}^{n_{k+1}} \lambda_i^k e_i\right) = (0, 0, \ldots 0, \alpha_1\lambda_{n_1+1}^1, \ldots, \alpha_1\lambda_{n_2}^1, \alpha_2\lambda_{n_2+1}^2, \ldots$$
$$\ldots \alpha_2\lambda_{n_3}^2, \ldots)$$

is in $\ell^{p'}$ (respectively, $c_0$). Hence

$$\sum_k |\alpha_k||a(z_k)| \leq \sum_k |\alpha_k| \sum_{i=n_k+1}^{n_{k+1}} |\lambda_i^k||a(x_i)| < \infty.$$

Using that $\ell^p = (\ell^{p'})^\times$ (if $1 < p < \infty$) and $\ell^1 = c_0^\times$, it follows that $(a(z_k)) \in \ell^p$ for all $a \in X^*$. Hence $(z_k) \in \ell_w^p(X)$, whereby $\|z_k\| \not\to 0$. We have a contradiction! $\square$

## 3.1 Sectional convergence of weak $\Lambda^{\times}$-sequences in Banach spaces

Obviously, we would like to generalise our result in the previous section in order to obtain a result for arbitrary $BK$ spaces with the $AK$ property. Let $\Lambda$ be a $DAK$ space and assume $\Lambda_w^{\times}(X) \subseteq c_0\{X\}$, whereas $\Lambda_w^{\times}(X) \neq \Lambda_c^{\times}(X)$. A careful glance at the proof of the previous result reveals that we may proceed as in the proof of (3.0.1) to construct the sequence $z_k := \sum_{i=n_k+1}^{n_{k+1}} sgn(\lambda_i^k a_k(x_i))\lambda_i^k x_i$ (with $\|z_k\| \geq \epsilon$ for all $\epsilon > 0$). Let $u_k := \sum_{i=n_k+1}^{n_{k+1}} \lambda_i^k e_i$. Then $\{u_k : k = 1, 2, \ldots\}$ is a normalised ($\|u_k\|_\Lambda = 1$ for $k = 1, 2, \ldots$) block basis of the standard unit vector basis $\{e_k : k \in \mathbb{N}\}$ of $\Lambda$. In order to show that the sequence

$$\sum_k \alpha_k \left( \sum_{i=n_k+1}^{n_{k+1}} \lambda_i^k e_i \right) = (0, 0, \ldots 0, \alpha_1 \lambda_{n_1+1}^1, \ldots, \alpha_1 \lambda_{n_2}^1, \alpha_2 \lambda_{n_2+1}^2, \ldots, \alpha_2 \lambda_{n_3}^2, \ldots)$$

is in $\Lambda$, we formulate the result as follows:

**Theorem 3.1.1** *Let $X$ be a Banach space. Let $\Lambda$ be a $DAK$ space such that each normalised block basis of the standard basis is equivalent to the standard coordinate basis. Then $\Lambda_w^{\times}(X) = \Lambda_c^{\times}(X)$ if and only if $\Lambda_w^{\times}(X) \subseteq c_0\{X\}$.*

The conditions of Theorem 3.1.1 call for a result which will indicate to us when we can expect to have $\Lambda_w^{\times}(X) \subseteq c_0\{X\}$.

**Theorem 3.1.2** *Let $\Lambda$ be a $DAK$ space. Suppose $\Sigma$ is a $BK$ space with $AK$ property such that every block basis of the standard basis $\{e_n : n \in \mathbb{N}\}$ in $\Sigma$ is equivalent to $\{e_n : n \in \mathbb{N}\}$. Let $\Sigma^{\times} \nsubseteq \Lambda^{\times}$. Then $\Lambda_w^{\times}(\Sigma) \subseteq c_0\{\Sigma\}$.*

**Proof** By contradiction. Suppose $\Lambda_w^{\times}(\Sigma) \nsubseteq c_0\{\Sigma\}$. Hence, let $((\alpha_{i,j})_i)_j \in \Lambda_w^{\times}(\Sigma)$ such that $\|(\alpha_{i,j})_i\|_\Sigma \nrightarrow 0$ if $j \to \infty$. Since $\Lambda^{\times}$ has $AK$ we have $\Lambda^{\times} \subseteq c_0$. For each $(\beta_i) \in \Sigma^{\times}$ we have $(\langle (\alpha_{i,j})_i, (\beta_i) \rangle)_j \in \Lambda^{\times}$. That is $(\alpha_{i,j})_i \xrightarrow{w} 0$ if $j \to \infty$. By the property of Pelczyński ([31], Proposition 1.a.12) we have a subsequence of $((\alpha_{i,j})_i)_j$ which is equivalent to a block basis of the standard basis $\{e_n : n \in \mathbb{N}\}$. However, the subsequence is also in $\Lambda_w^{\times}(\Sigma)$. Hence this implies that $\{e_n : n \in \mathbb{N}\} \in \Lambda_w^{\times}(\Sigma)$. This, however is impossible since if we take $(\delta_i) \in \Sigma^{\times} \backslash \Lambda^{\times}$, then we have $(\langle e_n, (\delta_i) \rangle)_n = (\delta_n)_n \notin \Lambda^{\times}$. We have a contradiction! Hence $\Lambda_w^{\times}(\Sigma) \subseteq c_0\{\Sigma\}$. □

**Corollary 3.1.3** *If $\Lambda = c_0$ or $\Lambda = \ell^p$ $(1 < p < \infty)$ and $\Sigma = \ell^r$ with $1 \leq r < p < \infty$, then $\Lambda_w^{\times}(\Sigma) \subseteq c_0\{\Sigma\}$.*

**Proof** $\Lambda$ is a $DAK$ space. Every block basis of the standard basis $\{e_n : n \in \mathbb{N}\}$ in $\Sigma$ is equivalent to $\{e_n : n \in \mathbb{N}\}$. Also $\Sigma^{\times} = \ell^{r'}$ (where $\frac{1}{r} + \frac{1}{r'} = 1$) and $\Lambda^{\times} \subsetneq \ell^{r'}$. Hence from Theorem 3.1.2 we have $\Lambda_w^{\times}(\ell^r) \subseteq c_0\{\ell^r\}$. □

Our result 3.0.1 leads to easy and short proofs of the above results (A) and (B). Moreover one may extend the results to a more general setting of operators on $BK$ spaces and also consider similar results for certain Orlicz sequence spaces.

Notice that the cases $\Lambda = c_0$ and $\Lambda = \ell^p(1 < p < \infty)$ are included in Theorem 0.0.1. It follows from Theorem 0.0.1 that if both $\Lambda$ and $\Lambda^\times$ have $AK$, then $K(\Lambda, E) = L(\Lambda, E)$ if and only if $\Lambda_w^\times(E) = \Lambda_c^\times(E)$. Hence the results (A) and (B) above are direct consequences of Theorem 3.0.1 and Theorem 0.0.1. Moreover, it follows from 3.1.1 and 0.0.1 that:

**Proposition 3.1.5** *If $\Lambda$ satisfies the conditions of Proposition 3.1.1, then $K(\Lambda, E) = L(\Lambda, E)$ if and only if $\Lambda_w^\times(E) \subset c_0\{E\}$.*

## 3.2 Orlicz sequence spaces

The results of the previous section may also be considered in the context of Orlicz spaces. We recall (from [31]) the definition of Orlicz function and some basic facts. A continuous non-decreasing convex function $M : [0, \infty) \to [0, \infty)$ is called on Orlicz function if $M(0) = 0$ and $\lim_{t\to\infty} M(t) = \infty$. For a given Orlicz function $M$, $\ell_M$ denotes the vector space of all sequences of scalars $(\alpha_i)$ such that

$$\sum_{n=1}^{\infty} M\left(\frac{|\alpha_n|}{\rho}\right) < \infty \text{ for some } \rho > 0.$$

The Banach space $\ell_M$, when equipped with the norm

$$\|(\alpha_i)\|_M := \inf\{\rho > 0 : \sum_{n=1}^{\infty} M\left(\frac{|\alpha_n|}{\rho}\right) \leq 1\},$$

is called an Orlicz sequence space. The unit vectors form a symmetric basic sequence in $\ell_M$. The vector space $h_M$ consisting of those sequences $(\alpha_i) \in \ell_M$ for which

$$\sum_{n=1}^{\infty} M\left(\frac{|\alpha_n|}{\rho}\right) < \infty$$

for every $\rho > 0$ is a closed subspace of $\ell_M$ and the unit vectors $\{e_n\}_{n=1}^{\infty}$ form a symmetric basis for $h_M$; in particular, $h_M$ has the $AK$-property. An Orlicz function $M$ is said to satisfy the $\Delta_2$-condition at zero if $\lim_{t\to\infty} \frac{M(2t)}{M(t)} < \infty$.

In this case $\ell_M = h_M$. Recall that if $M$ and $M^*$ are complementary Orlicz functions, then $(h_M)^*$ is isomorphic to $\ell_{M^*}$ and that $h_M$ is reflexive if and only if both $M$ and $M^*$ satisfy the $\Delta_2$ condition at zero.

Two Orlicz functions $M_1$ and $M_2$ are equivalent at zero if there exist constants $k > 0$, $K > 0$ and $t_0 > 0$ such that, for all $0 \leq t \leq t_0$, we have

$$K^{-1} M_2(k^{-1}t) \leq M_1(t) \leq K M_2(kt).$$

$M_1$ and $M_2$ are equivalent at zero if and only if $\ell_{M_1} = \ell_{M_2}$ (hence the two spaces consist of the same sequences and the identity mapping is an isomorphism between them) (cf [31]). Also, the unit vector bases of $h_{M_1}$ and $h_{M_2}$ are equivalent in this case. There are many instances where an Orlicz function $M$ is defined only in a neighbourhood of zero (for $t \leq t_0$ for instance). The function $M$ can then be extended for $t > t_0$ so that it becomes an Orlicz function on the entire positive line. The corresponding spaces $\ell_M$ and $h_M$ will be the same regardless of the way we have extended $M$, with the norms associated to two different extensions being equivalent. For example if $M(t) = t^p|\log t|^\alpha$ (for $1 < p < \infty$), then $M^*$ is equivalent to the function $t \mapsto t^q|\log t|^{\alpha(1-q)}$, where $p$ and $q$ are conjugate numbers.

An Orlicz sequence space $\ell_M$ is topologically isomorphic to $\ell^p (1 \leq p < \infty)$ if and only if $M$ is equivalent to $M_p$, such that $M_p(x) = x^p$. Notice that in this case

$$M_p(xy) = M_p(x)M_p(y)$$

for all $x, y \geq 0$. Thus $M_p$ is an example of an Orlicz function which is submultiplicative at 0. It is namely said that

**Definition 3.2.1** *An Orlicz function $M$ is submultiplicative at 0 if there are constants $K > 0$ and $\epsilon > 0$ such that*

$$M(st) \leq KM(s)M(t), \ \forall s, t \leq \epsilon.$$

Following are examples of Orlicz functions (which are discussed in [31] for different reasons) that are submultiplicative at 0:

(1) If $1 < p < \infty$ and $N_p(t) = t^p(1 + |\log t|)$, $N_p(0) = 0$ then

$$N_p(ts) \leq N_p(t)N_p(s) \ \forall s, t > 0.$$

(2) If $1 < p < \infty$, $\alpha > 0$ and $M(t) = t^p|\log t|^\alpha$, $M(0) = 0$ then $M$ is an Orlicz function on some interval $[0, t_0]$ and can be extended to an Orlicz function on $[0, \infty)$. In this case the space $\ell_M$ is reflexive. Moreover,

$$M(ts) \leq M(t)M(s), \text{ if } s, t \to 0.$$

(3) If $M$ is the function in (2), then the Orlicz function $M^*$ is equivalent to

$$N : t \mapsto t^q|\log t|^{\alpha(1-q)}, \ \frac{1}{p} + \frac{1}{q} = 1.$$

As before

$$N(ts) \leq N(t)N(s), \text{ if } s, t \to 0.$$

The proof of (3.0.1) will go through for Orlicz sequence spaces generated by Orlicz functions which are submultiplicative at 0. This fact is formally stated in

**Proposition 3.2.2** *Let $M$ be an Orlicz function which is submultiplicative at 0 and put $\Lambda = h_M$. If $\Lambda^\times$ has AK, then $\Lambda_w^\times(X) = \Lambda_c^\times(X)$ if and only if $\Lambda_w^\times(X) \subset c_0\{X\}$. In particular, if the Orlicz function $M$ is submultiplicative at 0 and $M$ and $M^*$ are complementary Orlicz functions such that $h_M^* = h_{M^*}$ - this is for instance the case when both $M$ and $M^*$ satisfy the $\Delta_2$-condition at zero - then $(h_{M^*})_w(X) = (h_{M^*})_c(X)$ if and only if $(h_{M^*})_w(X) \subset c_0\{X\}$.*

**Proof** The proof is similar to that of 3.0.1. In this case however we replace $(*)$ in the proof of 3.0.1 by

$$(**) \qquad M\left(\frac{|\alpha_N \lambda^N_{n_N+1}|}{\rho}\right) + \cdots + M\left(\frac{|\alpha_N \lambda^N_{n_{N+1}}|}{\rho}\right) +$$

$$M\left(\frac{|\alpha_{N+1} \lambda^{N+1}_{n_{N+1}+1}|}{\rho}\right) + \cdots + M\left(\frac{|\alpha_{N+1} \lambda^{N+1}_{n_{N+2}}|}{\rho}\right) + \cdots$$

$$\leq K \sum_{i=N}^{\infty} M\left(\frac{|\alpha_i|}{\rho}\right) < \infty$$

where $(\alpha_i) \in \Lambda = h_M$ and $K > 0$ and $N \in \mathbb{N}$ depend on the given $\rho > 0$. □

In the setting of Orlicz spaces one can actually formulate a more general version of (3.1.1). This will follow after the introduction of some notions.

We refer to the book of Lindenstrauss and Tzafriri (cf. [31]) page 141. Starting with an Orlicz function $M$ it is there showed how we can associate with any normalised block basis $\{u_j\}_{j=1}^\infty$ of the unit vectors in $\ell_M$ an Orlicz function $N$ such that $h_N$ is topologically isomorphic to the closed subspace of $\ell_M$ spanned by a suitable subsequence of $\{u_n\}$. Moreover, if

$$u_k := \sum_{i=n_k+1}^{n_{k+1}} \lambda_i^k e_i,$$

then the map $I_N^M : \ell_N \to \ell_M$ defined by

$$I_N^M((\alpha_i)) = (0, \ldots, 0, \alpha_1 \lambda^1_{n_1+1}, \alpha_1 \lambda^1_{n_1+2}, \ldots, \alpha_1 \lambda^1_{n_2}, \alpha_2 \lambda^2_{n_2+1}, \ldots, \alpha_2 \lambda^2_{n_3}, \ldots)$$

is an isomorphism from $\ell_N$ into $\ell_M$. We agree to say that the Orlicz sequence space $h_N$ is *generated by a normalised block basis of the unit vector basis of* $h_M$.

**Definition 3.2.3** *Suppose $h_N \approx [u_i]_{i=1}^\infty$, where $\{u_i\}$ is a suitable subsequence of a block basis of $\{e_n\}$ in an Orlicz sequence space $h_M$ and where the isomorphism is given by the mapping $I_M^N$ above (i.e. $h_N$ is generated by a block basis of the unit vector basis of $h_M$). We call $(y_i)$ an $N$-block sequence of $(x_i) \in X^{\mathbb{N}}$ if*

$$y_k = \sum_{i=n_k+1}^{n_{k+1}} \lambda_i^k x_i.$$

**Proposition 3.2.4** *Let $M$ be an Orlicz function such that both $M$ and $M^*$ satisfy the $\Delta_2$-condition at zero (equivalently, $h_M$ is reflexive). Then we have*

$$(h_M^\times)_w(X) = (h_M^\times)_c(X)$$

*if and only if for each Orlicz sequence space $h_N$ generated by a normalised block basis of the unit vector basis of $h_M$ and for each each $N$-block sequence $(y_k) \in (h_N^\times)_w(X)$ of some $(x_i) \in (h_M^\times)_w(X)$ we have $(y_k) \in c_0\{X\}$.*

**Proof** Suppose $(h_M^\times)_w(X) = (h_M^\times)_c(X)$. Let $h_N \approx [u_k]_{k=1}^\infty$ where the block basis $\{u_k\}_{k=1}^\infty$ and the isomorphism $I_N^M : h_N \to h_M$ are defined as before. Consider $(y_k) \in (h_N^\times)_w(X)$ with $y_k = \sum_{i=n_k+1}^{n_{k+1}} \lambda_i^k x_i$ for some $(x_i) \in (h_M^\times)_w(X)$. Then

$$\begin{aligned} \|y_k\| &= \sup_{\|a\|\le 1} |a(y_k)| \le \sup_{\|a\|\le 1} \sum_{i=n_k+1}^{n_{k+1}} |\lambda_i^k||a(x_i)| \\ &\le \sup_{\|a\|\le 1} \|(a(x_i))(n_k+1 \le i \le n_{k+1})\|_{h_M^\times} \\ &\le \epsilon_{h_M^\times}((x_i)(i \ge n_k)) \to 0 \text{ if } k \to \infty. \end{aligned}$$

Conversely, assume that to the contrary, there is $(x_i) \in (h_M^\times)_w(X)$ for which

$$\epsilon_{h_M^\times}((x_i)(\ge n)) \not\to 0.$$

That is there is an $\epsilon > 0$ such that $\epsilon_{h_M^\times}((x_i)(\ge n)) \ge 2\epsilon$ for all $n$. Using the $AK$ property of $h_{M^*} = h_M^\times$, we construct an increasing sequence $n_0 < n_1 < n_2 < \dots$ of natural numbers and a sequence $(z_k) \in X^{\mathbb{N}}$, with

$$z_k := \sum_{i=n_k+1}^{n_{k+1}} \lambda_i^k x_i,$$

such that $\|z_k\| \ge \epsilon$ for all $k \in \mathbb{N}$ and $\|(\lambda_i^k)(n_k < i \le n_{k+1})\|_{h_M^\times} = 1$ for all $k \in \mathbb{N}$. Now let $h_N$ be an Orlicz sequence space generated by the normalised block basis $\{u_k\}_{k=1}^\infty$, $u_k := \sum_{i=n_k+1}^{n_{k+1}} \lambda_i^k e_i$. We see that $(z_k)$ (or a subsequence of $(z_k)$ if necessary) is an $N$-block sequence of $(x_i)$. Furthermore, if $(\alpha_i) \in h_N$, then

$$(0,\dots,0,\alpha_1\lambda_{n_1+1}^1, \alpha_1\lambda_{n_1+2}^1, \dots, \alpha_1\lambda_{n_2}^1, \alpha_2\lambda_{n_2+1}^2, \dots, \alpha_2\lambda_{n_3}^2, \dots) \in h_M.$$

Hence we have

$$\sum_{k=1}^\infty |\alpha_k||a(z_k)| \le \sum_{k=1}^\infty |\alpha_k| \sum_{i=n_k+1}^{n_{k+1}} |\lambda_i^k||a(x_i)| < \infty.$$

Thus it follows that $(a(z_k)) \in h_N^\times$ for all $a \in X^*$. Hence $(z_k) \in (h_N^\times)_w(X)$, whereby $\|z_k\| \not\to 0$. We have a contradiction! □

**Remark** It is now clear that the results of Ansari (in the paper [2]) can also be considered in the context of $X$-valued operators on Orlicz sequences spaces. Once the conditions of Proposition 3.2.2 or Proposition 3.2.4 are satisfied, one may formulate (as in Proposition 3.1.5) a result which states when $K(h_M, X) = L(h_M, X)$.

# Chapter 4

# Absolutely summing multipliers

## 4.1 The sequence space $m(E)$

In the paper [14], Fourie introduced the concept of *absolutely summing multiplier* of a Banach space $E$ as follows:

**Definition 4.1.1** *A sequence $(\xi_i) \in \omega$ is called an absolutely summing multiplier of $E$ if $(\xi_i x_i)$ is absolutely summable in $E$ whenever $(x_i)$ is weakly absolutely summable in $E$; hence $(\xi_i x_i) \in \ell^1_s(E)$ for all $(x_i) \in \ell^1_w(E)$.*

The scalar sequence space of all absolutely summing multipliers of $E$ is denoted by $m(E)$. It is a vector subspace of $\ell^\infty$ such that $\ell^1 \subseteq m(E)$ (cf. the proof of Theorem 4.1.3 below). For each sequence $(x_i) \in \ell^1_w(E)$, the scalar sequence $(\|x_i\|)$ is by definition of the absolutely summing multipliers in the Köthe dual space $m(E)^\times$ of $m(E)$. Therefore it is clear that $m(E)^{\times\times} \subseteq m(E)$ – hence the scalar sequence space $m(E)$ is perfect. The space $m(E)$ is a complete normed space with respect to the (operator) norm

$$\|(\xi_i)\|_{1,1} := \sup\{\pi_1((\xi_i x_i)) : \epsilon_1((x_i)) \leq 1\}.$$

A proof for the completeness of a more general space of $(\Lambda, \Sigma)$-summing multipliers will be discussed in chapter 5. Clearly, if $E$ has finite dimension then $\ell^1_w(E) = \ell^1_s(E)$, so that $m(E) = \ell^\infty$.

Recall that a sequence $(x_n) \subset E$ is said to be *unconditionally summable* if $\sum_n x_{\sigma(n)}$ converges in $E$, regardless of the permutation $\sigma$ of the indices. We refer to [9] (Theorem 8, pp 45) for a proof of the fact that each $(x_i) \in \ell^1_w(E)$ is unconditionally summable if and only if $E$ does not contain a copy of $c_0$.

Using the well known Dvoretzky-Rogers Theorem, it is proved in [14] that for any infinite dimensional Banach space $E$ we have

$$\ell^1 \subseteq m(E) \subseteq \ell^2.$$

We recall the important

**Theorem 4.1.2** (Dvoretzky-Rogers Theorem, [11] pp 2) *Let $X$ be an infinite dimensional Banach space. Then no matter how we choose $(\lambda_n) \in \ell^2$ there is always an unconditionally summable sequence $(x_n)$ in $X$ with $\|x_n\| = |\lambda_n|$ for all $n$.*

**Theorem 4.1.3** ([14], Proposition 2.4) *Let $E$ be an infinite dimensional Banach space. Then $\ell^1 \subseteq m(E) \subseteq \ell^2$.*

**Proof** Let $(\alpha_n) \in \ell^1$. Since each $(x_n) \in \ell^1_w(E)$ is norm bounded in $E$, it follows that $\sum_{n=1}^{\infty} \|\alpha_n x_n\| < \infty$. Hence $(\alpha_n) \in m(E)$.

Conversely, let $(\alpha_n) \in m(E)$. For $(\beta_i) \in \ell^2$ there is by Theorem 4.1.2 a sequence $(x_i) \in \ell^1_w(E)$ such that $|\beta_i| = \|x_i\|$ for all $i = 1, 2, \ldots$. Then $(\alpha_i x_i) \in \ell^1_s(E)$. Hence it follows that $\sum_{i=1}^{\infty} |\alpha_i \beta_i| < \infty$. Since $(\beta_i) \in \ell^2$ was arbitrary, it is clear that $(\alpha_i) \in (\ell^2)^\times = \ell^2$. □

There are several Banach spaces for which the space of absolutely summing multipliers turns out to be $\ell^2$. In fact, it is easy to characterise the family of Banach spaces $E$ for which $m(E) = \ell^2$. First we recall the so called *Orlicz property*:

**Definition 4.1.4** (cf [9], pp 188) *A Banach space $E$ is said to have the Orlicz property if all unconditionally summable sequences in $E$ are in the space $\ell^2_s(E)$ of 2-absolutely summable sequences.*

**Example 1** There are numerous examples of Banach spaces having the Orlicz property. In fact, by a result of B. Maury (cf. [9], pp188) this property characterises Banach lattices with cotype 2 (we refer to the remark after Theorem 1.1.8 in this work for a definition of "cotype $p$"). All Banach spaces having cotype 2 have the Orlicz property. We refer to [11], pp 217 for the definition of type $p$ and the original definition of cotype $p$. It is known (cf. [11], pp 220) that $E^*$ has cotype 2 whenever $E$ has type 2. Hence if $E$ has type 2, then $E^*$ has the Orlicz property.

Since the Banach lattice $c_0$ has no finite cotype, it is clear from the above discussion that $c_0$ does not have the Orlicz property. Hence it is a necessary condition for a Banach space with the Orlicz property not to contain a copy of $c_0$.

We are now ready to characterise the Banach spaces for which the space of absolutely summing multipliers is $\ell^2$.

**Theorem 4.1.5** *Let $E$ be an infinite dimensional Banach space. Then $m(E) = \ell^2$ if and only if $E$ has the Orlicz property.*

**Proof** Let $m(E) = \ell^2$. Then if $(x_n) \in \ell^1_w(E)$, it follows that $\sum_n |\alpha_n| \|x_n\| < \infty$ for all $(\alpha_n) \in \ell_2$. Thus $(\|x_n\|) \in \ell_2^\times = \ell_2$.

Conversely, suppose $E$ has the Orlicz property. From the above discussion it is clear that each $(x_i) \in \ell^1_w(E)$ is unconditionally summable; thus $\ell^1_w(E) \subseteq \ell^2_s(E)$. Hence $\sum_{i=1}^{\infty} |\alpha_i| \|x_i\| < \infty$ for all $(x_i) \in \ell^1_w(E)$ and all $(\alpha_i) \in \ell^2$. Therefore we have $m(E) \supseteq \ell^2$. □

**Remark** Since both $m(E)$ and $\ell^2$ are $BK$-spaces, it follows in particular that they are also topologically isomorphic if $E$ has the Orlicz poperty.

**Lemma 4.1.6** *Suppose the Banach space $E$ is topologically isomorphic to a closed subspace of the Banach space $F$. Then*

$$m(F) \subseteq m(E).$$

**Proof** Let $I : E \to F$ be the topological isomorphism and take $K > 0$ a real number such that $\|x\| \leq K\|Ix\|$ for all $x \in E$. Let $(\alpha_i) \in m(F)$. For $(x_i) \in \ell^1_w(E)$ put $Ix_i = y_i$ for all $i$. For each $a \in F^*$ it follows that

$$\langle y_i, a\rangle = \langle Ix_i, a\rangle = \langle x_i, a \circ I\rangle$$

with $a \circ I \in E^*$. Hence it is clear that $(y_i) \in \ell^1_w(F)$. Finally it follows that

$$\sum_i |\alpha_i|\|x_i\| \leq K \sum_i |\alpha_i|\|y_i\| < \infty;$$

thus $(\alpha_i) \in m(E)$. □

**Example 2** The Banach spaces which contain isomorphic copies of $c_0$ are excluded by the Orlicz property. But it is easily verified (using the fact that $(e_i) \in \ell^1_w(c_0)$) that $m(c_0) = \ell^1$. Also (by the last lemma) $m(F) \subseteq m(E)$ if $E$ is topologically isomorphic to a subspace of $F$. Thus $m(E) = \ell^1$ for all Banach spaces $E$ which contain isomorphic copies of $c_0$.

Adjusting the proof of (4.1.5), we obtain conditions for $m(E) = \ell^q$ (with $1 \leq q < 2$) to hold:

**Proposition 4.1.7** *Let $1 \leq q < 2$. Suppose $E$ is an infinite dimensional Banach space such that*

*(i) there exists a real number $K > 0$ such that for all $(\alpha_i) \in \ell^p$ (with $\frac{1}{p} + \frac{1}{q} = 1$) there exists $(x_i) \in \ell^1_w(E)$ with $|\alpha_i| \leq K\|x_i\|$ for all $i$;*

*(ii) $\ell^1_w(E) \subseteq \ell^p_s(E)$.*

*Then $m(E) = \ell^q$. Being $BK$-spaces, it follows that the norms on $m(E)$ and $\ell^q$ are equivalent in this case.*

**Proof** Let $(\alpha_i) \in m(E)$ and take arbitrary $(\beta_i) \in \ell^p$. There exists (by (i)) a sequence $(x_i) \in \ell^1_w(E)$ such that

$$|\beta_i| \leq K\|x_i\| \text{ for all } i = 1, 2, \ldots.$$

Now, $(\alpha_i\|x_i\|) \in \ell^1$. Thus

$$\sum_i |\alpha_i\beta_i| \leq K \sum_i |\alpha_i|\|x_i\| < \infty.$$

Since $(\alpha_i)$ and $(\beta_i)$ were arbitrary chosen, it follows that

$$m(E) \subseteq (\ell^p)^\times = \ell^q.$$

Conversely, let $(\alpha_i) \in \ell^q$. Take $(x_i) \in \ell^1_w(E)$. Then by (ii) we have $(\|x_i\|) \in \ell^p$. Thus $\sum_i |\alpha_i|\|x_i\| < \infty$. This shows that $(\alpha_i) \in m(E)$. □

From a result of J.P. Kahane (cf. [9], pp 141) it follows that property (ii) is satisfed by Banach spaces of cotype $p$. The space $\ell^p$ (with $2 \leq p < \infty$) is easily seen to satisfy both the properties (i) and (ii). So we may conclude that $m(\ell^p) = \ell^q$ where $\frac{1}{p} + \frac{1}{q} = 1$. Moreover, we have

**Proposition 4.1.8** *Let $E$ be an infinite dimensional $L^p$-space, where $1 \leq p \leq \infty$. Then $m(L^p) = \ell^s$ where $s = min\{2, q\}$, with $\frac{1}{p} + \frac{1}{q} = 1$.*

**Proof** When $1 \leq p \leq 2$, the space $E$ has cotype 2. Thus by Theorem 4.1.5 we have $m(E) = \ell^2$. For $2 < p < \infty$ the property (ii) in (4.1.7) is satisfied by Kahane's result, since $E$ has cotype $p$ in this case. This also follows from Corollary 10.7 in [11], pp 200. Furthermore, $\ell^p$ is topologically isomorphic to a closed subspace of $E$; thus there is an isomorphism $I : \ell^p \to E$ into $E$ and a number $K > 0$ such that $\|(\alpha_i)\|_p \leq K\|I((\alpha_i))\|$ for all $(\alpha_i) \in \ell^p$. Let $(\alpha_i) \in \ell^p$. Put $x_i := I(\alpha_i e_i)$, for $i = 1, 2, \ldots$. Then $|\alpha_i| \leq K\|x_i\|$ for all $i$ and $(x_i) \in \ell^1_w(E)$ since $(\alpha_i e_i) \in \ell^1_w(\ell^p)$. Thus the property (i) in Proposition 4.1.7 is also satisfied. We have $m(L^p) = \ell^q$. When $p = \infty$, then $m(E) = \ell^1$, since then $E$ contains an isomorphic copy of $c_0$. □

We saw that $m(\ell^p) = \ell^q$ for $p \geq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$ and that $m(\ell^p) = \ell^2$ if $1 \leq p \leq 2$. Thus it follows that $m((\ell^p)^*) \subseteq \ell^p$ for $1 \leq p \leq \infty$. How general is this result for sequence spaces? Our next proposition shows that this is indeed a very general result:

**Proposition 4.1.9** *Let $\Lambda$ be a perfect sequence space satisfying*

*(a) There exists a norm $\|\cdot\|_\Lambda$ on $\Lambda$ such that $(\Lambda, \|\cdot\|_\Lambda)^* = \Lambda^\times$.*

*(b) $\liminf_{n\to\infty} \|e^*_n\|_{\Lambda^*} > 0$, being $e^*_n$ the sequence $(0, 0, \ldots, 1, 0, \ldots)$ (1 in the n-th place) and $\|\cdot\|_{\Lambda^*}$ the dual norm of $\|\cdot\|_\Lambda$.*

*Then $m(\Lambda^*) \subset \Lambda$.*

**Proof** If $(\delta_n) \in \Lambda^\times$, then $(\delta_n e^*_n) \in \ell^1_w(\Lambda^\times)$. In fact, for all $(x_n) \in \Lambda$, we have $\sum_{n=1}^\infty |\langle (x_n), \delta_n e^*_n \rangle| = \sum_{n=1}^\infty |\delta_n x_n| < \infty$. This implies $(\delta_n e^*_n) \in \ell^1_w(\Lambda^\times)$. Hence

$$\sum_{n=1}^\infty |\alpha_n| \|\delta_n e^*_n\| < \infty, \text{ for all } (\alpha_n) \in m(\Lambda^*).$$

Choose $c > 0$ and $n_0 \in \mathbb{N}$ so that $\|e_n^*\| \geq c > 0$, for all $n \geq n_0$. If $(\alpha_n) \in m(\Lambda^*)$, then

$$\frac{1}{c}\sum_{n\geq n_0} |\alpha_n| \, \|\delta_n e_n^*\| < \infty ,$$

so that

$$\infty > \sum_{n\geq n_0} |\alpha_n||\delta_n|\frac{\|e_n^*\|}{c} \geq \sum_{n\geq n_0} |\alpha_n||\delta_n|.$$

Hence $\sum_{n=1}^{\infty} |\alpha_n||\delta_n| < \infty$, $\forall\ (\delta_n) \in \Lambda^\times$. Thus $(\alpha_n) \in \Lambda^{\times\times} = \Lambda$. □

**Remark:** It is clear from the proof of (4.1.9) that for a non perfect sequence space $\Lambda$ such that the conditions of the proposition are satisfied, the inclusion becomes $m(\Lambda^*) \subseteq \Lambda^{\times\times}$.

From Lemma 4.1.6 follow some interesting observations, which we summarise in the following

**Proposition 4.1.10** *Let $E$ be a closed subspace of the Banach space $F$. Then*

*(a) If $E$ is complemented in $F$, then $m(F) \subseteq m(F/E)$.*

*(b) If $E$ is complemented in $F$, then $m(F^*) \subseteq m(E^*)$.*

*(c) $m(F^*) \subseteq m(E^\perp) = m((F/E)^*)$.*

**Proof** (a) The quotient space $F/E$ is isomorphic to a closed subspace of $F$ (which is complementary to $E$). Thus by Lemma 4.1.6 we have $m(F) \subseteq m(F/E)$.

(b) The normed space $E^*$ is (isometrically) isomorphic to the quotient space $F^*/E^\perp$, where the annihilator $E^\perp$ is a (closed) complemented subspace of $F^*$ (cf. [7], pp 132–133). Thus by part (a) we have

$$m(F^*) \subseteq m(F^*/E^\perp) = m(E^*).$$

(c) The annihilator $E^\perp$ of $E$ is a closed subspace of $F^*$. Thus by Lemma 4.1.6 we have $m(F^*) \subseteq m(E^\perp)$. Also, $(F/E)^*$ is (isometrically) isomorphic to $E^\perp$ (cf. [7], pp 132). □

The result in (4.1.10(b)) is not in general true for non-complemented subspaces $E$ of $F$. This is illustrated by the following example.

**Example 3** From the paper [38] we know about the existence of Banach spaces $X$ such that $X$ contains an (isometric) isomorphic copy $Y$ of $\ell^1$ and such that both $X$ and $X^*$ have cotype 2. For such a Banach space $X$ we have

$$m(X) = m(X^*) = \ell^2.$$

However, $m(Y^*) = m(\ell^\infty) = \ell^1$. Thus $m(X^*) \not\subseteq m(Y^*)$.

## 4.2 The sequence space $\lambda_X$ and its relation to $m(X^*)$

In the first definition of this section, the reader is reminded of the concept "range of a vector measure".

**Definition 4.2.1** *The range of a vector-valued measure is defined to be the set of the form*

$$rg(F) := \{F(E) : E \in \Sigma\}$$

*where $\Sigma$ is a $\sigma$-algebra of sets and $F$ is a countably additive measure on $\Sigma$ with values in an appropriate Banach space $X$.*

Let $A$ be a subset of a Banach space $X$. The phrases "$A$ is in the range of a vector measure (of bounded variation)" and "$A$ lies in the range of a vector measure (of bounded variation)" have different meanings. $A$ is in the range of a vector measure if there exists a $\sigma$-algebra $\Sigma$ and a vector measure $F : \Sigma \to X$ such that $A \subset rg(F)$, whereas "$A$ lies in the range of a vector measure (of bounded variation)" is defined as follows:

**Definition 4.2.2** (cf. [36], Piñeiro 1995) *Let $X$ be a Banach space. We say that a subset $A$ of $X$ lies in the range of a vector measure of bounded variation (we shorten to a vector bv-measure) provided there exist a Banach space $X_0$, an isometry $J : X \to X_0$ and a vector measure $F : \Sigma \to X_0$ with bounded variation so that $J(A) \subset rg(F)$.*

It is clear that to say a subset $A$ of a Banach space $X$ lies in the range of an $X$-valued measure is the same as to say that $A$ is in the range of a vector measure.

For a fixed Banach space $X$, the sequence space $\lambda_X$ is defined as follows:

**Definition 4.2.3** (Marchena and Piñeiro, [32]) *A scalar sequence $(\alpha_i)$ belongs to $\lambda_X$ if and only if for every null sequence $(x_i)$ in $X$, the sequence $(\alpha_i x_i)$ lies in the range of some $X$-valued measure with bounded variation.*

The sequence space $\lambda_X$ is easily seen to satisfy $\ell^1 \subseteq \lambda_X \subseteq \ell^\infty$. The paper [36] contains some very important information about sequences in the range of a vector measure. Following is a statement of one of the main results in that paper.

**Theorem 4.2.4** ([36], Piñeiro 1995, pp 3329) *Let $X$ be a Banach space. For a bounded sequence $(x_n)$ in $X$, consider the linear operator $T : (\alpha_n) \in \ell^1 \to \Sigma\alpha_n x_n \in X$. The following assertions hold:*

*(i) $(x_n)$ is in the range of an $X$-valued bv-measure if and only if $T$ is Pietsch-integral.*

*(ii) $(x_n)$ lies inside the range of a vector bv-measure if and only if $T$ is 1-summing.*

*(iii) $(x_n)$ lies in the range of an $X^{**}$-valued bv-measure if and only if $T$ is integral.*

**Definition 4.2.5** *A series $\sum_n x_n$ in a Banach space $X$ is called weakly unconditionally Cauchy(wuC) if, given any permutation $\pi$ of the natural numbers, $(\sum_{k=1}^{n} x_{\pi(k)})$ is a weakly Cauchy sequence. This is equivalent to $\sum_{n=1}^{\infty} |\langle x_n, x^* \rangle| < \infty$ for all $x^* \in X^*$.*

From the last definition it is clear that a series $\sum_n x_n^*$ in $X^*$ is weakly unconditionally Cauchy if $\sum_{n=1}^{\infty} |\langle x_n^*, x^{**} \rangle| < \infty$ , $\forall\, x^{**} \in X^{**}$. A sequence $(x_n^*)$ in $X^*$ is said to be weakly absolutely summable (or in $\ell_w^1(X^*)$) if $\sum_{n=1}^{\infty} |\langle x_n^*, x \rangle| < \infty\ \forall\, x \in X$. Alternatively, the series $\sum_{i=1}^{\infty} x_i^*$ is called weakly absolutely convergent in this case.

It is possibly general knowledge (at least for Analysts) that weakly absolutely convergent series in $X^*$ are indeed weakly unconditionally Cauchy (and the converse is of course true by definition). Since we do not find a reference for the proof of this fact in the literature, we present one here.

**Lemma 4.2.6** *Consider any Banach space $X$. Weakly absolutely convergent series in $X^*$ are unconditionally Cauchy.*

**Proof** Let $(a_i) \in \ell_w^1(X^*)$. Take any $x^{**} \in X^{**}$, $\|x^{**}\| \leq 1$. Then there exists a net $(x_\delta) \subset X$ with $\|x_\delta\| \leq 1$ for all $\delta$, such that $x_\delta \to x^{**}$ with respect to the weak* topology $\sigma(X^{**}, X^*)$. For $m \in \mathbb{N}$ we thus have,

$$\sum_{i=1}^{m} |\langle x^{**}, a_i \rangle| = \lim_{\delta} \sum_{i=1}^{m} |\langle x_\delta, a_i \rangle|.$$

But

$$\begin{aligned}\sum_{i=1}^{m} |\langle x_\delta, a_i \rangle| &\leq \sum_{i=1}^{\infty} |\langle x_\delta, a_i \rangle| \\ &\leq \sup_{\|x\| \leq 1} \sum_{i=1}^{\infty} |\langle x, a_i \rangle| = \epsilon_1((a_i)) < \infty, \quad \text{for all } m \in \mathbb{N},\ \forall\, \delta.\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{i=1}^{m} |\langle x^{**}, a_i \rangle| &\leq \sup_{\delta} \sum_{i=1}^{m} |\langle x_\delta, a_i \rangle| \\ &\leq \epsilon_1((a_i)) < \infty.\end{aligned}$$

Hence $\sum_{i=1}^{\infty} |\langle x^{**}, a_i \rangle| < \infty$. Therefore $\sum_i a_i$ is a weakly unconditionally Cauchy series in $X^*$.

□

Although the authors of the paper [32] do not mention the space $m(X)$ (which was introduced in the paper [14]), their main result in [32] gives in fact an important connection between the spaces $m(X^*)$ and $\lambda_X$. We recall both the result and its proof.

**Theorem 4.2.7** ([32], Marchena and Piñeiro, Theorem 1) *Let $X$ be a Banach space and let $(\alpha_n)$ be a bounded scalar sequence. Then $(\alpha_n) \in \lambda_X$ if and only if $\sum_{i=1}^{\infty} |\alpha_i| \|x_i^*\|$ converges for all weakly unconditionally Cauchy series $\sum_n x_n^*$ in $X^*$.*

**Proof** Let $(\alpha_n) \in \lambda_X$. By (4.2.4) we can define a linear map $U$ from $(c_0)_s(X)$ into $\mathcal{PI}(\ell^1, X)$ by

$$U((x_i)) = \sum_{i=1}^{\infty} \alpha_i e_i^* \otimes x_i.$$

$U$ is continuous, since it has closed graph. The space $\mathcal{N}(\ell^1, X)$ is isometric to a subspace of $\mathcal{PI}(\ell^1, X)$ (cf. [23], pp 410). Hence, since $U$ maps each finite sequence $(x_1, x_2, \ldots, x_n, 0, \ldots)$ onto a nuclear operator, the continuity of $U$ yields $U((c_0)_s(X)) \subset \mathcal{N}(\ell^1, X)$. Thus we consider $U$ as a bounded linear operator from $(c_0)_s(X)$ into $\mathcal{N}(\ell^1, X)$. Its dual operator $U^*$ takes $L(X, (\ell^1)^{**})$ into $\ell_s^1(X^*)$ (cf. [23], pp 449); in particular $U^*(L(X, \ell^1)) \subset \ell_s^1(X^*)$. By trace duality and the isometric identification of $L(X, \ell^1)$ with $\ell_w^1(X^*)$, we have:

$$\begin{aligned}
\langle U^*((x_i^*)), (x_n) \rangle &= \operatorname{tr}(\sum_i \alpha_i x_i^* \otimes x_i) \\
&= \sum_i \alpha_i \langle x_i, x_i^* \rangle \\
&= \langle (\alpha_i x_i^*), (x_i) \rangle
\end{aligned}$$

for all $(x_i) \in (c_0)_s(X)$. Hence $U^*((x_i^*)) = (\alpha_i x_i^*)$, for all $(x_i^*) \in \ell_w^1(X^*)$. We have obtained $\sum_i |\alpha_i| \|x_i^*\| < \infty$ for all $(x_i^*) \in \ell_w^1(X^*)$.

Conversely, suppose $\sum_{n=1}^{\infty} |\alpha_n| \|x_n^*\| < \infty$ for all $(x_n^*) \in \ell_w^1(X^*)$. We consider the operator $V$ defined by

$$V : \sum_{n=1}^{\infty} x_n^* \otimes e_n \in K(X, \ell_1) \to (\alpha_n x_n^*) \in \ell_s^1(X^*).$$

Dualising we obtain

$$V^* : \ell_s^{\infty}(X^{**}) \to \mathcal{I}(\ell^1, X^{**}).$$

Again we use the trace duality to show that

$$V^*((x_n^{**})) = \sum_{n=1}^{\infty} e_n^* \otimes \alpha_n x_n^{**} \in \mathcal{I}(\ell^1, X^{**})$$

for all $(x_n^{**}) \in \ell_s^{\infty}(X^{**})$. In particular,

$$V^*((x_n)) = \sum_{n=1}^{\infty} e_n^* \otimes \alpha_n x_n \in \mathcal{I}(\ell^1, X^{**})$$

for all $(x_n) \in (c_0)_s(X)$. By Theorem 8 in [10](pp 233), this implies that $V^*$ maps continuously $(c_0)_s(X)$ into $\mathcal{I}(\ell^1, X)$. As before, actually $V^*$ maps $(c_0)_s(X)$ into $\mathcal{N}(\ell^1, X) \subseteq \mathcal{PI}(\ell^1, X)$. By Theorem 4.2.4 this concludes the proof. □

It clearly follows from Theorem 4.2.7 that

**Theorem 4.2.8** *For any Banach space $X$ we have $m(X^*) = \lambda_X$.*

**Corollary 4.2.9** *The bounded scalar sequence $(\alpha_i)$ is in $m(X^*)$, if and only if for all null sequences $(x_i)$ in $X$ the sequence $(\alpha_i x_i)$ is in the range of some $X$-valued measure with bounded variation.*

Realising the relationship in Theorem 4.2.8, several results in the paper [32] now follow easily from corresponding results (with easy proofs) in the previous section. For example:

**Proposition 4.2.10** *If $X$ is an infinite dimensional Banach space, then $\lambda_X$ is perfect and $\ell^1 \subseteq \lambda_X \subseteq \ell^2$.*

**Proof** This is so since $m(X^*)$ is perfect and by Theorem 4.1.2 we have $\ell^1 \subseteq m(X^*) \subseteq \ell^2$. □

**Proposition 4.2.11** *If $X$ is an infinite dimensional Banach space, then*

*(i) $\lambda_X = \ell^2$ if and only if $X^*$ has the Orlicz property.*

*(ii) $\lambda_X = \ell^1$ if $X$ is a Banach space whose dual $X^*$ contains an isomorphic copy of $c_0$.*

**Proof** (i) This is so since $m(X^*) = \ell^2$ if and only if $X^*$ has the Orlicz property by Theorem 4.1.5.

(ii) This is so because $m(X^*) = \ell^1$ for all Banach spaces $X$ for which $X^*$ contains an isomorphic copy of $c_0$ – this was discussed in Example 2 of the previous section. □

**Proposition 4.2.12** *Let $p \in [1, \infty]$ and suppose $X$ is an infinite dimensional $L^p$-space. Then $\lambda_X = \ell^s$, where $s = min\{p, 2\}$.*

**Proof** For $p \geq 2$, the result follows from (4.2.11) since $X^*$ has cotype 2. For $1 \leq p < 2$ the result follows from Proposition 4.1.8, since $X^*$ is an infinite dimensional $L^q$-space, with $\frac{1}{p} + \frac{1}{q} = 1$ and $m(X^*) = \ell^p$. □

**Proposition 4.2.13** *Let $X$ be a Banach space. If $Y$ is a closed subspace of $X$, then $\lambda_X \subseteq \lambda_{X/Y}$.*

**Proof** By Proposition 4.1.10 we have

$$m(X^*) \subseteq m((X/Y)^*).$$

□

The inclusion $\lambda_{X^{**}} \subseteq \lambda_X$ is obvious (it is also clear from Lemma 4.1.6, using that $X^*$ is a closed subspace of $X^{***}$). On the other hand, the authors in [32] use the fact that for $(\alpha_i) \in \lambda_X$ the operator $\sum_n e_n^* \otimes x_n^{**}$ is in $\mathcal{I}(\ell^1, X^{**})$ to see that $(\alpha_i) \in \lambda_{X^{**}}$. Thus they prove that $\lambda_X = \lambda_{X^{**}}$. Now we may apply their result to conclude that

**Proposition 4.2.14** *Let $X$ be a Banach space. Then*

$$m(X^*) = m(X^{***}).$$

A close look at the first part of the proof of Theorem 4.2.7 reveales that if $(\alpha_i) \in \lambda_X$, then $\sum_{i=1}^{\infty} \alpha_i e_i^* \otimes x_i \in \mathcal{N}(\ell^1, X)$ for all $(x_i) \in (c_0)_s(X)$. Conversely, if $\sum_{i=1}^{\infty} \alpha_i e_i^* \otimes x_i \in \mathcal{N}(\ell^1, X) \subseteq \mathcal{PI}(\ell^1, X)$ for all $(x_i) \in (c_0)_s(X)$, then by Theorem 4.2.4 the sequence $(\alpha_i x_i)$ is in the range of an $X$-valued $bv$-measure for all $(x_i) \in (c_0)_s(X)$ – thus $(\alpha_i) \in \lambda_X$ in this case. Thus we have:

**Corollary 4.2.15** *Let $X$ be a Banach space. Then*

$$(\alpha_i) \in m(X^*) \iff \sum_{i=1}^{\infty} \alpha_i e_i^* \otimes x_i \in \mathcal{N}(\ell^1, X)$$

*for all $(x_i) \in (c_0)_s(X)$.*

In particular, this says that

**Corollary 4.2.16** *Let $X$ be a Banach space. Then*

$$(\alpha_i) \in m(X^{**}) \iff \sum_{i=1}^{\infty} \alpha_i e_i^* \otimes x_i^* \in \mathcal{N}(\ell^1, X^*)$$

*for all $(x_i^*) \in (c_0)_s(X^*)$.*

We know from Lemma 4.1.6 that $m(X^{**}) \subseteq m(X)$. Trying to prove that in fact the equality $m(X^{**}) = m(X)$ holds, will clearly need some effort – it is not possible to obtain this result as a direct conclusion from the work of Piñeiro and Marchena. Our next lemma will be the key to proving the desired equality.

**Lemma 4.2.17** *Let $(\alpha_i) \in m(X)$. Consider the bounded linear operator*

$$P : K(c_0, X) \to \ell_s^1(X) : \sum_{n=1}^{\infty} e_n^* \otimes x_n \mapsto (\alpha_n x_n).$$

*$P^*$ maps $\ell_s^\infty(X^*)$ into $\mathcal{I}(X, c_0)$. Moreover, $P^*$ maps $(c_0)_s(X^*)$ into $\mathcal{N}(X, c_0)$ and*

$$(P^*((x_i^*)))^* = \sum_{i=1}^{\infty} \alpha_i e_i^* \otimes x_i^*$$

*for all $(x_i^*) \in (c_0)_s(X^*)$.*

**Proof** The linear operator $P$ is clearly bounded, since

$$\pi_1((\alpha_i x_i)) \leq \|(\alpha_i)\|_{1,1} \epsilon_1((x_i)).$$

Hence

$$P^* : \ell_s^\infty(X^*) \to K(c_0, X)^* \cong \mathcal{I}(X, c_0^{**})$$

is also bounded. The isometry $K(c_0, X)^* \cong \mathcal{I}(X, c_0^{**})$ is defined by trace duality (cf. [23], pp 449). Also,

$$\langle P^*((x_i^*)), \sum_{n=1}^{\infty} e_n^* \otimes x_n \rangle = \langle (x_i^*), (\alpha_n x_n) \rangle = \sum_{i=1}^{\infty} \alpha_i x_i^*(x_i).$$

Fix $(x_i^*) \in \ell_s^\infty(X^*)$ and let

$$T_k : X \to \ell^\infty : x \mapsto (\alpha_n x_n^*(x))(\leq k).$$

For each $k \in \mathbb{N}$, $T_k$ is bounded and $T_k = \sum_{n=1}^{k} \alpha_n x_n^* \otimes e_n$. Now

$$\sum_{j=1}^{k} \alpha_j (x_j^*(x_i))_i \otimes e_j = T_k \circ \sum_{n=1}^{\infty} e_n^* \otimes x_n.$$

Thus

$$\text{tr}(T_k \circ \sum_{n=1}^{\infty} e_n^* \otimes x_n) = \sum_{j=1}^{k} \alpha_j x_j^*(x_j).$$

Hence

$$\langle P^*((x_i^*)), \sum_{n=1}^{\infty} e_n^* \otimes x_n \rangle = \sum_{i=1}^{\infty} \alpha_i x_i^*(x_i) = \lim_k \langle T_k, \sum_{n=1}^{\infty} e_n^* \otimes x_n \rangle$$

for all $(x_i) \in \ell_c^1(X)$. This shows that

$$P^*((x_i^*))(x) = \lim_k (\sum_{n=1}^{k} \alpha_n x_n^* \otimes e_n)(x)$$

for all $x \in X$. Using that $(\alpha_n) \in c_0$, it is easily seen that $(\sum_{n=1}^{k} \alpha_n x_n^*(x) e_n)_k$ is a Cauchy sequence in $c_0$. We conclude that

$$P^*((x_i^*))(x) = (\sum_{n=1}^{\infty} \alpha_n x_n^* \otimes e_n)(x) \in c_0.$$

It follows (by Theorem 8 in [10], pp 233) that $P^*((x_i^*)) \in \mathcal{I}(X, c_0)$. Since $c_0$ has the (metric) approximation property, $\mathcal{N}(X, c_0)$ is isometric to a subspace of $\mathcal{I}(X, c_0)$ (cf [23], pp 410). For $(x_i^*) \in (c_0)_s(X^*)$, the continuity of $P^*$ thus implies that

$$\lim_n P^*((x_i^*)(\leq n)) = P^*((x_i^*)) \in \mathcal{N}(X, c_0),$$

since each $P^*((x_i^*)(\leq n)) = \sum_{i=1}^{n} \alpha_i x_i^* \otimes e_i$ is in $\mathcal{N}(X, c_0)$. The dual operator is also nuclear (cf. [23], pp 379); thus

$$(P^*((x_i^*)))^* \in \mathcal{N}(\ell^1, X^*).$$

Moreover,

$$\langle (P^*((x_i^*))^*((\gamma_i)), x\rangle = \langle (\gamma_i), (\alpha_i x_i^*(x))_i\rangle = (\sum_n \gamma_n \alpha_n x_n^*)(x)$$

for all $x \in X$ and all $(\gamma_i) \in \ell^1$. On the other hand,

$$\langle (\sum_{i=1}^{\infty} \alpha_i e_i^* \otimes x_i^*)((\gamma_i)), x\rangle = (\sum_n \gamma_n \alpha_n x_n^*)(x)$$

for all $x \in X$ and all $(\gamma_i) \in \ell^1$. Finally we may conclude that

$$(P^*((x_i^*)))^* = \sum_{i=1}^{\infty} \alpha_i e_i^* \otimes x_i^*.$$

□

**Theorem 4.2.18** *Let $X$ be Banach space. Then*

$$m(X) = m(X^{**}).$$

**Proof** We need only prove that $m(X) \subseteq m(X^{**})$. Let $(\alpha_i) \in m(X)$. It follows from Lemma 4.2.17 that

$$\sum_{i=1}^{\infty} \alpha_i e_i^* \otimes x_i^* \in \mathcal{N}(\ell^1, X^*)$$

for all $(x_i^*) \in (c_0)_s(X^*)$. Hence $(\alpha_i) \in m(X^{**})$ by Corollary 4.2.16. □

**Corollary 4.2.19** *Let $X$ be a Banach space. The following are equivalent:*

*(i)* $(\alpha_i) \in m(X)$

*(ii)* $\sum_{i=1}^{\infty} \alpha_i e_i^* \otimes x_i^* \in \mathcal{N}(\ell^1, X^*)$, *for all* $(x_i^*) \in (c_0)_s(X^*)$

*(iii)* $(\alpha_i) \in \lambda_{X^*}$

*(iv)* *For every null sequence* $(x_i^*)$ *in* $X^*$, *the sequence* $(\alpha_i x_i^*)$ *lies in the range of some* $X^*$*-valued measure with bounded variation.*

**Remark:** It is interesting to notice that from our Lemma 4.2.17 it follows that for all $(\alpha_i) \in m(X)$ and $(x_i^*) \in (c_0)_s(X^*)$ holds

$$\begin{aligned} \nu(\sum_{i=1}^{\infty} \alpha_i e_i^* \otimes x_i^*) &= \nu(P^*((x_i^*))) \leq \|P^*\| \pi_\infty((x_i^*)) \\ &= (\sup_i \|x_i^*\|)\|P\| = (\sup_i \|x_i^*\|) \sup\{\sum_{i=1}^{\infty} |\alpha_i| \|x_i\| : \epsilon_1((x_i)) \leq 1\} \\ &= (\sup_i \|x_i^*\|) \|(\alpha_i)\|_{1,1}. \end{aligned}$$

Hence we conclude that

**Corollary 4.2.20** *The bilinear operator*

$$m(X) \times (c_0)_s(X^*) \to \mathcal{N}(\ell^1, X^*) : ((\alpha_i), (x_i^*)) \mapsto \sum_{i=1}^{\infty} \alpha_i e_i^* \otimes x_i^*$$

*is continuous.*

Question?
using principle of local reflexivity

$m(X) = m(X^{**})$

# Chapter 5

# General families of summing multipliers

## 5.1 $p$-Summing multipliers

Extending the notion of absolutely summing multiplier, we now define the notion of *p-summing multiplier* as follows:

**Definition 5.1.1** *Let $1 \leq p \leq \infty$. A scalar sequence $(\alpha_i)$ is called a p-summing multiplier for a Banach space $X$, if $\sum_{n=1}^{\infty} \|\alpha_n x_n\|^p < \infty$ for all sequences $(x_n) \in \ell^p_w(X)$. Put*

$$m_p(X) = \{(\alpha_n) \in \omega : \sum_{n=1}^{\infty} \|\alpha_n x_n\|^p < \infty \ , \forall (x_n) \in \ell^p_w(X)\}.$$

In the next section (on more general families of summing multipliers) we show that $m_p(X) \subseteq \ell^\infty$. Since each $(x_i) \in \ell^p_w(X)$ is a norm bounded sequence in $X$, it is also clear that $\ell^p \subseteq m_p(X)$.

On the vector space $m_p(X)$ we define a norm

$$\|(\alpha_i)\|_{p,p} := \sup_{\epsilon_p((x_i)) \leq 1} \left( \sum_{n=1}^{\infty} |\alpha_n|^p \|x_n\|^p \right)^{1/p},$$

which is well defined because for each $(\alpha_i) \in m_p(X)$ this is the operator norm of the bounded (having closed graph) linear operator

$$T_\alpha : \ell^p_w(X) \to \ell^p_s(X) :: (x_i) \mapsto (\alpha_i x_i).$$

It will follow from the more general case in the next section that the space $m_p(X)$ is a complete normed space with respect to the above operator norm.

We first prove an inclusion relation between the different $p$-summing multiplier spaces of a fixed Banach space.

**Theorem 5.1.2** *If* $1 < p < q < \infty$ *then* $m_p(X) \subseteq m_q(X)$. *Moreover, if* $(\alpha_i) \in m_p(X)$, *then* $\|(\alpha_i)\|_{q,q} \leq \|(\alpha_i)\|_{p,p}$.

**Proof** Let $(\alpha_n) \in m_p(X)$. Take any arbitrary $(x_i) \in \ell_w^q(X)$. Then

$$\left(\sum_{i=1}^{n} |\alpha_i|^q \|x_i\|^q\right)^{1/p} = \left(\sum_{i=1}^{n} |\alpha_i|^p \|\lambda_i x_i\|^p\right)^{1/p}$$

where

$$\lambda_i = |\alpha_i|^{(q/p)-1} \|x_i\|^{(q/p)-1}.$$

Now $q > p$ and $\frac{1}{q/p} + \frac{1}{q/(q-p)} = \frac{p}{q} + \frac{q-p}{q} = \frac{q}{q} = 1$. We have

$$\begin{aligned}
\left(\sum_{i=1}^{n} |\alpha_i|^q \|x_i\|^q\right)^{1/p} &= \left(\sum_{i=1}^{n} |\alpha_i|^p \|\lambda_i x_i\|^p\right)^{1/p} \\
&\leq \|(\alpha_i)\|_{p,p} \sup_{\|x^*\| \leq 1} \left(\sum_{i=1}^{n} \lambda_i^p |\langle x_i, x^* \rangle|^p\right)^{1/p} \\
&\leq \|(\alpha_i)\|_{p,p} \sup_{\|x^*\| \leq 1} \left[\left(\sum_{i=1}^{n} (\lambda_i^p)^{q/(q-p)}\right)^{(q-p)/q} \left(\sum_{i=1}^{n} (|\langle x_i, x^* \rangle|^p)^{q/p}\right)^{p/q}\right]^{1/p} \\
&= \|(\alpha_i)\|_{p,p} \sup_{\|x^*\| \leq 1} \left(\sum_{i=1}^{n} |\langle x_i, x^* \rangle|^q\right)^{1/q} \left(\sum_{i=1}^{n} \lambda_i^{pq/(q-p)}\right)^{(q-p)/pq} \\
&= \|(\alpha_i)\|_{p,p}\, \epsilon_q((x_i)(\leq n)) \left(\sum_{i=1}^{n} |\alpha_i|^q \|x_i\|^q\right)^{(1/p)-(1/q)}
\end{aligned}$$

Dividing both sides by

$$\left(\sum_{i=1}^{n} |\alpha_i|^q \|x_i\|^q\right)^{(1/p)-(1/q)}$$

yields

$$\begin{aligned}
\left(\sum_{i=1}^{n} |\alpha_i|^q \|x_i\|^q\right)^{1/q} &\leq \|(\alpha_i)\|_{p,p}\, \epsilon_q((x_i)(\leq n)) \\
&\leq \|(\alpha_i)\|_{p,p}\, \epsilon_q((x_i)) \quad \text{for all } n \in \mathbb{N}.
\end{aligned}$$

Therefore

$$\left(\sum_{i=1}^{\infty} |\alpha_i|^q \|x_i\|^q\right)^{1/q} \leq \|(\alpha_i)\|_{p,p}\, \epsilon_q((x_i)) < \infty$$

for all $(x_i) \in \ell_w^q(X)$. Thus $(\alpha_i) \in m_q(X)$. The norm inequality for $(\alpha_i) \in m_p(X)$ is also clear from the last inequality. □

As in Lemma 4.1.6, using a similar argument, one easily verifies that

**Lemma 5.1.3** *If a Banach space $X$ is topologically isomorphic to a closed subspace of a Banach space $Y$, then*

$$m_p(Y) \subseteq m_p(X).$$

**Theorem 5.1.4** *Let $(\alpha_i)$ be a bounded scalar sequence. Then $(\alpha_i) \in m_p(X^*)$ if and only if $\ell^p \to X : (\beta_i) \mapsto \sum_{i=1}^{\infty} \beta_i \alpha_i x_i$ is integral for all sequences $(x_i) \in \ell_s^q(X)$. Here $1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1$.*

**Proof** Let $(\alpha_i) \in m_p(X^*)$. Using $K(X, \ell^p) \cong \ell_c^p(X^*)$, define $P : K(X, \ell^p) \to \ell_s^p(X^*)$ by

$$P(\sum_{n=1}^{\infty} x_n^* \otimes e_n) = (\alpha_n x_n^*).$$

Then $P$ is linear and bounded with

$$\begin{aligned} \|P(\sum_{n=1}^{\infty} x_n^* \otimes e_n)\| &= \|(\alpha_n x_n^*)\| = \left(\sum_{n=1}^{\infty} |\alpha_n|^p \|x_n^*\|^p\right)^{1/p} \\ &\le \|(\alpha_n)\|_{p,p}\, \epsilon_p((x_n^*)) = \|(\alpha_n)\|_{p,p} \|\sum_{n=1}^{\infty} x_n^* \otimes e_n\|. \end{aligned}$$

Consider $P^* : \ell_s^q(X^{**}) \to K(X, \ell^p)^*$. Since $\ell^p$ has the metric approximation property

$$P^* : \ell_s^q(X^{**}) \to K(X, \ell^p)^* = \mathcal{I}(\ell^p, X^{**}) \quad \text{(cf. [23], pp 449)}$$

is bounded . Also,

$$\begin{aligned} \left\langle P^*((x_n^{**})), \sum_{n=1}^{\infty} x_n^* \otimes e_n \right\rangle &= \left\langle (x_n^{**}), P\left(\sum_{n=1}^{\infty} x_n^* \otimes e_n\right)\right\rangle \\ &= \langle (x_n^{**}), (\alpha_n x_n^*)\rangle \\ &= \sum_{n=1}^{\infty} \alpha_n \langle x_n^*, x_n^{**}\rangle \\ &= \sum_{n=1}^{\infty} \alpha_n x_n^{**}(x_n^*). \end{aligned} \tag{5.1}$$

Consider $\sum_{n=1}^{\infty} x_n^* \otimes e_n$ a bounded linear operator from $X^{**}$ into $\ell^p$ in the obvious way (the extension to $X^{**}$ of the given compact operator on $X$) and notice that

$$\left(\sum_{n=1}^{k} x_n^* \otimes e_n\right) \circ \left(\sum_{n=1}^{\infty} e_n^* \otimes \alpha_n x_n^{**}\right) = \sum_{n=1}^{k} (\alpha_j x_n^*(x_j^{**}))_j \otimes e_n \in \mathcal{N}(\ell^p, \ell^p),\ \forall k \in \mathbb{N}.$$

Then by

$$\Phi\left(\sum_{n=1}^{\infty} x_n^* \otimes e_n\right) = \lim_k \operatorname{tr}\left(\sum_{n=1}^{k} (\alpha_j x_n^*(x_j^{**}))_j \otimes e_n\right)$$
$$= \sum_{n=1}^{\infty} \langle (\alpha_j x_n^*(x_j^{**}))_j, e_n\rangle = \sum_{n=1}^{\infty} \alpha_n x_n^*(x_n^{**}), \tag{5.2}$$

we define a bounded linear functional on $K(X, \ell^p)$ with

$$\left|\Phi\left(\sum_{n=1}^{\infty} x_n^* \otimes e_n\right)\right| \le \pi_q((x_n^{**}))\|(\alpha_i)\|_{p,p} \left\|\sum_{n=1}^{\infty} x_n^* \otimes e_n\right\|.$$

From (5.1) and (5.2) we have

$$P^*((x_n^{**})) = \sum_{n=1}^{\infty} e_n^* \otimes \alpha_n x_n^{**}.$$

Hence $\sum_n e_n^* \otimes \alpha_n x_n^{**} \in \mathcal{I}(\ell^p, X^{**})$ for all $(x_n^{**}) \in \ell_s^q(X^{**})$.

In particular $P^*((x_n)) = \sum_{n=1}^{\infty} e_n^* \otimes \alpha_n x_n \in \mathcal{I}(\ell^p, X^{**})$ for all $(x_n) \in \ell_s^q(X)$. However, $(\sum_n e_n^* \otimes \alpha_n x_n)((\beta_n)) \in X$ for each $(\beta_i) \in \ell^p$; i.e. $(\sum_n e_n^* \otimes \alpha_n x_n)(\ell^p) \subset X$. Thus

$$\sum_n e_n^* \otimes \alpha_n x_n \in \mathcal{I}(\ell^p, X),\ \forall\ (x_n) \in \ell_s^q(X) \quad \text{(cf [10], pp 233).}$$

Conversely, let $(\alpha_i) \in \ell^\infty$ be given. Suppose

$$\ell^p \to X : (\beta_i) \mapsto \sum_i \beta_i \alpha_i x_i$$

is integral for all $(x_i) \in \ell_s^q(X)$. Define

$$Q : \ell_s^q(X) \to \mathcal{I}(\ell^p, X) :: Q((x_n)) = \sum_n e_n^* \otimes \alpha_n x_n \in \mathcal{I}(\ell^p, X).$$

Then $Q$ has closed graph. In fact, let $(x_i^n) \xrightarrow[\infty]{n} (x_i) \in \ell_s^q(X)$, and $\sum_i e_i^* \otimes \alpha_i x_i^n \xrightarrow[\infty]{n} T$. Then $T((\beta_i)) = \lim_n \sum_i \alpha_i \beta_i x_i^n$, and

$$\|\sum_i \alpha_i\beta_i x_i - \sum_i \alpha_i \beta_i x_i^n\| \le (\sup_i |\alpha_i|)\sum_i |\beta_i|\|x_i - x_i^n\|$$
$$\le (\sup_i |\alpha_i|)\left(\sum_i |\beta_i|^p\right)^{1/p}\left(\sum_i \|x_i - x_i^n\|^q\right)^{1/q} \xrightarrow[\infty]{n} 0.$$

Since $\ell^q = (\ell^p)^*$ has the metric approximation property, it follows (cf [23] pp 410) that $\mathcal{N}(\ell^p, X)$ is isometric to a subspace of $\mathcal{I}(\ell^p, X)$. Now

$$Q((x_1, x_2 \ldots x_n, 0, 0 \ldots)) = \sum_{i=1}^{n} e_i^* \otimes \alpha_i x_i$$

is a nuclear operator for all $n \in \mathbb{N}$ and $(x_i) \in \ell_s^q(X)$. From the continuity of $Q$ (having closed graph) and the fact that $(x_1, x_2 \ldots x_n, 0, 0 \ldots) \underset{\infty}{\overset{n}{\to}} (x_i)$ in the norm of $\ell_s^q(X)$, it follows that $Q(\ell_s^q(X)) \subset \mathcal{N}(\ell^p, X)$. Hence $Q^* : L(X, \ell^p)(= \mathcal{N}(\ell^p, X)^*) \to \ell_s^p(X^*)$ is bounded.

Using $\ell_w^p(X^*) \overset{\text{isom}}{\cong} L(X, \ell^p)$, it follows that

$$\begin{aligned}
& \langle Q^*((x_n^*)), (x_n)\rangle = \langle (x_n^*), Q((x_n))\rangle \\
= \ & \langle (x_n^*), \sum_n e_n^* \otimes \alpha_n x_n\rangle \\
= \ & \langle \sum_n x_n^* \otimes e_n, \sum_n e_n^* \otimes \alpha_n x_n\rangle \\
= \ & \operatorname{tr}\left(\left(\sum_n e_n^* \otimes \alpha_n x_n\right) \circ \left(\sum_n x_n^* \otimes e_n\right)\right) \\
= \ & \operatorname{tr}\left(\sum_n x_n^* \otimes \alpha_n x_n\right) = \sum_n \alpha_n x_n^*(x_n) \\
= \ & \langle (\alpha_n x_n^*), (x_n)\rangle \text{ for all } (x_n) \in \ell_s^q(X).
\end{aligned}$$

Hence,

$$Q^*((x_n^*)) = (\alpha_n x_n^*).$$

Therefore $\sum_n |\alpha_n|^p \|x_n^*\|^p < \infty$, $\forall\ (x_n^*) \in \ell_w^p(X^*)$. Thus $(\alpha_n) \in m_p(X^*)$. □

It is clear from the above proof of (5.1.4) that

**Theorem 5.1.5** *Let $(\alpha_n) \in \ell^\infty$ and $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. The following are equivalent:*

*(a)* $(\alpha_n) \in m_p(X^*)$.

*(b)* $\sum_n e_n^* \otimes \alpha_n x_n : \ell^p \to X$ *is integral for all* $(x_i) \in \ell_s^q(X)$.

*(c)* $\sum_n e_n^* \otimes \alpha_n x_n : \ell^p \to X$ *is nuclear for all* $(x_i) \in \ell_s^q(X)$.

**Corollary 5.1.6** *Let $(\alpha_n) \in \ell^\infty$ and $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. The following are equivalent:*

*(a)* $(\alpha_n) \in m_p(X^{**})$.

*(b)* $\sum_n e_n^* \otimes \alpha_n x_n^* : \ell^p \to X^*$ *is integral for all* $(x_i^*) \in \ell_s^q(X^*)$.

*(c)* $\sum_n e_n^* \otimes \alpha_n x_n^* : \ell^p \to X^*$ *is nuclear for all* $(x_i^*) \in \ell_s^q(X^*)$.

**Lemma 5.1.7** *Let $(\alpha_i) \in m_p(X)$ and let $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p, q < \infty$. Consider the bounded linear operator*

$$P : K(\ell^q, X) \to \ell_s^p(X) : \sum_{n=1}^{\infty} e_n^* \otimes x_n \mapsto (\alpha_n x_n).$$

*$P^*$ maps $\ell^q_s(X^*)$ into $\mathcal{N}(X, \ell^q)$ and*

$$(P^*((x_i^*)))^* = \sum_{i=1}^{\infty} \alpha_i e_i^* \otimes x_i^*$$

*for all $(x_i^*) \in \ell^q_s(X^*)$.*

**Proof** The linear operator $P$ is clearly bounded, since

$$\pi_p((\alpha_i x_i)) \le \|(\alpha_i)\|_{p,p}\, \epsilon_p((x_i)).$$

Hence

$$P^* : \ell^q_s(X^*) \to K(\ell^q, X)^* \cong \mathcal{I}(X, \ell^q)$$

is also bounded. The isometry $K(\ell^q, X)^* \cong \mathcal{I}(X, \ell^q)$ is defined by trace duality (cf. [23], pp 449). Also,

$$\langle P^*((x_i^*)), \sum_{n=1}^{\infty} e_n^* \otimes x_n \rangle = \langle (x_i^*), (\alpha_n x_n) \rangle = \sum_{i=1}^{\infty} \alpha_i x_i^*(x_i),$$

for all $(x_i) \in \ell^p_c(X)$. Fix $(x_i^*) \in \ell^q_s(X^*)$ and let

$$T_k : X \to \ell^q : x \mapsto (\alpha_n x_n^*(x))(\le k).$$

For each $k \in \mathbb{N}$, $T_k$ is bounded and $T_k = \sum_{n=1}^{k} \alpha_n x_n^* \otimes e_n$. Now

$$\sum_{j=1}^{k} \alpha_j (x_j^*(x_i))_i \otimes e_j = T_k \circ \sum_{n=1}^{\infty} e_n^* \otimes x_n.$$

Thus

$$\mathrm{tr}(T_k \circ \sum_{n=1}^{\infty} e_n^* \otimes x_n) = \sum_{j=1}^{k} \alpha_j x_j^*(x_j).$$

Hence

$$\langle P^*((x_i^*)), \sum_{n=1}^{\infty} e_n^* \otimes x_n \rangle = \sum_{i=1}^{\infty} \alpha_i x_i^*(x_i) = \lim_k \langle T_k, \sum_{n=1}^{\infty} e_n^* \otimes x_n \rangle$$

for all $(x_i) \in \ell^p_c(X)$. This shows that

$$P^*((x_i^*))(x) = \lim_k (\sum_{n=1}^{k} \alpha_n x_n^* \otimes e_n)(x)$$

for all $x \in X$, i.e. that

$$P^*((x_i^*))(x) = (\sum_{n=1}^{\infty} \alpha_n x_n^* \otimes e_n)(x) \in \ell^q.$$

Since $\ell^q$ has the (metric) approximation property, $\mathcal{N}(X, \ell^q)$ is isometric to a subspace of $\mathcal{I}(X, \ell^q)$ (cf [23], pp 410). The continuity of $P^*$ thus implies that

$$\lim_n P^*((x_i^*)(\leq n)) = P^*((x_i^*)) \in \mathcal{N}(X, \ell^q),$$

since each $P^*((x_i^*)(\leq n)) = \sum_{i=1}^n \alpha_i x_i^* \otimes e_i$ is in $\mathcal{N}(X, \ell^q)$. The dual operator is also nuclear (cf. [23], pp 379); thus

$$(P^*((x_i^*)))^* \in \mathcal{N}(\ell^p, X^*).$$

Moreover,

$$\langle (P^*((x_i^*))^*((\gamma_i)), x\rangle = \langle (\gamma_i), (\alpha_i x_i^*(x))_i\rangle = (\sum_n \gamma_n \alpha_n x_n^*)(x)$$

for all $x \in X$ and all $(\gamma_i) \in \ell^p$. On the other hand,

$$\langle (\sum_{i=1}^{\infty} \alpha_i e_i^* \otimes x_i^*)((\gamma_i)), x\rangle = (\sum_n \gamma_n \alpha_n x_n^*)(x)$$

for all $x \in X$ and all $(\gamma_i) \in \ell^p$. Hence

$$(P^*((x_i^*)))^* = \sum_{i=1}^{\infty} \alpha_i e_i^* \otimes x_i^*.$$

□

**Theorem 5.1.8** *Let $X$ be a Banach space. Then*

$$m_p(X) = m_p(X^{**}).$$

**Proof** By Lemma 5.1.3 we need only prove that $m_p(X) \subseteq m_p(X^{**})$. Let $(\alpha_i) \in m_p(X)$. It follows from Lemma 5.1.7 that

$$\sum_{i=1}^{\infty} \alpha_i e_i^* \otimes x_i^* \in \mathcal{N}(\ell^p, X^*)$$

for all $(x_i^*) \in \ell_s^q(X^*)$. Hence $(\alpha_i) \in m_p(X^{**})$ by Corollary 5.1.6. □

## 5.2 $(\Lambda, \Sigma)$-summing multipliers

Throughout this section we assume that the scalar sequence spaces $\Lambda$ and $\Sigma$ are normal $BK$ spaces with the $AK$ property.

**Definition 5.2.1** *Let $\Lambda$ and $\Sigma$ be $BK$-spaces with $AK$. A scalar sequence $(\xi_i)$ is said to be a $(\Lambda, \Sigma)$-summing multiplier for a Banach space $E$ if $(\xi_i x_i) \in \Sigma_s(E)$ for all $(x_i) \in \Lambda_w(E)$.*

Put

$$\begin{aligned} m_{\Lambda,\Sigma}(E) &= \{(\xi_i) \in w : (\xi_i x_i) \in \Sigma_s(E) \ , \ \forall \ (x_i) \in \Lambda_w(E)\} \\ &= \{(\xi_i) \in w : (\|\xi_i x_i\|) \in \Sigma \ , \ \forall \ (x_i) \in \Lambda_w(E)\}. \end{aligned}$$

To see that $m_{\Lambda,\Sigma}(E) \subseteq \ell^\infty$, consider arbitrary $(\alpha_i) \in m_{\Lambda,\Sigma}(E)$ and let

$$T_n : \Lambda_w(E) \to \Sigma_s(E) :: (x_i) \mapsto (\alpha_i x_i)(\leq n).$$

Each $T_n$ has closed graph, hence is a bounded linear operator. And

$$\pi_\Sigma((T_n((x_i)))) = \|(|\alpha_i|\|x_i\|)(\leq n)\|_\Sigma \leq \|(|\alpha_i|\|x_i\|)\|_\Sigma$$

for all $n$. The set $\{T_n : n = 1, 2, \ldots\}$ is thus pointwise bounded, hence also uniformly bounded. There exists $M > 0$ such that

$$\pi_\Sigma((T_n((x_i)))) \leq M\epsilon_\Lambda((x_i))$$

for all $n$. In particular, for any $x \in E$ such that $\|x\| = 1$ we have

$$|\alpha_i| = \|(0, \ldots, 0, |\alpha_i|\|x\|, 0, 0, \ldots)\|_\Sigma \leq M$$

for all $i = 1, 2, \ldots$. Since the sequences in $\Lambda_w(E)$ are norm bounded in $E$, it is easy to see that $\Sigma \subseteq m_{\Lambda,\Sigma}(E)$.

On the vector space $m_{\Lambda,\Sigma}(E)$ we define a norm by

$$\begin{aligned} \|(\xi_i)\|_{\Lambda,\Sigma} &= \sup\{\pi_\Sigma((\xi_i x_i)) : \epsilon_\Lambda((x_i)) \leq 1\} \\ &= \sup\{\|(\|\xi_i x_i\|)\|_\Sigma : \epsilon_\Lambda((x_i)) \leq 1\}. \end{aligned}$$

**Theorem 5.2.2** $(m_{\Lambda,\Sigma}(E), \|\cdot\|_{\Lambda,\Sigma})$ is a complete normed space.

**Proof** Let $((\alpha_{m,i})_i)_m$ be a Cauchy sequence in $m_{\Lambda,\Sigma}(E)$. Then for every $\epsilon > 0$ there is an $N > 0$ such that for all $(x_i)$ in the unit ball of $\Lambda_w(E)$ and all $m, n > N$

$$\|(\|(\alpha_{m,i} - \alpha_{n,i})x_i\|)_i\|_\Sigma < \epsilon. \tag{5.3}$$

So for every $i = 1, 2, \ldots$ we have

$$|\alpha_{m,i} - \alpha_{n,i}| < \epsilon \qquad (\forall m, n > N). \tag{5.4}$$

From (5.4) we see that for each fixed $i$, $(\alpha_{1,i}, \alpha_{2,i}, \ldots)$ is a Cauchy sequence of numbers. It converges since $\mathbb{R}$ and $\mathbb{C}$ are complete. Say $\alpha_{m,i} \to \alpha_i$ as $m \to \infty$. We show that $(\alpha_i) \in m_{\Lambda,\Sigma}(E)$ and $(\alpha_{m,i})_i \to (\alpha_i)$ as $m \to \infty$. From (5.3), we have for all $\epsilon_\Lambda((x_i)) \leq 1$, all $m, n > N$ and all natural numbers $k$ that

$$\|(|\alpha_{m,i} - \alpha_{n,i}|\|x_i\|)(\leq k)\|_\Sigma < \epsilon.$$

Letting $n \to \infty$, we obtain for $m > N$

$$\|(|\alpha_{m,i} - \alpha_i| \|x_i\|)_i(\leq k)\|_\Sigma < \epsilon.$$

Now let $k \to \infty$, then for $m > N$

$$\|(|\alpha_{m,i} - \alpha_i| \|x_i\|)\|_\Sigma \leq \epsilon \tag{5.5}$$

for all $(x_i)$ in the unit ball of $\Lambda_w(E)$. This shows that $(\alpha_{m,i} - \alpha_i) \in m_{\Lambda,\Sigma}(E)$. Since $(\alpha_{m,i})$ belongs to the vector space $m_{\Lambda,\Sigma}(E)$, it follows that

$$(\alpha_i) = (\alpha_{m,i}) + (\alpha_i - \alpha_{m,i}) \in m_{\Lambda,\Sigma}(E).$$

Furthermore it follows from (5.5) that $(\alpha_{m,i}) \to (\alpha_i)$ if $m \to \infty$. Since $((\alpha_{m,i})_i)_m$ was an arbitrary Cauchy sequence in $m_{\Lambda,\Sigma}(E)$, this proves the completeness of $m_{\Lambda,\Sigma}(E)$. □

The proof of the following generalisation of Theorem 5.1.4 will not be discussed in full detail (since it is similar to the proof of Theorem 5.1.4.), but for the sake of completeness we choose to present an outline of the proof in this general context.

**Theorem 5.2.3** *Let $(\alpha_i)$ be a bounded scalar sequence. Let $\Lambda$ be a reflexive BK-space with AK. Then $(\alpha_i) \in m_{\Lambda,\Lambda}(X^*)$ if and only if $\Lambda \to X : (\beta_i) \to \sum_{i=1}^{\infty} \beta_i \alpha_i x_i$ is integral for all sequences $(x_i) \in \Lambda_s^\times(X)$.*

**Proof** Let $(\alpha_i) \in m_{\Lambda,\Lambda}(X^*)$. Then $(|\alpha_i| \|x_i^*\|) \in \Lambda$ for all $(x_i^*) \in \Lambda_w(X^*)$. Define

$$P : K(X, \Lambda) \to \Lambda_s(X^*) :: P\left(\sum_{n=1}^{\infty} x_n^* \otimes e_n\right) = (\alpha_n x_n^*),$$

using the isometry $K(X, \Lambda) \cong \Lambda_c(X^*)$ (cf. Chapter 0). Then $P$ is linear and bounded with

$$\pi_\Lambda\left(P\left(\sum_{n=1}^{\infty} x_n^* \otimes e_n\right)\right) = \pi_\Lambda((\alpha_n x_n^*)) \leq \|(\alpha_i)\|_{\Lambda,\Lambda} \left\|\sum_{n=1}^{\infty} x_n^* \otimes e_n\right\|.$$

Consider $P^* : \Lambda_s^\times(X^{**}) \to K(X, \Lambda)^* = \mathcal{I}(\Lambda, X^{**})$. $P^*$ is bounded and as before (refer to the proof of Theorem 5.1.4) we have by trace duality that

$$P^*((x_n^{**})) = \sum_n e_n^* \otimes \alpha_n x_n^{**} \in \mathcal{I}(\Lambda, X^{**})$$

for all $(x_n^{**}) \in \Lambda_s^\times(X^{**})$. In particular

$$P^*((x_n)) = \sum_n e_n^* \otimes \alpha_n x_n \in \mathcal{I}(\Lambda, X^{**})$$

for all $(x_n) \in \Lambda_s^\times(X)$. However, $\left(\sum_n e_n^* \otimes \alpha_n x_n\right)(\Lambda) \subseteq X$ for all $(x_n) \in \Lambda_s^\times(X)$. Thus

$$\sum_n e_n^* \otimes \alpha_n x_n \in \mathcal{I}(\Lambda, X), \ \forall\ (x_n) \in \Lambda_s^\times(X).$$

Conversely, suppose $\Lambda \to X : (\beta_i) \mapsto \sum_i \beta_i \alpha_i x_i$ is integral for all $(x_i) \in \Lambda_s^\times(X)$ and let

$$Q : \Lambda_s^\times(X) \to \mathcal{I}(\Lambda, X) :: Q((x_n)) = \sum_n e_n^* \otimes \alpha_n x_n \in \mathcal{I}(\Lambda, X).$$

As before (proof of Theorem 5.1.4) $Q$ is continuous. Since $\Lambda^\times$ has the metric approximation property (it has a Schauder basis!), it follows that $\mathcal{N}(\Lambda, X)$ is isometric to a subspace of $\mathcal{I}(\Lambda, X)$ (cf [23], pp 410). Now $Q((x_1, x_2, \ldots x_n, 0, 0 \ldots)) = \sum_{i=1}^n e_i^* \otimes \alpha_i x_i$ is a nuclear operator for all $n \in \mathbb{N}$. Because of the $AK$ property of $\Lambda^\times$, $(x_i) = \lim_n (x_i)(\leq n)$ in the norm topology of $\Lambda_s^\times(X)$. Thus $Q(\Lambda_s^\times(X)) \subset \mathcal{N}(\Lambda, X)$. Hence

$$Q : \Lambda_s^\times(X) \to \mathcal{N}(\Lambda, X) :: (x_i) \mapsto \sum_n e_n^* \otimes \alpha_n x_n$$

is continuous, so that also

$$Q^* : L(X, \Lambda) \to \Lambda_s(X^*)$$

is bounded. Using the trace duality and the fact that $\Lambda_w(X^*) \cong L(X, \Lambda)$ (cf. Chapter 0), we have

$$(\alpha_n x_n^*) = Q^*((x_n^*)) \in \Lambda_s(X^*)$$

for all $(x_n^*) \in \Lambda_w(X^*)$. Thus $(\alpha_i) \in m_{\Lambda,\Lambda}(X^*)$.

□

We need only use the unit ball of the closed subspace $\Lambda_c(E)$ of $\Lambda_w(E)$ to define the norm on $m_{\Lambda,\Sigma}(E)$, as is explained in the following

**Lemma 5.2.4** *Let* $(\xi_i) \in m_{\Lambda,\Sigma}(E)$. *Then*

$$\|(\xi_i)\|_{\Lambda,\Sigma} = \sup\{\pi_\Sigma((\xi_i x_i)) : (x_i) \in \Lambda_c(E), \epsilon_\Lambda((x_i)) \leq 1\}.$$

**Proof** For $\epsilon > 0$, let $(x_i) \in \Lambda_w(E)$ such that $\epsilon_\Lambda((x_i)) \leq 1$ and

$$\|(\alpha_i)\|_{\Lambda,\Sigma} < \|(|\alpha_i|\|x_i\|)\|_\Sigma + \epsilon/2.$$

Using the $AK$ property of $\Sigma$, let $n_0 \in \mathbb{N}$ such that $\|(|\alpha_i|\|x_i\|)(> n_0 + 1)\|_\Sigma < \epsilon/2$. It follows that

$$\begin{aligned} \|(\alpha_i)\|_{\Lambda,\Sigma} &< \|(|\alpha_i|\|x_i\|)(\leq n_0)\|_\Sigma + \epsilon \\ &\leq \sup\{\|(|\alpha_i|\|y_i\|)\|_\Sigma : (y_i) \in \Lambda_c(E), \epsilon_\Lambda((y_i)) \leq 1\} + \epsilon. \end{aligned}$$

□

In general the $BK$-space $m_{\Lambda,\Sigma}(E)$ may not have the $AK$ property. The last part of this section is devoted to an attempt to find some conditions on $E$ and the relevant sequence spaces that will ensure that $m_{\Lambda,\Sigma}(E)$ has $AK$. Let us denote the unit balls in $\Lambda_c(E)$ and $(\Sigma_s(E))^* = \Sigma_s^\times(E^*)$ by $B_\Lambda^c$ and $B_{\Sigma^\times}^*$, respectively. The unit ball in $m_{\Lambda,\Sigma}(E)$ will be denoted by $B_{\Lambda,\Sigma}$.

**Theorem 5.2.5** *$m_{\Lambda,\Sigma}(E)$ has the AK-property if and only if the set*

$$A := \{(\langle x_i, a_i\rangle) : ((x_i), (a_i)) \in B_\Lambda^c \times B_{\Sigma^\times}^*\}$$

*is $\sigma(m_{\Lambda,\Sigma}(E)^\times, m_{\Lambda,\Sigma}(E))$ relatively compact.*

**Proof** First we show that $A \subseteq m_{\Lambda,\Sigma}(E)^\times$. Let $(\alpha_i) \in m_{\Lambda,\Sigma}(E)$ and $(\langle x_i, a_i\rangle) \in A$. Then

$$\begin{aligned}\left|\sum_{i=1}^{\infty} \alpha_i\langle x_i, a_i\rangle\right| &= \left|\sum_{i=1}^{\infty}\langle \alpha_i x_i, a_i\rangle\right| = |\langle(\alpha_i x_i), (a_i)\rangle| \\ &\leq \pi_\Sigma((\alpha_i x_i))\pi_{\Sigma^\times}((a_i)) < \infty.\end{aligned}$$

Since $(\alpha_i) \in m_{\Lambda,\Sigma}(E)$ was arbitrary chosen, it follows that $A \subseteq m_{\Lambda,\Sigma}(E)^\times$.

Next we show that $A^\circ = B_{\Lambda,\Sigma}$ . Let $(\alpha_i) \in A^\circ$. For any $(x_i) \in B_\Lambda^c$ we have

$$\begin{aligned}\pi_\Sigma((\alpha_i x_i)) &= \sup_{\pi_{\Sigma^\times}((a_i))\leq 1} |\langle(\alpha_i x_i), (a_i)\rangle| \\ &= \sup_{\pi_{\Sigma^\times}((a_i))\leq 1} \left|\sum_{i=1}^{\infty} \alpha_i\langle x_i, a_i\rangle\right| \leq 1.\end{aligned}$$

This holds for all $(x_i) \in B_\Lambda^c$. Thus $\|(\alpha_i)\|_{\Lambda,\Sigma} \leq 1$. Hence $A^\circ \subseteq B_{\Lambda,\Sigma}$.

On the other hand, if $(\alpha_i) \in B_{\Lambda,\Sigma}$ and $(\langle x_i, a_i\rangle) \in A$, then

$$\begin{aligned}\left|\sum_{i=1}^{\infty} \alpha_i\langle x_i, a_i\rangle\right| &= |\langle(\alpha_i x_i), (a_i)\rangle| \\ &\leq \pi_\Sigma((\alpha_i x_i))\pi_{\Sigma^\times}((a_i)) \leq 1.\end{aligned}$$

Thus it follows that $B_{\Lambda,\Sigma} \subset A^\circ$.

Assume that $A$ is *weak** compact (in the Köthe duality). Since $A^\circ$ is a $\delta(m_{\Lambda,\Sigma}(E), m_{\Lambda,\Sigma}(E)^\times)$ neighbourhood of the origin (where $\delta$ denotes the topology of uniform convergence on the weak* compact sets), the equality $A^\circ = B_{\Lambda,\Sigma}$ implies that the norm topology is weaker than the $\delta$-topology on $m_{\Lambda,\Sigma}(E)$. By a result of G. Bennett (cf. [30], Theorem 2.1, pp 188) the normal sequence space $m_{\Lambda,\Sigma}(E)$ has *AK* with respect to this $\delta$-topology. Hence $m_{\Lambda,\Sigma}(E)$ has *AK* with respect to its (weaker) norm topology. In particular, this shows that $m_{\Lambda,\Sigma}(E)^* = m_{\Lambda,\Sigma}(E)^\times$.

Conversely, let $m_{\Lambda,\Sigma}(E)$ have the *AK* property. Then $m_{\Lambda,\Sigma}(E)^* = m_{\Lambda,\Sigma}(E)^\times$. Since $A^\circ = B_{\Lambda,\Sigma}$, we have $A \subseteq B_{\Lambda,\Sigma}^\circ$, which implies that $A$ is equicontinuous. Thus $A$ is $\sigma(m_{\Lambda,\Sigma}(E)^\times, m_{\Lambda,\Sigma}(E))$ relatively compact. □

Consider the bilinear mapping

$$\Phi : B_\Lambda^c \times B_{\Sigma^\times}^* \to m_{\Lambda,\Sigma}(E)^\times :: ((x_i), (a_i)) \mapsto (\langle x_i, a_i\rangle).$$

On $B_\Lambda^c$ and $B_{\Sigma^\times}^*$ consider the restrictions of the $\sigma(\Lambda_c(E), \Lambda_c(E)^*)$ and $\sigma(\Sigma_s^\times(E^*), \Sigma_s(E))$ topologies, respectively and on $m_{\Lambda,\Sigma}(E)^\times$ consider the $\sigma(m_{\Lambda,\Sigma}(E)^\times, m_{\Lambda,\Sigma}(E))$ topology. We show that $\Phi$ is separately continuous. Let $(x_i^\delta)_i \to (x_i)_i$ in $B_\Lambda^c$ weakly. For fixed $(a_i) \in B_{\Sigma^\times}^*$ and $(\lambda_i) \in m_{\Lambda,\Sigma}(E)$ we have

$$\sum_i \lambda_i \langle x_i^\delta, a_i \rangle = \sum_i \langle x_i^\delta, \lambda_i a_i \rangle.$$

By a result in [20] (refer to Chapter 0), $(\lambda_i a_i) \in \Lambda_c(E)^*$. This follows from

$$\left| \sum_{i=1}^\infty \langle y_i, \lambda_i a_i \rangle \right| \leq \sum_{i=1}^\infty |\langle y_i, \lambda_i a_i \rangle| = \sum_{i=1}^\infty |\langle \lambda_i y_i, a_i \rangle| < \infty$$

for all $(y_i) \in \Lambda_w(E)$ – recall that $(\lambda_i y_i) \in \Sigma_s(E)$ and $(a_i) \in \Sigma_s^\times(E^*)$ in this case. Therefore

$$\sum_i \lambda_i \langle x_i^\delta, a_i \rangle = \langle (x_i^\delta), (\lambda_i a_i) \rangle \xrightarrow{\delta} \langle (x_i), (\lambda_i a_i) \rangle = \sum_i \langle x_i, \lambda_i a_i \rangle.$$

Since this holds for all $(\lambda_i) \in m_{\Lambda,\Sigma}(E)$, we have that

$$\Phi((x_i^\delta), (a_i)) = (\langle x_i^\delta, a_i \rangle) \xrightarrow{\delta} (\langle x_i, a_i \rangle) = \Phi((x_i), (a_i))$$

in $m_{\Lambda,\Sigma}(E)^\times$ with the weak * topology. Therefore $\Phi$ is continuous in the first component. Similarly, let $(a_i^\delta)_i \to (a_i)_i$ in $B_{\Sigma^\times}^*$. For all $(\lambda_i) \in m_{\Lambda,\Sigma}(E)$ we have

$$\begin{aligned} \sum_i \lambda_i \langle x_i, a_i^\delta \rangle &= \langle (\lambda_i x_i), (a_i^\delta) \rangle \xrightarrow{\delta} \langle (\lambda_i x_i), (a_i) \rangle \\ &= \sum_i \langle \lambda_i x_i, a_i \rangle. \end{aligned}$$

Thus it follows that

$$\Phi((x_i), (a_i^\delta)) = (\langle x_i, a_i^\delta \rangle) \xrightarrow{\delta} (\langle x_i, a_i \rangle) = \Phi((x_i), (a_i))$$

in the weak * topology. Therefore $\Phi$ is continuous in the second component.

The separately continuous $\Phi$ (w.r.t. the topologies mentioned before) maps compact sets of the form $K_1 \times K_2$, with both $K_1$ and $K_2$ compact, onto compact sets. If $\Lambda_c(E)$ is reflexive, then the unit ball $B_\Lambda^c$ is weakly compact. The set $B_{\Sigma^\times}^*$ is weak * compact. So $A = \Phi(B_\Lambda^c \times B_{\Sigma^\times}^*)$ is weak * compact. Thus from Theorem 5.2.5 it is clear that

**Theorem 5.2.6** *$m_{\Lambda,\Sigma}(E)$ has AK if one of the following holds:*

*a) $\Lambda_c(E)$ is reflexive.*

*b) The dual pair $(m_{\Lambda,\Sigma}(E), m_{\Lambda,\Sigma}(E)^\times)$ is barrelled.*

If both $\Lambda$ and $E$ are reflexive Banach spaces, then we know that $\Lambda_w^\times(E) = L(\Lambda, E)$ is reflexive if and only if $L(\Lambda, E) = K(\Lambda, E)$ (refer to Chapter 0). Let $B_{\Lambda^\times}^c$ denote the unit ball in $\Lambda_c^\times(E)$. It follows that:

**Lemma 5.2.7** *Let $\Lambda$ be a reflexive BK space with AK. The set $B^{c}_{\Lambda^{\times}}$ is weakly compact $\iff \Lambda^{\times}_{c}(E)$ is reflexive $\iff \Lambda^{\times}_{c}(E) = \Lambda^{\times}_{w}(E)$ and $E$ is reflexive.*

Thus we conclude that:

**Theorem 5.2.8** *If $E$ and the BK space $\Lambda$ (with AK), are both reflexive Banach spaces such that $L(\Lambda, E) = K(\Lambda, E)$, then $m_{\Lambda^{\times},\Sigma}(E)$ has the AK property.*

For $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, it follows from Theorem 3.0.1 that

$$L(\ell^{p}, E) = K(\ell^{p}, E) \iff \ell^{q}_{w}(E) = \ell^{q}_{c}(E) \iff \ell^{q}_{w}(E) \subset (c_0)_s(E).$$

**Corollary 5.2.9** *Let $1 < p < \infty$ and suppose $\frac{1}{p} + \frac{1}{q} = 1$. If $E$ is reflexive and $\ell^{q}_{w}(E) \subset (c_0)_s(E)$, then $m_{p,r}(E)$ has the AK property for all $1 \leq r < \infty$.*

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