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NOTIONS OF CONTINUITY FOR FINITE RANK MAPS IN THE SPACE OF NORMAL OPERATORS

J. A. OTIENO[1], D. O. AMBOGO, AND F. O. NYAMWALA

ABSTRACT. We show that a natural extension of a continuous finite rank operator to an arbitrary Hilbert space is continuous. We also give sufficient conditions to calculate delta- epsilon numbers in all the domains of T. In addition, we characterize the concept of uniform continuity in terms of delta- epsilon function and finally show that finite rank operators preserve Cauchyness.

1. INTRODUCTION

It is well known that a mapping $T : H \to H$ is continuous if and only if it is bounded (see [6] and [5]). Most results in mathematical analysis use the concept of continuity directly or indirectly in order to extend a property of a function that is satisfied at a point p to a property satisfied in a neighborhood of p. It is known (see [4]) that the radius of the open ball depends on the norm of the linear mapping $[T^{'}p]^{-1}$ and also on a positive number delta appearing in the definition of continuity of a mapping $x \mapsto T^{'}x$ at the point p. Also [4] shows that for $2\lambda\|[T^{-1}p]^{-1}\| = 1$, then δ is such that if $\|x - p\| < \delta$, then $\|T^{'}x - T^{'}p\| < \lambda$. In this regard, the use of $\epsilon - \delta$ criterion in characterizing continuity is intriguing. In addition, uniform continuity has been studied by several researchers for instance [7] dealt with the characterization of uniform continuity for maps between unit balls of real Banach spaces in terms of universal properties. In [8],

[1]*corresponding author*

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the authors discussed the continuity of bounded linear operators on normed linear spaces showing that they are uniformly continuous. In [10], the authors described the basic properties of uniform continuity of functions on normed linear spaces. We have given results on sequential continuity as well. A mapping f between metric spaces is sequentially continuous if $x_n \to x$ implies that $fx_n \to fx$. It is well known in classical mathematics that sequentially continuous mapping between metric spaces is continuous as all proofs of this result involve the law of excluded middle [1]. Classically, for a linear mapping boundedness and sequential continuity are equivalent. For more details on sequential continuity see [1], [2] and the references therein.

2. PRELIMINARIES

We outline preliminary concepts which are useful to this sequel.

Definition 2.1. *[6] A bounded linear operator T on a Hilbert space H is said to be normal if it commutes with its adjoint i.e $TT^* = T^*T$. The space of all normal operators is denoted by $N(H)$.*

Definition 2.2. *[9] Let $T : H \to H$ be a continuous linear operator on a Hilbert space H. A Hilbert subspace H_0 is T-stable or T-invariant if $T_x \in H_0$ for all $x \in H_0$. In other words H_0 is invariant under T if $T|_{H_0}$ is an operator on H_0.*

Definition 2.3. *[3] A function $f : H \to \mathbb{R}$ with $H \subseteq \mathbb{R}$ is continuous at $x_0 \in H$ if and only if for any $\epsilon > 0$ there is a $\delta > 0$ such that $|fx - fx_0| < \epsilon$ holds for all $x \in H$ with $|x - x_0| < \delta$.*

Definition 2.4. *[3] Let $T : H \to H$ be a continuous map at $p \in H_1$ and $\epsilon > 0$. A positive number δ is said to be a delta-epsilon number for T at p, if δ satisfies the $\epsilon - \delta$ definition of continuity of T at the point p. In other words, δ is such that if $x \in H$ and $\|x - p\|_2 < \delta$, then $\|Tx - Tp\|_{H_1} < \epsilon$, which implies that $\forall\epsilon > 0 \; \exists \, \delta > 0 \; \forall x \in H$ such that $|x - x_0| < \delta \Rightarrow |fx - fx_0| < \epsilon$.*

Definition 2.5. *[7] A function f is said to be uniformly continuous on A if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|fx - fy| < \epsilon$ whenever $x, y \in A$ and $|x - y| < \delta$.*

Definition 2.6. *[2] A mapping $T : H \to H$ is sequentially continuous if for each sequence x_n converging to $x \in H,\ Tx_n$ converges to Tx that is if $x_n \to x \Rightarrow Tx_n \to Tx$.*

3. Continuity of finite rank operators

We now present the main results of this paper.

Proposition 3.1. *Let $T \in N(H)$ be a finite rank operator. Then T is continuous if and only if its usual operator norm is finite.*

Proof. The usual operator norm of a linear map $T : H \to H$ is given by

$$\begin{aligned} \|T\| &= \inf\{c \geq 0 : \|Tx\| \leq c\|x\| \ \forall x \in H\} \\ &= \sup\{\|Tx\| : x \in H \text{ with} \|x\| \leq 1\} \\ &= \sup\{\|Tx\| : x \in H \text{ with } \|x\| = 1\} \\ &= \sup\{\frac{\|Tx\|}{\|x\|} : x \in H \text{ with } x \neq 0\} \end{aligned}$$

We note that for all $x \in H, \|Tx\| \leq \|T\|.\|x\|$. In fact, $\|T\|$ is the smallest constant with this property: $\|T\| = \min\{c \geq 0 : \|Tx\| \leq c\|x\|, \forall x \in H\}$. □

Proposition 3.2. *Let $T \in N(H)$ be continuous finite rank operator, then T has a unique adjoint T^*.*

Proof. For each $x \in H$, the map $\pi_x : H \to \mathbb{C}$ given by $\pi_x(w) = \langle Tw, x\rangle$ is continuous on H. By Riesz-Fischer representation theorem, there is a unique $w_x \in H$ so that $\langle Tw, x\rangle = \pi_x(w) = \langle w, w_x\rangle$. We define adjoint T^* by $T^*x = w_x$ which makes the map to be well defined from H to H and has the adjoint property $\langle Tw, x\rangle_H = \langle w, T^*x\rangle_H$. To show that T^* is continuous, we only show that it is bounded. Applying Cauchy- Schwarz- Bunyakowsky inequality,

$$\|T^*x\|^2 = \|\langle T^*x, T^*x\rangle\| = \|\langle x, TT^*x\rangle\| \leq \|x\|.\|TT^*x\| \leq \|x\|.\|T\|.\|T^*x\|, \tag{3.1}$$

where $\|T\|$ is the usual operator norm. From inequality (3.1), we obtain

$$\|T^*x\|^2 \leq \|x\|\|T\|\|T^*x\|. \tag{3.2}$$

Dividing (3.2) by a nonzero T^*x, we obtain $\|T^*\| \leq \|T\|$. In particular, T^* is bounded. Since $(T^*)^* = T$ by symmetry $\|T\| = \|T^*\|$ and also T^* is linear. □

Characterization in the space of normal operators for continuous finite rank operators under T-stable subspace of H, follows.

Theorem 3.1. *Let $T \in N(H)$ be a continuous finite rank operator. If H_1 is T-stable subspace of H, then $H^{\perp}$ is T^*-stable. Moreover, if T is self-adjoint then both H and $H^{\perp}$ are T-stable.*

Proof. For $y \in H^{\perp}$ and $x \in H$, $\langle T^*y, x\rangle = \langle y, T^{**}x\rangle = \langle y, Tx\rangle$ for continuous linear map and $T^{**} = T$. Since H is T-stable, $Tw \in H$, and this inner product is 0. Hence $T^*y \in H^{\perp}$. □

Characterization of norm continuity of finite rank maps is given below.

Proposition 3.3. *Suppose that ι is a finite rank map on H, then the following are equivalent:*

(i) *ι is (norm) continuous.*

(ii) *there is a sequence $p_n(a, b)$ of non-commutative polynomials such that $\|p_n(T, T^* - \iota(T))\| \to 0$ uniformly in T on bounded subsets of $N(H)$.*

Proof. $(i) \to (ii)$. Suppose ι is continuous, $T \in N(H)$ and also that U_n is a sequence of unitary operators such that $\|U_nT - TU_n\| \to 0$. Then $U_n^*TU_n \to T$ and we find that $U_n^*\iota(T)U_n = \iota(U_n^*TU_n) \to \iota(T)$ and thus $\|U_n\iota(T) - \iota(T)U_n\| \to 0$. Hence $\iota(T) \in C^*(T)$. Let S be unitarily equivalent to direct sum of finite matrices so that $\|S\| = 1$. For each integer n where $n \geq 0$ then $\iota(nS) \in C^*(nS)$ hence there is a noncommutative polynomial $p_n(a, b)$ such that $\|p_n(nS, nS^*) - \iota(nS)\| \leq \frac{1}{n}$. If $T \in N(H), n \geq \|T\|$ then $\|p_n(T, T^*) - \iota(T)\| = \|\pi(p_n(nS, nS^*) - \iota(nS)\| \leq \frac{1}{n}$. $(ii) \to (i)$. Let $\|p_n(T, T^*) - \iota(T)\| = \|p_n(nS, nS^*) - \iota(nS)\| \leq \frac{1}{n}$, for all $n \in \mathbb{N}$. As $n \to \infty$, since U^* is unitary and S is unitarily equivalent to the direct sum of finite matrices, then we have $\|p_n(nS, nS^*) - \iota(nS)\| \to 0$ hence $\|U_n\iota(T) - \iota(T)U_n\| \to 0$, therefore ι is norm continuous. □

Remark 3.1. *Part (ii) implies that a natural extension of a continuous finite rank operator to an arbitrary Hilbert space is continuous.*

Theorem 3.2. *Suppose that $T \in N(H)$. If a mapping $\pi : C^*(T) \to Cf(\Sigma(T))$ is defined by $\pi(\lambda(T)) = \lambda|\Sigma(T)$, for each continuous finite rank map on H then λ is an isomorphism.*

Proof. Since π acts on $C^*(T)$ then it is an isometric $*$- homomorphism. Now, we show that π is onto. Suppose $\kappa \in Cf(\Sigma(T))$, if U_n is a sequence of unitary operators on H such that $\|U_nT - TU_n\| = \|U_n^*TU_n - T\| \to 0$, then by Proposition 3.3, $\|U_n\kappa(T) - \kappa(T)U_n\| = \|\kappa(U_n^*TU_n) - \kappa(T)\| \to 0$ since $U_n^*TU_n \in \Sigma(T)$, for each n. So $\kappa(T) \in C^*(T)$. Choose a continuous finite rank map λ so that $\lambda(T) = \kappa(T)$. Also $\lambda(S) = \kappa(S)$, for every S in the closure of the unitary equivalence class $U(T)$ of T. Since each operator is a sub operator of an operator in $U(T)^-$, it follows that $\lambda|\Sigma(t) = \kappa$. □

Proposition 3.4. *If α, κ are continuous finite rank and $T \in N(H)$, then $\alpha(T) = \kappa(T)$ if and only if $\alpha(A) = \kappa(A)$ for every irreducible operator A in $\Sigma(T)$.*

Proof. There is an operator S in $\Sigma(T)$ and a sequence U_n of unitary operators such that S is a direct sum of irreducible operators and $\|U_n^*SU_n - T\| \to 0$. It follows that $\alpha(S) = \kappa(S)$, and that $\alpha(T) = \lim \alpha(U_n^*SU_n) = \lim U_n^*\alpha(S)U_n) = \lim U_n^*\kappa(S)U_n) = \lim U_n^*\kappa(S)U_n) = \lim \kappa(U_n^*SU_n) = \kappa(T)$. □

At this point we focus on characterization of finite rank preserver maps on spaces of normal operators using the $\epsilon - \delta$ criterion for continuity.

Proposition 3.5. *Let $T : H \to H$ be a finite rank continuous map, $p \in H$, and $\epsilon > 0$.*

(i) *If $T^{-1}(S[Tp, \epsilon]) \neq \emptyset$, the quantity $\delta(p, \epsilon) = dist(p, T^{-1}(S[Tp, \epsilon]))$ is well defined and represents a positive number with $S[Tp, \epsilon]$ representing a sphere with center at Tp and radius ϵ. Then*

$$S[Tp, \epsilon] = r \in H \textit{such that} \|Tp - r\|_{H_1} = \epsilon.$$

(ii) *In addition, if the open ball $B(p, \delta(p, \epsilon))$ is path-connected then the number $\delta(p, \epsilon)$ is a delta-epsilon number for T at p.*

(iii) *$\delta(p, \epsilon)$ is the greatest delta-epsilon number at p.*

(iv) *Define the set $\{\mathcal{K}p, \epsilon\}$ as:*

$$\{\mathcal{K}p, \epsilon\} = \{\beta \in \mathbb{R}^+ : \|x - p\|_{H_1} < \beta \Rightarrow \|Tx, Tp\|_H < \epsilon, \forall x \in H\},$$

then $\delta(p, \epsilon) = \max\{\mathcal{K}p, \epsilon\}$ and of course $\{\mathcal{K}p, \epsilon\} = (0, \delta(p, \epsilon)]$.

Proof. For the proof we have:

(i) Since $T^{-1}(S[Tp, \epsilon])$ is a nonempty set, then the number

$$\delta(p, \epsilon) = \inf\{\|x - p\|_H : x \in H, \|Tx - Tp\|_{H_1} = \epsilon\}$$

is well defined. If $\delta(p,\epsilon) = 0$, then there exists a sequence $x_n \in H$ such that $\lim \|x_n - p\|_H = 0$ with $\lim \|Tx_n - Tp\|_{H_1} = \epsilon$. Being that T is continuous at p then $\lim \|Tx_n - Tp\|_{H_1} = 0$, since $\epsilon > 0$ hence a contradiction. Thus, $\delta(p,\epsilon)$ must be a positive number.

(ii) We use contradiction to show that if $\|x - p\|_H < \delta(p,\epsilon)$, then we have $\|Tx - Tp\|_{H_1} < \epsilon$. By definition of $\delta(p,\epsilon)$, we have $\|Tx - Tp\|_{H_1} \neq \epsilon$ and so the inequality $\|Tx - Tp\|_{H_1} > \epsilon$ is not possible. If $\|Tx - Tp\|_{H_1} > \epsilon$, and since the open ball $B(p, \delta(p,\epsilon))$ is path-connected, there exists a continuous map $\gamma : [0,1] \to B(p, \delta(p,\epsilon))$ such that $\gamma(0) = p$ and $\gamma(1) = x$. Considering a map $g : [0,1] \to \mathbb{R}$ given by $g(t) = \|T\gamma t - Tp\|_{H_1}$ is continuous, it also satisfies $g(0) = 0$ and $g(1) > \epsilon$ and therefore there exists $t_0 \in (0,1)$ by the intermediate value theorem. Then $g(t_0) = \|T\gamma t_0 - Tp\|_{H_1} = \epsilon$. Satisfying γt_0 that $\|\gamma t_0 - p\|_H < \delta(p,\epsilon)$ and $\|T\gamma t_0 - Tp\|_{H_1} = \epsilon$, hence a contradiction to the definition of $\delta(p,\epsilon)$. So $\delta(p,\epsilon)$ is a delta-epsilon number for T at p.

(iii) For φ is such that $\delta(p,\epsilon) < \varphi$, then there exists $x \in H$ such that $\delta(p,\epsilon) \leq \|x - p\|_H < \varphi$ with $\|Tx - Tp\|_{H_1} = \epsilon$. So, φ is not a delta-epsilon number for T at p.

(iv) In order to prove this, we look back at (i) and (ii) where we deduced that $\delta(p,\epsilon) \in \{\mathcal{K}p, \epsilon\}$. For (iii) we obtained that any other number greater that $\delta(p,\epsilon)$ is not in $\{\mathcal{K}p, \epsilon\}$. Then we can conclude that $\delta(p,\epsilon) = \max\{\mathcal{K}p, \epsilon\}$.

□

Proposition 3.6. *Let $T : H \to H$ be a finite rank continuous map and suppose that there exists $p, x \in H$ such that $\|Tx - Tp\|_{H_1} = \beta > 0$ and there be a path connecting the two points p and x. Then for every ϵ such that $0 < \epsilon \leq \beta$ we have $T^{-1}(S[Tp,\epsilon]) \neq \emptyset$ and $T^{-1}(S[Tx,\epsilon]) \neq \emptyset$. Also for every ϵ satisfying $0 < \epsilon \leq \beta$, the numbers $\delta(p,\epsilon)$ and $\delta(x,\epsilon)$ are well defined and positive.*

Proof. From the statement of the proposition, a path connecting p and x given as $\gamma : [0,1] \to H$ exists. A map $g(t) : [0,1] \to \mathbb{R}$ defined by $\|T\gamma t - Tp\|_{H_1}$ is continuous and satisfying $\|T\gamma t_0 - Tp\|_{H_1} = \epsilon$ that proves that $T^{-1}(S[Tp,\epsilon]) \neq \emptyset$. The rest follows from Proposition 3.5. □

In order to compute delta-epsilon numbers in a neighborhood of a point p, we introduce the next lemma.

Lemma 3.1. *Let $T : H \to H$ be a finite rank continuous map and there exists $p, x \in H$ such that $\|Tx - Tp\|_{H_1} = \beta > 0$. If an open ball $B(p, \delta(p, \beta))$ is path-connected and suppose that p and x are also path-connected, then for every ϵ with $0 < \epsilon < \beta$, there exists δ satisfying $0 < \delta \leq \delta(p, \beta)$, such that if $\|q - p\|_H < \delta$ the numbers $\delta(q, \epsilon)$ are path-connected and for all $q \in B(p, \delta)$ the number $\delta(q, \epsilon)$ are delta-epsilon numbers.*

Proof. First, we show that there is a δ with $0 < \delta \leq \delta(p, \beta)$ such that if $\|q-p\|_H < \delta$, then $\epsilon < \|Tx - Tq\|_{H_1}$. From Proposition 3.5 and since $T^{-1}(S[Tp, \beta]) \neq \emptyset$ and open ball $B(p, \delta(p, \beta))$ is path-connected, we conclude that $\delta(p, \beta)$ is the maximum delta-epsilon number at p. On the other hand, T is continuous at p and since $\beta - \epsilon$ is positive, there exists $\delta > 0$ such that if $\|q - p\|_H < \delta$ then $\|Tq - Tp\|_{H_1} < \beta - \epsilon < \beta$. Now, $\delta(p, \beta)$ is the maximum delta-epsilon number at p, then $\delta \leq \delta(p, \beta)$. Using triangle inequality and having $q \in B(p, \delta)$ we find that

$$\begin{aligned} \beta &= \|Tx - Tp\|_{H_1} \\ &\leq \|Tx - Tq\|_{H_1} + \|Tq - Tp\|_{H_1} \\ &< \|Tx - Tq\|_{H_1} + \beta - \epsilon. \end{aligned}$$

This implies that if $\|q-p\|_H < \delta$, then $\epsilon < \|Tx-Tq\|_{H_1}$, which is what we wanted to show. Furthermore, as each point $q \in B(p, \delta)$ can be path-connected to x and $\epsilon < \|Tx - Tq\|_H$, then by Proposition 3.6, we conclude that $T^{-1}(S[Tq, \epsilon]) \neq \emptyset$. The numbers $\delta(q, \epsilon)$ are well defined in the ball $B(p, \delta)$. Being that the open ball is path-connected then by Proposition 3.5 (ii), the numbers $\delta(q, \epsilon)$ are delta-epsilon numbers. □

The next theorem gives sufficient conditions to calculate delta-epsilon numbers in all the domains of T.

Theorem 3.3. *Let $T : H \to H$ be a finite rank nonconstant continuous map. For all $p \in H$ and $r > 0$, the open ball $B(p, r)$ is path-connected and there exists $\beta > 0$ such that $\|Tp - Tx\|_{H_1} = \beta$, whereby, the delta-epsilon numbers $\delta(p, \epsilon)$ are well defined.*

Proof. It is necessary to find a positive number β so that for every $p \in H$ there is $x \in H$ such that $\|Tp - Tx\|_{H_1} = \beta$. Being that T is a nonconstant map, then the diameter of $T(H)$ is positive, namely $diam(T(H)) > R$ for some $R > 0$. So, there exists $a, b \in H$ with $\frac{R}{2} < \|Ta - Tb\|_{H_1}$. Now, for $p \in H$ then, $\frac{R}{2} <$

$\|Ta - Tb\|_{H_1} \leq \|Ta - Tp\|_{H_1} + \|Tp - Tb\|_{H_1}$, thus either $\frac{R}{4} < \|Ta - Tp\|_{H_1}$ or $\frac{R}{4} < \|Tp - Tb\|_{H_1}$. Alternatively, since H is path-connected, there exists $x \in H$ such that $\|Tx - Tp\|_{H_1} = \frac{R}{4}$. By direct application of Proposition 3.6 and Lemma 3.1 and taking $\beta := \frac{R}{4}$ the proof follows. □

Looking at uniform continuity in a bid to characterize finite rank linear maps, the next theorem shows us that a continuous mapping T that admits a family is uniformly continuous.

Theorem 3.4. *Let H be nonempty and $T : H \to H$ a finite rank continuous map. Then T is uniformly continuous on H if and only if there exists a family $\{g_\epsilon\}_{\epsilon>0}$ of delta-epsilon mappings for T such that:*

$$\eta_\epsilon := \inf_{x \in H} g_\epsilon(x) > 0, \tag{3.3}$$

for every $\epsilon > 0$.

Proof. If $T : H \to H$ is uniformly continuous and $\epsilon > 0$, then there exists $\delta > 0$ such that for every $a, b \in H$ with $\|a - b\|_H < \delta$, then $\|Tx - Ty\|_{H_1} < \epsilon$. A constant function $g_\epsilon : H \to \mathbb{R}^+, g_\epsilon(a) = \delta$, is a delta-epsilon function for T that clearly satisfies Equation (3.3). Conversely, let $\{g_\epsilon\}_{\epsilon>0}$ be a family of delta-epsilon mappings for continuous operator T that satisfies the Equation (3.3), then for every $\epsilon > 0$ and $a, b \in H$, we have that $\|a - b\| < \eta_\epsilon \leq g_\epsilon(a)$ since T is continuous at a and $g_\epsilon(a)$ satisfies the continuity definition at a. Hence, we conclude that $\|Ta - Tb\|_{H_1} < \epsilon$. □

We can now give characterization of the concept of uniform continuity in terms of delta-epsilon function.

Theorem 3.5. *Let $T : H \to H$ be a finite rank nonconstant continuous map. Suppose that for all $s \in H$ and $r > 0$ the open ball $B(s, r)$ is path-connected. Then the following conditions are equivalent:*

- (i) *T is not uniformly continuous on H.*
- (ii) *There exists ϵ_0 such that, $\inf_{x \in H} \delta(x, \epsilon_0) = 0$.*
- (iii) *There exists ϵ_0 and sequences $x_n, y_n \in H$, such that,* $\lim_{n\to\infty} \|x_n - y_n\|_H = 0$ *and* $\|Tx_n - Ty_n\|_{H_1} = \epsilon_0$.

Proof.

$(i) \Rightarrow (ii)$. T is not uniformly continuous on H, then by Theorem 3.4, the family of delta-epsilon $\{\delta(.,\epsilon)\}_{\epsilon\in(0,\beta)}$ must have an element satisfying condition (ii).

$(ii) \Rightarrow (iii)$. Since $\inf_{x\in H}\delta(x,\epsilon_0) = 0$, then for all $n \in \mathbb{N}$, there exists $x_n \in H$ such that $0 < \delta(x_n,\epsilon_0) < \frac{1}{n}$. By definition of $\delta(x_n,\epsilon_0)$ there is $y_n \in H$ satisfying, $0 < \delta(x_n,\epsilon_0) \le \|x_n - y_n\|_H < \frac{1}{n}$ and $\|Tx_n - Ty_n\|_{H_1} = \epsilon_0$. Getting two sequences of elements $x_n, y_n \in H$ such that $\lim_{n\to\infty}\|x_n - y_n\|_H = 0$ and $\|Tx_n - Ty_n\|_{H_1} = \epsilon_0$.

Lastly, $(iii) \Rightarrow (i)$. If (iii) holds, then $0 < \delta(x_n,\epsilon_0) \le \|x_n - y_n\|_H$. Hence $\lim_{n\to\infty}\delta(x_n,\epsilon_0) = 0$, which implies that $\inf_{x\in H}\delta(x,\epsilon_0) = 0$. Let $\{\rho_\epsilon\}_{\epsilon>0}$ be a family of delta-epsilon for T. Then we have that $\rho_\epsilon(x) \le \delta(x,\epsilon) \le \delta(x,\epsilon_0)$, for all $x \in H$, where $0 < \epsilon \le \epsilon_0$. We obtain that $\inf_{x\in H}\rho_\epsilon(x) = 0$. Then T is not uniformly continuous on H. □

For the final characterization, we turn to sequentially continuous finite rank operators.

Proposition 3.7. *Let $T \in N(H)$ be a sequentially continuous finite rank operator and x_m a Cauchy sequence in H with $0 < \Omega < \pi$. Then there exists m such that $\|Tx_m\| > \Omega$.*

Proof. Being that T is a linear map, we assume that $\pi - \Omega > 1$. Choosing a strictly increasing sequence $(N_l)_{l=1}^{\infty}$ of positive integers such that $\|x_m - x_n\| < 2^{-3l}$, for all $n, m \ge N_l$, write $s_l = \max\{\|Tx_m\| : 1 \le m \le N_l\}$. We construct an increasing binary sequence $(\beta_l)_{l=1}^{\infty}$ such that

$$\beta_l = 0 \Rightarrow \forall j \le l(sj < \pi - 2^{-2j}),$$

$$\beta_l = 1 \Rightarrow \exists j \le l(sj < \pi - 2^{-2j+1}).$$

We may assume that $\beta_1 = \beta_2 = 0$. Next we construct a sequence w_l in H as follows: If $\beta_{l+1} = 0$ or if $\beta_{l+1} = \beta_l = 1$, set $w_l = 0$. If $\beta_{l+1} = 1$ and $\beta_l = 0$, then $\|Tx_{Nl}\| \le s_l < \pi - 2^{-2j}$ and $s_{l+1} > \pi - 2^{-2l-1}$, so we can choose l such that $N_l < j < N_{l+1}$ and $\|Tx_l\| > \pi - 2^{-2l-1}$, setting $w_l = 2^{2l}(x_j - x_{Nl})$, we see that $\|w_l\|^* < 2^{-l}$ and

$$\begin{aligned}\|Tw_l\| &= 2^{2l}\|Tx_l - Tx_{Nl}\| \ge 2^{2l}(\|Tx_l\| - \|Tx_{Nl}\|)\\ &> 2^{2l}(\pi - 2^{-2l-1} - (\pi - 2^{-2l})) = \frac{1}{2}.\end{aligned}$$

This completes the construction of a sequence w_l converging to 0 in H. By sequential continuity of T, $\lim_{l\to\infty} Tw_l = 0$. We choose K such that $\|Tx_l\| < \frac{1}{2}$ for all $l \geq K$, so that $\beta_l \neq 1 - \beta_l$ for all $l \geq K$. If $\beta_l = 1$, then there exists $n \leq N_K$ such that $\|Tw_l\| > \pi - 2^{-2n+1} > \Omega$. If $\beta_K = 0$, then $\beta_l = 0$, for all $l \geq K$ so $\|Tw_l\| < \pi$, for all l. The rest is clear. □

Proposition 3.8. *Let $T \in N(H)$ be a sequentially continuous finite rank operator and x_m a Cauchy sequence in H, then $\sup_{m\geq 1} \|Tx_m\|$ exists.*

Proof. We show that the sequence x_m is bounded. We first choose $M > 0$ such that $\|x_m\| M$ for all m. Taking $\Omega = 1$ and $\pi = 2$ in Proposition 3.7, we assume that there is m_1 such that $\|Tx_{m_1}\| > 1$. Set $\beta_1 = 0$. Applying Proposition 3.7 repeatedly, we now construct an increasing binary sequence β_m, and an increasing sequence $(m_l)_{l=1}^{\infty}$ of positive integers, such that $\beta_1 = 0 \Rightarrow \|Tx_{m_l}\| > l$ and $m_l > m_{l-1}, \beta_1 = 1 \Rightarrow Tx_m$ is a bounded sequence and $m_{l+1} = m_l$. Suppose we have found β_l and m_l and if $\beta_1 = 1$, we set $\beta_{l+1} = \beta_l$ and $m_{l+1} = m_l$. If $\beta_l = 0$, then $\|Tx_{m_j}\| > j$, for all $j \leq l$. Applying Proposition 3.7 to Cauchy sequence $(x_l)_{l>m_l}$, we obtain $m_{l+1} > m_l$ such that $\|Tx_{m_{l+1}}\| > l+1$ or else $\|Tx_j\| < l+2$ for all $j > m_l$. If we set $\beta_{l+1} = 0$ for the first case and $\beta_{l+1} = 1$ for the second, then $(Tx_m)_{m=1}^{\infty}$ is bounded and $m_{l+1} = m_l$. If $\beta_l = 0$, set $w_l = l^{-1}x_{m_1}$; if $\beta_k = 1$, set $w_l = 0$. Then $\|w_l\| \leq Ml^{-1}$ for each l, so $w_l \to 0$ and therefore, by the sequential continuity of T, $T(w_l) \to 0$. We choose M such that $\|Tw_l\| < 1$ for all $l \geq M$. If $\beta_M = 0$, then $\|Tw_l\| = l^{-1}\|Tx_{m_l}\| > 1$, a contradiction. Hence, $\beta_M = 1$ so $(\|Tx_m\|)_{m=1}^{\infty}$ is bounded. It then follows that $\sup_{m\geq 1} \|Tx_m\|$ exists. □

Finally, we extend a finite rank sequentially continuous linear map to the completion of its domain by the theorem below.

Theorem 3.6. *A linear mapping $T : H \to H$ is finite rank sequentially continuous if and only if it maps Cauchy sequences to Cauchy sequences.*

Proof. Taking T to be sequentially continuous and given a Cauchy sequence x_m in H, we can choose a strictly increasing sequence $(N_l)_{l=1}^{\infty}$ of positive integers such that $\|x_n - x_m\| < 2^{-l}$ for all $n, m \geq N_l$. Next, we consider the existence of the supremum. For each l let $s_l = \sup_{m\geq N_l} \|Tx_m - Tx_{N_l}\|$ exists. We show that $s_l < \epsilon$ for some l given $\epsilon > 0$. Next we construct an increasing binary sequence β_m such that $\beta_l = 0 \Rightarrow s_l > \frac{\epsilon}{4}$ and $\beta_l = 1 \Rightarrow s_l < \frac{\epsilon}{2}$. Assume that $\beta_1 = 0$. If $\beta_l = 0$, choose $j \geq N_l$ such that $\|Tx_j - Tx_{N_l}\| > \frac{\epsilon}{4}$ and set $w_l = x_j - x_{N_l}$. If

$\beta_l = 1$, set $w_l = 0$. Then for each l we have $\|w_l\|2^{-l}$ so $w_l \to 0$. Having taken T as sequentially continuous, $Tw_l \to 0$ and we choose M so that $\|Tw_l\| < \frac{\epsilon}{4}$, for all $l \geq M$. Next if $\beta_l \neq 0$ it implies that $\|Tw_l\| > \frac{\epsilon}{4}$ which is absurd. Then $\beta_l = 1$ and thus $s_l < \frac{\epsilon}{2}$. For all $j, l \geq N_l$ we have that

$$\begin{aligned} \|Tx_j - Tx_l\| &\leq \|Tx_j - Tx_{N_l}\| + \|Tx_l - Tx_{N_l}\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Being that ϵ is arbitrary, $(Tx_m)_{m=1}^{\infty}$ is a Cauchy sequence in H. Conversely, assume that T maps Cauchy sequences to Cauchy sequences. If x_m is a converging sequence to 0 in H, then $(Tx_m)_{m=1}^{\infty}$ is a Cauchy sequence in H. We then find a subsequence that converges to 0 as well for the proof that $(Tx_m)_{m=1}^{\infty}$ converges to 0. Let $(x_{m_l})_{l=1}^{\infty}$ be a subsequence of x_m such that $\|x_{m_l}\| < \frac{1}{l^2}$, for each l. We note that $(lx_{m_l})_{l=1}^{\infty}$ converges to 0 in H, so that $(Tlx_{m_l})_{l=1}^{\infty}$ is a Cauchy sequence in H. Then there exists $K > 0$ such that for each l, $\|lTx_{m_l}\| \leq K$ and hence $\|Tx_{m_l}\| \leq \frac{K}{l}$. Therefore, $\lim_{l\to\infty} Tx_{m_l} = 0$. □

Corollary 3.1. *Let T be a sequentially continuous finite rank linear mapping of H into H. Then T extends to a sequentially continuous linear mapping of $H^\sharp$ into H, where $H^\sharp$ is the completion of H.*

Proof. Let x_m, x_m' be sequences in H that converges to the same limit $x \in H^\sharp$. From Theorem 3.6, $(Tx_1, Tx_1', Tx_2, Tx_2', ...)$ is a Cauchy sequence in H. Being that H is complete, this Cauchy sequence converges to a limit $y \in H$. Then each of the sequences Tx_m and Tx_m' converges to y so $T^\sharp x \equiv \lim_{m\to\infty} Tx_m$ does not depend on the sequence x_m of elements of H converging to x. Then $T^\sharp$ is linear and coincides with T on H. Now, let x_m be any sequence in $H^\sharp$ converging to 0. By the definition of $T^\sharp$, for each m there exists $x_m' \in H$ such that $\|x_m' - x_m\| \leq \frac{1}{m}$ and $\|Tx_m' - Tx_m\| < \frac{1}{m}$. Then $\lim_{m\to 0} x_m' = 0$, so $0 = T^\sharp 0 = \lim_{m\to\infty} Tx_m' = \lim_{m\to\infty} Tx_m$, hence $T^\sharp$ is sequentially continuous. □

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DEPARTMENT OF MATHEMATICS, PHYSICS AND COMPUTING
MOI UNIVERSITY
P.O BOX 3900-30100, ELDORET-KENYA
Email address: jacjec2@gmail.com

DEPARTMENT OF MATHEMATICS
MASENO UNIVERSITY
P.O BOX PRIVATE BAG, MASENO-KENYA
Email address: otivoe@yahoo.com

DEPARTMENT OF MATHEMATICS, PHYSICS AND COMPUTING
MOI UNIVERSITY
P.O BOX 3900-30100, ELDORET-KENYA
Email address: nyamwalafredrick27@gmail.com