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SPECTRA OF COMPOSITION GROUPS ON THE WEIGHTED DIRICHLET SPACE OF THE UPPER HALF-PLANE*

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Abstract We prove that the group of weighted composition operators induced by continuous automorphism groups of the upper half plane $\mathbb{U}$ is strongly continuous on the weighted Dirichlet space of $\mathbb{U}$, $\mathcal{D}_\alpha(\mathbb{U})$. Further, we investigate when they are isometries on $\mathcal{D}_\alpha(\mathbb{U})$. In each case, we determine the semigroup properties while in the case that the induced composition group is an isometry, we apply similarity theory to determine the spectral properties of the group.

Key words one-parameter semigroup; composition operator semigroups; strong continuity; infinitesimal generator; spectra

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1 Introduction

For an open subset Ω of $\mathbb{C}$, let $\mathcal{H}(\Omega)$ denotes the Fréchet space of analytic functions on Ω endowed with the topology of uniform convergence on compact subsets of Ω. In this note, Ω can be either the open unit disk $\mathbb{D}$ or the upper half plane $\mathbb{U}$. If φ is a self analytic map on Ω, then the induced composition operator C_φ acting on $\mathcal{H}(\Omega)$ is defined by $C_\varphi f = f\circ\varphi$, with the corresponding weighted composition operator on $\mathcal{H}(\Omega)$ given by $S_\varphi = (\varphi')^\gamma C_\varphi$ for some appropriately chosen weight γ.

Composition operators on spaces of analytic functions on the unit disc $\mathcal{H}(\mathbb{D})$ have been extensively studied in the literature comparatively to their counterparts on the analytic spaces of the upper half plane $\mathcal{H}(\mathbb{U})$. Even though there are isomorphisms between the corresponding spaces of $\mathbb{D}$ and of $\mathbb{U}$, composition operators act differently in the two cases. For instance, unlike the case of Hardy or Bergman spaces of $\mathbb{D}$, not every composition operator is bounded on Hardy or Bergman spaces of $\mathbb{U}$, see [1, 2]. It has also been proved in [3, 4] that there are no non-trivial (i.e. with symbol not constant) compact composition operators on the Hardy space $H^2(\mathbb{U})$ or the weighted Bergman space $L^2_a(\mathbb{U},\mu_\alpha)$ which is not the case for $H^2(\mathbb{D})$ or $L^2_a(\mathbb{D},m_\alpha)$.

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Unlike the Hardy and Bergman spaces of $\mathbb{D}$ cases, composition operators on the (weighted) Dirichlet space of the unit disk $\mathcal{D}_\alpha(\mathbb{D})$ are not necessarily bounded; an indication that the action of composition operators on the weighted Dirichlet space of the upper half plane $\mathcal{D}_\alpha(\mathbb{U})$ is very much complicated. Recent attempts to study composition operators on $\mathcal{D}_\alpha(\mathbb{U})$ can be found in [5, 6]. Schroderus [5] obtained the spectrum of composition operators induced by linear fractional transformations (LFTs) of the upper half plane $\mathbb{U}$; while Sharma, Sharma and Raj [6] characterized boundedness and compactness of composition operators on $\mathcal{D}_0(\mathbb{U})$. In particular, it is proved in [6] that every LFT of $\mathbb{U}$ induces a bounded composition operator on $\mathcal{D}_0(\mathbb{U})$. This therefore implies that continuous groups $(\varphi_t)_{t\in\mathbb{R}}$ of automorphisms of the upper half plane may induce a bounded group of (weighted) composition operators $T_t := S_{\varphi_t}$ on $\mathcal{D}_\alpha(\mathbb{U})$. It is important to note that the study of composition semigroups on the Dirichlet spaces has scantly been considered with the only reference [7] being on the Dirichlet space of the unit disk.

In this paper, we extend the study carried out by the second author with his co-authors in [8] on Hardy and weighted Bergman spaces of $\mathbb{U}$ to the setting of the weighted Dirichlet space of $\mathbb{U}$. In particular, we prove that the group $(T_t)_{t\in\mathbb{R}}$ of weighted composition operators is strongly continuous on $\mathcal{D}_\alpha(\mathbb{U})$. Using the classification theorem of continuous groups of automorphisms of $\mathbb{U}$ [8, Proposition 2.3], we consider the corresponding weighted composition operator groups $(T_t)_{t\in\mathbb{R}}$ on $\mathcal{D}_\alpha(\mathbb{U})$ and investigate when they are isometries. It turns out that the scaling and translation groups do not induce weighted composition operator groups on $\mathcal{D}_\alpha(\mathbb{U})$ which are isometries as is the case for Hardy and Bergman spaces of $\mathbb{U}$ [8]. The infinitesimal generators of the composition groups are calculated and their properties discussed. For the rotation automorphism group, $(T_t)_{t\in\mathbb{R}}$ turns out to be a group of isometries on $\mathcal{D}_\alpha(\mathbb{D})$. In this case therefore, we determine both its semigroup and spectral properties.

2 Preliminaries and Definitions

The set $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is called the open unit disk. Let $\mathrm{d}A(z)$ denotes the area measure on $\mathbb{D}$, normalized so that the area of $\mathbb{D}$ is 1. In terms of rectangular and polar coordinates, we have $\mathrm{d}A(z) = \frac{1}{\pi}\mathrm{d}x\mathrm{d}y = \frac{r}{\pi}\mathrm{d}r\mathrm{d}\theta$ where $z = x + \mathrm{i}y = r\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{D}$. For $\alpha \in \mathbb{R}$, $\alpha > -1$, we define a positive Borel measure $\mathrm{d}m_\alpha$ on $\mathbb{D}$ by $\mathrm{d}m_\alpha(z) = (1-|z|^2)^\alpha \mathrm{d}A(z)$. On the other hand, the set $\mathbb{U} := \{\omega \in \mathbb{C} : \Im(\omega) > 0\}$ denotes the upper half of the complex plane $\mathbb{C}$, where $\Im(w)$ denotes the imaginary part of a complex number ω. For $\alpha > -1$, we define a weighted measure on $\mathbb{U}$ by $\mathrm{d}\mu_\alpha(\omega) = (\Im(\omega))^\alpha \mathrm{d}A(\omega)$, for each $\omega \in \mathbb{U}$ and where $\mathrm{d}A(\omega)$ denotes the Lebesgue measure on $\mathbb{U}$. The function $\psi(z) = \frac{\mathrm{i}(1+z)}{1-z}$ is referred to as the Cayley transform and maps the unit disk $\mathbb{D}$ conformally onto the upper half-plane $\mathbb{U}$ with the inverse $\psi^{-1}(\omega) = \frac{\omega-\mathrm{i}}{\omega+\mathrm{i}}$ mapping $\mathbb{U}$ onto $\mathbb{D}$. See [9] for details. Let $\{V_1, V_2\} = \{\mathbb{D}, \mathbb{U}\}$, and let $LF(V_i, V_j)$ denote the collection of all linear fractional transformations (LFTs) from V_i onto V_j. Then $LF(V_i, V_i) = \mathrm{Aut}(V_i)$, the group of automorphisms on V_i.

For $1 \le p < \infty$, $\alpha > -1$, the weighted Bergman spaces of the upper half plane, $L^p_a(\mathbb{U}, \mu_\alpha)$, are defined by

$$L^p_a(\mathbb{U}, \mu_\alpha) := \left\{ f \in \mathcal{H}(\mathbb{U}) : \|f\|_{L^p_a(\mathbb{U},\mu_\alpha)} = \left(\int_{\mathbb{U}} |f(z)|^p \,\mathrm{d}\mu_\alpha(z) \right)^{\frac{1}{p}} < \infty \right\},$$

while the corresponding weighted Bergman spaces of the unit disc, $L^p_a(\mathbb{D}, m_\alpha)$, by

$$L^p_a(\mathbb{D}, m_\alpha) := \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_{L^p_a(\mathbb{D},m_\alpha)} = \left(\int_{\mathbb{D}} |f(z)|^p \, \mathrm{d}m_\alpha(z) \right)^{\frac{1}{p}} < \infty \right\}.$$

Take note that $L^p_a(.)$ is a Banach space and that $L^2_a(.)$ is a Hilbert space. For a comprehensive theory of Bergman spaces, we refer to [9–11].

For $\alpha > -1$, the weighted Dirichlet space of the unit disk, $\mathcal{D}_\alpha(\mathbb{D})$, is defined by

$$\mathcal{D}_\alpha(\mathbb{D}) := \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|^2_{\mathcal{D}_{\alpha,1}(\mathbb{D})} = \int_{\mathbb{D}} |f'(z)|^2 \mathrm{d}m_\alpha(z) < \infty \right\}$$

with the norm given as:

$$\|f\|_{\mathcal{D}_\alpha(\mathbb{D})} = \left(|f(0)|^2 + \|f\|^2_{\mathcal{D}_{\alpha,1}(\mathbb{D})} \right)^{\frac{1}{2}}.$$

The corresponding weighted Dirichlet space of the upper half-plane $\mathbb{U}$ is given by

$$\mathcal{D}_\alpha(\mathbb{U}) := \left\{ f \in \mathcal{H}(\mathbb{U}) : \|f\|^2_{\mathcal{D}_{\alpha,1}(\mathbb{U})} = \int_{\mathbb{U}} |f'(\omega)|^2 \mathrm{d}\mu_\alpha(\omega) < \infty \right\}$$

with the norm given as:

$$\|f\|_{\mathcal{D}_\alpha(\mathbb{U})} = \left(|f(i)|^2 + \|f\|^2_{\mathcal{D}_{\alpha,1}(\mathbb{U})} \right)^{\frac{1}{2}}.$$

Again, we note that $\|\cdot\|_{\mathcal{D}_{\alpha,1}(\cdot)}$ is a seminorm and $\mathcal{D}_\alpha(\cdot)$ is a Banach space with respect to the norm $\|\cdot\|_{\mathcal{D}_\alpha}$. Moreover, $\mathcal{D}_\alpha(\cdot)$ is a Hilbert space. By definition, it is easy to see that $f \in \mathcal{D}_\alpha(\mathbb{U})$ if and only if $f' \in L^2_a(\mathbb{U}, \mu_\alpha)$. Indeed, by definition, $f \in \mathcal{D}_\alpha(\mathbb{U})$ if and only if

$$\|f\|_{\mathcal{D}_{\alpha,1}(\mathbb{U})} = \int_{\mathbb{U}} |f'(z)|^2 \mathrm{d}\mu_\alpha(z) < \infty.$$

But again

$$\int_{\mathbb{U}} |f'(z)|^2 \mathrm{d}\mu_\alpha(z) = \|f'\|^2_{L^2_a(\mathbb{U},\mu_\alpha)},$$

which means that $f \in \mathcal{D}_\alpha(\mathbb{U})$ if and only if $f' \in L^2_a(\mathbb{U}, \mu_\alpha)$, as claimed. Similarly, $f \in \mathcal{D}_\alpha(\mathbb{D})$ if and only if $f' \in L^2_a(\mathbb{D}, m_\alpha)$.

For $f \in \mathcal{D}_\alpha(\mathbb{D})$, then f satisfies the growth condition:

$$|f(z)| \le c\|f\|_{\mathcal{D}_\alpha(\mathbb{D})} \sqrt{\log \frac{1}{1-|z|^2}}. \tag{2.1}$$

Very little is known about $\mathcal{D}_\alpha(\mathbb{U})$. For instance, the growth condition for $\mathcal{D}_\alpha(\mathbb{U})$ is not well captured in the literature. We refer to [12–14] for a comprehensive theory of Dirichlet spaces.

Let X be an arbitrary Banach space. A semigroup $(T_t)_{t\ge0}$ of bounded linear operators on X is strongly continuous if $\lim\limits_{t\to0^+} T_t x = x$, that is,

$$\lim_{t\to0^+} \|T_t x - x\| = 0 \quad \text{for all} \quad x \in X.$$

The infinitesimal generator Γ of a strongly continuous semigroup $(T_t)_{t\ge0}$ is defined by

$$\Gamma x = \lim_{t\to0^+} \frac{T_t x - x}{t} = \frac{\partial}{\partial t}(T_t x)\Big|_{t=0}$$

for each $x \in \mathrm{dom}(\Gamma)$, where the domain of Γ is given by

$$\mathrm{dom}(\Gamma) = \left\{ x \in X : \lim_{t\to0^+} \frac{T_t x - x}{t} \text{ exists} \right\}.$$

We refer to [15–17] for details on the theory of semigroups.

If X and Y are arbitrary Banach spaces, we denote the Banach space of bounded linear operators from X to Y by $\mathcal{L}(X,Y)$. We shall write $\mathcal{L}(X)$ instead of $\mathcal{L}(X,X)$. Let T be a closed operator on X. The resolvent set $\rho(T)$ of T is given by $\rho(T)=\{\lambda\in\mathbb{C}:\lambda I-T \text{ is invertible}\}$ and its spectrum is $\sigma(T)=\mathbb{C}\setminus\rho(T)$. The spectral radius of T is defined by $r(T)=\sup\{|\lambda|:\lambda\in\sigma(T)\}$ and satisfies the relation $r(T)\le\|T\|$. The point spectrum $\sigma_p(T)=\{\lambda\in\mathbb{C}:Tx=\lambda x$ for some $0\neq x\in\operatorname{dom}(T)\}$. For $\lambda\in\rho(T)$, the operator $R(\lambda,T):=(\lambda I-T)^{-1}$ is by the closed graph theorem a bounded operator on X and is called the resolvent of T at the point λ or simply the resolvent operator. See [15, 16, 18] for details.

If X and Y are arbitrary Banach spaces and $U\in\mathcal{L}(X,Y)$ is an invertible operator, then clearly $(A_t)_{t\in\mathbb{R}}\subset\mathcal{L}(X)$ is a strongly continuous group if and only if $B_t:=UA_tU^{-1}$, $t\in\mathbb{R}$, is a strongly continuous group in $\mathcal{L}(Y)$. In this case, if $(A_t)_{t\in\mathbb{R}}$ has a generator Γ then $(B_t)_{t\in\mathbb{R}}$ has generator $\Delta=U\Gamma U^{-1}$ with domain

$$D(\Delta)=UD(\Gamma):=\{y\in Y:Uy\in D(\Gamma)\}.$$

Moreover, $\sigma_p(\Delta,Y)=\sigma_p(\Gamma,X)$ and $\sigma(\Delta,Y)=\sigma(\Gamma,X)$. If λ is in the resolvent set $\rho(\Gamma,X):=\mathbb{C}\setminus\sigma(\Gamma,X)$, we have that $R(\lambda,\Delta)=UR(\lambda,\Gamma)U^{-1}$. For more details on the theory of similar semigroups, we refer to [15–17].

All self analytic maps of $\mathbb{U}$ were identified and classified in [8] into three distinct groups according to the location of their fixed points, namely; scaling, translation and rotation groups. Specifically we give the following classification theorem:

Theorem 2.1 (see [8, Proposition 2.3]) Let $\varphi:\mathbb{R}\to\operatorname{Aut}(\mathbb{U})$ be a nontrivial continuous group homomorphism. Then exactly one of the following cases holds:

1. There exists $k>0$, $k\neq1$, and $g\in\operatorname{Aut}(\mathbb{U})$ so that $\varphi_t(z)=g^{-1}(k^tg(z))$ for all $z\in\mathbb{U}$ and $t\in\mathbb{R}$.

2. There exists $k\in\mathbb{R}$, $k\neq0$, and $g\in\operatorname{Aut}(\mathbb{U})$ so that $\varphi_t(z)=g^{-1}(g(z)+kt)$ for all $z\in\mathbb{U}$ and $t\in\mathbb{R}$.

3. There exists $k\in\mathbb{R}$, $k\neq0$, and a conformal mapping g of $\mathbb{U}$ onto $\mathbb{D}$ such that $\varphi_t(z)=g^{-1}(\mathrm{e}^{\mathrm{i}kt}g(z))$ for all $z\in\mathbb{U}$ and $t\in\mathbb{R}$. Equivalently, there exist $\theta\in\mathbb{R}\setminus0$ and $h\in\operatorname{Aut}(\mathbb{U})$ so that:

$$\varphi_t(z)=h^{-1}\left[\frac{h(z)\cos(\theta t)-\sin(\theta t)}{h(z)\sin(\theta t)+\cos(\theta t)}\right].$$

Corollary 2.2 Let $(\varphi_t)_{t\in\mathbb{R}}$ be defined as in Theorem 2.1 above. Then it follows that $\varphi_t'(z)\xrightarrow[t\to0]{}1$ and $\varphi_t''(z)\xrightarrow[t\to0]{}0$ uniformly on compact subsets of $\mathbb{U}$.

Proof Let g and k be as defined in Theorem 2.1 for each case. Then for case 1, $\varphi_t(z)=g^{-1}(k^tg(z))$ for all $z\in\mathbb{U}$ and $t\in\mathbb{R}$, implying that $g(\varphi_t(z))=k^tg(z)$. Taking derivative on both sides yields

$$g'(\varphi_t(z))\varphi_t'(z)=k^tg'(z). \tag{2.2}$$

Taking limits as $t\to0$ on both sides of equation (2.2), we get

$$g'(z)\lim_{t\to0}\varphi_t'(z)=g'(z)\ \Rightarrow\ \lim_{t\to0}\varphi_t'(z)=1,$$

since $g'(z)\neq0$ (g is an automorphism).

Now from (2.2) again, we obtain

$$g''(\varphi_t(z))\varphi_t'(z)\varphi_t'(z) + g'(\varphi_t(z))\varphi_t''(z) = k^t g''(z)$$

$$\Rightarrow g''(z) + g'(z)\lim_{t\to 0}\varphi_t''(z) = g''(z)$$
$$\Rightarrow \lim_{t\to 0}\varphi_t''(z) = 0.$$

The proofs for cases 2 and 3 are similar. We omit the details. □

The assertions 1, 2 and 3 of Theorem 2.1 above corresponds to the automorphism groups: scaling, translation and rotation groups respectively. In this study and as noted in [8], we shall consider without loss of generality, the following special cases:

I. For the scaling group, we consider $\varphi_t(z) = \mathrm{e}^{-t}z, z \in \mathbb{U}$.

II. For the translation group, we consider $\varphi_t(z) = z + t, z \in \mathbb{U}$.

III. For the rotation group, we consider $\varphi_t(z) = \mathrm{e}^{\mathrm{i}kt}z, z \in \mathbb{D}$.

3 Scaling and Translation Groups

3.1 Strong continuity

We prove that the groups of composition operators induced by the automorphism groups I and II above are strongly continuous on the Dirichlet space of the upper half plane $\mathcal{D}_\alpha(\mathbb{U})$. The strong continuity of composition groups on the Dirichlet space of the unit disc $\mathbb{D}$ was proved by Siskakis in [7]. For $\alpha > -1$, let $\mathcal{D}^i_\alpha(\mathbb{U})$ denotes the space consisting of $f \in \mathcal{H}(\mathbb{U})$ for which $\int_{\mathbb{U}} |f'(\omega)|\,\mathrm{d}\mu_\alpha(\omega) < \infty$ with the norm

$$\|f\|_{\mathcal{D}^i_\alpha(\mathbb{U})} = \left(\int_{\mathbb{U}} |f'(\omega)|\,\mathrm{d}\mu_\alpha(\omega)\right)^{\frac{1}{2}}.$$

In particular, the space $\mathcal{D}^i_\alpha(\mathbb{U})$ is given by $\mathcal{D}^i_\alpha(\mathbb{U}) := \{f \in \mathcal{D}_\alpha(\mathbb{U}) : f(i) = 0\}$. The space $\mathcal{D}^i_\alpha(\mathbb{U})$ was discussed in [19, Section 6] and recently considered in [5, Section 5]. Using a version of the Paley-Wiener theorem for the weighted Dirichlet space as well as some spectral results, Schroderus [5] established the relation

$$\mathcal{D}^i_\alpha(\mathbb{U}) = L^2_a(\mathbb{U}, \mu_\tau), \tag{3.1}$$

where $\tau = \alpha - 2$. With the help of equation (3.1), we prove the following result.

Theorem 3.1 Let $(\varphi_t)_{t\in\mathbb{R}}$ be the continuous groups in $\mathrm{Aut}(\mathbb{U})$ given by I and II above, and let $(T_t)_{t\in\mathbb{R}}$ be the induced group of weighted composition operators on $\mathcal{D}_\alpha(\mathbb{U})$, where $T_tf = (\varphi_t')^\gamma(f\circ\varphi_t)$ and $\gamma = \frac{\alpha+2}{2}$. Then $(T_t)_{t\in\mathbb{R}}$ is strongly continuous.

Proof To prove strong continuity of $(T_t)_{t\in\mathbb{R}}$, it suffices to show that for any $f \in \mathcal{D}_\alpha(\mathbb{U})$:

$$\lim_{t\to 0^+}\|T_tf - f\|_{\mathcal{D}_\alpha(\mathbb{U})} = 0.$$

But following the above remarks as well as equation (3.1),

$$\begin{aligned}\|T_tf - f\|^2_{\mathcal{D}_\alpha(\mathbb{U})} &= |(T_tf - f)(i)|^2 + \int_{\mathbb{U}} |(T_tf - f)'(z)|^2\,\mathrm{d}\mu_\alpha(z)\\ &= |(T_tf - f)(i)|^2 + \|T_tf - f\|^2_{\mathcal{D}^i_\alpha(\mathbb{U})}\\ &= |(T_tf - f)(i)|^2 + \|T_tf - f\|^2_{L^2_a(\mathbb{U},\mu_\tau)}.\end{aligned}$$

Thus, we need to show that: $|(T_t f - f)(i)|^2 \to 0$ and

$$\|T_t f - f\|^2_{L^2_a(\mathbb{U},\mu_\tau)} = \int_{\mathbb{U}} |(T_t f - f)(z)|^2 \,\mathrm{d}\mu_\tau(z) \to 0 \text{ as } t \to 0,$$

where $\gamma_\tau = \gamma + 1$ and $\gamma := \gamma_\alpha = \frac{\alpha+2}{2}$. By using Corollary 2.2, it is clear that

$$|(T_t f - f)(i)| \to |(T_0 f - f)(i)| = 0 \text{ as } t \to 0.$$

Next, suppose that $t_n \to 0$ in $\mathbb{R}$. Let $f_n = T_{t_n} f$. Then $f_n(z) \to f(z)$ uniformly on compact subsets of $\mathbb{U}$, by (2.1) and Corollary 2.2. Moreover, we have:

$$\|T_{t_n} f\|_{L^2_a(\mathbb{U},\mu_\tau)} = \|f\|_{L^2_a(\mathbb{U},\mu_\tau)}. \tag{3.2}$$

Indeed, for the specific case I (scaling group), we have by change of variables

$$\begin{aligned}\|T_t f\|^2_{L^2_a(\mathbb{U},\mu_\tau)} &= \int_{\mathbb{U}} |T_t f(z)|^2 \,\mathrm{d}\mu_\tau(z) \\ &= \int_{\mathbb{U}} \left|\mathrm{e}^{-t\gamma_\tau} f(\mathrm{e}^{-t} z)\right|^2 \,\mathrm{d}\mu_\tau(z) \\ &= \int_{\mathbb{U}} |f(\omega)|^2 \,\mathrm{d}\mu_\tau(\omega) = \|f\|^2_{L^2_a(\mathbb{U},\mu_\tau)}.\end{aligned}$$

For the specific case II, again by change of variables we have

$$\|T_t f\|^2_{L^2_a(\mathbb{U},\mu_\tau)} = \int_{\mathbb{U}} |f(z+t)|^2 \,\mathrm{d}\mu_\tau(z) = \int_{\mathbb{U}} |f(\omega)|^2 \,\mathrm{d}\mu_\tau(\omega) = \|f\|^2_{L^2_a(\mathbb{U},\mu_\tau)}.$$

Therefore (3.2) holds for both the I and II cases. Now, let $g_n := 2(|f|^2 + |f_n|^2) - |f - f_n|^2$. Then $g_n \geq 0$ and $g_n(z) \to 2^2|f(z)|^2$ on $L^2_a(\mathbb{U},\mu_\tau)$ as $n \to \infty$. By Fatou's lemma, we have

$$\begin{aligned}\int_{\mathbb{U}} 2^2|f|^2 \mathrm{d}\mu_\tau &= \int_{\mathbb{U}} \liminf_n g_n \mathrm{d}\mu_\tau \\ &\leq \liminf_n \int_{\mathbb{U}} g_n \mathrm{d}\mu_\tau \\ &= \liminf_n \int_{\mathbb{U}} \left(2(|f|^2 + |f_n|^2) - |f - f_n|^2\right)\mathrm{d}\mu_\tau \\ &= 2\int_{\mathbb{U}} |f|^2 \mathrm{d}\mu_\tau + 2\int_{\mathbb{U}} |f|^2 \mathrm{d}\mu_\tau - \limsup_n \int_{\mathbb{U}} |f - f_n|^2 \mathrm{d}\mu_\tau \\ &= 4\int_{\mathbb{U}} |f|^2 \mathrm{d}\mu_\tau - \limsup_n \int_{\mathbb{U}} |f - f_n|^2 \mathrm{d}\mu_\tau.\end{aligned}$$

Thus, $0 \leq -\limsup_n \int_{\mathbb{U}} |f - f_n|^2 \,\mathrm{d}\mu_\tau \leq 0$. This implies that $\limsup_n \int_{\mathbb{U}} |f - f_n|^2 \mathrm{d}\mu_\tau = 0$ and hence $\lim_n \|f_n - f\|^2_{L^2_a(\mathbb{U},\mu_\tau)} = 0$. Therefore, $\|T_t f - f\|_{\mathcal{D}_\alpha(\mathbb{U})} \to 0$ as $t \to 0$ as desired. □

3.2 Scaling group

Here, we consider the self analytic maps on $\mathbb{U}$ of the form $\varphi_t(z) = \mathrm{e}^{-t} z$, $z \in \mathbb{U}$.
The induced groups of weighted composition operators on $\mathcal{D}_\alpha(\mathbb{U})$ are therefore given by

$$T_t f(z) = \mathrm{e}^{-t\gamma} f(\mathrm{e}^{-t} z) \tag{3.3}$$

for all $f \in \mathcal{D}_\alpha(\mathbb{U})$, where $\gamma = \frac{\alpha+2}{2}$. By Theorem 3.1, this group is strongly continuous and we compute its infinitesimal generator Γ in the following theorem.

Theorem 3.2 Let $(T_t)_{t\in\mathbb{R}}$ be the group of weighted composition operators given by equation (3.3). The infinitesimal generator Γ of $(T_t)_{t\in\mathbb{R}}$ is given by $\Gamma f(z) = -\gamma f(z) - z f'(z)$ with domain $\mathrm{dom}(\Gamma) = \{f \in \mathcal{D}_\alpha(\mathbb{U}) : z f'(z) \in \mathcal{D}_\alpha(\mathbb{U})\}$.

Proof For $f \in \mathcal{D}_\alpha(\mathbb{U}))$, we have by definition;

$$\begin{aligned}\Gamma f(z) &= \lim_{t\to 0^+} \frac{\mathrm{e}^{-\gamma t} f(\mathrm{e}^{-t}z) - f(z)}{t} = \frac{\partial}{\partial t}(\mathrm{e}^{-\gamma t} f(\mathrm{e}^{-t}z))|_{t=0} \\ &= -\gamma \mathrm{e}^{-\gamma t} f(\mathrm{e}^{-t}z) - z\mathrm{e}^{-\gamma t} f'(\mathrm{e}^{-t}z)|_{t=0} = -\gamma f(z) - zf'(z).\end{aligned}$$

Therefore, $\mathrm{dom}(\Gamma) \subset \{f \in \mathcal{D}_\alpha(\mathbb{U}) : zf'(z) \in \mathcal{D}_\alpha(\mathbb{U})\}$.

Conversely, let $f \in \mathcal{D}_\alpha(\mathbb{U})$ be such that $zf'(z) \in \mathcal{D}_\alpha(\mathbb{U})$. Then for $z \in \mathbb{U}$, we have

$$\begin{aligned}T_t f(z) - f(z) &= \int_0^t \frac{\partial}{\partial s}(\mathrm{e}^{-\gamma s} f(\varphi_s(z))\mathrm{d}s \\ &= \int_0^t (-\mathrm{e}^{-\gamma s}\varphi_s(z) f'(\varphi_s(z)) - \gamma \mathrm{e}^{-\gamma s} f(\varphi_s(z)))\mathrm{d}s \\ &= \int_0^t T_s F(z)\mathrm{d}s,\end{aligned}$$

where $F(z) = -\gamma f(z) - zf'(z)$.

Thus, $\lim\limits_{t\to 0} \frac{T_t(f)-f}{t} = \lim\limits_{t\to 0} \frac{1}{t}\int_0^t T_s F \mathrm{d}s$ and strong continuity of $(T_s)_{s\in\mathbb{R}}$ implies that

$$\frac{1}{t}\int_0^t \|T_s F - F\|\mathrm{d}s \to 0 \text{ as } t \to 0$$

since $F \in \mathcal{D}_\alpha(\mathbb{U})$ by hypothesis. Thus

$$\mathrm{dom}(\Gamma) \supseteq \{f \in \mathcal{D}_\alpha(\mathbb{U}) : zf'(z) \in \mathcal{D}_\alpha(\mathbb{U})\}.$$

□

In the next theorem, we show that this group fails to be an isometry on $\mathcal{D}_\alpha(\mathbb{U})$.

Theorem 3.3 The group $(T_t)_{t\in\mathbb{R}}$ given by equation (3.3) is not a group of isometries on $\mathcal{D}_\alpha(\mathbb{U})$.

Proof By change of variables, we have

$$\begin{aligned}\|T_t f\|^2_{\mathcal{D}_\alpha(\mathbb{U})} &= |T_t f(i)|^2 + \int_{\mathbb{U}} |(T_t f)'(\omega)|^2 \,\mathrm{d}\mu_\alpha(\omega) \\ &= \mathrm{e}^{-2t\gamma}|f(\mathrm{e}^{-t}i)|^2 + \int_{\mathbb{U}} \mathrm{e}^{-2t\gamma}\mathrm{e}^{-2t}|f'(\mathrm{e}^{-t}\omega)|^2 \Im(\omega)^\alpha \,\mathrm{d}A(\omega) \\ &= \mathrm{e}^{-2t\gamma}|f(\mathrm{e}^{-t}i)|^2 + \int_{\mathbb{U}} \mathrm{e}^{-2t\gamma}\mathrm{e}^{-2t}|f'(z)|^2 \mathrm{e}^{t\alpha}\Im(z)^\alpha \mathrm{e}^{2t} \,\mathrm{d}A(z) \\ &= \mathrm{e}^{-2t\gamma}\left(|f(\mathrm{e}^{-t}i)|^2 + \mathrm{e}^{t\alpha}\int_{\mathbb{U}} |f'(z)|^2 \mathrm{d}\mu_\alpha(z)\right). \qquad (3.4)\end{aligned}$$

The RHS of equation (3.4) is not equal to the norm $\|f\|^2_{\mathcal{D}_\alpha(\mathbb{U})}$ for all $f \in \mathcal{D}_\alpha(\mathbb{U})$. This implies that the weighted composition operator T_t is not an isometry on $\mathcal{D}_\alpha(\mathbb{U})$. □

Remark 3.4 The fact that the weighted composition operator T_t fails to be an isometry on $\mathcal{D}_\alpha(\mathbb{U})$ complicates the spectral analysis of the group $(T_t)_{t\in\mathbb{R}}$. This is because the theory of spectra of semigroups of linear operators are easily applied when we can identify exactly what the spectrum of $(T_t)_{t\in\mathbb{R}}$ is. For the case when $(T_t)_{t\in\mathbb{R}}$ is an isometry, then spectral mapping theorem for semigroups readily gives the spectrum of $(T_t)_{t\in\mathbb{R}}$ and together with Hille-Yosida theorem, a complete spectral analysis of the infinitesimal generator as well as the resulting resolvents can be easily carried out. For this composition group therefore, we shall only determine the point spectrum of the infinitesimal generator Γ on $\mathcal{D}_\alpha(\mathbb{U})$, but first we state the following Lemma.

Lemma 3.5 (see [8, Lemma 3.2]) Let X denote one of the spaces $H^p(\mathbb{U})$ or $L^p_a(\mathbb{U},\mu_\alpha)$, $1 \le p < \infty$ and $\alpha > -1$ ($\alpha = -1$ if $X = H^p(\mathbb{U})$) and let $\gamma = (\alpha+2)/p$. If $c \in \mathbb{R}$ and $\lambda, \nu \in \mathbb{C}$, then;

1. $f(\omega) = (\omega - c)^\lambda(\omega + \mathrm{i})^\nu \in X$ if and only if $\Re(\lambda+\nu) < -\gamma < \Re(\lambda)$. In particular: $(\omega-c)^\lambda \notin X$ for any $\lambda \in \mathbb{C}$ and $(\omega+\mathrm{i})^\nu \in X$ if and only if $\Re\nu < -\gamma$.

2. $f(\omega) = \mathrm{e}^\omega/\omega^c \in X$ if and only if $1/p < c < \gamma$. In particular, $\mathrm{e}^\omega/\omega^c \notin H^p(\mathbb{U})$ for any $c \in \mathbb{R}$.

The point spectrum of the infinitesimal generator Γ is given in the following theorem:

Proposition 3.6 Let Γ be the infinitesimal generator of the group $(T_t)_{t\in\mathbb{R}}$ given by equation (3.3). Then the point spectrum of Γ is empty, that is, $\sigma_p(\Gamma) = \emptyset$.

Proof Let $\lambda \in \sigma_p(\Gamma)$. Then there exists $0 \ne f \in \mathcal{D}_\alpha(\mathbb{U})$ such that $\Gamma f = \lambda f$. This implies that $-\gamma f(z) - zf'(z) = \lambda f(z)$ which simplifies to $\frac{f'(z)}{f(z)} = -(\gamma+\lambda)\frac{1}{z}$. Integrating both sides yield

$$\ln f(z) = -(\gamma+\lambda)\ln z + C_1,$$

which is equivalent to

$$f(z) = Cz^{-(\gamma+\lambda)},$$

where C is a constant.

It remains to determine for which $\lambda' s$ is $f \in \mathcal{D}_\alpha(\mathbb{U})$ given that $f(z) = Cz^{-(\gamma+\lambda)}$. Now by definition, $f \in \mathcal{D}_\alpha(\mathbb{U})$ if and only if $f' \in L^2_a(\mathbb{U},\mu_\alpha)$. By calculation, $f'(z) = -C(\gamma+\lambda)z^{-(\gamma+\lambda+1)}$ which is not in $L^2_a(\mathbb{U},\mu_\alpha)$ by Lemma 3.5. Thus, $\sigma_p(\Gamma) = \emptyset$. □

3.3 Translation group

For this group, we consider the self analytic maps on $\mathbb{U}$ of the form $\varphi_t(z) = z + t$, $z \in \mathbb{U}$ with the induced group of composition operators defined on $\mathcal{D}_\alpha(\mathbb{U})$ given by;

$$T_t f(z) = f(z+t) \quad \text{for all } f \in \mathcal{D}_\alpha(\mathbb{U}). \tag{3.5}$$

This group is strongly continuous by Theorem 3.1 and we determine its infinitesimal generator Γ in the following theorem:

Theorem 3.7 Let Γ be the infinitesimal generator of the group $(T_t)_{t\in\mathbb{R}}$ given by equation (3.5) on $\mathcal{D}_\alpha(\mathbb{U})$. Then $\Gamma f(z) = f'(z)$ with domain $\mathrm{dom}(\Gamma) = \{f \in \mathcal{D}_\alpha(\mathbb{U}) : f' \in \mathcal{D}_\alpha(\mathbb{U})\}$.

Proof If $f \in \mathrm{dom}(\Gamma)$, then by definition

$$\Gamma f(z) = \lim_{t\to 0^+}\left(\frac{(\varphi_t'(z))^\gamma f(\varphi_t(z)) - f(z)}{t}\right) = \frac{\partial}{\partial t}\left(f(z+t)\right)\Big|_{t=0} = f'(z).$$

Thus, $\mathrm{dom}(\Gamma) \subset \{f \in \mathcal{D}_\alpha(\mathbb{U}) : f' \in \mathcal{D}_\alpha(\mathbb{U})\}$.

Conversely, if $f \in \mathcal{D}_\alpha(\mathbb{U})$ is such that $f' \in \mathcal{D}_\alpha(\mathbb{U})$, then for $z \in \mathbb{U}$, we have

$$\frac{T_t f - f}{t} = \int_0^t \frac{\partial}{\partial s}\left(T_s f\right)\mathrm{d}s = \int_0^t f'(z+s)\mathrm{d}s = \int_0^t T_s F(z)\mathrm{d}s,$$

where $F(z) = f'(z)$.

Thus,

$$\left\|\frac{T_t f}{t} - f'\right\| \leq \frac{1}{t}\int_0^t \|T_s f' - f'\| \,\partial s \to 0 \text{ as } t \to 0$$

by strong continuity since $f' \in \mathcal{D}_\alpha(\mathbb{U})$. Hence, $\mathrm{dom}(\Gamma) \supseteq \{f \in \mathcal{D}_\alpha(\mathbb{U}) : f' \in \mathcal{D}_\alpha(\mathbb{U})\}$. □

In the next theorem we show that this group also fails to be a group of isometries on $\mathcal{D}_\alpha(\mathbb{U})$.

Theorem 3.8 The group $(T_t)_{t\in\mathbb{R}}$ given by equation (3.5) is not a group of isometries on $\mathcal{D}_\alpha(\mathbb{U})$.

Proof Again by change of variables, we have

$$\begin{aligned}\|T_t f\|^2_{\mathcal{D}_\alpha(\mathbb{U})} &= |T_t f(i)|^2 + \int_{\mathbb{U}} |(T_t f)'(\omega)|^2 \,\mathrm{d}\mu_\alpha(\omega)\\ &= |f(i+t)|^2 + \int_{\mathbb{U}} |f'(\omega+t)|^2 \,\Im(\omega)^\alpha \,\mathrm{d}A(\omega)\\ &= |f(i+t)|^2 + \int_{\mathbb{U}} |f'(z)|^2 \,\Im(z)^\alpha \,\mathrm{d}A(z)\\ &= |f(i+t)|^2 + \int_{\mathbb{U}} |f'(z)|^2 \,\mathrm{d}\mu_\alpha(z),\end{aligned}$$

which cannot be equal to $\|f\|^2_{\mathcal{D}_\alpha(\mathbb{U})}$ for all $f \in \mathcal{D}_\alpha(\mathbb{U})$ and $t \neq 0$. □

In the next theorem, we determine the point spectrum of the infinitesimal generator Γ,

Theorem 3.9 Let Γ be the infinitesimal generator of the group $(T_t)_{t\in\mathbb{R}}$ given by equation (3.5). Then $\sigma_p(\Gamma) = \emptyset$.

Proof Let $\lambda \in \mathbb{C}$ be such that $\lambda \in \sigma_p(\Gamma)$. Then $\Gamma f = \lambda f$ for some $0 \neq f \in \mathcal{D}_\alpha(\mathbb{U})$, which is equivalent to the differential equation;

$$f'(z) = \lambda f(z) \text{ for all } z \in \mathbb{U}, \tag{3.6}$$

and whose solution is given as $f(z) = C\mathrm{e}^{\lambda z}$, where C is a constant. It remains to check for which $\lambda' s$ is $f \in \mathcal{D}_\alpha(\mathbb{U})$.

Recall that $f \in \mathcal{D}_\alpha(\mathbb{U})$ if and only if $f' \in L^2_a(\mathbb{U}, \mu_\alpha)$. Now, $f'(z) = \lambda C\mathrm{e}^{\lambda z}$. Using Lemma 3.5, it follows that $f'(z) = \lambda C\mathrm{e}^{\lambda z} \in L^2_a(\mathbb{U}, \mu_\alpha)$ if and only if $\frac{1}{2} < 0 < \frac{\alpha+2}{2}$ which is impossible and therefore no such $\lambda \in \mathbb{C}$ exists. Hence $\sigma_\rho(\Gamma) = \emptyset$, as claimed. □

4 Rotation Group

For the rotation group, the self-analytic maps are given by $\varphi_t(z) = \mathrm{e}^{\mathrm{i}kt} z$ for $z \in \mathbb{D}$ and we consider the induced groups of weighted composition operators on $\mathcal{D}_\alpha(\mathbb{D})$ of the form

$$S_{\varphi_t} f(z) = \mathrm{e}^{\mathrm{i}ct} f(\mathrm{e}^{\mathrm{i}kt} z) \tag{4.1}$$

for all $f \in \mathcal{D}_\alpha(\mathbb{D})$ and $c, k \in \mathbb{R}$ with $k \neq 0$. Arguing as in Theorem 3.1, we see that $(S_{\varphi_t})_{t\in\mathbb{R}}$ is a strongly continuous group on $\mathcal{D}_\alpha(\mathbb{D})$; and as we prove in the next theorem, this group is a group of isometries. Let $\Gamma_{c,k}$ be the generator of the group $(S_{\varphi_t})_{t\in\mathbb{R}}$ given by equation (4.1). Then as remarked in [20], to analyze the group $(S_{\varphi_t})_{t\in\mathbb{R}}$, it is sufficient to consider the case when $c = 0$ and $k = 1$ since the properties are related in the following ways.

Theorem 4.1 Let $(S_{\varphi_t})_{t\in\mathbb{R}}$ be a group of weighted composition operators defined on $\mathcal{D}_\alpha(\mathbb{D})$ by $S_{\varphi_t} f(z) = \mathrm{e}^{\mathrm{i}ct} f(\mathrm{e}^{\mathrm{i}kt} z)$ and let $\Gamma_{c,k}$ be its infinitesimal generator. Then:

1. $(S_{\varphi_t})_{t\in\mathbb{R}}$ is an isometry on $\mathcal{D}_\alpha(\mathbb{D})$.
2. $S_{\varphi_t} = C_{\varphi_t}$ for $c = 0, k = 1$.
3. $\Gamma_{c,k} = \mathrm{i}c + k\Gamma_{0,1}$ with the domain $\mathrm{dom}(\Gamma_{c,k}) = \mathrm{dom}(\Gamma_{0,1}) = \{f \in \mathcal{D}_\alpha(\mathbb{D}) : f' \in \mathcal{D}_\alpha(\mathbb{D})\}$.
4. $\sigma(\Gamma_{c,k}) = \{\mathrm{i}c + k\sigma(\Gamma_{0,1})\}$, and $\sigma_\rho(\Gamma_{c,k}) = \{\mathrm{i}c + k\sigma_\rho(\Gamma_{0,1})\}$.
5. $\lambda \in \sigma(\Gamma_{0,1})$ if and only if $\mathrm{i}c + k\lambda \in \sigma(\Gamma_{c,k})$ and

$$R(\mathrm{i}c + k\lambda, \Gamma_{c,k}) = \frac{1}{k} R(\lambda, \Gamma_{0,1}).$$

Proof We begin by proving that $(S_{\varphi_t})_{t\in\mathbb{R}}$ is an isometry on $\mathcal{D}_\alpha(\mathbb{D})$. A change of variables argument will yield

$$\begin{aligned}
\|S_{\varphi_t} f\|^2_{\mathcal{D}_\alpha(\mathbb{D})} &= |S_{\varphi_t} f(0)|^2 + \int_{\mathbb{D}} |\left(S_{\varphi_t} f\right)'(z)|^2 \,\mathrm{d}m_\alpha(z) \\
&= |\mathrm{e}^{\mathrm{i}ct} f(0)|^2 + \int_{\mathbb{D}} |\mathrm{e}^{\mathrm{i}ct}\mathrm{e}^{\mathrm{i}kt} f'(\mathrm{e}^{\mathrm{i}kt} z)|^2 \,\mathrm{d}m_\alpha(z) \\
&= |f(0)|^2 + \int_{\mathbb{D}} |f'(\mathrm{e}^{\mathrm{i}kt} z)|^2 \,\mathrm{d}m_\alpha(z) \\
&= |f(0)|^2 + \int_{\mathbb{D}} |f'(\omega)|^2 \,\mathrm{d}m_\alpha(\omega) \\
&= \|f\|^2_{\mathcal{D}_\alpha(\mathbb{D})},
\end{aligned}$$

as desired. Now, from the definition, $S_{\varphi_t} f(z) = \mathrm{e}^{\mathrm{i}ct} f(\mathrm{e}^{\mathrm{i}kt} z)$. Taking $c = 0$ and $k = 1$, we get

$$S_{\varphi_t} f(z) = f(\mathrm{e}^{\mathrm{i}kt} z) = C_{\varphi_t} f(z),$$

where $\varphi_t(z) = \mathrm{e}^{\mathrm{i}kt} z$ for $z \in \mathbb{D}$. This proves assertion 1. The rest of the proof is similar to the proof of [20, Lemma 4.3] but taking note that $f \in \mathcal{D}_\alpha(\mathbb{D})$ in this setting. We omit the details. $\square$

Because of Theorem 4.1, we shall therefore restrict our attention to the group $C_{\varphi_t} f(z) = f(\mathrm{e}^{\mathrm{i}t} z)$ for all $f \in \mathcal{D}_\alpha(\mathbb{D})$ whose infinitesimal generator is $\Gamma_{0,1}$, and using similarity theory of semigroups, we carry out a complete analysis of both the semigroup and spectral properties of $(C_{\varphi_t})_{t\in\mathbb{R}}$.

Now, define $C_{\varphi_t} : \mathcal{D}_\alpha(\mathbb{D}) \to \mathcal{D}_\alpha(\mathbb{D})$ by $C_{\varphi_t} f(z) = f(\mathrm{e}^{\mathrm{i}t} z)$ for all $t \in \mathbb{R}$, $z \in \mathbb{D}$ and $f \in \mathcal{D}_\alpha(\mathbb{D})$, and where $\varphi_t(z) = \mathrm{e}^{\mathrm{i}t} z$. Then we have the following proposition.

Proposition 4.2 Let $C_{\varphi_t} : \mathcal{D}_\alpha(\mathbb{D}) \to \mathcal{D}_\alpha(\mathbb{D})$ be given by $C_{\varphi_t} f(z) = f(\mathrm{e}^{\mathrm{i}t} z)$. Then $C_{\tau_t} := \varphi_t' C_{\varphi_t}$ is a group of composition operators on $L^2_a(\mathbb{D}, m_\alpha)$.

Proof By definition $f \in \mathcal{D}_\alpha(\mathbb{D})$ if and only if $f' \in L^2_a(\mathbb{D}, m_\alpha)$. Now, let $f \in \mathcal{D}_\alpha(\mathbb{D})$. Then $C_{\varphi_t} f = f \circ \varphi_t$ and so;

$$(C_{\varphi_t} f)' = (f \circ \varphi_t)' = \varphi_t' f'(\varphi_t) = \varphi_t' f' \circ \varphi_t \in L^2_a(\mathbb{D}, m_\alpha)$$

since $f \in \mathcal{D}_\alpha(\mathbb{D})$ if and only if $f' \in L^2_a(\mathbb{D}, m_\alpha)$.

Next, we need to show that the family $(C_{\tau_t})_{t\in\mathbb{R}}$ defines a group of weighted composition operators on $L^2_a(\mathbb{D}, m_\alpha)$. Since $\varphi_t(z) = \mathrm{e}^{\mathrm{i}t} z$, for $z \in \mathbb{D}$, it follows that for all $f \in L^2_a(\mathbb{D}, m_\alpha)$,

$$C_{\tau_t} f(z) = \varphi_t' C_{\varphi_t} f(z) = \varphi_t' f(\varphi_t)(z) = \mathrm{e}^{\mathrm{i}t} f(\mathrm{e}^{\mathrm{i}t} z)$$

Now, $C_{\tau_0} f(z) = \mathrm{e}^0 f(\mathrm{e}^0 z) = f(z)$ and therefore $C_{\tau_0} = I$, the identity operator on $L^2_a(\mathbb{D}, m_\alpha)$. For $t, s \in \mathbb{R}$, we have

$$C_{\tau_t} \circ C_{\tau_s} f(z) = C_{\tau_t}(C_{\tau_s} f(z)) = \varphi_t' C_{\varphi_t}(C_{\tau_s} f(z)) = \varphi_t' C_{\varphi_t}\left(\varphi_s' C_{\varphi_s} f(z)\right),$$

which on further simplification gives

$$\begin{aligned}C_{\tau_t}\circ C_{\tau_s}f(z) &= \mathrm{e}^{\mathrm{i}t}\mathrm{e}^{\mathrm{i}s}C_{\varphi_t}(f(\varphi_s(z))) = \mathrm{e}^{\mathrm{i}(t+s)}f(\varphi_s(\varphi_t(z)))\\ &= \mathrm{e}^{\mathrm{i}(t+s)}f(\mathrm{e}^{\mathrm{i}(t+s)}z) = C_{\tau_{s+t}}f(z).\end{aligned}$$

Thus, $C_{\tau_t}\circ C_{\tau_s} = C_{\tau_{s+t}}$, as desired. Hence, $(C_{\tau_t})_{t\in\mathbb{R}}$ is a group on $L^2_a(\mathbb{D}, m_\alpha)$. □

Now, we define the set $\mathcal{D}^*_\alpha(\mathbb{D}) = \{f\in\mathcal{D}_\alpha(\mathbb{D}) : f(0)=0\}$. Then $\mathcal{D}^*_\alpha(\mathbb{D})$ is a closed subspace of $\mathcal{D}_\alpha(\mathbb{D})$ and is therefore a Banach space with respect to the norm $\|\cdot\|_{\mathcal{D}^*_\alpha(\mathbb{D})} = \|\cdot\|_{\mathcal{D}_{\alpha,1}(\mathbb{D})}$. Then we give the following proposition.

Proposition 4.3 Let $U : \mathcal{D}^*_\alpha(\mathbb{D}) \to L^2_a(\mathbb{D}, m_\alpha)$ be given by $Uf = f'$. Then U is unitary.

Proof For all $f\in\mathcal{D}^*_\alpha(\mathbb{D})$, we have by norm definition,

$$\|Uf\|^2_{L^2_a(\mathbb{D},m_\alpha)} = \|f'\|^2_{L^2_a(\mathbb{D},m_\alpha)} = \int_{\mathbb{D}}|f'(z)|^2\,\mathrm{d}m_\alpha(z) = \|f\|^2_{\mathcal{D}^*_\alpha(\mathbb{D})}.$$

Therefore U is an isometry. Moreover U is invertible since every $g\in L^2_a(\mathbb{D},m_\alpha)$ has a (unique) primitive vanishing at 0, which is, by definition, in $\mathcal{D}^*_\alpha(\mathbb{D})$. Since $\mathcal{D}^*_\alpha(\mathbb{D})$ and $L^2_a(\mathbb{D},m_\alpha)$ are Hilbert spaces, it follows that U is unitary, as claimed. □

We can therefore summarize the actions of the mappings U, U^{-1} and C_{τ_t} as we give below;

$$\mathcal{D}^*_\alpha(\mathbb{D}) \xrightarrow{U} L^2_a(\mathbb{D},m_\alpha) \xrightarrow{C_{\tau_t}} L^2_a(\mathbb{D},m_\alpha) \xrightarrow{U^{-1}} \mathcal{D}^*_\alpha(\mathbb{D}). \tag{4.2}$$

It is therefore apparent from (4.2) that $C_{\varphi_t} = U^{-1}C_{\tau_t}U$ and since U is unitary, we conclude that $C_{\tau_t} = UC_{\varphi_t}U^{-1}$.

Theorem 4.4 Let $C_{\varphi_t} : \mathcal{D}^*_\alpha(\mathbb{D}) \to \mathcal{D}^*_\alpha(\mathbb{D})$ be given by $C_{\varphi_t}f(z) = f(\mathrm{e}^{\mathrm{i}t}z)$, and $C_{\tau_t} : L^2_a(\mathbb{D},m_\alpha)\to L^2_a(\mathbb{D},m_\alpha)$ by $C_{\tau_t} := \varphi'_t C_{\varphi_t}$. Then C_{φ_t} and C_{τ_t} are similar.

Proof Let $g\in L^2_a(\mathbb{D},m_\alpha)$. Then $f := U^{-1}g$ is in $\mathcal{D}^*_\alpha(\mathbb{D})$ and

$$\begin{aligned}UC_{\varphi_t}U^{-1}g &= UC_{\varphi_t}f = U(f\circ\varphi_t) = (f\circ\varphi_t)' = \varphi'_t\,(f'\circ\varphi_t)\\ &= \varphi'_t\,(C_{\varphi_t}f') = C_{\tau_t}f' = C_{\tau_t}Uf = C_{\tau_t}g.\end{aligned}$$

Therefore, $(C_{\varphi_t})_t$ and $(C_{\tau_t})_t$ are similar semigroups. □

Before we state the main results of this section, recall that the multiplication operator M_z given by $M_zf(z) := zf(z)$ is bounded and bounded below on the space $L^p_a(\mathbb{D},m_\alpha)$, $1\le p<\infty$ and $\alpha>-1$, with

$$\operatorname{ran}(M_z) = \{f\in L^p_a(\mathbb{D},m_\alpha) : f(0)=0\}.$$

The left inverse of M_z is the operator $Qf(z) := \frac{f(z)-f(0)}{z}$. For every $m\in\mathbb{N}$, $L^p_a(\mathbb{D},m_\alpha) = \operatorname{ran}(M^m_z)\oplus\operatorname{span}\{z^n : n\in\mathbb{Z}_+, n<m\}$, and $P_m = M^m_zQ^m$ is the projection of $L^p_a(\mathbb{D},m_\alpha)$ onto $\operatorname{ran}(M^m_z)$ with kernel span $\{z^n : n\in\mathbb{Z}_+, n<m\}$. We then give the following Lemma.

Lemma 4.5 1. The infinitesimal generator of $(C_{\tau_t})_{t\ge0}\subset\mathcal{L}(L^2_a(\mathbb{D},m_\alpha))$ is given by $\Gamma f = i\,(f(z)+zf'(z))$ with domain given by

$$\operatorname{dom}(\Gamma) = \left\{f\in L^2_a(\mathbb{D},m_\alpha) : f'\in L^2_a(\mathbb{D},m_\alpha)\right\}.$$

2. $\sigma(\Gamma, L^2_a(\mathbb{D},m_\alpha)) = \sigma_p(\Gamma, L^2_a(\mathbb{D},m_\alpha)) = \{\mathrm{i}(n+1) : n\in\mathbb{Z}_+\}$ and for each $n\ge0$, $\ker(\mathrm{i}(n+1)-\Gamma) = \operatorname{span}(z^n)$.

3. If $\lambda \in \rho(\Gamma)$, then $\operatorname{ran}(M_z^m)$ is $R(\lambda, \Gamma)$ - invariant for every $m \in \mathbb{Z}_+$ such that $m+1 > \Im(\lambda)$. In fact, if $h \in \operatorname{ran}(M_z^m)$, then

$$R(\lambda, \Gamma)h(z) = z^{-(1+\lambda \mathrm{i})} \int_0^z \omega^{\mathrm{i}} h(\omega) \mathrm{i}\omega = z^m \int_0^1 t^{m+\lambda \mathrm{i}} Q^m h(tz) \mathrm{d}t.$$

Proof Take note that $C_{\tau_t} f(z) = \mathrm{e}^{\mathrm{i}t} f(\mathrm{e}^{\mathrm{i}t} z)$ for all $f \in L_a^2(\mathbb{D}, m_\alpha)$. The result therefore follows from [8, Theorem 5.1] by taking $c = k = 1$ and $p = 2$. □

Our main result which characterizes the properties of the group $(C_{\varphi_t})_{t\in\mathbb{R}}$ on $\mathcal{D}_\alpha^*(\mathbb{D})$ is the following.

Theorem 4.6 Let $C_{\varphi_t} : \mathcal{D}_\alpha^*(\mathbb{D}) \to \mathcal{D}_\alpha^*(\mathbb{D})$ and Δ be its generator. Then the following hold:

1. $\Delta h(z) = \mathrm{i} z h'(z)$ with $\operatorname{dom}(\Delta) = \{h \in \mathcal{D}_\alpha^*(\mathbb{D}) : h' \in \operatorname{dom}(\Gamma)\}$.
2. $\sigma_p(\Delta) = \sigma(\Delta) = \{\mathrm{i}(n+1) : n \in \mathbb{Z}_+\}$, and for each $n \geq 0$, $\ker(\mathrm{i}(n+1) - \Delta) = \operatorname{span}(z^n)$.
3. If $\lambda \in \rho(\Delta)$, then $\operatorname{ran}(M_z^m)$ is $R(\lambda, \Gamma)$-invariant for every $m \in \mathbb{Z}_+$ such that $m+1 > \Im(\lambda)$. In fact, if $h \in \operatorname{ran}(M_z^m)$, then

$$R(\lambda, \Delta)h(z) = \frac{\mathrm{i}}{\lambda} \left(-h(z) + \frac{1}{z^{\lambda \mathrm{i}}} \int_0^z \omega^{\lambda \mathrm{i}} h'(\omega)\, \mathrm{i}\omega \right).$$

4. $R(\lambda, \Delta)$ is compact on $\mathcal{D}_\alpha^*(\mathbb{D})$.
5.
$$\sigma\left(R(\lambda, \Delta)\right) = \sigma_p\left(R(\lambda, \Delta)\right) = \left\{ \omega \in \mathbb{C} : \left| \omega - \frac{1}{2\Re\lambda} \right| = \frac{1}{2\Re\lambda} \right\}.$$

Moreover,
$$r\left(R(\lambda, \Delta)\right) = \|R(\lambda, \Delta)\| = \frac{1}{|\Re(\lambda)|}.$$

Proof Since $C_{\tau_t} = U C_{\varphi_t} U^{-1}$, it follows that $C_{\varphi_t} = U^{-1} C_{\tau_t} U$. Using the similarity theory of semigroups, it follows that if Γ is the generator of the group $(C_{\tau_t})_{t\in\mathbb{R}}$ and Δ is the generator of the group $(C_{\varphi_t})_{t\in\mathbb{R}}$, then $\Delta = U^{-1}\Gamma U$ with the domain $\operatorname{dom}(\Delta) = U^{-1}\operatorname{dom}(\Gamma)$.

Now, let $f' \in L_a^2(\mathbb{D}, m_\alpha)$. Then $f \in \operatorname{dom}(\Gamma)$ and $h := U^{-1} f$ belongs to $\operatorname{dom}(\Delta)$ with $f = Uh$. Then;

$$\begin{aligned} \triangle\left(h(z)\right) &= U^{-1}\Gamma U h(z) = U^{-1}\Gamma f(z) = U^{-1}(\mathrm{i} f(z) + z f'(z)) \\ &= \mathrm{i}\left(U^{-1} f(z) + U^{-1}(z f'(z))\right) = \mathrm{i}\left(h(z) + z h'(z) - h(z)\right) \\ &= \mathrm{i} z h'(z), \end{aligned}$$

as desired. Moreover,

$$\operatorname{dom}(\Delta) = U^{-1}\operatorname{dom}(\Gamma) = \left\{ U^{-1} f : f \in \operatorname{dom}(\Gamma) \right\}$$

Now, $h \in \operatorname{dom}(\Delta) \Leftrightarrow Uh \in \operatorname{dom}(\Gamma) \Leftrightarrow h' \in \operatorname{dom}(\Gamma)$, and thus

$$\operatorname{dom}(\Delta) = \left\{ h \in \mathcal{D}_\alpha^*(\mathbb{D}) : h' \in \operatorname{dom}(\Gamma) \right\}.$$

This proves assertion 1.

The proof of 2 follows from Lemma 4.5 and the fact that the operators $C_{\varphi_t} : \mathcal{D}_\alpha^*(\mathbb{D}) \to \mathcal{D}_\alpha^*(\mathbb{D})$ and $C_{\tau_t} : L_a^2(\mathbb{D}, m_\alpha) \to L_a^2(\mathbb{D}, m_\alpha)$ are similar; in which case,

$$\sigma_p(\triangle) = \sigma(\Delta) = \left\{ \mathrm{i}(n+1) : n \in \mathbb{Z}_+ \right\},$$

and for each $n \geq 0$, $\ker(\mathrm{i}(n+1) - \Delta) = \mathrm{span}(z^n)$.

To prove assertion 3, we recall that by the similarity of the corresponding semigroups, we have that if $\lambda \in \rho(\Delta)$, and since $\rho(\Delta) = \rho(\Gamma)$, then for $m \in \mathbb{Z}_+$, $m+1 > \Im(\lambda)$, and if $h \in \mathrm{ran}(M_z^m)$, then by Lemma 4.5,

$$\begin{aligned} R(\lambda, \Delta)h(z) &= U^{-1}R(\lambda, \Gamma)Uh(z) = U^{-1}R(\lambda, \Gamma)h'(z) \\ &= U^{-1}\left(z^{-(1+\lambda \mathrm{i})}\int_0^z \omega^{\lambda i}h'(\omega)\mathrm{d}\omega\right) \\ &= \int \left(z^{-(1+\lambda \mathrm{i})}\int_0^z \omega^{\lambda \mathrm{i}}h'(\omega)\mathrm{d}\omega\right)\mathrm{d}z \\ &= \frac{\mathrm{i}}{\lambda}\left(-h(z) + \frac{1}{z^{\lambda \mathrm{i}}}\int_0^z \omega^{\lambda \mathrm{i}}h'(\omega)\mathrm{d}\omega\right) \text{as claimed.} \end{aligned}$$

Compactness of the resolvent $R(\lambda, \Delta)$ follows from the compactness of the resolvent $R(\lambda, \Gamma)$ (see [8, Theorem 5.2]) since the relation $R(\lambda, \Gamma) = U^{-1}R(\lambda, \Gamma)U$ preserves compactness.

For assertion 5, the spectral mapping theorem for resolvents and the assertion 2 above imply that for all $\lambda \in \rho(\Delta)$,

$$\begin{aligned} \sigma(R(\lambda, \Delta)) &= \left\{\frac{1}{\lambda - z} : z \in \sigma(\Delta)\right\} \cup \{0\} \\ &= \left\{\frac{1}{\lambda - (n+1)\mathrm{i}} : n \in \mathbb{Z}_+\right\} \cup \{0\} \\ &= \left\{\omega \in \mathbb{C} : \left|\omega - \frac{1}{2\Re(\lambda)}\right| = \frac{1}{2\Re(\lambda)}\right\}. \end{aligned}$$

Since $R(\lambda, \Delta)$ is compact on $\mathcal{D}_\alpha^*(\mathbb{D})$, we have by [16, Corollary V.1.15] that

$$\sigma(R(\lambda, \Delta)) = \sigma_\rho(R(\lambda, \Delta)).$$

Now from the obtained spectrum $\sigma(R(\lambda, \Delta))$, it follows that the spectral radius of the resolvent is given by $r(R(\lambda, \Delta)) = \frac{1}{|\Re(\lambda)|}$. Finally, the boundedness of the spectral radius $r(R(\lambda, \Delta))$ by the norm $\|R(\lambda, \Delta)\|$ as well as the Hille-Yosida theorem immediately yield

$$r(R(\lambda, \Delta)) = \frac{1}{|\Re(\lambda)|} \leq \|R(\lambda, \Delta)\| \leq \frac{1}{|\Re(\lambda)|}.$$

This completes the proof. □

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