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Approximations of Ruin Probabilities Under Financial Constraints

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Abstract

In this paper, we investigate the approximate ruin probabilities under financial constraints (interest rate, inflation, and taxation). We formulate a risk process whose premium inflow is influenced by the economic effects of inflation and interest rate. Thereafter we invoke the Albrecher-Hipp loss-carried-forward tax scheme from which an exact formula for the ruin probability for exponentially distributed claims is derived. Finally, an explicit asymptotic formula when the claims have sub-exponential distribution is also derived using the Pollaczek-Khintchine formula.

Keywords: Ruin probability, Risk Process, Financial Constraints, Sub-exponential distribution.

1 Introduction

The study of the risk theory was pioneered by Lundberg in 1903 whose work was later reviewed and refined by Cramér around 1930 and 1955. This marked the foundation of actuarial risk theory. Taylor (1979) studied the probability of ruin when the classical risk model was modified to take into account inflationary conditions and interest rates as outlined in [1]. Albrecher and Hipp (2007) studied the classical risk model by including taxation in the model. They investigated how tax influences the infinite time ruin probability as indicated in [2]. Wei (2009) in [7] investigated the ruin probability in which the Albrecher-Hipp tax identity was studied as a special case in the presence of both constant forces of interest and the loss-carried-forward tax payments. Finally, Dbicki et al.,(2015) investigated the effect of financial factors (inflation and interest rates) on Gaussian risk models as outlined in [6]. In the present paper, we consider a more general risk process which takes into account the three financial constraints (economic factors).

1.1 The Cramér-Lundberg Model

The surplus process of an insurance portfolio for the classical risk model is described as

$$U(t) = u + ct - S(t), \quad t \geq 0 \tag{1}$$

Where, $u = U(0) \geq 0$ is the initial capital, $(ct,\ c \geq 0)$ is the premium income collected up to time t. The aggregate claims process $\{S(t),\ t \geq 0\}$, is a compound Poi(λ) process. The distribution function of the X_k's is henceforth denoted by $(F(x),\ F(0) = 0)$ and expected value μ. Infinite horizon ruin probability is given by

$$\Psi(u, \infty) = \mathtt{P}r\left(U(t) < 0 \ \textbf{ for some } \ t \geq 0 | U(0) = u\right) \tag{2}$$

and for finite horizon it is described as finite time ruin probability.

1.2 Sub-exponential Claims

Sub-exponential class $\mathcal{S}$ is one of the most important class of heavy-tailed distributions. It is important to note that this section follows closely the description from [3], [5], and [7]. By definition, a distribution F on $\mathtt{R}_{[0,\infty)}$ is said to be sub-exponential if $\lim_{x \longrightarrow \infty} \frac{\bar{F^{n*}}(x)}{\bar{F}(x)} = n,$ for $n \geq 2$. Where $\bar{F}(x) = 1 - F(x) > 0$ for all $x \geq 0$ is the tail of the distribution and $F^{n*}(x) = \mathtt{P}\left(\sum_{i=1}^{n} X_i \leq x\right)$ is the n-fold convolutions of F. Also, for any $n \geq 1, \bar{F^{n*}}(x) \sim n\bar{F}$ as $x \longrightarrow \infty$. Finally, from the principle of single big jump for the sum of independent random variables $X_1, ..., X_n$ with common distribution function

$$\mathtt{P}(S_n = X_1 + X_2 \cdots + X_n > x) \sim \mathtt{P}(M_n = \max(X_1, X_2, ..., X_n) > x) \quad \text{as} \quad x \to \infty$$

2 Ruin Probabilities Under Financial Constraints

2.1 First Main Result

Theorem 2.1 *In the presence of all the financial constraints, the surplus-dependent ruin probability is expressed as*

$$\Psi_{\gamma,i,r}(u) = [1 - \Phi_{i,r}(u)]^{(1-\gamma)^{-1}} \tag{3}$$

where $\gamma \approx [0, 1)$ is the loss-carried-forward taxation rates as according to Albrecher and Hipp tax scheme, $\Phi_{i,r}(u)$ is the probability of ruin in the presence of inflation and interest rate, i is the rate of inflation, and r is the rate of interest.

To prove above theorem we need the following.
Consider the Fisher relation. Fisher equation gives the mathematical relationship between real, nominal, and inflation rates as follows

$$r_r = \frac{1 + n_r}{1 + i_r} - 1 \tag{4}$$

where, r_r is the real rate of interest, n_r is the nominal rate of interest, and i_r is the inflation/deflation rate of interest.

We modify Equation 1 by assuming that inflation and interest rates influence the premium inflow. We also assume that the effects of inflation and return on capital do not cancel out each other exactly. The resulting risk process after adjustment and discounting the reserve back to zero is given by,

$$\hat{U(t)} = u + \frac{f(t)}{A(t)}ct - S(t), \qquad t \geq 0 \tag{5}$$

The premium values at time t is inflated/deflated by the factor $f(t) > 0$. The function $A(t)$ ensures that investments at time zero accumulates at time t. Note that $f(0) = A(0) = 1$.

Lemma 2.2 *Under the Equation 5, the constant premium inflow is given by*

$$c = (1 + \theta_\beta)\frac{\mu\lambda}{\beta} \tag{6}$$

where, $\theta_\theta > 0$ is the premium loading factor in the presence of inflation and interest rate, λt is the expected value of a poison process at any time t, μ is the expected value of claim sizes, and $\beta > 0$ is defined as any arbitrary constant at this level.

Definition 2.3 *Let f be a function defined on some interval (a, ∞). Then* $\lim_{x \longrightarrow \infty} f(x) = \beta$

Definition 2.4 *The line $y = \beta$ is called the horizontal asymptote of the curve $y = f(x)$ if either* $\lim_{x\longrightarrow\infty} f(x) = \beta$ or $\lim_{x\to-\infty} f(x) = \beta$

Lemma 2.5 *As outlined in [4], the Strong Law of Large Numbers for the Poisson states that*

$$\lim_{t\longrightarrow\infty} \frac{N(t)}{t} = \lambda \quad \lim_{t\longrightarrow\infty} N(t) = \infty \quad \text{a.s} \tag{7}$$

To prove Lemma 2.2

Proof. We invoke the Strong Law of Large Numbers for a Poisson process in [4] when the surplus process is modeled by Equation 5 and by taking the limits as t tends to infinity on both sides as given by the definitions 2.3 and 2.4 as shown below,

$$\lim_{t\longrightarrow\infty} \frac{\hat{U(t)}}{t} = \lim_{t\longrightarrow\infty} \left[\frac{u}{t} + \frac{f(t)}{A(t)}c - \frac{N(t)}{t} \frac{\sum_{k=1}^{N(t)} X_k}{N(t)} \right] \tag{8}$$

From Equation 7, and Definitions 2.3 & 2.4 we obtain the following relation,

$$\beta c - \mu\lambda > 0 \ \text{ a.s} \tag{9}$$

Recall that Equation 9 follow from the fact that

$$\mathtt{P}\left(\lim_{t\longrightarrow\infty} \hat{U(t)} = \infty \right) = 1 \quad \textbf{and} \quad \mathtt{P}\left(\liminf_{t\longrightarrow\infty} \hat{U(t)} \longrightarrow -\infty \right) = 1 \ \text{ a.s} \tag{10}$$

which is the said sufficient condition as illustrated in [4]. Further, we note that financial ruin in the classical ruin model will occur if the surplus of the insurance company drops below zero with a probability of 1. To include the so-called safety/premium loading factor, we rewrite Equation 9 as $c \geq \frac{\mu\lambda}{\beta}$. Hence, we obtain,

$$c = (1 + \theta_\beta)\frac{\mu\lambda}{\beta} \Longrightarrow \theta_\beta = \frac{\beta c}{\mu\lambda} - 1 \tag{11}$$

Thus we obtain,

$$\Psi_{r,i}(u) = \mathtt{P}\left[\hat{U(t)} < 0, \text{ for some } \ t \geq 0 \ \mid \ \hat{U(0)} = u \right], \qquad u \geq 0 \tag{12}$$

From Albrecher and Hipp (2007) in [2], a loss carried-forward-tax scheme after extending the classical risk model to incorporate taxation was established. It was established that;

$$\Psi_\gamma(u) = [\Psi_0(u)]^{(1-\gamma)^{-1}} \tag{13}$$

where, $\Psi_\gamma(u)$ is the ruin probability in the influence of taxation, $\Psi_0(u)$ is the classical ruin probability, and $\gamma \in [0, 1)$ is the taxation rate interval. The reader is referred to [2] for more details.
Thus we obtain,

$$\Psi_{\gamma,i,r}(u) = [\Psi_{i,r}(u)]^{(1-\gamma)^{-1}} \tag{14}$$

which completes the proof of Theorem 2.1 which is the first main result.

2.1.1 Exponentially Distributed Claims

Suppose that the claim-sizes exhibit an exponential distribution $F(x) = 1 - e^{\frac{-x}{\mu}},\ x \geq 0$ for some $\mu > 0$. We obtain the first main result which is a proof of Theorem 2.1 above. In the presence of inflation and interest rates, the said non-ruin probability is given by

$$\Psi_{i,r}(u) = \frac{1}{1+\theta_\beta} \exp\left(-\frac{\theta u}{\mu(1+\theta_\beta)}\right) \tag{15}$$

Finally, in the presence of the three economic factors, we obtain the non-ruin probability as

$$\Psi_{i,r,\gamma}(u) = [\Psi_{i,r}(u)]^{(1-\gamma)^{-1}} \qquad \text{where} \quad \gamma = [0, 1) \tag{16}$$

□

2.2 Second Main Result

Theorem 2.6 *The probability of ruin for the claims with sub-exponential distribution in the presence of inflation and interest rates, with net profit condition is given by*

$$\Psi_{i,r}(u) \sim \frac{1}{\theta_\beta}\bar{F}(u) \qquad \text{as} \qquad u \longrightarrow \infty \tag{17}$$

and including the taxation effects we further obtain

$$\Psi_{i,r,\gamma}(u) = 1 - (1 - \Psi_{i,r}(u))^{(1-\gamma)^{-1}} \sim \frac{1}{1-\gamma}\Psi_{i,r}(u) = \frac{1}{\theta_\beta(1-\gamma)}\bar{F}(u) \tag{18}$$

where $\gamma = [0, 1)$, $i > 0$, and $r > 0$; are the taxation, inflation, and interest rates respectively, and then for a density function with a finite mean μ, θ_β is the premium loading factor in the presence of interest rates and inflation.

$$F(x) = \frac{1}{\mu}\int_0^x (1 - F(y))\, dy \tag{19}$$

To prove the above theorem, we need the following; First we state without proof Lemma F.6 in Appendix of [5].

Lemma 2.7 *let $F \in \mathcal{S}$. Then for any $\epsilon > 0$, there exists a $D \in \mathtt{R}$ such that* $\frac{1-F^{n*}(x)}{1-F(x)} \leq D(1+\epsilon)^n \quad \forall \quad x > 0 \quad$ and $\quad n \in \mathtt{N}$

Theorem 2.8 *Theorem 1.3 (Pollaczek-Khintchine formula) in [4]. It states that suppose* $\frac{\lambda\mu}{c} < 1$*. For all* $u \geq 0$

$$\Phi(u) = \left(1 - \frac{\lambda\mu}{c}\right) \sum_{k\geq 0} \left(\frac{\lambda\mu}{c}\right)^k \eta^{*k}(u)$$

where $\eta(u) = \frac{1}{\mu}\int_0^u (1 - F(y))\, dy, \quad u \geq 0$ *and for* $k \geq 0$*. It is understood that* η^{*k} *is the k-fold convolutions of* η *with the special understanding that* $\eta^{*0}(du) = \delta_0(du)$

Proof. From Equation 19 and choosing $\epsilon > 0$ such that $\lambda\mu(1+\epsilon) < \beta c$. By Lemma 2.7, there exist a D such that

$$\frac{1 - F^{n*}(x)}{1 - F(x)} \leq D(1+\epsilon)^n \tag{20}$$

From Pollaczek-Khintchine formula, we obtain

$$\frac{\Psi_{i,r}(u)}{1 - F(u)} = \left(1 - \frac{\lambda\mu}{\beta c}\right) \sum_{n=1}^{\infty} \left(\frac{\lambda\mu}{\beta c}\right)^n \frac{1 - F^{n*}(u)}{1 - F(u)}$$

$$\leq D\left(1 - \frac{\lambda\mu}{\beta c}\right) \sum_{n=1}^{\infty} \left(\frac{\lambda\mu}{\beta c}\right)^n (1+\epsilon)^n < \infty$$

Interchanging sum and limits and recalling that $\lim_{u\longrightarrow\infty} \frac{1-F^{n*}(u)}{1-F(u)} = n$ We have,

$$\lim_{u\longrightarrow\infty} \frac{\Psi_{i,r}(u)}{1 - F(u)} = \left(1 - \frac{\lambda\mu}{\beta c}\right) \sum_{n=1}^{\infty} n \left(\frac{\lambda\mu}{\beta c}\right)^n = \left(1 - \frac{\lambda\mu}{\beta c}\right) \sum_{n=1}^{\infty}\sum_{m=1}^{n} \left(\frac{\lambda\mu}{\beta c}\right)^n$$

$$= \left(1 - \frac{\lambda\mu}{\beta c}\right) \sum_{n=1}^{\infty}\sum_{n=m}^{\infty} \left(\frac{\lambda\mu}{\beta c}\right)^n = \sum_{m=1}^{\infty} \left(\frac{\lambda\mu}{\beta c}\right)^m$$

This is a sum of infinite series such that from $s_\infty = \frac{a}{1-r}$ we obtain,

$$\lim_{u\longrightarrow\infty} \frac{\Psi_{i,r}(u)}{1 - F(u)} = \frac{\lambda\mu/\beta c}{1 - \frac{\lambda\mu}{\beta c}} = \frac{\lambda\mu}{\beta c - \lambda\mu}$$

We defined the net profit condition or the premium loading factor as $\theta_\beta = \frac{\beta c}{\lambda\mu} - 1$. Thus, it implies that

$$\lim_{u\longrightarrow\infty} \frac{\Psi_{i,r}(u)}{1 - F(u)} = \frac{1}{\theta_\beta} \iff \lim_{u\longrightarrow\infty} \Psi_{i,r}(u) = \frac{\bar{F}(u)}{\theta_\beta}$$

Table 1: Approx. Ruin Probabilities for Expo. Dist. Claims ($\gamma_1 = 0.1$).

u	$\Psi(u)$	$\Psi_{i,r}(u)$	$\Psi_{i,r,\gamma}(u)$	$\lvert\Psi(u) - \Psi_{i,r}(u)\rvert$	$\lvert\Psi(u) - \Psi_{i,r,\gamma}(u)\rvert$
25,000	0.02059	0.00147	0.00071	0.01912	0.01988
30,000	0.00965	0.00041	0.00017	0.00924	0.00948

Table 2: Approx. Ruin Probabilities for Expo. Dist. Claims ($\gamma_2 = 0.2$).

u	$\Psi(u)$	$\Psi_{i,r}(u)$	$\Psi_{i,r,\gamma}(u)$	$\lvert\Psi(u) - \Psi_{i,r}(u)\rvert$	$\lvert\Psi(u) - \Psi_{i,r,\gamma}(u)\rvert$
25,000	0.02059	0.00147	0.00029	0.01912	0.02030
30,000	0.00965	0.00041	0.00006	0.00924	0.00959

so that $\Psi_{i,r}(u) \sim \frac{\bar{F}(u)}{\theta_\beta}$ as $u \longrightarrow \infty$
Now in the presence of all the three financial constraints, and also from [2] and [7]; we obtain the following second main result as follows;

$$\Psi_{i,r,\gamma}(u) = 1 - (1 - \Psi_{i,r}(u))^{(1-\gamma)^{-1}} \sim \frac{1}{\theta_\beta(1-\gamma)}\bar{F}(u) \tag{21}$$

□

3 Simulation and Numerical Results

3.1 Numerical results for claims with exponential distribution

The following assumptions are necessary; $\lambda = 20$, $\mu = 600$, $n_r = 13.66\%$, $i_r = 5.8\%$, $\beta = 1.0743$, $c = 13,200$, $\theta = 0.1$, $\theta_\beta = 0.18$,$\gamma_1 = 0.1$, $\gamma_2 = 0.2$, and $\gamma_3 = 0.3$. The values of u from 0 to 30,000 in an interval of 100 are simulated using R. .

3.2 Numerical results for claims with Pareto distribution

The tail of Pareto distribution with parameters $\alpha(2) > 0$, $b(600) > 0$ is given by $1 - F(u) = \left(\frac{b}{b+u}\right)^\alpha$. The values of u from 25,000 to 30,000 in an interval of 100 are simulated using R.

Table 3: Approx. Ruin Probabilities for Expo. Dist. Claims ($\gamma_3 = 0.3$).

u	$\Psi(u)$	$\Psi_{i,r}(u)$	$\Psi_{i,r,\gamma}(u)$	$\lvert\Psi(u) - \Psi_{i,r}(u)\rvert$	$\lvert\Psi(u) - \Psi_{i,r,\gamma}(u)\rvert$
25,000	0.02059	0.00147	0.00009	0.01912	0.02050
30,000	0.00965	0.00041	0.00001	0.00924	0.00964

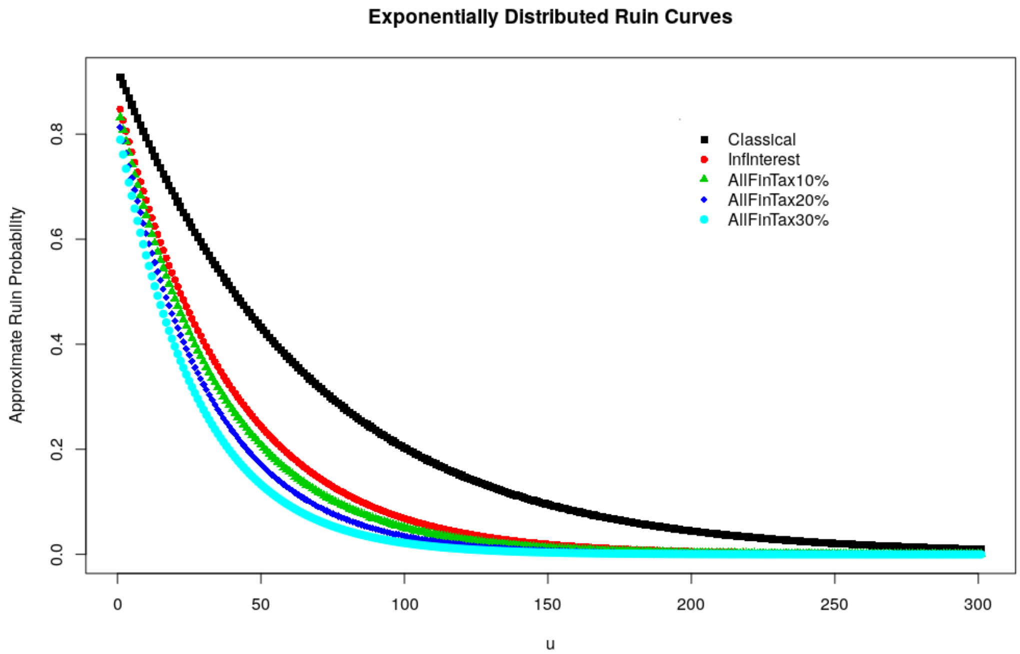

Figure 1: Exponential Ruin Scatter-plot for the All Financial Constraints

Table 4: Approx. Ruin Probabilities for claims with Pareto Dist. ($\gamma_1 = 0.1$).

u	$\Psi(u)$	$\Psi_{i,r}(u)$	$\Psi_{i,r,\gamma}(u)$	$\lvert\Psi(u) - \Psi_{i,r}(u)\rvert$	$\lvert\Psi(u) - \Psi_{i,r,\gamma}(u)\rvert$
25,000	0.00549	0.00305	0.00339	0.00244	0.00210
30,000	0.00384	0.00214	0.00237	0.00170	0.00147

In conclusion, the model developed in this paper gives less approximate ruin probabilities in comparison to those of the classical ruin model. Nevertheless, it can be improved to take into consideration a stochastic real interest rate.

Table 5: Approx. Ruin Probabilities for claims with Pareto Dist. ($\gamma_2 = 0.2$).

u	$\Psi(u)$	$\Psi_{i,r}(u)$	$\Psi_{i,r,\gamma}(u)$	$\lvert\Psi(u) - \Psi_{i,r}(u)\rvert$	$\lvert\Psi(u) - \Psi_{i,r,\gamma}(u)\rvert$
25,000	0.00549	0.00305	0.00381	0.00244	0.00168
30,000	0.00384	0.00214	0.00267	0.00170	0.00117

Table 6: Approx. Ruin Probabilities for claims with Pareto Dist. ($\gamma_3 = 0.3$).

u	$\Psi(u)$	$\Psi_{i,r}(u)$	$\Psi_{i,r,\gamma}(u)$	$\|\Psi(u) - \Psi_{i,r}(u)\|$	$\|\Psi(u) - \Psi_{i,r,\gamma}(u)\|$
25,000	0.00549	0.00305	0.00436	0.00154	0.00023
30,000	0.00384	0.00214	0.00305	0.00170	0.00079

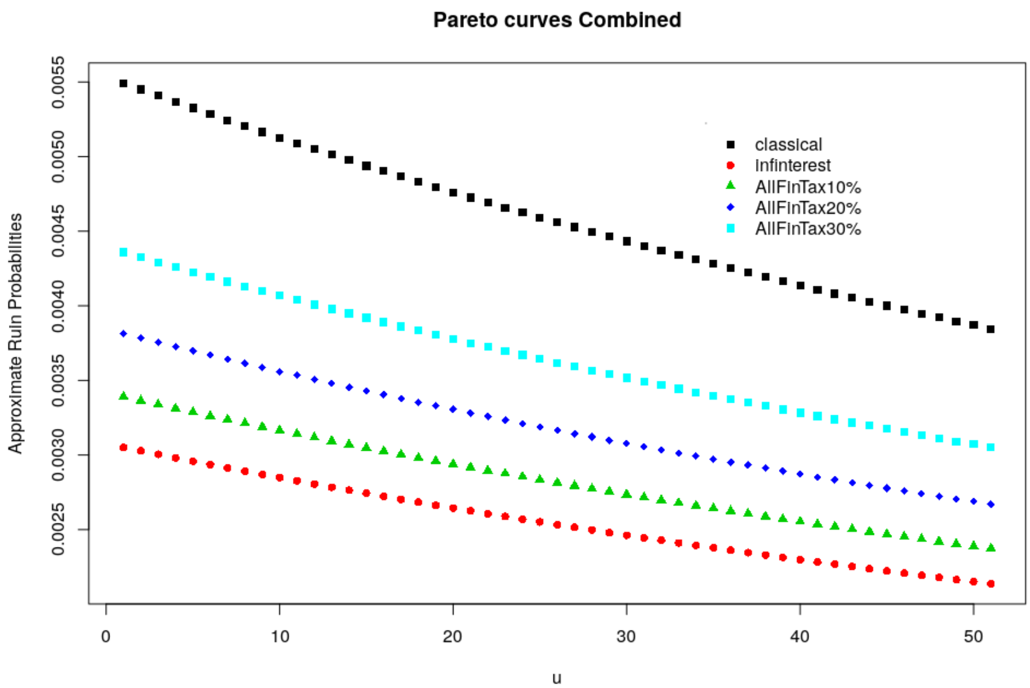

Figure 2: Pareto Scatter-plot for the Ruin Probabilities in Presence of All Financial Constraints

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