"STATISTICAL MODELLING OF A BENEVOLENT SCHEME"

(CASE STUDY-MASENO UNIVERSITY Burial and Benevolent Fund)

A thesis submitted in partial fulfillment of the requirements of the award of the degree of

Master of Science in Applied Statistics

BY

ARORI OMWANSA WILFRED
(PG/MSC/043/2001)

Department of Applied Statistics,
Faculty of Science,
Maseno University,
P.O. Box 333,
Maseno, Kenya.

ABSTRACT

The area of study is on a Benevolent Scheme. Here the insured contributes premiums to the insuring company and is compensated in the event of death of self or his or her dependant(s). The problem normally experienced by a number of insuring companies is how to determine the appropriate premium size to be paid by the insured, such that the company does not incur losses.

In this study statistical models of a *one-dependant* and an *m-dependant* scheme, have been formulated.

Using the formulated models, the expected expenditure and consequently profits or loses accrued to the insuring company have been calculated. Consequently the appropriate premium size that will give the insurer modest profit has been determined.

Properties of *Markov chains* and *Markov states* have been applied in determining the probabilities of transition from one state to another, in n-steps (years). Steady state transition probabilities have also been derived. Finally, correlation of the Exponential *probability distribution* with the benevolent scheme model has been established.

CHAPTER ONE

1. INTRODUCTION

In this introductory chapter, a brief theory of some of the basic statistical principles that are needed in the formulation and analysis of the models is given. An introduction of the binomial, and exponential distribution is also given. The principles of the hazard model and Markov Chains are highlighted too. It is in this chapter that the objectives, significance of the study, a review of related literature and a statement of the problem are given. Here the essence of statistical modelling and the types of models that can be built are introduced. The basic principles on the mode of operation of the Benevolent Scheme are also highlighted.

1.1: INTRODUCTION.

1.1.1: Statistical Modelling.

Statistical Modelling is the science of conversion of statements to statistical formulae to be used in the solution of physical problems. An acceptable model ought to either validate previous methodologies of modelling or be the foundation to new advances in such methodologies.

Modelling of physical situations is an old phenomenon. The basis of this activity is the diverse questions that require statistical redress. For example, the government may wish to know how to resuscitate the economy. Among the many questions they would require answers for are:

i. How much capital should we input to the agricultural sector, which is the backbone of our economy?

ii. What magnitude of revenue do we expect to earn from the input?

iii. Is it therefore profitable to fund agricultural projects as a priority?

iv. There are other options that could earn the government modest revenue (tourism being an example). Is it relatively more profitable to invest more on the tourism industry than in agriculture?

Answers to these questions could best be presented in mathematical equations. Such equations are *Empirical*, *Deterministic*, or *Stochastic*. *Empirical* models allow for collected data to be analysed in order to understand the nature of a given process. *Deterministic* models are meant for the qualitative analysis of the processes and are therefore not based on data. They are hinged on trends and rates of the activities in question. *Stochastic* models are applied in case the process being defined is random. The process given as an example above (on revamping the economy) is best fitted with a deterministic model.

Various statistical models have been improvised in an attempt to explain and give solutions to a wide range of analysable situations. This work is on insurance related modelling. In particular, we consider the *Benevolent Fund*. The formulated model is

Stochastic since the ensuing process is random. This is a discrete-time stochastic process due to the fact that the events involved are in steps of years.

1.1.2. The Benevolent Scheme.

A number of cooperative societies incorporate the Benevolent Scheme. This is where members contribute premiums to the cooperative society and are compensated in the event of death of self or that of any of their dependants. Different levels of compensation are made in the event of death of a family member. These levels depend on the individual involved. For instance, a member may pay on monthly basis a premium x. On the event of his or her death, the dependants are paid a sum of y_1 shillings. If the dependant dies, then the contributor is paid y_2 and in case both die, the cooperative society pays y_3. After one year, the expected spending by the cooperative society is

$$S = p_2 q_1 y_1 + p_1 q_2 y_2 + q_1 q_2 y_3. \tag{1.1}$$

where p_1 is the probability of the contributor surviving, p_2 is the probability of survival of the dependant, $q_1 = 1 - p_1$ and $q_2 = 1 - p_2$ are the probabilities of not surviving for the contributor and dependant respectively. The gain by the cooperative society is

$$G = 12 p_1 x - S. \tag{1.2}$$

From the results obtained using such a model, the average value for x that gives us modest profit can be estimated by

$$\hat{x} = \frac{G+S}{12 p_1} \tag{1.3}$$

The given case only considers one-dependant and only one year of existence. Generally, many families have more than one dependant. Therefore, it is necessary to model the Scheme to allow for the incorporation of more dependants and extend the period of sensor to n years. In such a model we consider the case where there are m dependants.

The Maseno University Burial and Benevolent Fund (BBF) was started in 1995 under the Maseno University SACCO. The main aim of introducing the scheme was to provide a common pool of funds that could assist members instantly on bereavement without having to wait on loan application procedures. The amount contributed is banked, lent out or invested in other income generating activities.

Those eligible for membership are the registered members of the Maseno University SACCO who have paid KShs. 100 non-refundable Registration fee, and contribute (from their salaries) a premium of KShs. 200 per month. The amount is the same for every member whether he/she has one, two, three, four or even five dependants.

The major source of capital for the fund was member monthly subscription fee. To ensure that there were sufficient funds reserved for payment, the management introduced a condition that there would be no compensation within the first six months of contribution.

The fund compensates a member in case of his/her death or that of a parent or child. In case of an extra wife the member contributes KShs. 150 more. Only up to five children may be registered as beneficiaries to the scheme. Claims can only be made when a registered beneficiary dies. There is an age limit after which a child ceases to be recognised as a beneficiary. For a sibling who is 18 years old and above, it is assumed that he or she is independent of the parents unless still at school or college.

Ever since its inception the scheme has realised tremendous growth in membership due to the benefits that members have already experienced. The initial registration was 100 individuals while the target was 300 for viability. This grew to 450 members by the year 2000 only to decline sharply to about 350 due to retrenchment.

1.1.3. Notations and Symbols.

The following are some of the notations and symbols used in this research.

P	Premium size (amount per annum).
x	Premium size (amount per month).
${}_m\xi_n$	The expected spending on one contributor with m dependants in the n^{th} year.
p_i	probability of survival of the i^{th} individual.
q_i	probability of death of the i^{th} individual.
A^b	A power b.
$L(.,.,.)$	The likelihood function of the probability density function f $(.,.,.)$.
$\frac{\partial L}{\partial p}$	Partial differential of the likelihood function with respect to p.
$M_x(t)$	The moment generating function of the p.d.f $f(x)$.
P_{ij}^n	Probability of transition from i to j in n steps.

f_{ij}^{n}	Probability of first passage of state *i* through *j* in the nth step.
μ_j	Mean recurrence time of state *j*.
h_{ki}	Number of children contributor k has in the i^{th} year.
$\bar{c}$	The average number of children a contributor has.
D_k	Compensation due to the death of the k^{th} member of the family.
Φ_k	Expected compensation in a k -dependant Scheme.
${}_m\Pi_n$	Expected cumulative profit for an *m*-dependant model in the n^{th} year.
$\bigcup_{i=1}^{g}$	Union of Sets.
$\prod_{k=2}^{m+1}$	Normal multiplication of elements.
$\rightarrow$	Tends to.
$\Rightarrow$	Implies.
$\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$	A matrix of sub-matrices.
$\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$	A simple matrix.
$\forall$	For all.
$\in$	Member of.

1.2: BASIC STATISTICAL CONCEPTS.

The following are the basic concepts used in the research and modelling of the Benevolent Scheme.

1.2.1: Binomial Distribution.

In a sequence of *Bernoulli* trials, one is interested in finding the total number of successes in the n trials and not in the series of successes (or failures). In this case, the random variable of interest is;

$$Y = y_1 + \ldots\ldots + y_n = \sum_{1 \le i \le n} y_i \tag{1.3}$$

For example, consider a redundant group of n independent units operating in parallel. The group operates successfully if the number of operating (or functioning) units is not less than m. Let y_i be one if the i^{th} unit is functioning at some chosen time, and zero otherwise. Then Y is the number of successfully operating units in the group. Thus, the group is operating successfully as long as $Y \ge m$, where m is some positive number, the threshold number.

Y has a *Binomial distribution* with parameters n and p, n being the number of units and p the probability of a unit remaining in operating status. By well known theorems of probability theory, for any set of random variables y_i,

$$E\left\{\sum_{1 \le i \le n} y_i\right\} = \sum_{1 \le i \le n} E\{y_i\} \tag{1.4}$$

In this particular case,

$$E\{Y\} = np \tag{1.5}$$

For independent random variables of Y, the variance is expressed as

$$\operatorname{var}\left\{\sum_{1 \le i \le n} y_i\right\} = \sum_{1 \le i \le n} \operatorname{var}\left\{y_i\right\} \tag{1.6}$$

For identically independently distributed *Bernoulli* random variables,

$$\operatorname{var}\{Y\} = npq \tag{1.7}$$

where q is the probability of failure.

The probability density function of a binomial random variable is given as:

$$f(y) = prob(y = k) = \binom{n}{k} p^k q^{n-k}. \tag{1.8}$$

This distribution has the moment generating function.

$$M_y(t) = \left(q + pe^t\right)^n \tag{1.9}$$

The binomial distribution has already been introduced. Following is an explanation of the maximum likelihood estimator of the parameter p. For a random variable x that has a binomial distribution,

$$f(x) = \binom{n}{x} p^x q^{n-x}. \tag{1.10}$$

The likelihood function of this distribution is

$$L(x_i, n, p) = \prod_{i=1}^{n} \binom{n}{x_i} p^{x_i} (1-p)^{n-x_i}.$$

$$= \prod_{i=1}^{n} \binom{n}{x_i} * p^{\left(\sum_{i=1}^{n} x_i\right)} * (1-p)^{n^2 - \left(\sum_{i=1}^{n} x_i\right)} \tag{1.11}$$

here * means multiplication.

We need to find p such that L (x, n, p) is maximised. This is the same as finding p such that the natural log of the likelihood function **ln** *{L (x, n, p)}* is maximised.

$$\ln L(x,n,p) = \sum_{i=1}^{n} \binom{n}{x_i} + \left(\sum_{i=1}^{n} x_i\right) \ln p + \left(n^2 - \sum_{i=1}^{n} x_i\right) \ln(1-p). \tag{1.12}$$

At maximum **ln** *{L (x, n, p)}*,

$$\frac{\partial L(x,n,p)}{\partial p} = 0. \tag{1.13}$$

Now

$$\frac{\partial L(x,n,p)}{\partial p} = \frac{\sum_{i=1}^{n} x_i}{p} - \frac{\left(n^2 - \sum_{i=1}^{n} x_i\right)}{1-p} = 0. \tag{1.14}$$

From which

$$\left(\frac{p}{1-p}\right)=\frac{\sum_{i=1}^{n} x_i}{\left(n^2-\sum_{i=1}^{n} x_i\right)}=\frac{\bar{x}}{n-\bar{x}}=\frac{\bar{x}/n}{1-\bar{x}/n}. \tag{1.15}$$

Thus we have the maximum likelihood estimator of p as

$$\hat{p}=\frac{\bar{x}}{n} \tag{1.16}$$

Hence, that of q is

$$\hat{q}=1-\frac{\bar{x}}{n}=\frac{n-\bar{x}}{n}. \tag{1.17}$$

1.2.2: Exponential Distribution

This distribution is most popular and commonly used in reliability theory and engineering. Many mathematical researchers prefer using the exponential distribution because they can obtain a lot of elegant results with it. Although it is principally impossible to find a natural process that is exactly described by a mathematical model, the exponential distribution under certain conditions is best placed than other distributions.

The distribution is often used to describe the failure process of electronic equipment. Failure of such equipment occurs mostly because of the appearance of extreme conditions during their operation. The probability density function of an exponential random variable is

$$f(t)=\lambda e^{-\lambda t} \quad \text{with} \quad \lambda, t>0 \tag{1.18}$$

Its mean and standard deviation respectively are

$$E\{x\} = 1/\lambda$$

and

$$\sigma = \sqrt{\operatorname{var}(x)} = 1/\lambda \qquad (1.19)$$

It has the moment generating function

$$M_x(t) = \frac{\lambda}{\lambda + t} \qquad (1.20)$$

1.2.3: The Hazard Model.

Excessive work has been carried out in the use of the *hazard model* in various areas of study dealing with population dynamics. These kinds of models have advantage over static models since they account for time. The time dependent variable in the hazard model is the time spent by the individual under study in the 'healthy' group (alive, operational or in good condition).

As explained in Gnedenko (1995) and Murray (1990), the Hazard models also incorporate time varying covariates or explanatory random variables that change with time. These models also produce more efficient out of sample forecasts by utilising much more data. Hazard models require selection of parametric forms such as the Weibull distribution, Exponential distribution, Extreme value and Gompertz distribution for baseline duration dependence.

The Exponential distribution plays a central role in survival analysis. Though few systems have exponentially distributed lifetimes, most of the useful survival distributions are closely related to it.

As earlier noted, the density function of the exponential random variable T for survival time t is

$$f(t) = prob\{t < T < t + dt\}/dt = \frac{1}{\mu}e^{-t/\mu}; \mu, t > 0. \qquad (1.21)$$

Where μ is the mean of the distribution. The cumulative density function of the distribution is

$$F(t) = 1 - e^{-t/\mu}. \qquad (1.22)$$

A fundamental concept of survival analysis is that of the hazard function $h\ (t)$ with the conditional density function at time t given by survival up to time t being

$$h(t) = prob\{t < T \leq t + dt / T > t\} = f(t)dt/[1 - F(t)]. \qquad (1.23)$$

The Survivor function is defined as

$$S(t) = prob\{t < T\} = 1 - F(t) \qquad (1.24)$$

and

$$h(t) = f(t)/S(t)$$

is the hazard function. It can be interpreted as the instantaneous failure rate at time t. Any continuous probability distribution can be specified equivalently by its density function, survivor function or hazard function. For any distribution, the functions mentioned above are given as

$$f(t) = h(t)\exp\{-H(t)\}$$

$$S(t) = \exp\{-H(t)\}$$

and

$$h(t) = f(t)/S(t) \qquad (1.25)$$

where the function

$$H(t) = \int_0^t h(u)du. \tag{1.26}$$

is the integrated hazard function. In particular, for a random variable T with the exponential distribution,

$$h(t) = \lambda.$$

$$H(t) = \int_0^t h(u)du.$$

$$= \int_0^t \lambda du. = \lambda t. \tag{1.27}$$

hence

$$S(t) = e^{-\lambda t}.$$

and finally

$$f(t) = \lambda e^{-\lambda t}.$$

as earlier stated. From this we deduce that the Exponential distribution has constant hazard rate λ.

Davis (1993) defines the hazard rate as a function satisfying the condition:

$$prob\{T \in (s, s+t)/T > s\} = h(s)\delta + 0(\delta) \tag{1.28}$$

where $0(\delta)$ is a function such that

$$0(\delta) \to 0 \text{ as } \delta \to 0.$$

$h(s)\delta$ expresses, to the first order, the probability that T occurs 'now' given that it has not occurred 'so far'. The memorylessnes property of the distribution of T regardless the time elapsed s is seen from the equation

$$prob\{T > t+s/T > s\} = \frac{F(t+s)}{F(s)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = e^{-\lambda t} .. t \geq 0. \qquad (1.29)$$

Thus the probability of survival to time t is not dependent on the length of the previous lifetime. The Exponential distribution represents the lifetime distribution of an item that does not age or wear. Instantaneous failure is the same no matter how long the item has survived already.

The survivor function, $S(t)$, which is the probability of surviving to at least time t, is most commonly estimated (for censored data) by the ***Kaplan-Meier*** or ***product limit*** estimate,

$$\hat{S}(t) = \prod_{j=1}^{i} \left(\frac{n_j - d_j}{n_j} \right), \qquad t_i \leq t < t_{i+1}. \qquad (1.30)$$

where d_j is the number of failures occurring at time t_j out of n_j surviving to time t_j. This is a step function with steps at each failure time but not at censored times. As $S(t) = e^{-H(t)}$, the cumulative hazard rate can be estimated by $\hat{H}(t) = -\log[\hat{S}(t)]$.

A plot of $\hat{H}(t)$ or log $(\hat{H}(t))$ against t or $\log$ (t) is often useful in identifying a suitable parametric model for the survivor times. The following relationships as given in Gross and Clark (1975) can be used in the identification.

(a) Exponential distribution: $H(t) = \lambda t$.

(b) Weibull distribution: $\log[H(t)] = \log \lambda + \gamma \log t$.

(c) Gompertz distribution: $\log[H(t)] = \log \lambda + \gamma t$.

(*d*) Extreme value (smallest) distribution: $\log[H(t)] = \lambda(t-\gamma)$.

In the case of the exponential, Weibull and extreme value distribution, the proportional hazard model can be fitted to censored data using the method described by Aitkin and Clayton (1980), which uses a generalised linear model with Poisson errors. Other possible models are the gamma and lognormal distributions.

1.2.4: Markov Chain

Definition 1:

Let E_1, E_2, ...E_n be n possible outcomes of sequence of trails. The sequence is a *Markov Chain* if the probability of the sample space satisfies the equation,

$$prob\left\{\left(E_{j_0}, E_{j_1}, \ldots, E_{j_n}\right)\right\} = a_{j_0} p_{j_0, j_1} p_{j_1, j_2} \ldots p_{j_{n-1}, j_n} \tag{1.31}$$

where P_{rs} is the conditional probability of occurrence of event E_s given than event E_r has occurred. a_{j_0} is the initial probability distribution. Examples of Markov chains are Random walks, Branching processes and urn models. The transition probabilities P_{rs} are arranged in matrix form as below.

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix} \tag{1.32}$$

This is a Markov Chain with n states.

The first subscript stands for row and the second for column. P is a square matrix with non-negative elements. If the sum of each row of elements is unity, then such a matrix (finite or infinite) is called a *stochastic* matrix. Any stochastic matrix can serve as a matrix of transition probabilities. This, together with the initial distribution constitutes a Markov chain.

Example 1:

Let an experiment have two possible outcomes following the two events E_1 and E_2. These are the states the experiment can assume. If the experiment remains in state 1 with probability α, then it moves to state 2 with probability $1-\alpha$. Conversely, if it is in state 2 and remains in it with probability β, then it moves back to state 1 with probability 1-β. The corresponding stochastic matrix of transition probabilities is given as,

$$P=\begin{pmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{pmatrix} \tag{1.33}$$

A Markov chain is doubly stochastic if each of both the rows and columns of the matrix of transition probabilities independently sums to one. In example 1, if $\alpha=\beta$ then P is *doubly stochastic.*

Transient states.

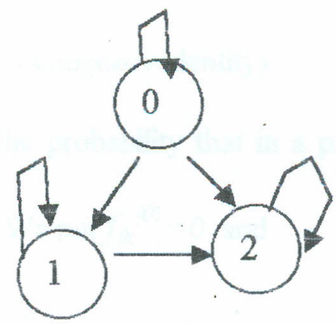
A state E_r is transient if for some (or all) other states E_s, $E_r \rightarrow E_s$ but $E_s \not\rightarrow E_r$. ($\rightarrow$ Implies communicability, and $/\rightarrow$ denotes incommunicability). Thus $P_{rs}>0$ but $P_{sr}=0$ for some fixed r.

Example 2.

Let P be a transition probability matrix with state space E={0,1,2}.

$$P=\begin{pmatrix}1/3 & 1/3 & 1/3\\ 0 & 1/2 & 1/2\\ 0 & 0 & 1\end{pmatrix} \qquad (1.34)$$

Then diagrammatically the states can be represented as



State 0 is transient.

Absorbing states.

A state E_r is absorbing if for some (or all) other states in the system, $E_s \rightarrow E_r$ but $E_r \not\rightarrow E_s$. This implies that $P_{rs}=0$ and $P_{sr}>0$ for some fixed r. thus $E_r \rightarrow E_r$ and $p_{rr}=1$. In example 2 above, state 2 is absorbing.

Higher Transition Probabilities.

Let $P_{jk}^{(n)}$ denote the probability of transition from E_j to E_k in exactly n steps, then we have,

$$P_{jk}^{(1)} = P_{jk}.$$

$$P_{jk}^{(2)} = \sum_{v} P_{jv} P_{vk}. \tag{1.35}$$

....

$$P_{jk}^{(n+1)} = \sum_{v} P_{jv} P_{vk}^{(n)}.$$

(General recursion formula).

$$P_{jk}^{(n+m)} = \sum_{v} P_{jv}^{(m)} P_{vk}^{(n)}. \tag{1.36}$$

(A special case of the Chapman Kolmogorov identity).

Now let $f_{jk}^{(n)}$ stand for the probability that in a process starting from E_j the first entry to E_k occurs in the n^{th} step. We put $f_{jk}^{(0)}=0$ and

$$f_{jk} = \sum_{n=1}^{\infty} f_{jk}^{(n)}. \tag{1.37}$$

Thus $f_{jk} \leq 1$. When $f_{jk}=1$, then $\left\{ f_{jk}^{(n)} \right\}$ is a proper probability distribution (first passage distribution) for E_k. If $f_{jj}=1$, then return to E_j is certain. The mean recurrence time μ_j is given by

$$\mu_j = \sum_{n=1}^{\infty} n f_{jj}^{(n)} \tag{1.38}$$

Definition 2.

The state E_j is persistent if $f_{jj}=1$ and transient if $f_{jj}<1$. A persistent state E_j is called null state if its mean recurrence time $\mu_j = \infty$.

1.3: OBJECTIVES OF THE STUDY.

Our main objective of this study is to formulate a Statistical Model that can be used in calculating the spending and profit a cooperative society expects due to the Benevolent Scheme. By use of this model, it is expected that the right premium to be contributed will be determined. Application of the model to the Maseno University SACCO is to be made with the aim of forecasting the financial status of the cooperative society, hence improving its services to members.

1.4: SIGNIFICANCE OF THE STUDY.

At the moment, arbitrary premiums are paid to most SACCO societies. This could result in either losses or abnormal profits to the cooperative society. The results we envisage to obtain will be usefully in the SACCO to obtain the average premium to be paid by members of the Benevolent Scheme. This premium size is to be such that the cooperative society earns a modest profit.

1.5: REVIEW OF RELATED LITERATURE.

Many authors have studied statistical modelling of premiums. Cummins and Chang (1983) studied the rate of return on insurance companies' equity and the premiums insurance companies should charge.

Ferrari (1968) developed a descriptive model, which allows an algebraic expression for the rate of return of equity as a function of the premiums charged to be derived. Combining this work with the Capital Asset Pricing Model (CAPM) meant that an equilibrium value for the return on equity and the corresponding level of premiums could be found. This model is known as the insurance CAPM. Cooper (1974), Biger and Kahane (1978), Fairley (1979), Clements (1982) and Hill (1979) also studied the CAPM model.

Myers and Cohn (1987) suggested an adaptation of the adjusted present value method for calculating the price of insurance. They determined the premium by discounting the expected cash flows associated with insurance at the appropriate discount rates.

The National Council on Compensation Insurance ((NCCI) (1987)) used the internal rate of return methodology. This involves finding the discount rate such that the net discount cash flow is zero. The fair premium is one in which this rate of discount is equal to the opportunity cost of capital.

Pratt (1964) addressed the problem of the maximum risk premium. This is the maximum premium that the insured is willing to pay the insurer. Henri (1991) examined the insurer-reinsurer interface in relation to this risk.

Analysis of survival, reliability and failure time data has also been carried out. Kalbfleisch and Prentice (1980) highlighted the most commonly used statistical techniques in the analysis of such data to be the Hazard Model. Gross and Clark (1975) derived the relationships used in the identification of the baseline distribution to use together with the hazard model.

Aitkin and Clayton (1980) described the method by which the exponential, Weibull and extreme value distribution could be fitted to censored data using a generalised linear model with Poisson errors.

Rather than using a specified form for the hazard function, Cox (1972b) considered the case when $\lambda_0(t)$ (the baseline hazard function) was an unspecified function of time. To fit such a model, assuming fixed covariates, a marginal likelihood was used. More literature on the Cox's proportional hazard model and the other earlier mentioned models can be sought from Kalbfleisch and Prentice (1980).

Feller (1968), Stirzaker (1994) and Grimmet (1992) have a comprehensive overview of Markov chains and various probability distributions, which are to be used in our formulation, and analysis of the statistical model of the Benevolent Scheme.

1.6: STATEMENT OF THE PROBLEM.

It is expected that a formulation of a statistical model that is to help in the calculation of the expected expenditure, gains and loses a cooperative society incurs due to the Benevolent Fund will be met. Using the results, an expression for the estimation of the right premium size to be charged by such a cooperative society is to be derived. In line with this, one-dependant and m-dependant models of the Benevolent Scheme are to be formulated using Branching and Markov models.

CHAPTER TWO.

2. MODEL FORMULATION.

In this chapter, discussed are the methods intended for use in the calculation of Survival Probabilities and the basis of the one-dependant model of the Benevolent Scheme. This leads to the formulation of the model. Data from the Maseno University SACCO is applied to the Model as a case study. Finally, the implications of the results are read by use of line graphs.

2.1. METHODOLOGY.

In section 1.2, basic probability principles of the binomial distribution were highlighted. A brief theory on the Hazard Model was also given. These are the tools used in the formulation of this model.

The Maximum Likelihood Estimator (of p in the binomial distribution) approach is employed in the calculation of Survival Probabilities of the contributor and dependant. To find the probability of survival p we consider the death rate or rate of "departure" from the scheme.

Fundamental probability rules are applied in the calculation of the expected expenditure in compensation to a beneficiary due to death of a family member of the contributor. This is done for various stages (years a contributor remains a member of the scheme). Data from one of the co-operatives that has the scheme is collected and analyzed using the derived model.

The required data is summarized in the table below. Here h_{ki} is the number of children contributor k has in the ith year.

Year (1) 1998		Year (2) 1999		Year (3) 2000		Year (4) 2001		Year(5)2002	
Σh_{k1}		Σh_{k2}		Σh_{k3}		Σh_{k4}		Σh_{k5}	
Jan	Dec	Jan	Dec	Jan	Dec	Jan	Dec	Jan	Dec
X_{11}	X_{12}	X_{21}	X_{22}	X_{31}	X_{32}	X_{41}	X_{42}	X_{51}	X_{52}

Table 2.1: A table of the data we expect to collect.

2.1.1: Calculation of Survival Probabilities.

In table (2.1), X_{ij} represents the number of contributors present in the i^{th} year and j^{th} month. j=1 for the month January.

j=2 for the month December.

Assuming that the death rate remains constant over n (in this case 5) calendar years, the probability q_1 that the contributor does not survive to the end of the first year is estimated to be,

$$q_1 = \frac{1}{n}\left\{\sum_{i=1}^{n} \frac{x_{i1}-x_{i2}}{x_{i1}}\right\} \tag{2.1}$$

This is the same for all the five years under consideration.

We know from Sampling Theory that sample mean is an unbiased estimator of population mean. So is the variance. The use of the estimator q is therefore statistically justified.

Assuming that the first dependant M is the wife or husband of the contributor, then we can let $q_1 = q_2$ since death is relatively equally likely in their age bracket. This is checked from the data we expect to collect. Other dependants (siblings) have their survival probabilities $p_3 \ldots p_4 \ldots$ calculated from past records using the following data set.

1998	1999	2000	2001	2002
Y_1	Y_2	Y_3	Y_4	Y_5
N_1	N_2	N_3	N_4	N_5

Table 2.2: A table of compensation data related to child death.

Y_i are numbers of claims in year i of compensation related to child death.

$$N_i = \left\{ \frac{x_{i1} + x_{i2}}{2} \right\} \tag{2.2}$$

N_i is the average number of contributors in year i. Then assuming that each child has equal probability of survival, the average number of children the contributor has is,

$$\bar{c} = \frac{1}{n}\left\{\sum_{i=1}^{n}\left(\frac{\sum_{k=1}^{X_{i1}} h_{ki}}{X_{i1}}\right)\right\}, \tag{2.3}$$

Where h_{ki} is the number of children contributor k has. The average probability of not surviving in the case of siblings of the contributor is estimated by,

$$q_3 = q_4 = \ldots\ldots = q_{\bar{c}+2} = \frac{1}{n}\left\{\sum_{i=1}^{n}\left(\frac{y_i}{\bar{c}N_i}\right)\right\} \tag{2.4}$$

y_i is the number of claims in the i^{th} year due to child death, $\bar{c}N_i$ is the average number of children in the i^{th} year, and the subscript $\bar{c}+2$ is the total number of beneficiaries that is, the average number children per contributor plus the contributor and spouse.

In chapter one, other methods of determining survival probabilities as the Kaplan Meier product estimator were enumerated. We compare results of use of the estimator to those of the Maximum Likelihood Estimator. Incidentally, it is not possible to predict future survival probabilities using the Kaplan-Meier or product limit estimator. This is because it requires data up to the end of the time of study. This is mainly why the Maximum Likelihood Estimator method is applied. It gives us an average estimate to be used throughout the period of study.

2.1.2: The Flow Chart and Tree Diagram.

The following figures are useful in explaining the transitions involved in the one-dependant model. Figure (2.1) shows the flow chart of possible events in the transactions of the Benevolent Scheme.

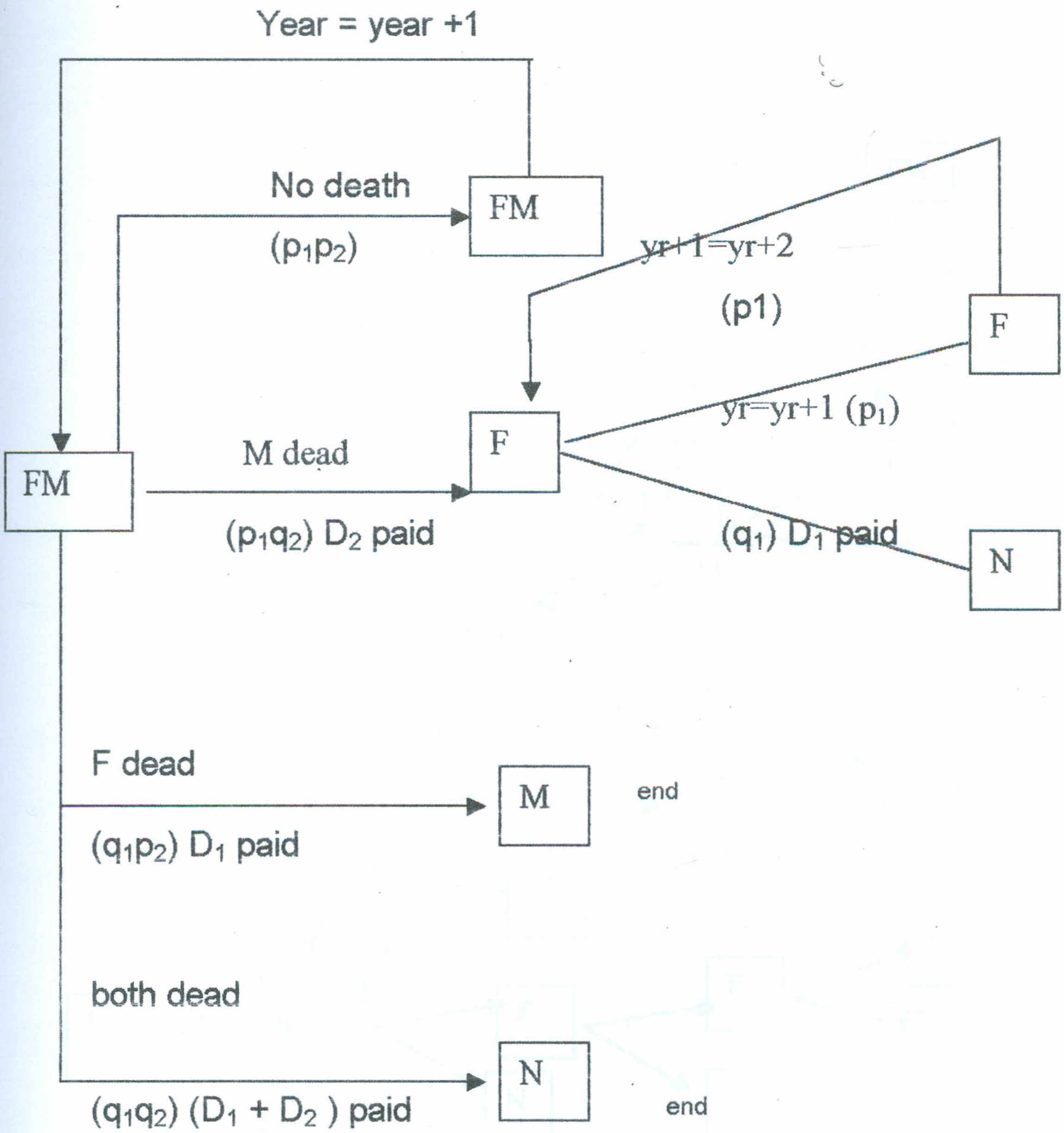


Figure 2.1: A flow chart of events in the one dependant Model.

The tree diagram below best gives the branching process of the model

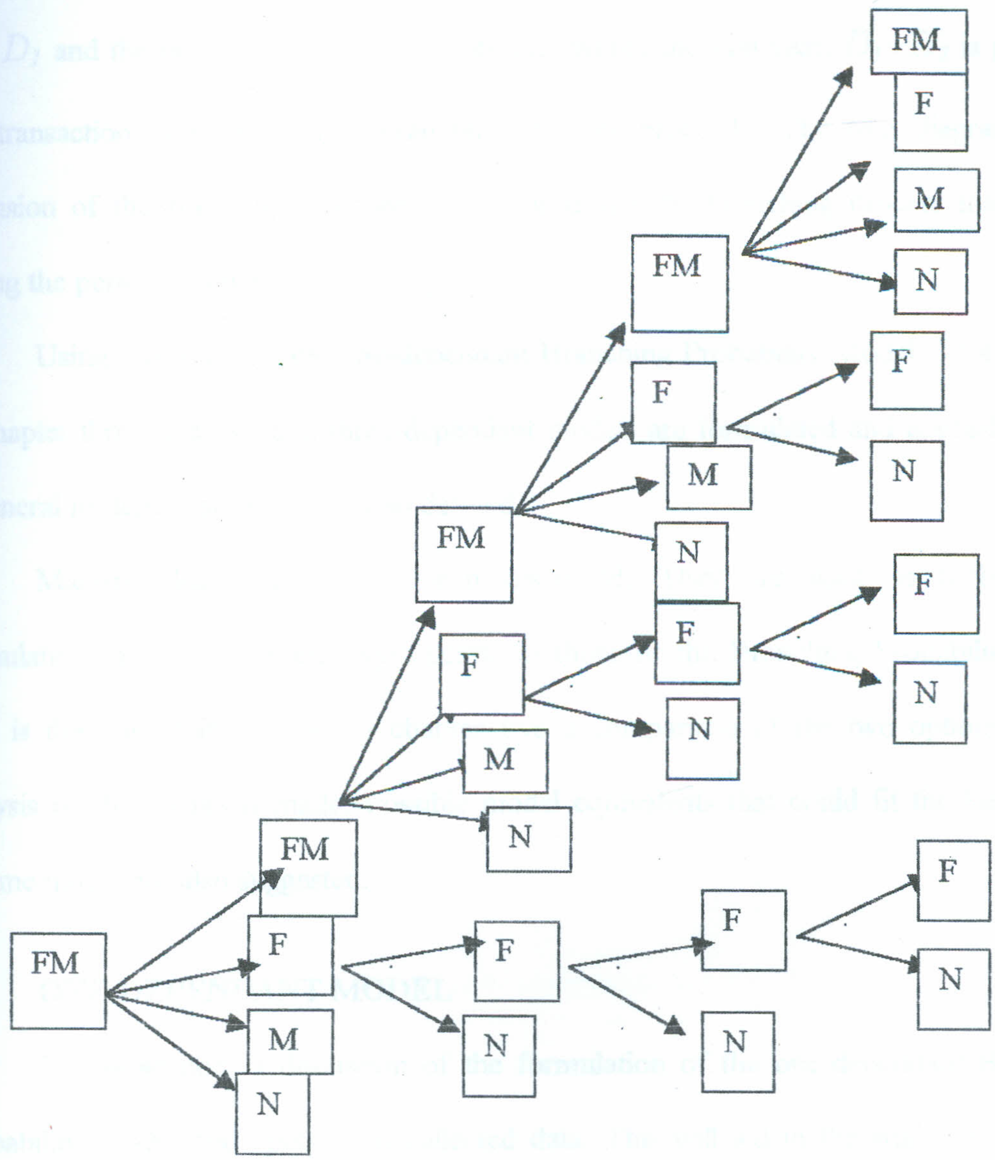


Figure 2.2: A tree diagram for the model.

From figure (2.1) we see that if both F and M survive, the insurer pays no compensation. The process proceeds to the second year. If only M dies, amount D_2 is paid to F and the process proceeds to the second year. In the event of death of F, M is paid D_1 and the process ends. In case both die, within the first year, D_1+D_2 is paid and the transaction ends here. The branching process proceeds. The tree diagram is an extension of the flow chart in figure (2.1). It shows the branching process that ensues during the period of study.

Using figure (2.2), the one-dependant Branching Probability Model is formulated. In chapter three, the two and three-dependant models are formulated and applied to data. A general m-dependant model is also derived.

Markov chains have also been discussed. They are used as tools in the formulation of Markov model equivalents, to those of the Branching Probabilistic type. This is done in chapter four. In chapter five a comparison of the two options and an analysis of the results is made. Possible model equivalents that could fit the Benevolent scheme model are also suggested.

2.2. ONE-DEPENDANT MODEL

In this section, a discussion of the formulation of the one-dependant Branching Probability Model and apply it to collected data. This will aid in the study of the trend and expectations of an insurer with a member who has only one dependant.

2.2.1. Introduction.

In the formulation of this model, the assumptions are as follows:

i The contributor has only one dependant.

ii The survival probabilities of the contributor and dependant are different.

iii Claims are made towards the end of a calendar year.

iv Since if both die at the same time the amount paid as compensation is not equivalent to the sum of separate compensations, we assume that the probability of both dying at the same time is negligible.

v. Survival probability is independent of age and sex.

There are four possibilities at the end of the year or at the time of observation. That is, starting with two individuals, the contributor F and spouse M.

i. Both of them survive by the end of the year.

ii. Only F survives.

iii. Only M. survives.

iv. None survives.

The following are the figures of compensation we require from cooperatives that have the Benevolent Scheme.

X: is contribution per year.

$D_1, D_2, D_3 \ldots \ldots D_{c+2}$, which are amounts of compensation for each death.

Here;

D_1: is the amount compensated for the death of the contributor F.

D_2: is the compensated for the death of the contributor's spouse M.

$D_3 \ldots \ldots D_{c+2}$: are the amounts compensated for the death of the contributors' children.

The probabilities assigned to the events i, ii, iii, and iv are summarized in table (2.3).

Event	probability	compensation
i	p_1p_2	0
ii	p_1q_2	D_2
iii	q_1p_2	D_1
iv	q_1q_2	D_1+D_2

Table 2.3: A table of events, probabilities and respective compensations.

We now show that the total probability is unit.

$$\sum prob = p_1p_2 + p_1q_2 + q_1p_2 + q_1q_2 = p_1(p_2 + q_2) + q_1(p_2 + q_2)$$

$$= p_1 + q_1$$

$$=1.$$

The observation stops in case either M survives or both do not survive. This is because the contributor no longer exists, and consequently the transaction. On the other hand, there is continuity in case the contributor or both survive. In case both survive in the first year, then in the second year events (i) to (iv) are probable. Meanwhile if only the contributor survives, then in the second year he may survive again (with probability p_1) or die (with probability q_1). This branching process replicates as in figure (2.2) at each stage of observation.

Consider the case of detectability in estimation of biological populations. Let the probability of correct observation be p_a and let Y be a random variable representing the number of animals present. The total population of the species under observation is T_a. Then Y has a Binomial distribution with mean

$$E\,(Y)=T_a p_a \tag{2.5}$$

and variance

$$Var\,(Y)=T_a p_a\,(1-p_a). \tag{2.6}$$

In the case of calculation of the expected expenditure using the probabilities that have been modeled, we have the actual payment multiplied by the probability of the event (for which payment is made) occurring.

2.2.2: Modelling expected claim amount.

Let ${}_1\xi_n$ represent the expected claim amount in the n[th] year for the one-dependant model. Let F denote the contributor and M the dependant. Also let the probability of survival of F be p_1 and that of M be p_2. Then using figure (2.1) and (2.2), the calculations of expected cooperative expenditure and profit from a single contributor are made. We are assuming that there is only one dependant, the spouse.

By the end of the first year, using achieved probabilities (table (2.3)) and figure (2.2), the expected claim amount to the insuring company by the contributor is:

$$\begin{aligned} {}_1\xi_1 &= p_1p_2\,(0)+p_1q_2\,(D_2)+q_1p_2\,(D_1)+q_1q_2\,(D_1+D_2) \\ &= p_1q_2\,(D_2)+q_1q_2\,(D_2)+q_1p_2\,(D_1)+q_1q_2\,(D_1) \\ &= q_1D_1+q_2D_2. \end{aligned} \tag{2.7}$$

Let $q_1D_1+q_2D_2$ be denoted by Φ_1. Thus we have:

$$ {}_1\xi_1 = \Phi_1. \tag{2.8} $$

For the second year, the expected claim amount is calculated in the case of the contributor or both having survived by the end of the first year. There is no compensation for the death of the dependant in case the contributor had previously died.

$$ \begin{aligned} {}_1\xi_2 &= p_1p_2\{p_1p_2(0)+p_1q_2D_2+q_1p_2D_1+q_1q_1(D_1+D_2)\}+p_1q_2\{q_1D_1\} \\ &= p_1p_2\,\Phi_1+p_1q_2\{q_1D_1\} \\ &= p_1\{p_2\,\Phi_1+q_2q_1D_1\} \\ &= p_1\{p_2\,\Phi_1+q_2\,\Phi_{1\setminus 2}\} \end{aligned} \tag{2.9} $$

We have introduced the notation $\Phi_{1\setminus 2}$, which represents q_1D_1, and essentially is

$$ \Phi_1 - q_2D_2. \tag{2.10} $$

Generally ,

$$ \Phi_{r\setminus s} = \{q_1D_1+q_2D_2+\ldots+q_rD_r+q_{r+1}D_{r+1}\} - q_sD_s. \tag{2.11} $$

Similarly, for the third year,

$$ \begin{aligned} {}_1\xi_3 &= p_1^2p_2^2\,\Phi_1+p_1p_2\{p_1q_2q_1D_1\}+p_1q_2p_1q_1D_1. \\ &= p_1^2p_2^2\,\Phi_1+p_1^2p_2q_1q_2D_1+p_1^2q_1q_2D_1 \\ &= p_1^2\{p_2^2\,\Phi_1+q_1q_2D_1(1+p_2)\} \\ &= p_1^2\{p_2^2\,\Phi_1+(1-p_2^2)\,\Phi_{1\setminus 2}\}. \end{aligned} \tag{2.12} $$

Further we see that

$$ {}_1\xi_4 = p_1^3p_2^3\,\Phi_1+p_1^3p_2^2q_2q_1D_1+p_1^3p_2q_2q_1D_1+p_1^3q_2q_1D_1 $$

$$=p_1^3\{p_2^3\,\Phi_1+q_2q_1D_1(1+p_2+p_2^2)\}$$

$$=p_1^3\{p_2^3\,\Phi_1+q_1D_1(1-p_2^3)\}$$

$$=p_1^3\{p_2^3\,\Phi_1+(1-p_2^3)\,\Phi_{1\backslash 2}\} \tag{2.13}$$

We can now generalize the expected expenditure on the contributor's family as:

$$_1\xi_n\ =p_1^{n-1}\,\{p_2^{n-1}\Phi_1+(1-p_2^{n-1})\,\Phi_{1\backslash 2}\} \tag{2.14}$$

To prove this for n=1 and n=4 we have

$$_1\xi_1\ =p_1^{1-1}\,\{p_2^{1-1}\Phi_1+(1-p_2^{1-1})\,\Phi_{1\backslash 2}\}$$

$$=\Phi_1 \tag{2.15}$$

and

$$_1\xi_4\ =p_1^{4-1}\,\{p_2^{4-1}\Phi_1+(1-p_2^{4-1})\,\Phi_{1\backslash 2}\}$$

$$=p_1^3\{p_2^3\,\Phi_1+(1-p_2^3)\,\Phi_{1\backslash 2}\} \tag{2.16}$$

as before.

2.2.3: Calculating Profit.

Let $_1\Pi_n$ represent the amount of profit the fund gets from one contributor in the nth year of membership. Annually the contributor is expected to give a premium of size P to the insuring company. So, in the n^{th} year, the contributor is expected to have given a total of Kshs.

$$P\sum_{k=1}^{n} p_1^k. \tag{2.17}$$

Assuming that premiums once contributed are kept in a bank account, an interest at rate I is earned per annum. This incorporated in equation (2.17) gives the amount available due to the contributor as

$$P\left(1+\frac{I}{100}\right)\sum_{k=1}^{n} p_1^k. \tag{2.18}$$

It is assumed that premiums are paid at the beginning of the year but benefits are payable at the end of a year. So, at the end of the nth year, claims amounting to $\sum_{k=1}^{n} {}_1\xi_k$ will have been made. It is also important to note that with time, the deposits decline in value due to rise in the coast of living. We therefore introduce the factor V as in the equations below. The expected profit ${}_1\Pi_n$ is given as

$$\begin{aligned}
{}_1\Pi_n &= P\left(1+\frac{I}{100}\right)\sum_{k=1}^{n} V^{k-1} p_1^k - \sum_{k=1}^{n} V^k\, {}_1\xi_k. \\
&= \sum_{k=1}^{n}\left\{P\left(1+\frac{I}{100}\right)V^{k-1} p_1^k - V^k\, {}_1\xi_k\right\}. \\
&= \sum_{k=1}^{n} V^k \left\{\frac{P\left(1+\frac{I}{100}\right)}{V} p_1^k - {}_1\xi_k\right\}.
\end{aligned} \tag{2.19}$$

where $V = \dfrac{1}{1+i}$ is the discounting factor introduced due to inflation of currency at rate i per annum.

In the first year, the expected profit is

$${}_1\Pi_1 = V\left\{\frac{P\left(1+\frac{I}{100}\right)}{V} p_1 - {}_1\xi_1\right\}. \tag{2.20}$$

and by the end of the fifth year the expected profit is

$$
{}_1\Pi_5 = \sum_{k=1}^{5} V^k \left\{ \frac{P\left(1+\frac{I}{100}\right)}{V} p_1^k - {}_1\xi_k \right\} \qquad (2.21)
$$

2.2.4: Testing the Model with Collected Data.

The following data was collected from Maseno University SACCO and used in the testing of the Model.

Year (1) 1998		Year (2) 1999		Year (3) 2000		Year (4) 2001		Year (5) 2002	
$\sum h_{k1}=932$		$\sum h_{k2}=1230$		$\sum h_{k3}=1350$		$\sum h_{k4}=1177$		$\sum h_{k5}=1123$	
Jan	Dec	Jan	Dec	Jan	Dec	Jan	Dec	Jan	Dec
318	300	420	406	452	424	402	380	391	355

Table 2.4: A table of collected data.

1998	1999	2000	2001	2002
$Y_1=48$	$Y_2=62$	$Y_3=59$	$Y_4=61$	$Y_5=56$
309	413	438	391	373

Table 2.5: A table of the average number of contributors.

On calculating using the suggested equations (2.1), (2.2), (2.3), and (2.4), we find that $q_1 = q_2 = 0.06$, $q_3 = q_4 = q_5 = 0.15$. Other results are as in table (2.6).

ITEM	*VALUE*
D_1	20,000
D_2	10,000
$D_3, D_4, D_5, \ldots$	7,000
$q_1 = q_2$	0.06
$q_3 = q_4 = q_5 = \ldots$	0.15
X/12	200
$\bar{c}$	3

Table 2.6: A table of data and calculated results.

Using the Kaplan –Meier or product limit estimator for the survival function introduced in chapter one, the survival probability $p_1 = p_2$ is as in table (2.7). The results of this estimator are compared with those of the maximum likelihood estimator. This is possible for the fist five years for which data is available.

Year (t)	nj-dj	nj	nj-dj over nj	$S_1(t)$	p1 power t
1	300	318	0.943396	0.943396	0.9433
2	406	420	0.966667	0.91195	0.889815
3	422	450	0.931707	0.84967	0.839362
4	380	402	0.945274	0.803171	0.791771
5	355	391	0.907928	0.729222	0.746877

Table 2.7: A table of values of survival probability using the Kaplan-Meier and Maximum likelihood estimators.

The parameter $\mathbf{S_1(t)}$ is calculated using the formula

$$\hat{S}(t) = \prod_{j=1}^{i} \left(\frac{n_j - d_j}{n_j} \right), \qquad t_i \leq t < t_{i+1}.$$

We test the hypothesis;

H_0: $\quad \mathbf{S_1(t)} = p_1$ against

H_1: $\quad \mathbf{S_1(t)} \neq p_1$.

H_0: implies that the two methods give similar results while

H_1: implies that the two are not the same.

Using SPSS application package, the following are the results:

Pearson's Correlation

		$S_1(t)$	P_1
$S_1(t)$	Pearson Correlation	1.000	**.988
	Sig. (2-tailed)	.	.002
P_1	Pearson Correlation	**.988	1.000
	Sig. (2-tailed)	.002	.

Table 2.8: Correlation of survivor function vs. survival probability (Pearson's correlation).

*** **Correlation is significant at the 0.01 level (2-tailed).***

From table (2.8) we see that the p value is 0.002 for the two-tailed test. Pearson's correlation coefficient is 0.988.

Spearman's rho Correlation

			$S_1(t)$	P_1
Spearman's rho	$S_1(t)$	Correlation Coefficient	1.000	**1.000
		Sig. (2-tailed)	.	.
	P_1	Correlation Coefficient	**1.000	1.000
		Sig. (2-tailed)	.	.

Table 2.9: Correlation of survivor function vs. survival probability (Spearman's Rho Correlation coefficient).

***** Correlation is significant at the .01 level (2-tailed).***

Here the p value is negligible and the Spearman's Rho Correlation coefficient is 1.000.

From the results in tables (2.8) and (2.9) above, we conclude that at 0.01 level of significance, the two methods give the same results. Thus we accept H_o.

Using Ms-Excel Application Package, the parameters p_1 power (n-1), Claim amount, Cumulative Claim amount, Contribution, Cumulative contribution, and Profit result. These are tabulated in table (2.10). Let the value for inflation rate i be *10%*. Then the discounting factor V is

$$V = \frac{1}{1+\frac{10}{100}} = 0.9091$$

Using this in our expression for expected claim amount we have,

$$\sum_{k=1}^{n} V^{k}\,{}_1\xi_k = \sum_{k=1}^{n} (0.9091)^{k}\,{}_1\xi_k.$$

Year n	p_1 power (n)	Expected claim amount	discounted claim amount	cumulative claim amount	Expected contribution (discounted)	discounted cumulative contribution	growth in profit
1	0.94	1800	1636	1636	2527	2527	891
2	0.884	1658	1370	3006	2159	4686	1679.811
3	0.831	1529	1149	4155	1845	6531	2376.359
4	0.781	1411	963.5	5119	1577	8108	2989.642
5	0.734	1303	808.9	5927	1347	9456	3528.223
6	0.69	1204	679.6	6607	1151	10607	4000.113
7	0.648	1113	571.4	7178	984	11591	4412.723
8	0.61	1030	480.8	7659	840.9	12432	4772.834
9	0.573	954.4	404.8	8064	718.6	13151	5086.604
10	0.539	884.6	341.1	8405	614.1	13765	5359.586
11	0.506	820.4	287.6	8693	524.7	14289	5596.758
12	0.476	761.4	242.6	8935	448.4	14738	5802.563
13	0.447	707	204.8	9140	383.2	15121	5980.947
14	0.421	656.9	173	9313	327.5	15448	6135.405
15	0.395	610.7	146.2	9459	279.8	15728	6269.019
16	0.372	568.1	123.7	9583	239.1	15967	6384.502
17	0.349	528.7	104.6	9688	204.4	16172	6484.235
18	0.328	492.3	88.57	9776	174.6	16346	6570.303
19	0.309	458.7	75.01	9851	149.2	16496	6644.527
20	0.29	427.5	63.56	9915	127.5	16623	6708.498
21	0.273	398.6	53.88	9969	109	16732	6763.601
22	0.256	371.9	45.69	10014	93.13	16825	6811.039
23	0.241	347	38.76	10053	79.58	16905	6851.858
24	0.227	324	32.9	10086	68.01	16973	6886.966
25	0.213	302.6	27.93	10114	58.12	17031	6917.15
26	0.2	282.7	23.73	10138	49.66	17081	6943.089
27	0.188	264.2	20.16	10158	42.44	17123	6965.372
28	0.177	247	17.13	10175	36.27	17159	6984.508
29	0.166	231	14.56	10189	30.99	17190	7000.937
30	0.156	216.1	12.39	10202	26.49	17217	7015.036

Table 2.10: A table of Cumulative claim amount contributions and Profit (one-dependant model).

2.3: TREND AS $n \rightarrow \infty$

From the general formula derived earlier (equation (2.14)),

$$ {}_1\xi_n = p_1^{n-1} \{p_2^{n-1} \Phi_1 + (1-p_2^{n-1}) \Phi_{1\backslash 2}\} \tag{2.22}$$

Since p_1 and p_2 are fractions, as $n \rightarrow \infty$ we have

$$p_1^{n-1} \rightarrow 0, p_2^{n-1} \rightarrow 0$$

and consequently $\quad {}_1\xi_n \rightarrow 0 \{0+ (1-0) \Phi_{1\backslash 2}\},\quad$ which implies that

$$ {}_1\xi_n \rightarrow 0. \tag{2.23}$$

From the results that were arrived at in section 2.2.4 table (2.10), it is eminent that the trend above illustrated is justified. As time of stay in the scheme tends to ∞ the expected claim amount tends to zero. This is true because the contributor has a higher probability of death as time increases. Using SPSS program the data in table (2.10) above is graphically presented.

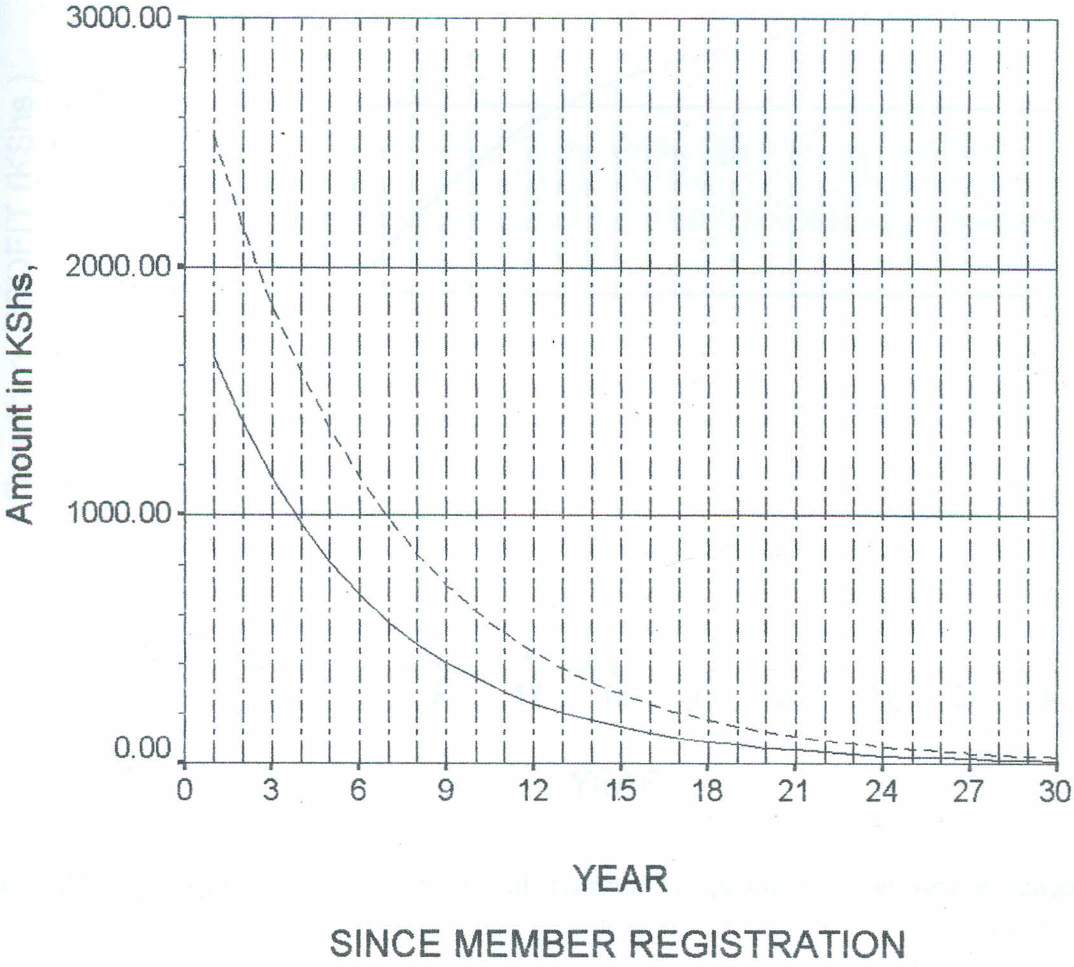
The graph in figure (2.3a) illustrates the trend in cumulative claim amount with increasing period. It is clear that with time the expected claim amount reduces exponentially to a minimum. If extrapolated there is a time when it becomes zero. This implies that at some stage, the membership ceases and hence compensation.

In the same grid (figure (2.3a)), the trend of growth in premium contribution is similar to that of compensation claim amount due to the same reason stated above. We expect contribution to cease at some time due to eminent departure from the scheme.

Lastly, Figure (2.3b) shows the growth in profit levels expected from one individual under the one-dependant model. It grows exponentially with time. At some stage, we expect a sudden stop since no profit is earned once a member ceases to contribute premiums.

EXPECTED CONTRIBUTION COMPARED WITH COMPENSATION CLAIM AMOUNT

One-dependant Model.



KEY

------- *EXPECTED CONTRIBUTION*

——— *EXPECTED CLAIM AMOUNT*

Figure 2.3a: A graph giving the trend of expected contribution and claim amount for the one-dependant model.

GROWTH IN PROFIT

One-dependant Model.

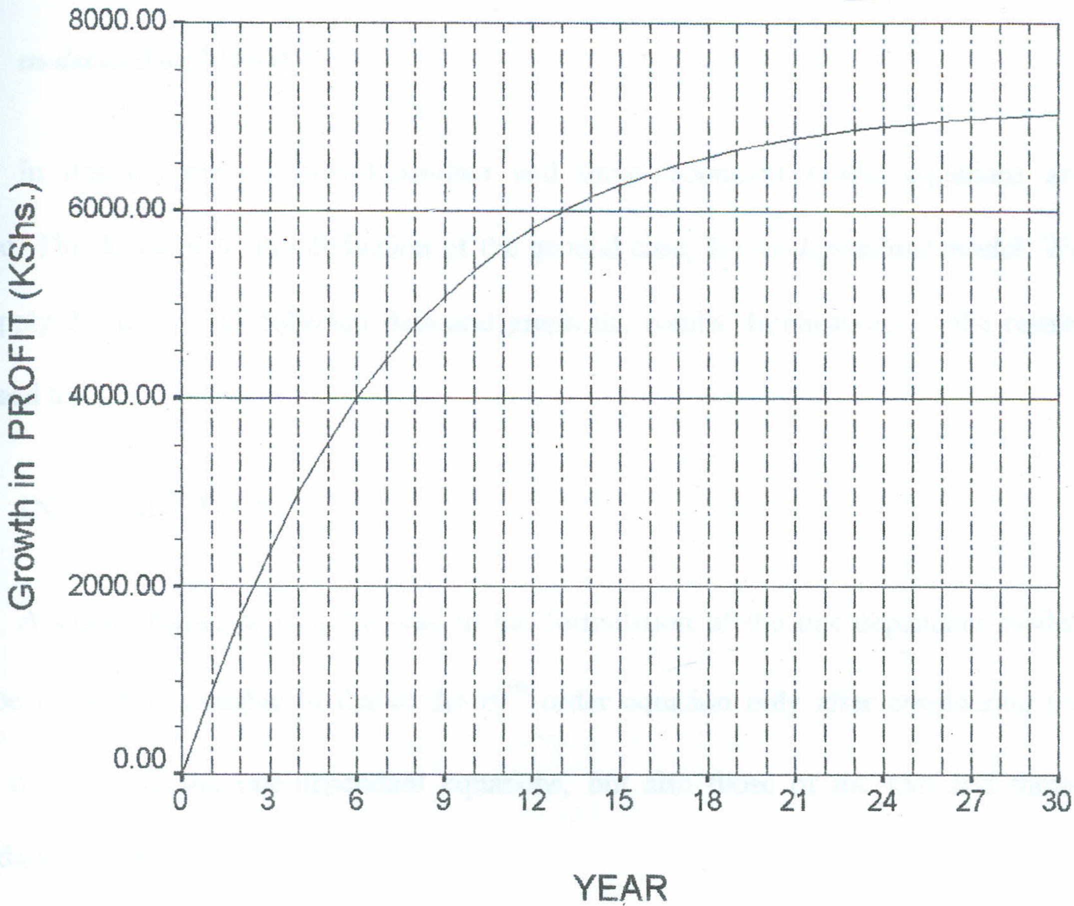


Figure 2.3b: A graph giving the trend of cumulative profit for the one-dependant model.

With the achieved one-dependant model we can now focus on the case of two, three and consequently the m-dependant case. This is done in the next chapter (chapter three).

CHAPTER THREE.

---

3. *m-dependant* MODEL.

In this chapter the two-dependant and three-dependant model equations are derived. This leads us to the derivation of the general case, the ***m-dependant model***. We also apply the models to collected data and graph the results. Implications of the results discussed too.

3.1: INTRODUCTION.

A similar trend, as was the case in the formulation of the one-dependant model, shall be used. It is possible to derive the m^{th} order equation only after considering the trend, not only of the one-dependant equations, but also those of the two and three-dependant models.

Let F and M take the already stated meanings. We introduce another dependant B, say the first born of the couple F and M. p_1 and p_2 take their earlier stated meanings. We include p_3, the probability that B survives. We also let the amount of compensation if B dies, to be D_3. The events of survival, their probabilities and related compensations are given in table (3.1).

EVENT (Survival of)	PROBABILITY	COMPENSATION KShs.
FMB	$p_1p_2p_3$	0
FM	$p_1p_2q_3$	D_3
FB	$p_1q_2p_3$	D_2
F	$p_1q_2q_3$	D_2+D_3
MB	$q_1p_2p_3$	D_1
M	$q_1p_2q_3$	D_1+D_3
B	$q_1q_2p_3$	D_1+D_2
N	$q_1q_2q_3$	$D_1+D_2+D_3$

Table 3.1: A table of possible events, their probabilities and compensation amounts therewith (two-dependant model).

The tree diagram (figure (3.1) at the appendix) shows that there are eight possibilities at the end of the first year. The first four events involve the survival of the contributor. For this reason, the branching process only extends on at these particular points. The remaining four options do not extend further because the contributor no longer exists, hence the end of the transaction.

The survival of either FMB, or FM, or FB, or F implies that at the end of the second year, there are eight other possibilities for FMB, four for FM, four for FB, and two for F. The process proceeds in a similar manner for as long as we wish to observe. Note that if all members of a family die, the next of kin are responsible for their burial. Thus they take the claimed amount.

It can be shown that the sample space has been exploited by proving that the sum of all probabilities involved is unit.

$$\sum prob = p_1p_2p_3 + p_1p_2q_3 + p_1q_2p_3 + p_1q_2q_3 + q_1p_2p_3 + q_1p_2q_3 + q_1q_2p_3 + q_1q_2q_3.$$

$$= p_1p_2\{p_3+q_3\} + p_1q_2\{p_3+q_3\} + q_1p_2\{p_3+q_3\} + q_1q_2\{p_3+q_3\}$$

$$= p_1\{p_2+q_2\} + q_1\{p_2+q_2\}$$

$$= p_1 + q_1 = 1.$$

The branching process together with the probabilities in table (3.1) are used in the formulation of the two-dependant model.

3.2: FORMULATING *a two-dependant* MODEL.

In this section the two-dependant model of the Benevolent Scheme is derived and applied using collected data in section 2.2.4.

3.2.1: Derivation of $_2\xi_n$.

Figure (3.1) together with table (3.1) are utilized in the construction of a two-dependant model for the Benevolent Scheme.

The expected claim amount by the end of the first year is found to be:

$$_2\xi_1 = p_1p_2p_3(0) + p_1p_2q_3(D_3) + p_1q_2p_3(D_2) + p_1q_2q_3(D_2+D_3) + q_1p_2p_3(D_1+D_2) + q_1p_2q_3(D_1+D_3) + q_1q_2p_3(D_1+D_2) + q_1q_2q_3(D_1+D_2+D_3).$$

$$= \{q_1p_2p_3 + q_1p_2q_3 + q_1q_2p_3 + q_1q_2q_3\} D_1 + \{p_1q_2p_3 + p_1q_2q_3 + q_1p_2p_3 + q_1q_2p_3 + q_1q_2q_3\} D_2 + \{p_1p_2q_3 + p_1q_2q_3 + p_1q_2q_3 + q_1p_2q_3 + q_1q_2q_3\} D_3.$$

$$\Rightarrow \quad {}_2\xi_1 = q_1(D_1) + q_2(D_2) + q_3(D_3).$$

Using equation (2.1) we have,

$$ {}_2\xi_1 = \Phi_2. \tag{3.1}$$

In the second year, the expected claim amount is:

$$ {}_2\xi_2 = p_1p_2p_3\,\Phi_2 + p_1p_2q_3\,\Phi_{2\backslash 3} + p_1q_2p_3\,\Phi_{2\backslash 2} + p_1q_2q_3\,\Phi_{2\backslash 2,3}$$

$$= p_1\{p_2p_3\,\Phi_2 + p_2q_3\,\Phi_{2\backslash 3} + q_2p_3\,\Phi_{2\backslash 2} + q_2q_3\,\Phi_{2\backslash 2,3}\}. \tag{3.2}$$

In the third year:

$$ {}_2\xi_3 = p_1^2p_2^2p_3^2\,\Phi_2 + p_1^2p_2^2p_3q_3\,\Phi_{2\backslash 3} + p_1^2p_2q_2p_3^2q_2\,\Phi_{2\backslash 2} + p_1^2p_2q_2q_3\,\Phi_{2\backslash 2,3} + p_1^2p_2^2q_3\,\Phi_{2\backslash 3} + p_1^2p_2q_2q_3\,\Phi_{2\backslash 2,3} + p_1^2q_2p_3^2\,\Phi_{2\backslash 2,3} + p_1^2q_2q_3p_3\,\Phi_{2\backslash 2,3} + p_1^2q_2q_3\,\Phi_{2\backslash 2,3}.$$

$$\Rightarrow {}_2\xi_3 = p_1^2\{p_2^2p_3^2\,\Phi_2 + p_2^2q_3(1+p_3)\,\Phi_{2\backslash 3} + q_2p_3^2(1+p_2)\,\Phi_{2\backslash 2} + q_2q_3(1+p_2+p_3+p_2p_3)\,\Phi_{2\backslash 2,3}\}. \tag{3.3}$$

Similarly we have;

$$ {}_2\xi_4 = p_1^3\{p_2^3p_3^3\,\Phi_2 + p_2^3q_3(1+p_3+p_3^2)\,\Phi_{2\backslash 3} + q_2p_3^3(1+p_2+p_2^2)\,\Phi_{2\backslash 2} + q_2q_3(1+p_2+p_3+p_2p_3+p_2^2+p_3^2+p_2^2p_3^2)\,\Phi_{2\backslash 2,3}\}.$$

$$= p_1^3\{p_2^3 p_3^3 \Phi_2 + p_2^3 q_3 (\sum_{k=0}^{2} p_3^k)\Phi_{2\backslash 3} + q_2 p_3^3 (\sum_{k=0}^{2} p_2^k)\Phi_{2\backslash 2} +$$

$$q_2 q_3 (\sum_{a_1=0}^{2} \sum_{a_2=0}^{2} p_2^{a_1} p_3^{a_2})\Phi_{2\backslash 2,3}.$$

Using the property

$$q_2 q_3 \ldots\ldots q_{m+1}\left(\sum_{f_1=0}^{n-2} \sum_{f_2=0}^{n-2} \ldots\ldots \sum_{f_m=0}^{n-2} (p_2^{f_1} p_3^{f_2} \ldots\ldots p_{m+1}^{f_m})\right)$$

$$=\left(1-p_2^{n-1}\right)\left(1-p_3^{n-1}\right)\ldots\ldots\left(1-p_{m+1}^{n-1}\right) \qquad (3.4)$$

where

$$qi=1\text{-}pi \qquad \forall i \in \mathrm{N},$$

we have

$$_2\xi_4 = p_1^3\{p_2^3 p_3^3\ \Phi_2 + p_2^3(1-p_3^3)\ \Phi_{2\backslash 3} + p_3^3(1-p_2^3)\ \Phi_{2\backslash 2} +$$

$$(1\text{-}p_2^3)(1-p_3^3)\ \Phi_{2\backslash 2,3}\}. \qquad (3.5)$$

We can now generalize this expression to a case of n years as;

$$_2\xi_n = p_1^{n-1}\{p_2^{n-1} p_3^{n-1}\ \Phi_2 + p_2^{n-1}(1-p_3^{n-1})\ \Phi_{2\backslash 3} + p_3^{n-1}(1-p_2^{n-1})\ \Phi_{2\backslash 2} +$$

$$(1\text{-}p_2^{n-1}) * (1-p_3^{n-1})\ \Phi_{2\backslash 2,3}\}. \qquad (3.6)$$

This general formula can be shown to generate the already derived equations of the two-dependant model. For example if n = 2, then ;

$$_2\xi_2 = p_1^{2-1}\{p_2^{2-1} p_3^{2-1}\ \Phi_2 + p_2^{2-1}(1-p_3^{2-1})\Phi_{2\backslash 3} + p_3^{2-1}(1-p_2^{2-1})\ \Phi_{2\backslash 2} +$$

$(1-p_2^{2-1})*(1-p_3^{2-1})\,\Phi_{2\backslash 2,3}\}$.

$= p_1\{p_2p_3\,\Phi_2 + p_2q_3\,\Phi_{2\backslash 3} + q_2p_3\,\Phi_{2\backslash 2} + q_2q_3\,\Phi_{2\backslash 2,3}\}$.

As derived before.

3.2.2: Derivation of ${}_2\Pi_n$.

As was the case with the one-dependant model, the expected profit ${}_2\Pi_n$ is given by;

$${}_2\Pi_n = P\left(1+\frac{I}{100}\right)\sum_{k=1}^{n} V^{k-1} p_1^k - \sum_{k=1}^{n} V^k\,{}_2\xi_k.$$

$$= \sum_{k=1}^{n} V^k \left\{ \frac{P\left(1+\frac{I}{100}\right)}{V} p_1^k - {}_2\xi_k \right\}. \qquad (3.7)$$

The trend of ${}_2\xi_n$ is

$$\lim_{n\to\infty} {}_2\xi_n = 0 \qquad (3.8)$$

That is, the expected expenditure tends to zero as time increases. Next the actual picture of events is given using collected data.

3.2.3: Model Simulation.

Data to be used is already given in tables (2.4), (2.5), and (2.6). Table (3.2) has the values of expected claim amount, contribution and profit under the two-dependant Model. These are used in the plotting of graphs that follow. The graphs (figure (3.2a) and

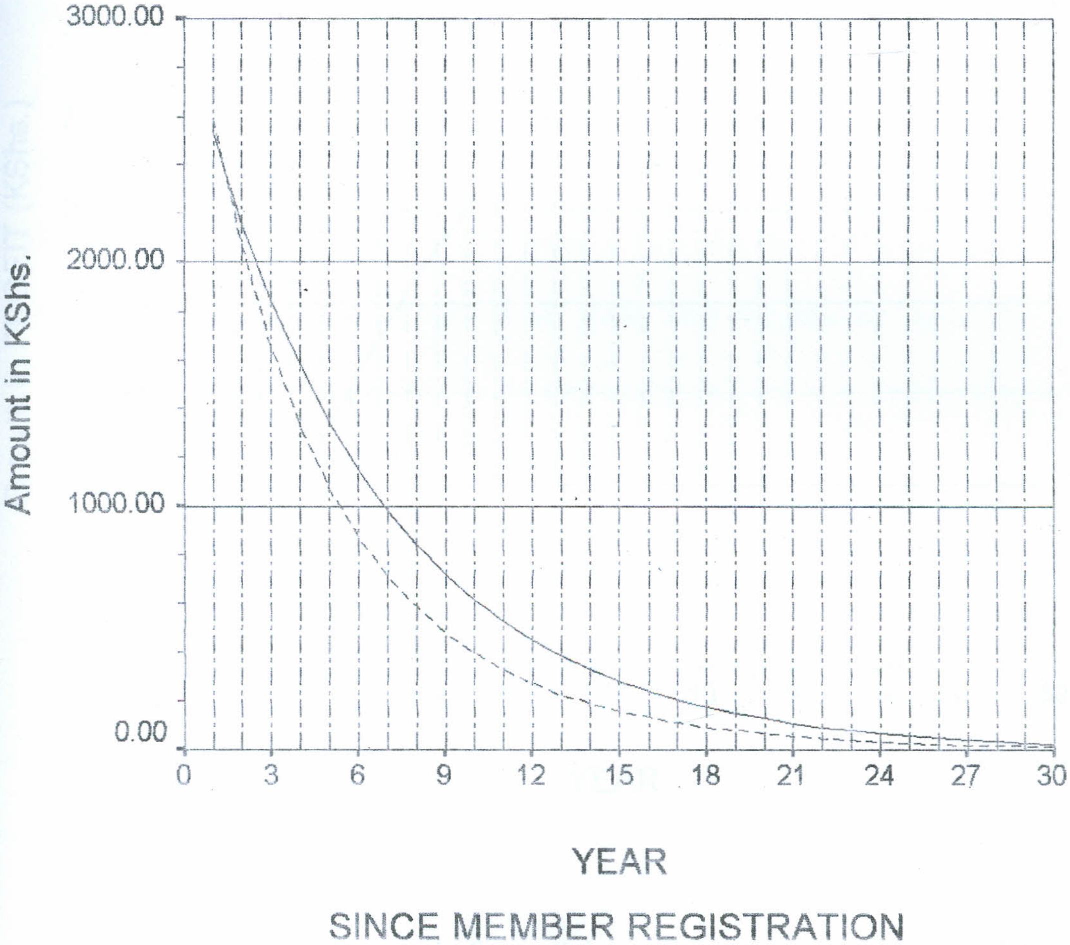
(3.2b)) illustrate the trends in claim amount, contribution and profit with increasing period.

Year *n*	*p_1 power (n)*	*Expected claim amount*	*discounted claim amount*	*cumulative claim amount*	*Expected contribution (discounted)*	*discounted cumulative contribution*	*growth in profit*
1	0.94	2850	2591	2591	2527	2527	-64
2	0.884	2497.11	2064	4655	2159	4686	31.45014
3	0.831	2199.09	1652	6307	1845	6531	224.3607
4	0.781	1946.21	1329	7636	1577	8108	471.8171
5	0.734	1730.57	1075	8711	1347	9456	744.6708
6	0.69	1545.77	872.6	9584	1151	10607	1023.545
7	0.648	1386.59	711.6	10295	984	11591	1295.954
8	0.61	1248.77	582.6	10878	840.9	12432	1554.227
9	0.573	1128.83	478.8	11357	718.6	13151	1794.025
10	0.539	1023.94	394.8	11751	614.1	13765	2013.276
11	0.506	931.743	326.6	12078	524.7	14289	2211.419
12	0.476	850.322	271	12349	448.4	14738	2388.875
13	0.447	778.085	225.4	12574	383.2	15121	2546.667
14	0.421	713.713	188	12762	327.5	15448	2686.167
15	0.395	656.11	157.1	12919	279.8	15728	2808.916
16	0.372	604.36	131.5	13051	239.1	15967	2916.508
17	0.349	557.699	110.4	13161	204.4	16172	3010.508
18	0.328	515.48	92.73	13254	174.6	16346	3092.412
19	0.309	477.159	78.03	13332	149.2	16496	3163.612
20	0.29	442.274	65.75	13398	127.5	16623	3225.386
21	0.273	410.431	55.47	13453	109	16732	3278.893
22	0.256	381.292	46.85	13500	93.13	16825	3325.172
23	0.241	354.568	39.61	13540	79.58	16905	3365.149
24	0.227	330.008	33.51	13573	68.01	16973	3399.646
25	0.213	307.393	28.38	13602	58.12	17031	3429.385
26	0.2	286.535	24.05	13626	49.66	17081	3455.001
27	0.188	267.267	20.39	13646	42.44	17123	3477.05
28	0.177	249.443	17.3	13663	36.27	17159	3496.016
29	0.166	232.932	14.69	13678	30.99	17190	3512.321
30	0.156	217.622	12.48	13691	26.49	17217	3526.331

Table 3.2: A table of data on claim amount, contributions and Profit for the two-dependant model.

EXPECTED CONTRIBUTION COMPARED WITH COMPENSATION CLAIM AMOUNT

Two-dependant Model.



KEY

.... *EXPECTED CLAIM AMOUNT*

______ *EXPECTED CONTRIBUTION*

Figure 3.2a: A graph giving the trend of expected contribution and claim amount for the two-dependant model.

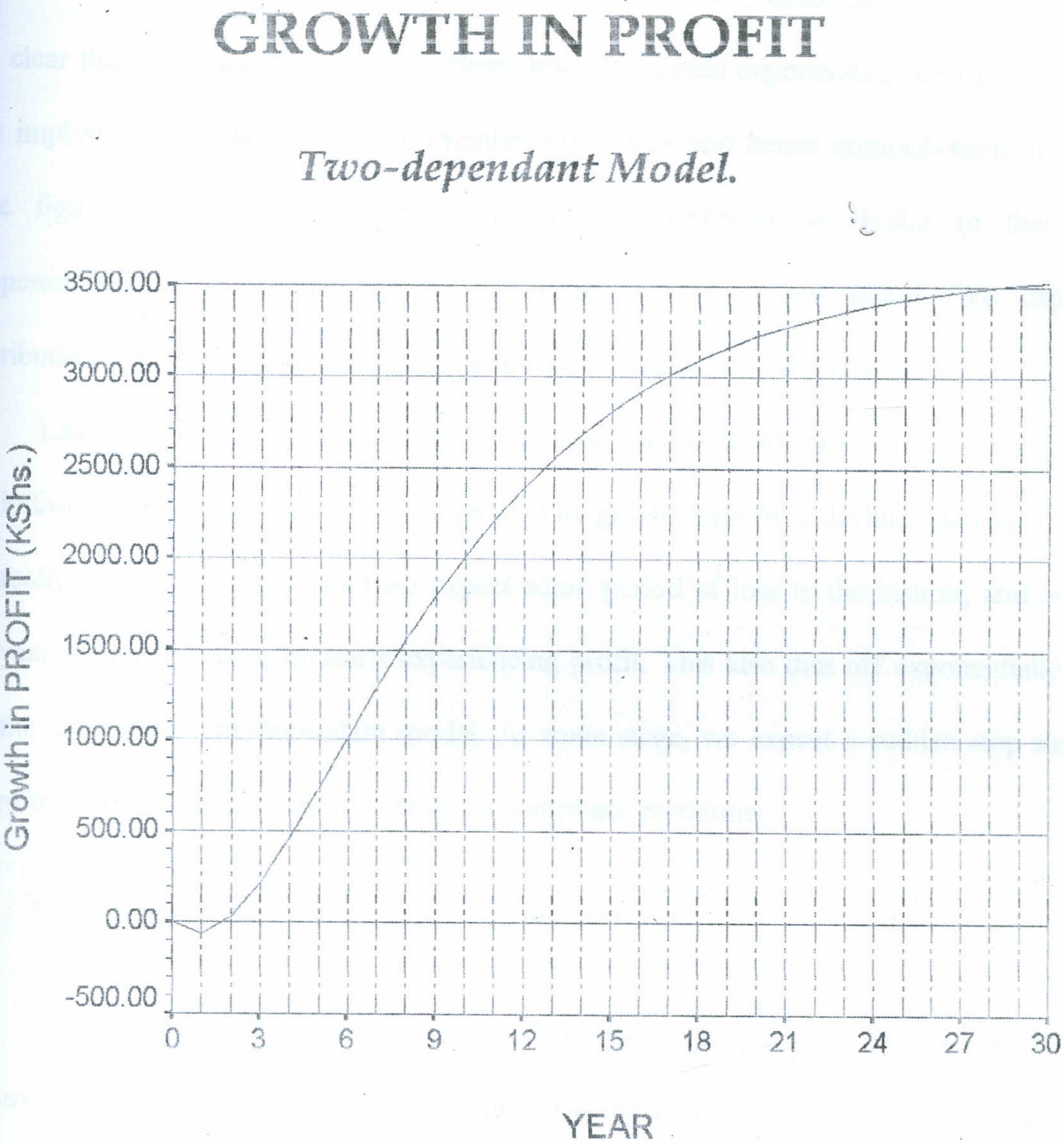


Figure 3.2b: A graph giving the trend of growth in profit in a two-dependant model.

The graph (figure (3.2a)) illustrates the trend of fall in claim amount with period. It is clear that with time the expected claim amount reduces exponentially to a minimum. This implies that at some stage, the membership ceases and hence compensation. In the same figure, the trend of decay in premium contribution is similar to that of

The graph (figure (3.2a)) illustrates the trend of fall in claim amount with period. It is clear that with time the expected claim amount reduces exponentially to a minimum. This implies that at some stage, the membership ceases and hence compensation. In the same figure, the trend of decay in premium contribution is similar to that of compensation claim amount due to the same reason already stated. We expect contribution to cease at some time due to eminent departure from the scheme.

Lastly, Figure (3.2b) shows the growth in accumulated profit levels expected from one individual under the two-pendant model. The graph starts by a decline, stabilizes and eventually rises. This implies that we expect some period of loss to the insurer, and after two years of contribution, he starts experiencing profit. This also dies off exponentially as was the case in the one-dependant model. At some stage, we expect a sudden stop since no profit is earned once a member ceases to contribute premiums.

3.3: FORMULATING A Three-Dependant MODEL.

After deriving the one and two-dependant benevolent scheme models, we need the three-dependant equation in order to get to a more realistic picture of the expected expenditure and benefits.

3.3.1: Derivation of ${}_3\xi_n$.

It is difficult to draw a tree diagram for this and higher order cases due to the numerous possibilities involved. In this case, another dependant G is introduced and assigned p_4 as the probability of his/her survival. We may conceptualize G as second born of the couple F and M. In summary, the ensuing events with their probabilities are as in table (3.3). D_4 is the compensation awarded in the event of G dying.

EVENT (Survival of)	**PROBABILITY**	**COMPENSATION** (In the event of death)
FMBG	$p_1 p_2 p_3 p_4$	0
FMB	$p_1 p_2 p_3 q_4$	D_4
FMG	$p_1 p_2 q_3 p_4$	D_3
FM	$p_1 p_2 q_3 q_4$	D_3+D_4
FBG	$p_1 q_2 p_3 p_4$	D_2
FB	$p_1 q_2 p_3 q_4$	D_2+D_4
FG	$p_1 q_2 q_3 p_4$	D_2+D_3
F	$p_1 q_2 q_3 q_4$	$D_2+D_3+D_4$
MBG	$q_1 p_2 p_3 p_4$	D_1
MB	$q_1 p_2 p_3 q_4$	D_1+D_4
MG	$q_1 p_2 q_3 p_4$	D_1+D_3
M	$q_1 p_2 q_3 q_4$	$D_1+D_3+D_4$
BG	$q_1 q_2 p_3 p_4$	D_1+D_2
B	$q_1 q_2 p_3 q_4$	$D_1+D_2+D_4$
G	$q_1 q_2 q_3 p_4$	$D_1+D_2+D_3$
N	$q_1 q_2 q_3 q_4$	$D_1+D_2+D_3+D_4$

Table 3.3: A table of possible events, their probabilities and compensation amounts therewith (three-dependant model).

We observe that there always are 2^{m+1} possibilities, where m is the number of dependants. By the end of the first year, the expected claim amount is;

$$
\begin{aligned}
{}_3\xi_1 &= p_1 p_2 p_3 p_4(0) + p_1 p_2 p_3 q_4(D_4) + p_1 p_2 q_3 p_4(D_3) + p_1 p_2 q_3 q_4(D_3+D_4) + \\
& p_1 q_2 p_3 p_4(D_2) + p_1 q_2 p_3 q_4(D_2+D_4) + p_1 q_2 q_3 p_4(D_2+D_3) + \\
& p_1 q_2 q_3 q_4(D_2+D_3+D_4) + q_1 p_2 p_3 p_4(D_1) + q_1 p_2 p_3 q_4(D_1+D_4) + \\
& q_1 p_2 q_3 p_4(D_1+D_3) + q_1 p_2 q_3 q_4(D_1+D_3+D_4) + q_1 q_2 p_3 p_4(D_1+D_2) + \\
& q_1 q_2 p_3 q_4(D_1+D_2+D_4) + q_1 q_2 q_3 p_4(D_1+D_2+D_3) + \\
& q_1 q_2 q_3 q_4(D_1+D_2+D_3+D_4)
\end{aligned}
$$

On calculation and simplifying we have,

$$ {}_3\xi_1 = q_1 (D_1) + q_2 (D_2) + q_3 (D_3) + q_4(D_4) \tag{3.9} $$

The sum on the R.H.S. is,

$$ q_1(D_1) + q_2 (D_2) + q_3 (D_3) + q_4 (D_4) = \Phi_3. \tag{3.10} $$

In the second year, the expected claim amount is found to be,

$$
\begin{aligned}
{}_3\xi_2 &= p_1p_2p_3p_4\Phi_3 + p_1p_2p_3q_4\Phi_{3\backslash 4} + p_1p_2q_3p_4\Phi_{3\backslash 3} + p_1q_2p_3p_4\Phi_{3\backslash 2} \\
&+ p_1p_2q_3q_4\Phi_{3\backslash 3,4} + p_1q_2p_3q_4\Phi_{3\backslash 2,4} + p_1q_2q_3p_4\Phi_{3\backslash 2,3} + p_1q_2q_3q_4\Phi_{3/2,3,4}
\end{aligned}
$$

Factoring out p_1 we have,

$$
\begin{aligned}
{}_3\xi_2 = p_1\Big\{ & p_2p_3p_4\Phi_3 + p_2p_3q_4\Phi_{3\backslash 4} + p_2q_3p_4\Phi_{3\backslash 3} + q_2p_3p_4\Phi_{3\backslash 2} \\
& + p_2q_3q_4\Phi_{3\backslash 3,4} + q_2p_3q_4\Phi_{3\backslash 2,4} + q_2q_3p_4\Phi_{3\backslash 2,3} + q_2q_3q_4\Phi_{3/2,3,4}\Big\}
\end{aligned} \tag{3.11}
$$

In the third year the expected expenditure is,

$$_3\xi_3 = p_1^2\left\{p_2^2 p_3^2 p_4^2 \Phi_3 + p_2^2 p_3^2 q_4 (1+p_4)\Phi_{3\backslash 4} + p_2^2 q_3 p_4^2 (1+p_3)\Phi_{3\backslash 3} +\right.$$

$$q_2 p_3^2 p_4^2 (1+p_2)\Phi_{3\backslash 2} + q_2 q_3 p_4^2 (1+p_2)(1+p_3)\Phi_{3\backslash 2,3} +$$

$$q_2 p_3^2 q_4 (1+p_2)(1+p_4)\Phi_{3\backslash 2,4} + p_2^2 q_3 q_4 (1+p_3)(1+p_4)\Phi_{3\backslash 2,3} +$$

$$\left. q_2 q_3 q_4 \left(1+p_2+p_3+p_4+p_2 p_3+p_2 p_4+p_3 p_4+p_2 p_3 p_4\right)\Phi_{3/2,3,4}\right\} \tag{3.12}$$

Higher order equations can similarly be derived despite the long calculations involved. Using the emerging trend, the case of year four gives,

$$_3\xi_4 = p_1^3\left\{p_2^3 p_3^3 p_4^3 \Phi_3 + p_2^3 p_3^3 q_4 \left(\sum_{a_1=0}^{2} p_4^{a_1}\right)\Phi_{3\backslash 4} + p_2^2 q_3 p_4^2 \left(\sum_{a_1=0}^{2} p_3^{a_1}\right)\Phi_{3\backslash 3} + \right.$$

$$\left. + \ldots + q_2 q_3 q_4 \sum_{a_1=0}^{2}\left\{\sum_{a_2=0}^{2}\left(\sum_{a_3=0}^{2} p_2^{a_1} p_3^{a_2} p_4^{a_3}\right)\right\}\Phi_{3/2,3,4}\right\}. \tag{3.13}$$

Generalizing this to n years ,

$$_3\xi_n = p_1^{n-1}\left\{p_2^{n-1} p_3^{n-1} p_4^{n-1} \Phi_3 + p_2^{n-1} p_3^{n-1} q_4 \left(\sum_{a_1=0}^{n-2} p_4^{a_1}\right)\Phi_{3\backslash 4} + \right.$$

$$+ p_2^{n-1} q_3 p_4^{n-1}\left(\sum_{a_1=0}^{n-2} p_3^{a_1}\right)\Phi_{3\backslash 3} + q_2 p_3^{n-1} p_4^{n-1}\left(\sum_{a_1=0}^{n-2} p_2^{a_1}\right)\Phi_{3\backslash 2} +$$

Using the property in equation (3.4) reduces equation (3.14) to

$$
{}_3\xi_n = p_1^{n-1}\{p_2^{n-1}p_3^{n-1}p_4^{n-1}\Phi_3 + p_2^{n-1}p_3^{n-1}\left(1-p_4^{n-1}\right)\Phi_{3\backslash 4} + p_2^{n-1}p_4^{n-1}\left(1-p_3^{n-1}\right)\Phi_{3\backslash 3} +
$$

$$
p_4^{n-1}p_3^{n-1}\left(1-p_2^{n-1}\right)\Phi_{3\backslash 2} + p_2^{n-1}(1-p_3^{n-1})\left(1-p_4^{n-1}\right)\Phi_{3\backslash 3,4} + (1-p_2^{n-1})p_3^{n-1}\left(1-p_4^{n-1}\right)\Phi_{3\backslash 2,4}
$$

$$
p_4^{n-1}(1-p_3^{n-1})\left(1-p_2^{n-1}\right)\Phi_{3\backslash 2,3} + (1-p_4^{n-1})(1-p_3^{n-1})\left(1-p_2^{n-1}\right)\Phi_{3\backslash 2,3,4}\}. \tag{3.15}
$$

3.3.2: Derivation of ${}_3\Pi_n$.

As was the case with the two-dependant model, the expected profit ${}_3\Pi_n$ is given by;

$$
{}_3\Pi_n = P\left(1+\frac{I}{100}\right)\sum_{k=1}^{n} V^{k-1}p_1^k - \sum_{k=1}^{n} V^k\,{}_3\xi_k.
$$

$$
= \sum_{k=1}^{n} V^k \left\{ \frac{P\left(1+\frac{I}{100}\right)}{V} p_1^k - {}_3\xi_k \right\}. \tag{3.16}
$$

The trend of ${}_3\xi_n$ is

$$
\lim_{n\to\infty} {}_3\xi_n = 0
$$

That is, the expected expenditure tends to zero as time increases.

3.2.3: Model Simulation.

We need the values of contribution and compensation in table (2.6) to calculate the expected claim amount and contribution under the three-dependant Model. The resultant data are recorded in table (3.4).

Year n	*p1 power (n)*	*Expected claim amount*	*Discounted claim amount*	*Cumulative claim amount*	*Expected contribution (discounted)*	*Discounted cumulative contribution*	*Growth in profit*
1	0.94	3900	3545	3545	2527	2527	-1018
2	0.884	3335.45	2757	6302	2159	4686	-1615.41
3	0.831	2865.88	2153	8455	1845	6531	-1923.49
4	0.781	2473.21	1689	10144	1577	8108	-2035.99
5	0.734	2143.86	1331	11475	1347	9456	-2019.77
6	0.69	1867.20	1054	12529	1151	10607	-1922.35
7	0.648	1634.57	838.9	13368	984	11591	-1777.21
8	0.61	1438.77	671.3	14040	840.9	12432	-1607.58
9	0.573	1273.76	540.2	14580	718.6	13151	-1429.25
10	0.539	1134.44	437.4	15017	614.1	13765	-1252.61
11	0.506	1016.48	356.3	15374	524.7	14289	-1084.17
12	0.476	916.280	292	15666	448.4	14738	-927.735
13	0.447	830.773	240.7	15906	383.2	15121	-785.207
14	0.421	757.428	199.5	16106	327.5	15448	-657.22
15	0.395	694.137	166.2	16272	279.8	15728	-543.575
16	0.372	639.155	139.1	16411	239.1	15967	-443.558
17	0.349	591.049	117	16528	204.4	16172	-356.157
18	0.328	548.641	98.7	16627	174.6	16346	-280.219
19	0.309	510.970	83.56	16710	149.2	16496	-214.548
20	0.29	477.251	70.95	16781	127.5	16623	-157.974
21	0.273	446.848	60.4	16842	109	16732	-109.389
22	0.256	419.241	51.51	16893	93.13	16825	-67.7735
23	0.241	394.011	44.01	16937	79.58	16905	-32.202
24	0.227	370.817	37.66	16975	68.01	16973	-1.84962
25	0.213	349.381	32.25	17007	58.12	17031	24.0130
26	0.2	329.479	27.65	17035	49.66	17081	46.0251
27	0.188	310.925	23.72	17058	42.44	17123	64.7429
28	0.177	293.570	20.36	17079	36.27	17159	80.6481
29	0.166	277.290	17.49	17096	30.99	17190	94.1558
30	0.156	261.982	15.02	17111	26.49	17217	105.622

Table 3.4: A table of Cumulative claim amount and contributions for a three-dependant model.

EXPECTED CONTRIBUTION COMPARED WITH COMPENSATION CLAIM AMOUNT

Three-dependant Model.

KEY

.... *EXPECTED CLAIM AMOUNT*

———— *EXPECTED CONTRIBUTION*

Figure 3.3a: A graph giving the trend of expected contribution and claim amount for the three-dependant model.

GROWTH IN PROFIT

Three-dependant Model.

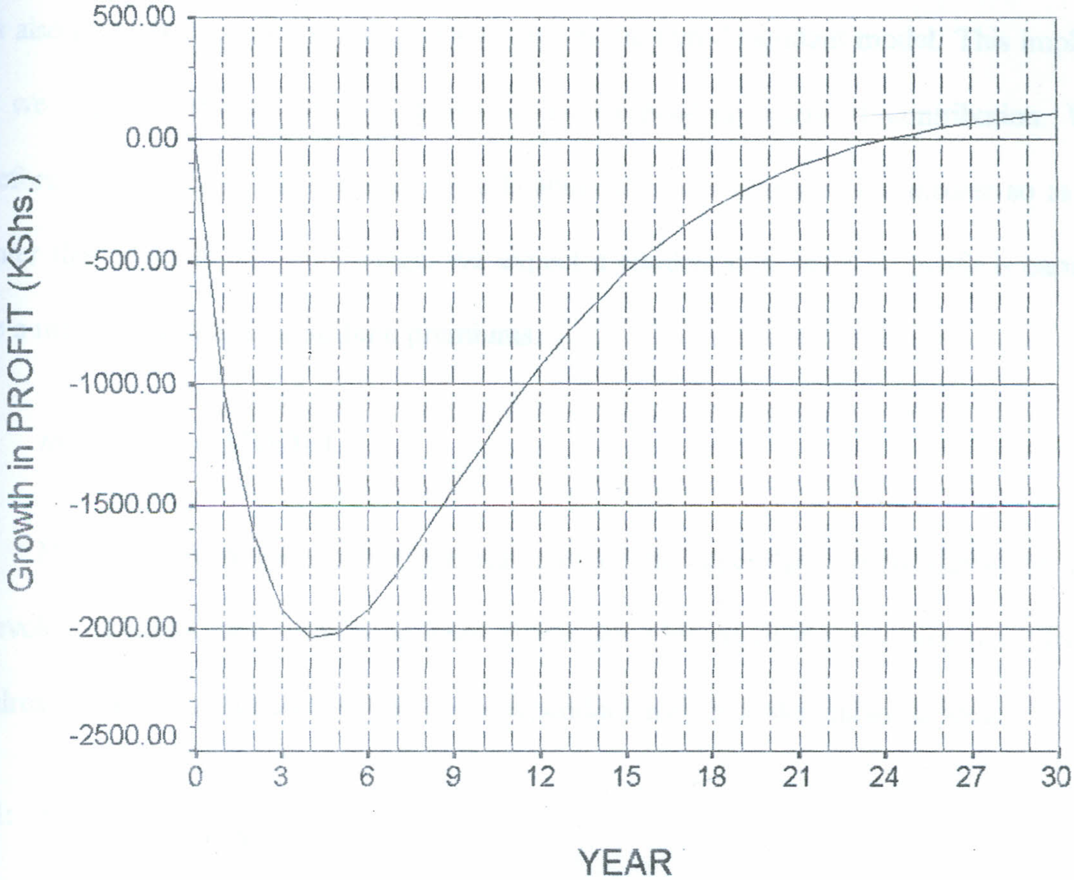


Figure 3.3b: A graph giving the trend of growth in profit in a three-dependant model.

The graph in figure (3.3a) illustrates the trend of fall in claim amount with period. It shows that with time the expected claim amount reduces exponentially. Thus at some stage, the membership ceases and hence compensation. In the same figure, the trend of decay in premium contribution is at a higher rate than that of compensation claim amount. We expect contribution to cease at some time due to eminent departure from the scheme.

Lastly, Figure (3.2b) shows the growth in accumulated profit levels expected from one individual under the three-pendant model. The graph starts by a decline for the first four years, stabilizes and eventually rises. This implies that we expect some period of loss to the insurer, and after twenty-four years of contribution, he starts experiencing profit. This also dies off exponentially as was the case in the two-dependant model. This implies that we expect total loss and no profit within twenty-four years of contribution. We therefore see reason as to why some rectification has to be done by the insurer so as to counter this anomaly. At some stage, we expect a sudden stop since no profit is earned once a member ceases to contribute premiums.

3.4. *m-dependant* MODEL.

Here the expression for a general case of m dependants participating in the Benevolent Scheme for n years is derived. This is done by use of the emerging trend from the already formulated one-dependant, two-dependant and three-dependant models.

3.4.1: Modelling ${}_{m}\xi_{n}$

The following formulae have already been derived:

i. $${}_{1}\xi_{n} = p_1^{n-1}\{p_2^{n-1}\,\Phi_1 + (1-p_2^{n-1})\,\Phi_{1\backslash 2}\}$$ (One-dependant model)

ii. $${}_{2}\xi_{n} = p_1^{n-1}\{p_2^{n-1}p_3^{n-1}\,\Phi_2 + p_2^{n-1}(1-p_3^{n-1})\,\Phi_{2\backslash 3} + p_3^{n-1}(1-p_2^{n-1})\,\Phi_{2\backslash 2} + (1-p_2^{n-1})*\,(1-p_3^{n-1})\,\Phi_{2\backslash 2,3}\}.$$ (Two-dependant model)

iii. $${}_3\xi_n = p_1^{n-1}\{p_2^{n-1}p_3^{n-1}p_4^{n-1}\Phi_3 + p_2^{n-1}p_3^{n-1}\left(1-p_4^{n-1}\right)\Phi_{3\backslash 4} +$$

$$p_2^{n-1}p_4^{n-1}\left(1-p_3^{n-1}\right)\Phi_{3\backslash 3} + p_4^{n-1}p_3^{n-1}\left(1-p_2^{n-1}\right)\Phi_{3\backslash 2} +$$

$$p_2^{n-1}(1-p_3^{n-1})\left(1-p_4^{n-1}\right)\Phi_{3\backslash 3,4} + (1-p_2^{n-1})p_3^{n-1}\left(1-p_4^{n-1}\right)\Phi_{3\backslash 2,4}$$

$$p_4^{n-1}(1-p_3^{n-1})\left(1-p_2^{n-1}\right)\Phi_{3\backslash 2,3} + (1-p_4^{n-1})(1-p_3^{n-1})\left(1-p_2^{n-1}\right)\Phi_{3\backslash 2,3,4}\}$$

(Three-dependant model)

Looking at the trend of the equations for the one, two and three dependant models above, we see that the first term is always

$$p_1^{n-1}. \tag{3.17}$$

The first term in the first bracket is

$$\left\{\left\{p_2^{n-1}p_3^{n-1}\ldots\ldots p_{m+1}^{n-1}\right\}\Phi_m\right\} \tag{3.18}$$

This can generally be presented as

$$\left\{\prod_{k=2}^{m+1} p_k^{n-1}\right\}\Phi_m \tag{3.19}$$

The middle terms of the equations being analyzed can also be generalized. Introducing the notation $\Psi_{m,n}$ as the middle term in the equation ${}_m\xi_n$, where

$$\Psi_{1,1} = \Psi_{1,2} = \Psi_{1,3} = \ldots = \Psi_{1,n} = 0$$

$$\Psi_{2,1} = 0,$$

$$\Psi_{2,2} = p_2(1-p_3)\Phi_{2\setminus 3} + p_3(1-p_2)\Phi_{2\setminus 2},$$

$$\Psi_{2,3} = p_2^{\,2}(1-p_3^{\,2})\Phi_{2\setminus 3} + p_3^{\,2}(1-p_2^{\,2})\Phi_{2\setminus 2}$$

....

$$\Psi_{2,n} = p_2^{\,n-1}(1-p_3^{\,n-1})\Phi_{2\setminus 3} + p_3^{\,n-1}(1-p_2^{\,n-1})\Phi_{2\setminus 2}$$

$$\Psi_{3,n} = p_2^{\,n-1}p_3^{\,n-1}(1-p_4^{\,n-1})\Phi_{3\setminus 4} + p_2^{\,n-1}p_4^{\,n-1}(1-p_3^{\,n-1})\Phi_{3\setminus 3}$$

$$+p_4^{\,n-1}p_3^{\,n-1}\left(1-p_2^{\,n-1}\right)\Phi_{3\setminus 2} + p_2^{\,n-1}(1-p_3^{\,n-1})\left(1-p_4^{\,n-1}\right)\Phi_{3\setminus 3,4} + (1-p_2^{\,n-1})p_3^{\,n-1}\left(1-p_4^{\,n-1}\right)\Phi_{3\setminus 2,4}$$

$$+p_4^{\,n-1}(1-p_3^{\,n-1})(1-p_2^{\,n-1})\Phi_{3\setminus 2,3}. \qquad (3.20)$$

Using the trend, we come up with the general expression as

$$\Psi_{mn} = \sum_{g=1}^{m-1}\left\{ \sum_{f_1=2}^{3}\sum_{f_2=3}^{4}\cdots\sum_{f_{g-1}=m-1}^{m}\sum_{f_g=m}^{m+1}\left[\left(\prod_{k=2}^{m+1} p_{k\setminus \bigcup_{i=1}^{g} f_i}\right)^{n-1}\left(\prod_{i=1}^{g}\left[1-p_{f_i}^{\,n-1}\right]\right)\right.\right.$$

$$\left.\left. \times \quad \Phi_{m\setminus \bigcup_{i=1}^{g} f_i}\right]\right\} \qquad (3.21)$$

$f_1 < f_2 < f_3 < \ldots f_{g-1} < f_g < \ldots f_{m-1}$ where f_i is an index taking values 2,3,4…

$m = 1, 2, 3, \ldots \qquad n = 1, 2, 3, \ldots$

Next we introduce another symbol $\Gamma_{m,n}$, with which the last term of each expression can also be summarized as in equation (3.22).

$$\Gamma_{m,n} = \left(\prod_{k=2}^{m+1} q_k\right)\left\{\sum_{a_1=0}^{n-2}\sum_{a_2=0}^{n-2}\cdots\sum_{a_3=0}^{n-2}\left(\prod_{k=2}^{m+1} p_k\right)\right\}\Phi_{m\backslash 2,3,\ldots,m+1}. \qquad (3.22)$$

The summations can be simplified so that we have equation (3.22) reducing to a shorter expression as;

$$\Gamma_{m,n} = \left(\prod_{k=2}^{m+1}\left(1-p_k^{n-1}\right)\right)\Phi_{m\backslash 2,3,\ldots,m+1} \qquad (3.23)$$

Finally, we now invoke equations (3.17), (3.19), (3.21), and (3.23) to get the ***m-dependant model*** as;

$$\boxed{{}_m\xi_n = p_1^{n-1}\left\{\prod_{k=2}^{m+1} p_k^{n-1}\Phi_m + \Psi_{m,n} + \Gamma_{m,n}\right\}} \qquad (3.24)$$

Where
$$\Phi_m = \sum_{k=1}^{m+1} q_k D_k \qquad (3.25)$$

3.4.2: Derivation of ${}_m\Pi_n$.

As was the case with the three-dependant model, the expected profit ${}_m\Pi_n$ is given by;

$${}_m\Pi_n = P\left(1+\frac{I}{100}\right)\sum_{k=1}^{n} V^{k-1} p_1^k - \sum_{k=1}^{n} V^k {}_m\xi_k.$$

$$= \sum_{k=1}^{n} V^k \left\{ \frac{P\left(1+\frac{I}{100}\right)}{V} p_1^k - {}_m\xi_k \right\}. \qquad (3.26)$$

3.4.3: Trend as n→ ∞.

From the general formula that was derived in section 3.4.1, the first term outside the brackets $p_1^{n-1} \to 0$ since p_1 is a fraction. Thus the trend is the same for whichever the number of dependants, m. this can be summarized as follows: The trend of ${}_m\xi_n$ is

$$\lim_{n\to\infty} {}_m\xi_n = 0 \qquad (3.27)$$

That is, the expected expenditure tends to zero with increase in time.

CHAPTER FOUR

4. MARKOV MODEL APPROACH.

In this chapter we use the Markov properties of the survival probabilities to give an alternative method of getting to the m-dependant model. We start by giving derivations of the one-dependant, two-dependant, and three-dependant models. Using the ensuing trend, we formulate the m-dependant model. For each of these models, we give the transition probabilities involved, classify the states, calculate the n-step transition probabilities and apply the model in the calculation of expected claim amount and profit.

4.1: One-Dependant MODEL.

Under the introductory chapter (Chapter one) in section 1.2.4, we gave a brief theory of the Markov chains and basic concepts of a stochastic process. We use the principles in this section.

4.1.1: Transition Probabilities Involved.

The events in this system can be illustrated by a matrix of transition probabilities. There are four states AB, A, B, and N, where AB represents survival of a member together with the spouse, A represents survival of the member only and B the survival of the spouse only while N denotes the death of both. We have the matrix of transition probabilities given as,

$$
\wp = \begin{pmatrix} p_1 p_2 & p_1 q_2 & q_1 p_2 & q_1 q_2 \\ 0 & p_1 & 0 & q_1 \\ 0 & 0 & p_2 & q_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (4.1)
$$

Using the definitions stated under section 2.2, we find that $\wp$ is a stochastic matrix. However, it is not doubly stochastic.

4.1.2: Classification of States.

Let the events E1, E2, E3, and E4 represent the following;

E1	→A and B Survive.
E2	→A alone Survives.
E3	→B alone Survives.
E4	→N (None) Survives.

The state space is {E1, E2, E3, E4}. We classify the states using the diagram in figure 4.1. In this figure, arrows represent accessibility of states in the Markov chain.

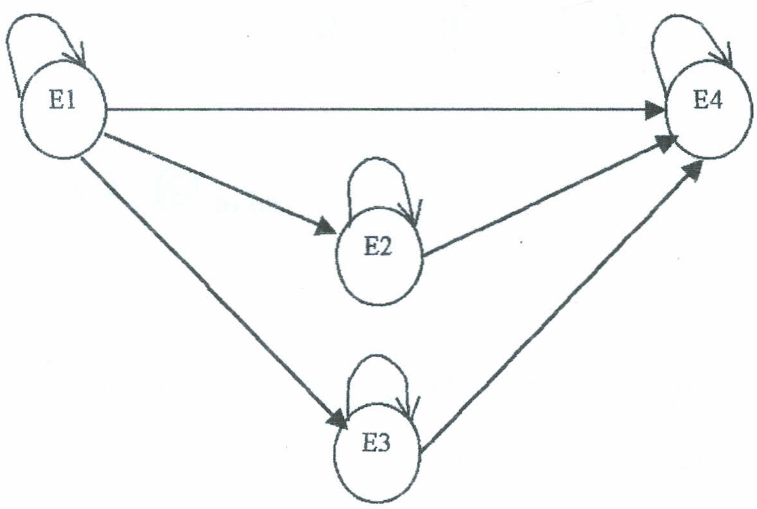


Figure 4.1: A diagram showing accessibility of states.

From the diagram, state E1 is transient. Once you are out of the state, you can never revisit it. Indeed, a dead person cannot become alive. E4 is absorbing. You cannot move out of the state once entered. Once both are dead, the transaction ends. No states communicate. No reverse is possible after any of the four probable transitions. The notable closed sets are {E4}, {E1, E2, E3, E4}, {E2, E3, E4}, {E3, E4}, and {E1, E2, E3, E4}.

4.1.3: n-Step Transition Probabilities and Steady State.

We now wish to calculate higher transition probabilities of such a Markov matrix. Considering a case where $p_1 = p_2$ (the probability of survival of the contributor is equal to that of the dependant), the matrix in equation (4.1) above becomes,

$$\wp = \begin{pmatrix} p^2 & pq & pq & q^2 \\ 0 & p & 0 & q \\ 0 & 0 & p & q \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} p^2 & p(1-p) & p(1-p) & (1-p)^2 \\ 0 & p & 0 & (1-p) \\ 0 & 0 & p & (1-p) \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (4.2)$$

We find $\wp^2 = \wp . \wp$ to be,

$$\wp^2 = \begin{pmatrix} p4 & p^2(1-p^2) & p^2(1-p^2) & (1-p^2)^2 \\ 0 & p^2 & 0 & (1-p^2) \\ 0 & 0 & p^2 & (1-p^2) \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (4.3)$$

$$\wp^3 = \begin{pmatrix} p^6 & p^3(1-p^3) & p^3(1-p^3) & (1-p^3)^2 \\ 0 & p^3 & 0 & (1-p^3) \\ 0 & 0 & p^3 & (1-p^3) \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.4}$$

and

$$\wp^4 = \begin{pmatrix} p^8 & p^4(1-p^4) & p^4(1-p^4) & (1-p^4)^2 \\ 0 & p^4 & 0 & (1-p^4) \\ 0 & 0 & p^4 & (1-p^4) \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.5}$$

We can now conclude without loss of generality that,

$$\wp^n = \begin{pmatrix} p^{2n} & p^n(1-p^n) & p^n(1-p^n) & (1-p^n)^2 \\ 0 & p^n & 0 & (1-p^n) \\ 0 & 0 & p^n & (1-p^n) \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.6}$$

Note that higher transition probability matrices are also Markov. They are also stochastic.

We now revert to our earlier case were p_1 is not equal to p_2. Equation (4.1) is the Markov matrix of interest. The higher transition probabilities are calculated as follows.

$$\wp = \begin{pmatrix} p_1p_2 & p_1(1-p_2) & p_2(1-p_1) & (1-p_1)(1-p_2) \\ 0 & p_1 & 0 & (1-p_1) \\ 0 & 0 & p_2 & (1-p_2) \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.7}$$

In the second step, we have the following transition probabilities.

$$\wp^2 = \begin{pmatrix} p_1^2 p_2^2 & p_1^2(1-p_2^2) & p_2^2(1-p_1^2) & (1-p_1^2)(1-p_2^2) \\ 0 & p_1^2 & 0 & (1-p_1^2) \\ 0 & 0 & p_2^2 & (1-p_2^2) \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (4.8)$$

And in the third step,

$$\wp^3 = \begin{pmatrix} p_1^3 p_2^3 & p_1^3(1-p_2^3) & p_2^3(1-p_1^3) & (1-p_1^3)(1-p_2^3) \\ 0 & p_1^3 & 0 & (1-p_1^3) \\ 0 & 0 & p_2^3 & (1-p_2^3) \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (4.9)$$

Using the trend taken by equations (4.7), (4.8), and (4.9), we have the general higher transition probability matrix as,

$$\wp^n = \begin{pmatrix} p_1^n p_2^n & p_1^n(1-p_2^n) & p_2^n(1-p_1^n) & (1-p_1^n)(1-p_2^n) \\ 0 & p_1^n & 0 & (1-p_1^n) \\ 0 & 0 & p_2^n & (1-p_2^n) \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (4.10)$$

Equation (4.10) is the n^{th}-step transition probability matrix.

Another useful tool in reading the physical implications of a Markov matrix is stationary distribution (or steady state) of the Markov chain. If $\wp$ is the matrix of transition probabilities and $\underline{\Pi}$ the Stationary distribution, then the following identities hold.

$$\text{i. } \underline{\Pi}\wp = \underline{\Pi}$$

$$\text{ii. } \underline{\Pi}\,\underline{1} = \underline{1} \qquad (4.11)$$

Where

$$\underline{\Pi} = [\Pi_1 \Pi_2 \Pi_3 \Pi_4]$$

Starting with equation (4.2) where $p_1 = p_2 = p$, and using the above identities, we have

$$[\Pi_1 \Pi_2 \Pi_3 \Pi_4]\begin{pmatrix} p_1 p_2 & p_1 q_2 & q_1 p_2 & q_1 q_2 \\ 0 & p_1 & 0 & q_1 \\ 0 & 0 & p_2 & q_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \Pi_1 \\ \Pi_2 \\ \Pi_3 \\ \Pi_4 \end{pmatrix} \qquad (4.12)$$

Multiplying through we have,

$$\Pi_1 p^2 = \Pi_1 \Rightarrow \Pi_1 = 0.$$

This is true because p^2 and hence p are strictly not equal to 1. Similarly,

$$\Pi_2 = \Pi_3 = 0.$$

Now using condition ii in equation (4.11) above, we have $\Pi_4 = 1$. Thus, the stationary distribution is,

$$\underline{\Pi} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (4.13)$$

Also from equation (4.6), with the limits as $n \to \infty$ we have

$$\wp^n = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.14}$$

This is similar to the result in equation (4.13) above. Equation (4.13) and (4.14) are stationary distributions (or steady state) transition probability matrices for the matrix in equation (4.1). In the case of the matrix in equation (4.7), we check the solution of equation (4.10) as $n \to \infty$ and find that the steady state transition probabilities are as given in equation (4.14) above.

4.1.4: Application to calculating ${}_1\xi_n$ and ${}_1\Pi_n$.

Given that

$$\wp = \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{pmatrix} \tag{4.15}$$

as was the case with the branching model, we require some of the transition probabilities from the matrix. These is used in the calculation of expected values of compensation. These are $\{p_{11}, p_{12}, p_{13}, p_{14}\}$ only, since we had already assumed that we have two beneficiaries at the start of the process. Thus we have the expected claim amount by the end of the first year being calculated as

$${}_1\xi_1 = \{p_{11}(0) + p_{12}(D_2) + p_{13}(D_1) + p_{14}(D_1 + D_2)\} \tag{4.16}$$

Substituting p_{ij}s with respective probabilities we obtain

$$_1\xi_1 = \{q_1(D_1)+q_2(D_2)\} = \Phi_1 \tag{4.17}$$

This is the same result as arrived at in equation (2.7) were we used a tree diagram.

We now wish to calculate $_1\xi_2$. To do this, we need the following transition probabilities.

$$P_{11}^{(2)}=p_1p_2 * p_1p_2 = p_1^2p_2^2$$
$$P_{12}^{(2)}=P_{11}P_{12} = p_1p_2 * p_1q_2 = p_1^2p_2q_2$$
$$P_{13}^{(2)}=P_{11}P_{13} = p_1p_2 * q_1p_2 = p_1p_2^2q_1$$
$$P_{14}^{(2)}=P_{11}P_{14} = p_1p_2 * q_1q_2 = p_1p_2q_1q_2$$

And $$P_{14}^{(2)}=P_{12}P_{24} = p_1q_2q_1 \tag{4.18}$$

Note that $p_{ij}^{(2)}$ does not imply $p_{ij}\cdot p_{ij}$.

Multiplying the probabilities above together with respective compensations we get the expected claim amount to be

$$_1\xi_2 = p_1\{p_2\Phi_1 + q_2\Phi_{1\backslash 2}\} \tag{4.19}$$

where

$$\Phi_{1\backslash 2} = q_1D_1$$

We need to come up with a much simpler method of using the Markov matrix directly rather than going for possible events, one after another. In chapter two and three, during the formulation of the one-dependant, two-dependant and three-dependant models, we assumed that all transitions start with the presence of the m dependants together with the contributor. We therefore only consider using elements of the first row of our transition probability matrix in the calculation of expected compensation claim amount.

Further we have been considering the event of survival of the contributor as the dependants die, until such a time that the contributor also dies. This implies that only elements with the probability $(p_1)^n$ as a factor is considered. This leaves us with the first half of the elements in the first row. In addition to these facts, we are only to consider the probability of survival up to right before the first "failure" or death. Thus the factors under consideration in the calculation of the expected claim amount in the nth year are those in the $(n\text{-}1)^{th}$ step transition probability matrix. This means, in order to calculate $_1\xi_n$, we need the matrix $\wp^{n-1}$. From the matrix, we pick the first row and use the first half of its elements as demonstrated next.

Case one. $(n=1)$.

In this case, $\wp^{n-1} = \wp^{1-1} = \wp^0 = \mathbf{I}$ (the identity matrix).

$$\wp^0 = \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.20}$$

We already have our factors with which the first two elements of the first row (first half of the four) are to be multiplied. These are

$\Phi_1 = q_1 D_1 + q_2 D_2$ (When both the contributor and dependant die).

$\Phi_{1\backslash 2} = q_1 D_1$ (When only the contributor dies).

Indeed in equation (4.1), the first element meant the survival of both and the second, the survival of the contributor only. Now, after one step, we are considering their death.

With this in mind therefore, the expected claim amount is

$$ {}_1\xi_1 = 1(\Phi_1) + 0(\Phi_{1\backslash 2}) $$

$$ = \Phi_1 \text{ As was the case in equation (4.17).} $$

Case two. $(n=2)$.

Here we use the matrix

$$ \wp^{n-1} = \wp^{1} = \begin{pmatrix} p_1 p_2 & p_1(1-p_2) & p_2(1-p_1) & (1-p_1)(1-p_2) \\ 0 & p_1 & 0 & (1-p_1) \\ 0 & 0 & p_2 & (1-p_2) \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.21} $$

From this, the elements of concern are, $p_1 p_2$ and $p_1(1-p_2)$. Multiplying as before, we have ${}_1\xi_2$ being given as

$$ {}_1\xi_2 = p_1 p_2 \Phi_1 + p_1(1-p_2)\Phi_{1\backslash 2} $$

$$ = p_1\{p_2\Phi_1 + q_2\Phi_{1\backslash 2}\} \text{ as was equation (4.19).} $$

Case three. $(n=\mathrm{k})$.

With tremendous confidence we can now generalize the calculation of expected claim amount as follows: We need the matrix of transition probabilities $\wp^{k-1}$ given as

$$ \wp^{n-1} = \begin{pmatrix} p_1^{k-1} p_2^{k-1} & p_1^{k-1}(1-p_2^{k-1}) & p_2^{k-1}(1-p_1^{k-1}) & (1-p_1^{k-1})(1-p_2^{k-1}) \\ 0 & p_1^{k-1} & 0 & (1-p_1^{k-1}) \\ 0 & 0 & p_2^{k-1} & (1-p_2^{k-1}) \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.22} $$

As in case one and two, the general formula is found to be

$$
\begin{aligned}
{}_1\xi_k &= (p_1 p_2)^{k-1}\Phi_1 + p_1^{\,k-1}(1-p_2)^{k-1}\Phi_{1\backslash 2} \\
&= p_1^{\,k-1}\{p_2^{\,k-1}\Phi_1 + q_2^{\,k-1}\Phi_{1\backslash 2}\}
\end{aligned}
$$

As was equation (2.20) in Chapter two. ${}_1\Pi_n$ is calculated just as was derived in part 2.2.3 in chapter two and is found to be

$$
\begin{aligned}
{}_1\Pi_n &= P\left(1+\frac{I}{100}\right)\sum_{k=1}^{n} V^{k-1} p_1^k - \sum_{k=1}^{n} V^k\, {}_1\xi_k . \\
&= \sum_{k=1}^{n} V^k \left\{ \frac{P\left(1+\dfrac{I}{100}\right)}{V} p_1^k - {}_1\xi_k \right\}.
\end{aligned}
$$

We can generally therefore conclude that both methods give similar results. In this case we only need to show higher transition probabilities of further transition probability matrices (for the two dependant and if possible, three dependant model). We can imagine how little we need to do to arrive at ${}_1\xi_k$ than was the case with the use of the tree diagram!

4.2: Two-dependant MARKOV MODEL.

In the case of two dependants, the state space increases. There are eight possible events namely, the event of survival of FMB, FM, FB, F, MB, M, B, and N. These form the state space. The letters denoting the state space take the same meanings, as was the case in the branching model.

4.2.1: Transition Probabilities Involved.

The matrix that represents the required transitions is an eight by eight Markov matrix with each of its rows adding up to one. However, the columns of the transition probability matrix do not add up to one. Hence we conclude that it is not doubly stochastic. As was the case with the one dependant model, the matrix is diagonal with all of its lower diagonal elements being zeros.

The matrix of transition probabilities involved for the two-dependant model is as in equation (4.23).

$$\wp = \begin{pmatrix} p_1p_2p_3 & p_1p_2q_3 & p_1q_2p_3 & p_1q_2q_3 & q_1p_2p_3 & q_1p_2q_3 & q_1q_2p_3 & q_1q_2q_3 \\ 0 & p_1p_2 & 0 & p_1q_2 & 0 & q_1p_2 & 0 & q_1q_2 \\ 0 & 0 & p_1p_3 & p_1q_3 & 0 & 0 & q_1p_3 & q_1q_3 \\ 0 & 0 & 0 & p_1 & 0 & 0 & 0 & q_1 \\ 0 & 0 & 0 & 0 & p_2p_3 & p_2q_3 & q_2p_3 & q_2q_3 \\ 0 & 0 & 0 & 0 & 0 & p_2 & 0 & q_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & p_3 & q_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.23}$$

This matrix can be partitioned to four square non-Markov matrices as in the equations that follow.

$$\wp = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{4.24}$$

where

$$A=\begin{pmatrix} p_1p_2p_3 & p_1p_2q_3 & p_1q_2p_3 & p_1q_2q_3 \\ 0 & p_1p_2 & 0 & p_1q_2 \\ 0 & 0 & p_1p_3 & p_1q_3 \\ 0 & 0 & 0 & p_1 \end{pmatrix}$$

and can be factorised as

$$A=p_1\begin{pmatrix} p_2p_3 & p_2q_3 & q_2p_3 & q_2q_3 \\ 0 & p_2 & 0 & q_2 \\ 0 & 0 & p_3 & q_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.25}$$

Equation (4.25) shows that A is a fraction of a Markov matrix. Next we have B given as,

$$B=\begin{pmatrix} q_1p_2p_3 & q_1p_2q_3 & q_1q_2p_3 & q_1q_2q_3 \\ 0 & q_1p_2 & 0 & q_1q_2 \\ 0 & 0 & q_1p_3 & q_1q_3 \\ 0 & 0 & 0 & q_1 \end{pmatrix}$$

This can also be represented as

$$B=q_1\begin{pmatrix} p_2p_3 & p_2q_3 & q_2p_3 & q_2q_3 \\ 0 & p_2 & 0 & q_2 \\ 0 & 0 & p_3 & q_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.26}$$

which is another fraction of the same Markov matrix. Similarly D is given as

$$D = \begin{pmatrix} p_2p_3 & p_2q_3 & q_2p_3 & q_2q_3 \\ 0 & p_2 & 0 & q_2 \\ 0 & 0 & p_3 & q_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.27}$$

This is Markov of the same kind as A and B but not a fraction as the previous cases.

Lastly,

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.28}$$

$\wp$ can therefore be presented as

$$\wp = \begin{bmatrix} p_1D & q_1D \\ 0 & D \end{bmatrix} \tag{4.29}$$

4.2.2: Classification of States.

Let the states in their order of presentation in equation (4.23) be labeled as E1, E2, E3, E4, E5, E6, E7, and E8. Then the state space S=: {E1, E2, E3, E4, E5, E6, E7, E8}. From equation (4.23) we notice that once you are out of the state, you can never revisit it. This correctly implies that a dead person cannot become alive. State E1 accesses E1, E2, E3, E4, E5, E6, E7, and E8. State E2 accesses E2, E4, E6, and E8. State E3 accesses E3, E4, E5, E6, E7, and E8. State E4 accesses E4, E5, E6, E7, and E8. State E5 accesses E5, E6, E7, and E8. State E6 accesses E6, E7, and E8. State E7 accesses E7, and E8, while State E8 only accesses itself.

It is notable that from any state one can access state E8. Thus the state is absorbing. You cannot move out of the state once entered. Once all the three are dead, the transaction ends. No states communicate. No reverse is possible after any of the eight probable transitions. The closed sets are {E1, E2, E3, E4, E5, E6, E7, E8}, {E2, E3, E4, E5, E6, E7, E8}, {E3, E4, E5, E6, E7, E8}, {E4, E5, E6, E7, E8}, {E5, E6, E7, E8}, {E6, E7, E8}, {E7, E8}, and {E8}. State E1 is transient.

4.2.3: n-Step Transition Probabilities and Steady State.

Higher order transition probability matrices are as follows.

$$\wp = \begin{bmatrix} p_1 D & (1-p_1)D \\ 0 & D \end{bmatrix} \tag{4.30}$$

$$\wp^2 = \begin{bmatrix} p_1^{\,2} D^2 & (1-p_1^{\,2})D^2 \\ 0 & D^2 \end{bmatrix} \tag{4.31}$$

$$\wp^3 = \begin{bmatrix} p_1^{\,3} D^3 & (1-p_1^{\,3})D^3 \\ 0 & D^3 \end{bmatrix} \tag{4.32}$$

....

$$\wp^n = \begin{bmatrix} p_1^{\,n} D^n & (1-p_1^{\,n})D^n \\ 0 & D^n \end{bmatrix} \tag{4.33}$$

where D is as given in equation (4.27),

$$D^2 = \begin{pmatrix} p_2^2 p_3^2 & p_2^2(1-p_3^2) & (1-p_2^2)p_3^2 & (1-p_2^2)(1-p_3^2) \\ 0 & p_2^2 & 0 & (1-p_2^2) \\ 0 & 0 & p_3^2 & (1-p_3^2) \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.34}$$

$$D^3 = \begin{pmatrix} p_2^3 p_3^3 & p_2^3(1-p_3^3) & (1-p_2^3)p_3^3 & (1-p_2^3)(1-p_3^3) \\ 0 & p_2^3 & 0 & (1-p_2^3) \\ 0 & 0 & p_3^3 & (1-p_3^3) \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.35}$$

Involving subscripts that represent the beneficiary's number, i.e.

$$D_{23}^n \Rightarrow D^n \, ,$$

then we generally have

$$D_{23}^n = \begin{pmatrix} p_2^n p_3^n & p_2^n(1-p_3^n) & (1-p_2^n)p_3^n & (1-p_2^n)(1-p_3^n) \\ 0 & p_2^n & 0 & (1-p_2^n) \\ 0 & 0 & p_3^n & (1-p_3^n) \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.36}$$

As an example, $\mathscr{S}^6$ is given by

$$\mathscr{S}^6 = \begin{bmatrix} p_1{}^6 D_{23}^6 & \left(1 - p_1{}^6\right) D_{23}^6 \\ 0 & D_{23}^6 \end{bmatrix} \tag{4.37}$$

where

$$D_{23}^{6} = \begin{pmatrix} p_2^6 p_3^6 & p_2^6(1-p_3^6) & (1-p_2^6)p_3^6 & (1-p_2^6)(1-p_3^6) \\ 0 & p_2^6 & 0 & (1-p_2^6) \\ 0 & 0 & p_3^6 & (1-p_3^6) \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (4.38)$$

An explicit expression for $\wp^6$ can be written but will be too large. Hence, we leave it as the combination of the two equations (4.34) and (4.35). Finally, steady state transition probabilities for the two-dependant transition probability matrix are given as

$$\wp^n = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (4.39)$$

as $n \to \infty$

4.2.4: Application to calculating ${}_2\xi_n$ and ${}_2\Pi_n$.

Proceeding in a similar manner as in section 4.1.4, we show how to calculate ${}_2\xi_n$ and consequently ${}_2\Pi_n$. Since in the quoted section we had explained the use of the first half of the first row of the (n-1)th order of transition probability matrix, we only need to apply this theory in the derivation of ${}_2\xi_n$. We have,

$$\wp^{n-1} = \begin{bmatrix} p_1^{\,n-1} D_{23}^{\,n-1} & \left(1 - p_1^{\,n-1}\right) D_{23}^{\,n-1} \\ 0 & D_{23}^{\,n-1} \end{bmatrix} \tag{4.40}$$

where

$$D_{23}^{n-1} = \begin{pmatrix} p_2^{n-1} p_3^{n-1} & p_2^{n-1}(1-p_3^{n-1}) & (1-p_2^{n-1})p_3^{n-1} & (1-p_2^{n-1})(1-p_3^{n-1}) \\ 0 & p_2^{n-1} & 0 & (1-p_2^{n-1}) \\ 0 & 0 & p_3^{n-1} & (1-p_3^{n-1}) \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.41}$$

The required elements of the matrix (equation (4.40)) are multiples of $p_1^{\,n-1}$. These are the elements in the first row of the sub-matrix $D_{23}^{\,n-1}$ given in equation (4.41). The factors to be multiplied by these required elements of the matrix are

$$\Phi_2 = q_1 D_1 + q_2 D_2 + q_3 D_3 \qquad \text{(All the three dead)}$$

$$\Phi_{1\backslash 3} = q_1 D_1 + q_2 D_2 \qquad \text{(B dead)}$$

$$\Phi_{1\backslash 2} = q_1 D_1 + q_3 D_3 \qquad \text{(M dead)}$$

and $$\Phi_{1\backslash 2,3} = q_1 D_1. \qquad \text{(F dead)}$$

Hence we calculate ${}_2\xi_n$ as

$$ {}_2\xi_n = p_1^{n-1} p_2^{n-1} p_2^{n-1} p_3^{n-1} \Phi_2 + p_1^{n-1} p_2^{n-1}(1 - p_3^{n-1})\Phi_{2\backslash 3} + p_1^{n-1}(1 - p_2^{n-1}) p_3^{n-1} \Phi_{2\backslash 2} + $$
$$ p_1^{n-1}(1 - p_2^{n-1})(1 - p_3^{n-1})\Phi_{2\backslash 2,3}. $$

This, on factorising out p_1^{n-1} gives us

$$ {}_{2}\xi_n = p_1^{n-1}\{p_2^{n-1}p_3^{n-1}\,\Phi_2 + p_2^{n-1}(1-p_3^{n-1})\,\Phi_{2\backslash 3} + p_3^{n-1}(1-p_2^{n-1}) $$

$$ *\Phi_{2\backslash 2} + (1-p_2^{n-1})*\,(1-p_3^{n-1})\,\Phi_{2\backslash 2,3}\}. $$

This was the case in equation (3.5) of Chapter three. The derivation of ${}_2\Pi_n$ is the same as in section 3.2.2 of the same chapter.

4.3: Three-dependant MARKOV MODEL.

In the Case of 3 dependants, the state space increases. There are sixteen possible events namely, the event of survival of FMBG, FMB, FMG, FM, FBG, FB, FG, F, MBG, MB, MG, M, BG, B, G and N. These form the state space.

4.3.1: Transition Probabilities Involved.

The matrix that represents the required transitions is sixteen by sixteen Markov matrix with each of its rows adding up to one. However, the columns of the transition probability matrix do not. Hence it is not doubly stochastic. As was the case with the two-dependant model, the matrix is diagonal with all of its lower diagonal elements being zeros.

The matrix for the three-dependant model is too large to completely write. However, we can simplify it as was the case with that of two dependants as shown in the following equations.

$$ \wp = \begin{bmatrix} p_1 D_{234} & (1-p_1)D_{234} \\ 0 & D_{234} \end{bmatrix} \quad (4.42) $$

where

$$D_{234} = \begin{bmatrix} p_2 D_{34} & q_2 D_{34} \\ 0 & D_{34} \end{bmatrix} \tag{4.43}$$

and

$$D_{34} = \begin{pmatrix} p_3 p_4 & p_3 q_4 & q_3 p_4 & q_3 q_4 \\ 0 & p_3 & 0 & q_3 \\ 0 & 0 & p_4 & q_4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.44}$$

4.3.2: Classification of States.

Let the states in their order of presentation in section 4.3.1 be labelled as E1, E2, E3, E4, E5, E6, E7, E8, E9, E10, E11, E12, E13, E14, E15, and E16. Then the state space S=: {E1, E2, E3, E4, E5, E6, E7, E8, E9, E10, E11, E12, E13, E14, E15, E16}.

Here again we notice that once you are out of any state, you can never revisit it. This, as before, implies that a dead person cannot become alive. State E1 accesses E1, E2, E3, E4, E5, E6, E7, E8, E9, E10, E11, E12, E13, E14, E15, and E16. State E2 accesses E2, E3, E4, E5, E6, E7, E8, E9, E10, E11, E12, E13, E14, E15, and E16. State E3 accesses E3, E4, E5, E6, E7, E8, E9, E10, E11, E12, E13, E14, E15, and E16. State E4 accesses E4, E5, E6, E7, E8, E9, E10, E11, E12, E13, E14, E15, and E16. State E5 accesses E5, E6, E7, E8, E9, E10, E11, E12, E13, E14, E15, and E16. State E6 accesses E6, E7, E8, E9, E10, E11, E12, E13, E14, E15, and E16. State E7 accesses E7, E8, E9, E10, E11, E12, E13, E14, E15, and E16. The trend proceeds as can be noticed. That is, each member accesses all super-ceding elements while State E16 only accesses itself.

We note that from any state one can access state E16. Thus the state is absorbing. You cannot move out of the state once entered. Once all the three are dead, the transaction ends. No states communicate. No reverse is possible after any of the eight probable transitions. The closed sets are {E1, E2, E3, E4, E5, E6, E7, E8, E9, E10, E11, E12, E13, E14, E15, E16}, {E2, E3, E4, E5, E6, E7, E8, E9, E10, E11, E12, E13, E14, E15, E16}, {E3, E4, E5, E6, E7, E8, E9, E10, E11, E12, E13, E14, E15, E16}, {E4, E5, E6, E7, E8, E9, E10, E11, E12, E13, E14, E15, E16}... {E15, E16} and {E16}. State E1 is transient as has always been the case.

4.3.3: n-Step Transition Probabilities and Steady State.

Higher order transition probability matrices are as follows.

$$\wp = \begin{bmatrix} p_1 D_{234} & (1-p_1)D_{234} \\ 0 & D_{234} \end{bmatrix} \quad (4.45)$$

$$\wp^2 = \begin{bmatrix} p_1^2 D_{234}^2 & (1-p_1^2)D_{234}^2 \\ 0 & D_{234}^2 \end{bmatrix} \quad (4.46)$$

$$\wp^3 = \begin{bmatrix} p_1^3 D_{234}^3 & (1-p_1^3)D_{234}^3 \\ 0 & D_{234}^3 \end{bmatrix} \quad (4.47)$$

....

$$\wp^n = \begin{bmatrix} p_1^n D_{234}^n & (1-p_1^n)D_{234}^n \\ 0 & D_{234}^n \end{bmatrix} \quad (4.48)$$

where

$$D_{234}^n = \begin{bmatrix} p_2^n D_{34}^n & (1-p_2^n)D_{34}^n \\ 0 & D_{34}^n \end{bmatrix} \quad (4.49)$$

and

$$D_{34}^{n}=\begin{pmatrix} p_3^n p_4^n & p_3^n(1-p_4^n) & (1-p_3^n)p_4^n & (1-p_3^n)(1-p_4^n) \\ 0 & p_3^n & 0 & (1-p_3^n) \\ 0 & 0 & p_4^n & (1-p_4^n) \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (4.50)$$

Steady state transition probabilities are found to be zeros for the first fifteen columns and ones for the last column.

4.3.4: Application to calculating ${}_3\xi_n$ and ${}_3\Pi_n$.

Following the trend explained in section 4.1.4 in order to derive ${}_3\xi_n$. In this case the required matrix is

$$\wp^{n-1}=\begin{bmatrix} p_1^{n-1}D_{234}^{n-1} & \left(1-p_1^{n-1}\right)D_{234}^{n-1} \\ 0 & D_{234}^{n-1} \end{bmatrix} \qquad (4.51)$$

and the first half of the first row of this matrix is p_1^{n-1} times the elements in the first row of the sub-matrix D_{234}^{n-1} which is given by,

$$D_{234}^{n-1}=\begin{bmatrix} p_2^{n-1}D_{34}^{n-1} & (1-p_2^{n-1})D_{34}^{n-1} \\ 0 & D_{34}^{n-1} \end{bmatrix} \qquad (4.52)$$

where

$$D_{34}^{n-1}=\begin{pmatrix} p_3^{n-1}p_4^{n-1} & p_3^{n-1}(1-p_4^{n-1}) & (1-p_3^{n-1})p_4^{n-1} & (1-p_3^{n-1})(1-p_4^{n-1}) \\ 0 & p_3^{n-1} & 0 & (1-p_3^{n-1}) \\ 0 & 0 & p_4^{n-1} & (1-p_4^{n-1}) \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (4.53)$$

The factors to be multiplied by this elements are Φ_3, $\Phi_{3\setminus 4}$, $\Phi_{3\setminus 3}$, $\Phi_{3\setminus 2}$, $\Phi_{3\setminus 3,4}$, $\Phi_{3\setminus 2,3}$, $\Phi_{3\setminus 2,4}$ and $\Phi_{3\setminus 2,3,4}$ respectively. Consequently, ${}_3\xi_n$ is found to be

$$\begin{aligned}{}_3\xi_n = p_1^{n-1}\{& p_2^{n-1}p_3^{n-1}p_4^{n-1}\Phi_3 + p_2^{n-1}p_3^{n-1}\left(1-p_4^{n-1}\right)\Phi_{3\setminus 4} + p_2^{n-1}p_4^{n-1}\left(1-p_3^{n-1}\right)\Phi_{3\setminus 3} + \\ & p_4^{n-1}p_3^{n-1}\left(1-p_2^{n-1}\right)\Phi_{3\setminus 2} + p_2^{n-1}(1-p_3^{n-1})\left(1-p_4^{n-1}\right)\Phi_{3\setminus 3,4} + (1-p_2^{n-1})p_3^{n-1}\left(1-p_4^{n-1}\right)\Phi_{3\setminus 2,4} \\ & p_4^{n-1}(1-p_3^{n-1})\left(1-p_2^{n-1}\right)\Phi_{3\setminus 2,3} + (1-p_4^{n-1})(1-p_3^{n-1})\left(1-p_2^{n-1}\right)\Phi_{3\setminus 2,3,4}\}. \end{aligned} \qquad (4.54)$$

As was in chapter three.

4.4: *m*-dependant MARKOV MODEL.

We now get the generalized form of the Markov model for the Benevolent Scheme. To do this, we use the trend brought out in the one-dependant, two-dependant and three-dependant models earlier derived.

4.4.1: Transition Probabilities involved.

It is more appropriate to apply the simplified Markov matrix in this case since, as we noticed earlier, the elements of the matrix can be too many (2^{m+1} times 2^{m+1}

elements) for large m. the number of rows, which are equal to the number of columns, are 2^{m+1} and are always divisible by 4. We can therefore partition each of the possible matrices of transition probabilities to 4 by 4 sub-matrices. This sub-matrices turn out to be either null (all elements being zeros) or fractions of similar Markov sub-matrices. Generally,

$$\wp = \begin{bmatrix} p_1(D_{2,3,\ldots,m+1}) & (1-p_1)(D_{2,3,\ldots,m+1}) \\ 0 & (D_{2,3,\ldots,m+1}) \end{bmatrix} \tag{4.55}$$

is the matrix of transition probabilities for the m-dependant markov matrix where

$$D_{23\,..\,m+1} =$$

$$\begin{pmatrix} p_2p_3(D_{45\,m+1}) & p_2(1-p_3)(D_{45\,m+1}) & (1-p_2)p_3(D_{45\,m+1}) & (1-p_2)(1-p_3^{n-1})(D_{45\,m+1}) \\ 0 & p_2(D_{45\,m+1}) & 0 & (1-p_2)(D_{45\,m+1}) \\ 0 & 0 & p_3(D_{45\,m+1}) & (1-p_3)(D_{45\,m+1}) \\ 0 & 0 & 0 & (D_{45\,m+1}) \end{pmatrix} \tag{4.56}$$

$$D_{45\,..\,m+1} =$$

$$\begin{pmatrix} p_4p_5(D_{67.m+1}) & p_4(1-p_5)(D_{67.m+1}) & (1-p_4)p_5(D_{67.m+1}) & (1-p_4)(1-p_5^{n-1})(D_{67.m+1}) \\ 0 & p_4(D_{67.m+1}) & 0 & (1-p_4)(D_{67.m+1}) \\ 0 & 0 & p_5(D_{67.m+1}) & (1-p_5)(D_{67.m+1}) \\ 0 & 0 & 0 & (D_{67.m+1}) \end{pmatrix} \tag{4.57}$$

And lastly,

$$D_{m,m+1} = \begin{pmatrix} p_m p_{m+1} & p_m(1-p_{m+1}) & p_{m+1}(1-p_m) & (1-p_m)(1-p_{m+1}) \\ 0 & p_m & 0 & (1-p_m) \\ 0 & 0 & p_{m+1} & (1-p_{m+1}) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(4.58)

As an example, for m=6 we have

$$P = \begin{bmatrix} p_1(D_{2,3,\ldots,7}) & (1-p_1)(D_{2,3,\ldots,7}) \\ 0 & (D_{2,3,\ldots,7}) \end{bmatrix} \tag{4.59}$$

where

$$D_{23\,..\,7} =$$

$$\begin{pmatrix} p_2p_3(D_{45.7}) & p_2(1-p_3)(D_{45.7}) & (1-p_2)p_3(D_{45.7}) & (1-p_2)(1-p_3^{n-1})(D_{45.7}) \\ 0 & p_2(D_{45.7}) & 0 & (1-p_2)(D_{45.7}) \\ 0 & 0 & p_3(D_{45.7}) & (1-p_3)(D_{45.7}) \\ 0 & 0 & 0 & (D_{45.7}) \end{pmatrix} \tag{4.60}$$

---

and

$$D_{6,7} = \begin{pmatrix} p_6p_7 & p_6(1-p_7) & p_7(1-p_6) & (1-p_6)(1-p_7) \\ 0 & p_6 & 0 & (1-p_6) \\ 0 & 0 & p_7 & (1-p_7) \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.61}$$

4.4.2: Classification of States.

Let the states in their order of presentation be labelled as E1, E2,..., and $E2^{m+1}$. Then the state space S=: {E1, E2, …, $E2^{m+1}$}. Once you are out of any of the states, you can never revisit it. This as before implies that a dead person cannot become alive. As before, each member accesses all super-ceding elements while State $E2^{m+1}$ only accesses itself.

Note that from any state one can access state $E2^{m+1}$. Thus the state is absorbing. You cannot move out of the state once entered. Once all the three are dead, the transaction ends. No states communicate. No reverse is possible after any of the 2^{m+1} probable transitions. State E1 is transient as has always been the case.

4.4.3: n-Step Transition Probabilities and Steady State.

Just as was the case in the one two and three-dependant model matrices, higher transition probability matrices for the general *m*- dependant model can be found. For $\wp$ given as in equation (4.55), we find

$$\wp^2 = \begin{bmatrix} p_1^2(D_{2,3,\ldots m+1})^2 & (1-p_1^2)(D_{2,3,\ldots m+1})^2 \\ 0 & (D_{2,3,\ldots m+1})^2 \end{bmatrix} \tag{4.62}$$

$$\wp^3 = \begin{bmatrix} p_1^3(D_{2,3,\ldots m+1})^3 & (1-p_1^3)(D_{2,3,\ldots m+1})^3 \\ 0 & (D_{2,3,\ldots m+1})^3 \end{bmatrix} \tag{4.63}$$

----

and

$$\wp^n = \begin{bmatrix} p_1^n(D_{2,3,\ldots m+1})^n & (1-p_1^n)(D_{2,3,\ldots m+1})^n \\ 0 & (D_{2,3,\ldots m+1})^n \end{bmatrix} \tag{4.64}$$

The steady state transition probabilities of the m-dependant Markov matrix are not different from the earlier derived cases of one two and three dependant matrices. The only difference is the number of elements in the stationary distribution matrix. There are 2^{m+1} columns with the 2^{m+1} th column having ones and the rest, zeros.

4.4.4: Application to calculating ${}_m\xi_n$ and ${}_m\Pi_n$.

Markov matrices have already been applied the in calculating ${}_1\xi_n$, ${}_2\xi_n$, and ${}_3\xi_n$. The following is the general Markov model for the Benevolent Scheme with m-dependants, in the n^{th} year. As before, we need

$$\wp^{n-1} = \begin{bmatrix} p_1^{n-1}(D_{2,3,\ldots,m+1})^{n-1} & (1-p_1^{n-1})(D_{2,3,\ldots,m+1})^{n-1} \\ 0 & (D_{2,3,\ldots,m+1})^{n-1} \end{bmatrix} \tag{4.65}$$

The required elements are p^{n-1} times the elements of the first row of the sub matrix $(D_{2,3,\ldots,m+1})^{n-1}$. These are multiplied by factors ranging from Φ_m to $\Phi_{m\backslash 234\ldots m+1}$. Essentially we will have 2^m such factors coinciding with 2^m required elements of the matrix in equation (4.65). Writing out the elements and multiplying them by appropriate factors gives us the equation

$$\boxed{{}_m\xi_n = p_1^{n-1}\left\{\prod_{k=2}^{m+1} p_k^{n-1}\Phi_m + \Psi_{m,n} + \Gamma_{m,n}\right\} \qquad (4.66)}$$

This is the same as equation (3.24). Profit is also found to be the same as was in chapter three, equation (3.25).

CHAPTER FIVE

5. ANALYSIS

Under this chapter, the results that were met in previous chapters are analyzed. In this chapter we also estimate the premium size that, on application, gives the insurer desirable profit. The estimated premium size is applied to a four-dependant Model (four being the average number of dependants on each contributor of the Maseno University Burial and Benevolent Fund (BBF)). Finally we suggest an estimate distribution to the Benevolent Scheme Model and derive its parameters.

5.1. GRAPHICAL ANALYSIS

In this section the trends of growth in claim amount and profit are compared. It is not necessary to compare contribution since they are the same for all models under consideration. We compare the one-dependent, two-dependant and three-dependant model, giving our observations and implications of emerging trends.

5.1.1. Comparing Claim amount.

Table (5.1) shows values of claim amount for the one, two and three-dependant models. The values have been extracted from tables (2.10), (3.2) and (3.4) in chapter two and three.

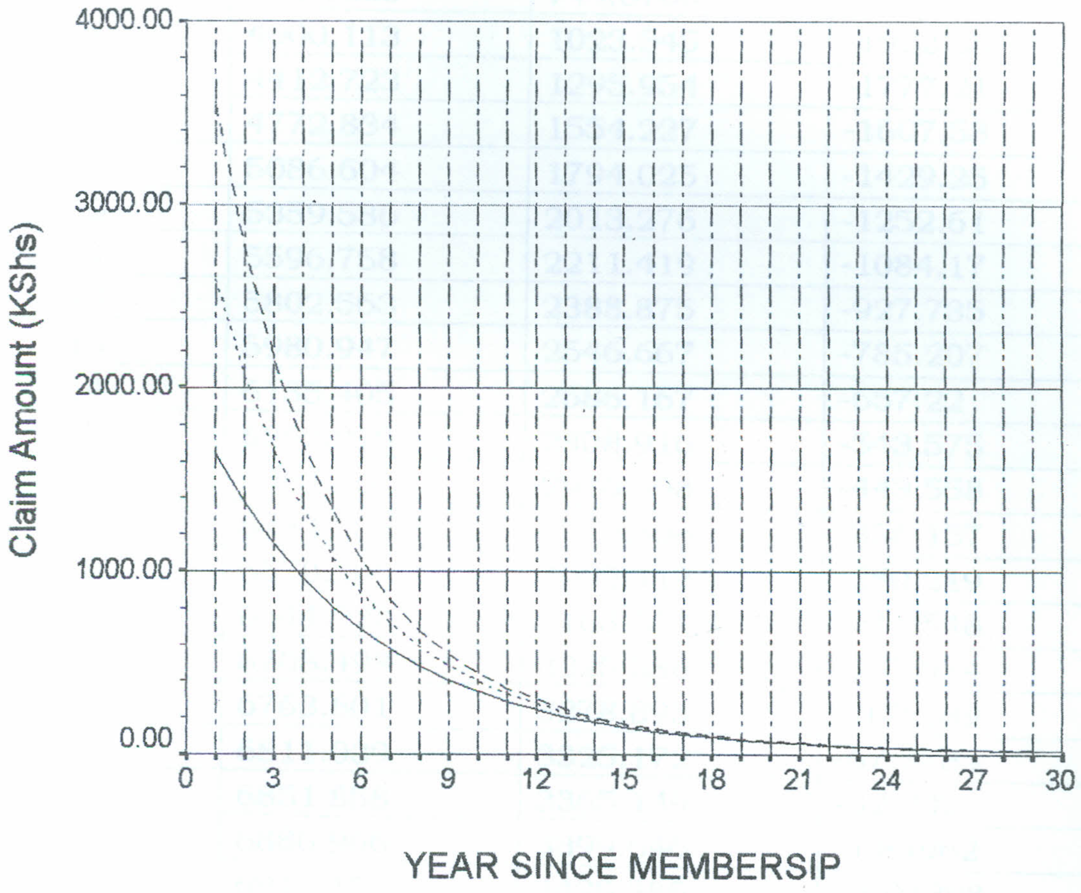
YEAR n	*CLAIM AMOUNT 1-dependant Model*	*CLAIM AMOUNT 2-dependant Model*	*CLAIM AMOUNT 3-dependant Model*
1	1636	2591	3545
2	1370	2064	2757
3	1149	1652	2153
4	963.5	1329	1689
5	808.9	1075	1331
6	679.6	872.6	1054
7	571.4	711.6	838.9
8	480.8	582.6	671.3
9	404.8	478.8	540.2
10	341.1	394.8	437.4
11	287.6	326.6	356.3
12	242.6	271	292
13	204.8	225.4	240.7
14	173	188	199.5
15	146.2	157.1	166.2
16	123.7	131.5	139.1
17	104.6	110.4	117
18	88.57	92.73	98.7
19	75.01	78.03	83.56
20	63.56	65.75	70.95
21	53.88	55.47	60.4
22	45.69	46.85	51.51
23	38.76	39.61	44.01
24	32.9	33.51	37.66
25	27.93	28.38	32.25
26	23.73	24.05	27.65
27	20.16	20.39	23.72
28	17.13	17.3	20.36
29	14.56	14.69	17.49
30	12.39	12.48	15.02

Table 5.1: A table of values of expected claim amount for one, two and three-dependant Models.

Using the data in table (5.1), the graphs in figure (5.1) are plotted. From the graphs we see that claim amount increases with increase in the number of dependants. The decrease of amounts with time is uniform since all three graphs take similar shape. It is expected that the line for the four-dependant case is right over that of the three-dependant case.

COMPARISON OF CLAIM AMOUNTS

One, Two, and Three-dependant models.



KEY:

——— Claim amount in the one-dependant model.

········ Claim amount in the two-dependant model.

– – – – Claim amount in the three-dependant model.

Figure 5.1: A graph representing the trends in claim amount for the one, two and three-dependent models.

5.1.2. Comparing Profits.

YEAR *n*	PROFIT *1-dependant Model*	PROFIT *2-dependant Model*	PROFIT *3-dependant Model*
1	891	-64	-1018
2	1679.811	31.45014	-1615.41
3	2376.359	224.3607	-1923.49
4	2989.642	471.8171	-2035.99
5	3528.223	744.6708	-2019.77
6	4000.113	1023.545	-1922.35
7	4412.723	1295.954	-1777.21
8	4772.834	1554.227	-1607.58
9	5086.604	1794.025	-1429.25
10	5359.586	2013.276	-1252.61
11	5596.758	2211.419	-1084.17
12	5802.563	2388.875	-927.735
13	5980.947	2546.667	-785.207
14	6135.405	2686.167	-657.22
15	6269.019	2808.916	-543.575
16	6384.502	2916.508	-443.558
17	6484.235	3010.508	-356.157
18	6570.303	3092.412	-280.219
19	6644.527	3163.612	-214.548
20	6708.498	3225.386	-157.974
21	6763.601	3278.893	-109.389
22	6811.039	3325.172	-67.7735
23	6851.858	3365.149	-32.202
24	6886.966	3399.646	-1.84962
25	6917.15	3429.385	24.01308
26	6943.089	3455.001	46.02517
27	6965.372	3477.05	64.74298
28	6984.508	3496.016	80.64814
29	7000.937	3512.321	94.15584
30	7015.036	3526.331	105.6227

Table 5.2: A table of values of cumulative profits for the one, two and three-dependant schemes.

COMPARISON OF CUMULATIVE PROFITS

One, Two, and Three-dependant models.

KEY:

CP1→ Cumulative profit for the one-dependant model.

CP2→ Cumulative profit for the two-dependant model.

CP3→ Cumulative profit for the three-dependant model.

Figure 5.2: A graph representing the trend in cumulative profits for the one, two and three-dependant models.

The lines in figure (5.2) elaborate the fact that the more the dependants the higher the loss than profit the insurer incurs. The average number of dependants a contributor has is four. One can therefore imagine the kind of loss the insurer is expected to undergo as far as such a contributor with four dependants is concerned.

5.2. ESTIMATION OF PREMIUM SIZE.

With the highlighted problem in section 5.1, we are faced with the task of estimating the appropriate premium size(s) that corrects abnormally large profits in the case of one and two dependants and abnormal loss in the case of three or more dependants (c.f. figure (5.2)).

5.2.1. Case of One–dependant.

From the graph in the figure (5.2), we have already seen that the insurer makes profit right from year one of contribution. This exploits the contributor. At this point, the estimate of premium size that allows for a balance between the two parties (the insurer and contributor) is considered. To do this chore, we need to know the average amount of time the contributor is expected to stay in the Scheme. This is found by averaging the length of time those who have quit the Scheme (without necessarily dying) had stayed on as contributors.

Let y be the number of years that a contributor is expected to remain as a member to the Scheme. It is desirable that the insurer starts earning profits at least by half of the period of stay of the contributor.

Thus at the $\left(\frac{y}{2}\right)^{th}$ year,

$$ {}_1\Pi_{\left(\frac{y}{2}\right)} = 0 \tag{5.1} $$

Since we have

$$ {}_1\Pi_n = P\left(1 + \frac{I}{100}\right)\sum_{k=1}^{n} V^{k-1} p_1^k - \sum_{k=1}^{n} V^k {}_1\xi_k . $$

Then letting profit be zero as equation (5.1) suggests, the following equations hold.

$$ P\left(1 + \frac{I}{100}\right)\sum_{k=1}^{n} V^{k-1} p_1^k - \sum_{k=1}^{n} V^k {}_1\xi_k = 0 $$

Hence

$$ P\left(1 + \frac{I}{100}\right)\sum_{k=1}^{n} V^{k-1} p_1^k = \sum_{k=1}^{n} V^k {}_1\xi_k \tag{5.2} $$

This implies that the estimate for premium size is

$$ \hat{P}(1) = \frac{\sum_{k=1}^{\left(\frac{y}{2}\right)} V^k {}_1\xi_k}{\left(1 + \frac{I}{100}\right)\sum_{k=1}^{\left(\frac{y}{2}\right)} V^{k-1} p_1^k} \tag{5.3} $$

here $\hat{P}$ is a function of the number of dependants, in this case one. As an example, let y =10. At year $\left(\frac{y}{2}\right) = 5$, we have

$$\hat{P}(1) = \frac{\sum_{k=1}^{5} V^{k} {}_{1}\xi_{k}}{\left(1 + \frac{I}{100}\right)\sum_{k=1}^{5} V^{k-1} p_{1}^{k}}$$

$$= \frac{5927}{1.12\sum_{k=1}^{5}(0.9091)^{k-1}(0.94)^{k}} = 1504.42$$

This amount is the premium size per annum. (Cf. Cumulative Claim amount at year 5 in table (2.10)). In a month we get the estimate to be KShs. 125.37. Hence we have $\hat{P} = 1504.42$ shillings per annum and approximately, $\hat{x} = 125$ shillings per month as the estimated premium size. This value is lower than the KShs. 200 per month currently being paid. Definitely it gives the contributor less burden and ensures reasonable profit to the insurer.

5.2.2. Case of m-dependants.

We now estimate the premium size desirable for the two-dependant model, three-dependant model and then generalize the expression to the m-dependant model.

Two-dependant model

As in section 5.2.1 we need P that minimizes ${}_2\Pi\left(\frac{y}{2}\right)$. This is found to be

$$\hat{P}(2) = \frac{\sum_{k=1}^{5} V^{k} {}_{2}\xi_{k}}{\left(1 + \frac{I}{100}\right)\sum_{k=1}^{5} V^{k-1} p_{1}^{k}} \tag{5.4}$$

We realize that the denominator is constant and is 3.939734.

$$= \frac{8711}{3.939734} = 2211.06 \qquad (5.5)$$

We have used y =10 $\Rightarrow \binom{y}{2} = 5$. The estimate of premium size for the two-dependant model is $\hat{P} = 2211.06$ shillings per annum and approximately $\hat{x} = 185$ shillings per month. This is still lesser than the KShs. 200 currently paid. With the estimated amount paid, we expect to see the required balance between the insurer's and the contributor's benefits.

Three-dependant

In this case we need P that minimizes ${}_3\Pi_n$. This is when $n = \binom{y}{2}$. This gives the estimated premium size for the three-dependant model as

$$\hat{P}(3) = \frac{\sum_{k=1}^{5} V^k {}_3\xi_k}{\left(1 + \frac{I}{100}\right) \sum_{k=1}^{5} V^{k-1} p_1^k} \qquad (5.6)$$

Once again let us assume that y =10 $\Rightarrow \binom{y}{2} = 5$ as an example. We obtain

$$= \frac{11475}{3.939734} = 2912.63 \qquad (5.7)$$

This gives the estimate of premium size for the three-dependant model as $\hat{P} = 2912.63$ shillings per annum and $\hat{x} = 245$ to the upper five shillings per month.

This is slightly higher than the KShs. 200 currently paid. We conclude that the amount paid is best suited for a three-dependant scheme. Unfortunately, the fund has an average of four dependants. This means that in case a flat rate of contributed amounts has to be set, then the figure should be slightly higher than 200. Next is to calculate the estimate of premium size for the four-dependant model for a more realistic rate.

m-dependant model

We now give a general expression for the estimate of premium size. In this case, premium size P that minimizes ${}_m\Pi_n$ is required. This is suggested to be when $n = \left(\frac{y}{2}\right)$. Thus we have

$$\hat{P}(m) = \frac{\sum_{k=1}^{\left(\frac{y}{2}\right)} V^k \, {}_m\xi_k}{\left(1 + \frac{I}{100}\right) \sum_{k=1}^{\left(\frac{y}{2}\right)} V^{k-1} p_1^k} \tag{5.8}$$

per annum. In a month, this is KShs. ($\hat{x} = \hat{P}/12$).

5.2.3. Four-dependant Model (the average case).

It has earlier been shown that the average number of children each contributor has is three. This together with the first dependant (the mother or father of the children) makes four dependants.

First we need the expression for ${}_4\xi_n$ which we use in the calculation of expected claim amount. Using the general formula for the m-dependant model (equation (3.24) in chapter three), the expected claim amount under the four-dependant model is found to be,

$$ {}_4\xi_n = p_1^{n-1}\left\{ \prod_{k=2}^{5} p_k^{n-1}\Phi_4 + \Psi_{4,n} + \Gamma_{4,n} \right\} \tag{5.9} $$

where

$$ \Phi_4 = \sum_{k=1}^{5} q_k D_k \tag{5.10} $$

$$ \Gamma_{4,n} = \left(\prod_{k=2}^{5} \left(1 - p_k^{n-1}\right)\right)\Phi_{4\backslash 2,3,\ldots 5} \tag{5.11} $$

and

$$ \Psi_{4,n} = \sum_{g=1}^{3}\left(\sum_{f_1=2}^{3}\sum_{f_2=3}^{4}\sum_{f_3=4}^{5} \left\{ \left(\prod_{k=2}^{5} p_{k\backslash \bigcup_{i=1}^{g} f_i} \right)^{n-1} * \left(\prod_{i=1}^{g}\left[1 - p_{f_i}^{n-1}\right]\right)\right\} \Phi_{4\backslash \bigcup_{i=1}^{g} f_i} \right) \tag{5.12} $$

This result to

$$ \begin{aligned} {}_4\xi_n = p_1^{n-1} \Big\{ & p_2^{n-1} p_3^{3(n-1)}\Phi_4 + p_2^{n-1}\Big(3p_3^{2(n-1)}(1-p_3^{n-1})\Phi_{4\backslash 3} + \\ & 3p_3^{(n-1)}(1-p_3^{n-1})^2\Phi_{4\backslash 3,3} + (1-p_3^{n-1})^3\Phi_{4\backslash 3,3,3}\Big) + \\ & p_3^{(n-1)}(1-p_2^{n-1})\Big(p_3^{2(n-1)}\Phi_{4\backslash 2} + 3p_3^{(n-1)}(1-p_3^{n-1})\Phi_{4\backslash 2,3} + \\ & 3(1-p_3^{n-1})^2\Phi_{4\backslash 2,3,3}\Big) + (1-p_2^{n-1})(1-p_2^{n-1})^3\Phi_{4\backslash 2,3,3,3}\Big\}. \end{aligned} \tag{5.13} $$

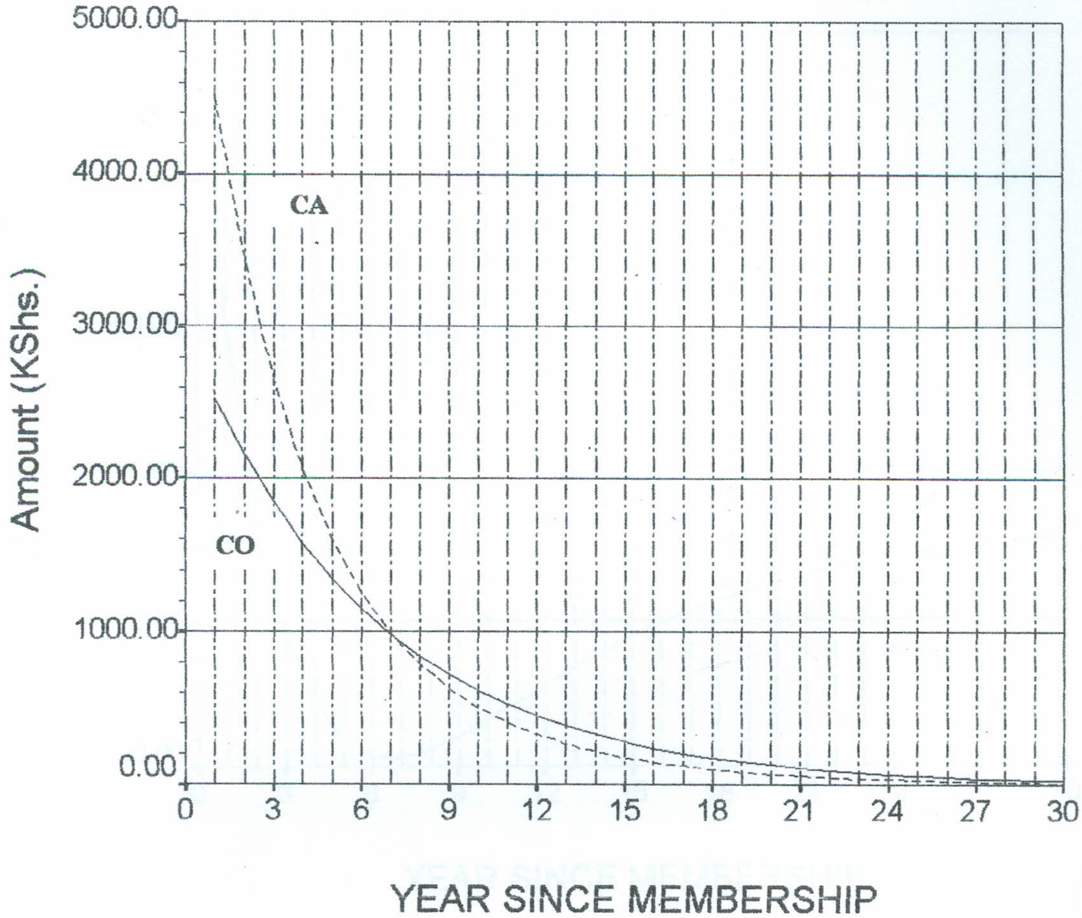
To calculate the values of ${}_4\xi_n$, for n= 1, 2, 3, 4, 5, …, 30, we use the Ms-Excel application package. Table (5.3) shows the resultant values.

Year n	*p1 power (n)*	*Expected claim amount*	*Discounted claim amount*	*Cumulative claim amount*	*Expected contribution (discounted)*	*Discounted cumulative contribution*	*Growth in profit*
1	0.94	4950	4500	4500	2527	2527	-1973
2	0.884	4175.01	3450	7950	2159	4686	-3264.27
3	0.831	3539.73	2660	10610	1845	6531	-4078.63
4	0.781	3017.38	2061	12671	1577	8108	-4562.83
5	0.734	2586.45	1606	14277	1347	9456	-4821.44
6	0.69	2229.61	1259	15536	1151	10607	-4928.59
7	0.648	1932.97	992	16528	984	11591	-4936.58
8	0.61	1685.33	786.3	17314	840.9	12432	-4881.99
9	0.573	1477.65	626.7	17941	718.6	13151	-4790.13
10	0.539	1302.64	502.3	18443	614.1	13765	-4678.34
11	0.506	1154.43	404.7	18848	524.7	14289	-4558.26
12	0.476	1028.24	327.7	19175	448.4	14738	-4437.5
13	0.447	920.25	266.6	19442	383.2	15121	-4320.89
14	0.421	827.3	217.9	19660	327.5	15448	-4211.31
15	0.395	746.87	178.8	19839	279.8	15728	-4110.29
16	0.372	676.87	147.3	19986	239.1	15967	-4018.48
17	0.349	615.64	121.8	20108	204.4	16172	-3935.95
18	0.328	561.77	101.1	20209	174.6	16346	-3862.37
19	0.309	514.15	84.08	20293	149.2	16496	-3797.22
20	0.29	471.83	70.15	20363	127.5	16623	-3739.84
21	0.273	434.04	58.66	20422	109	16732	-3689.52
22	0.256	400.16	49.17	20471	93.13	16825	-3645.56
23	0.241	369.64	41.29	20512	79.58	16905	-3607.27
24	0.227	342.05	34.74	20547	68.01	16973	-3574
25	0.213	317.02	29.27	20576	58.12	17031	-3545.15
26	0.2	294.23	24.69	20601	49.66	17081	-3520.18
27	0.188	273.41	20.86	20622	42.44	17123	-3498.6
28	0.177	254.35	17.64	20639	36.27	17159	-3479.97
29	0.166	236.85	14.94	20654	30.99	17190	-3463.91
30	0.156	220.76	12.66	20667	26.49	17217	-3450.08

Table 5.3. A table of expected claim amount contribution and profit for the four-dependant model.

EXPECTED CONTRIBUTION COMPARED WITH COMPENSATION CLAIM AMOUNT

Four-dependant Model.



KEY

CA ➔ *Claim Amount.*

CO ➔ *Contribution*

Figure 5.3a. A graph showing the trend in contribution and claim amount for the four-dependant model.

GROWTH IN PROFIT

Four-dependant Model.

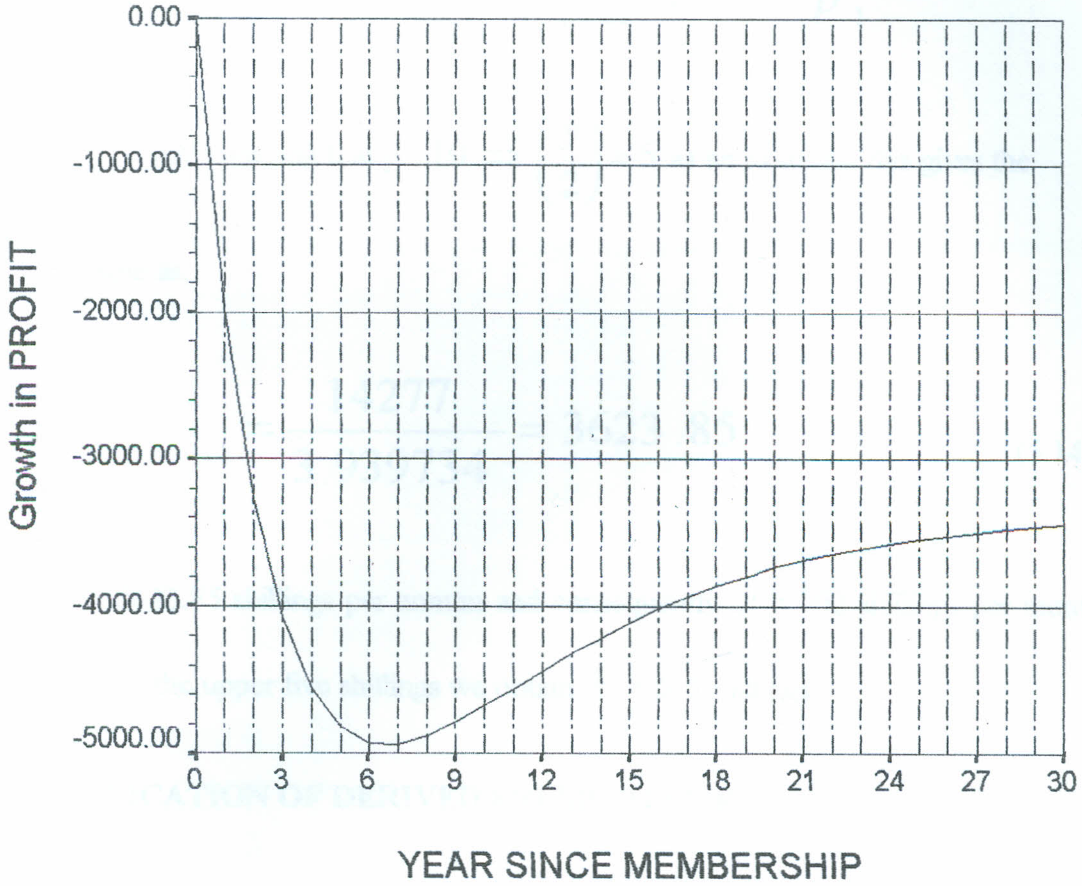


Figure 5.3b. A graph giving the trend of growth in profit for the four-dependant model.

The graphs in figure (5.3a) and (5.3b) take the same trend and have similar explanations as the equivalents of the one, two and three dependant models earlier given.

From table (5.3), the required claim amount, from which the estimate of premium size $\hat{P}$ and hence $\hat{x}$ is to be calculated, is found to be

$$\hat{P}(4) = \frac{\sum_{k=1}^{5} V^{k}\,{}_{4}\xi_{k}}{\left(1 + \frac{I}{100}\right)\sum_{k=1}^{5} V^{k-1}\,{}_{1}p^{k}}.$$

Still assuming that y =10 $\Rightarrow \binom{y}{2} = 5$ as an example, this gives the estimate as,

$$= \frac{14277}{3.939734} = 3623.85 \tag{5.14}$$

So that $\hat{P} = 3623.85$ shillings per annum and consequently $\hat{x} = 302$ shillings per month. Rounding up to the upper five shillings we obtain $\hat{x} = 305$ shillings.

5.3 APPLICATION OF DERIVED PREMIUM SIZE,

The estimate premiums of sizes were found to be:

$\hat{P}(1) = 1505$ (For the one-dependant model).

$\hat{P}(2) = 2211$ (For the two-dependant model).

$\hat{P}(3) = 2913$ (For the three-dependant model).

$\hat{P}(4) = 3624$ (For the four-dependant model).

5.3.1. One-dependant Model.

Once again we use Ms-Excel application package to calculate the following values in table (5.4). This is on replacement of the uniform premium sizes with the estimated value. Graphs of growth in contribution and claim amount, for each of the models are also plotted.

Year n	*Expected claim amount*	*Discounted claim amount*	*cumulative claim amount*	*Expected contribution (discounted)*	*Discounted cumulative contribution*	*Profit*	*Growth in profit*
1	1800	1636	1636	1584	1584	-52.526	-52
2	1658	1370	3006	1353	2937	-16.919	-68.9193
3	1529	1149	4155	1157	4094	8.00755	-60.9118
4	1411	963.5	5119	988.4	5083	24.8887	-36.0231
5	1303	808.9	5927	844.6	5927	35.7652	-0.25777
6	1204	679.6	6607	721.8	6649	42.2078	41.95003
7	1113	571.4	7178	616.8	7266	45.4227	87.37274
8	1030	480.8	7659	527.1	7793	46.3299	133.7027
9	954.4	404.8	8064	450.4	8243	45.6271	179.3298
10	884.6	341.1	8405	384.9	8628	43.8391	223.169

Table 5.4: Cumulative contribution, claim amount and profit using the estimated premium size (one-dependant model).

Values in table (5.4) have been used to plot the lines in figure (5.4). In figure (5.4), the dotted line shows the exponential rate of decay of expected claim amounts while the full line represents the trend of decrease in contribution with time. The lines meet between the second and third year of contribution. This is the time that the rate of contribution equals that of claims.

ESTIMATED PREMIUM SIZE

One-dependant Model.

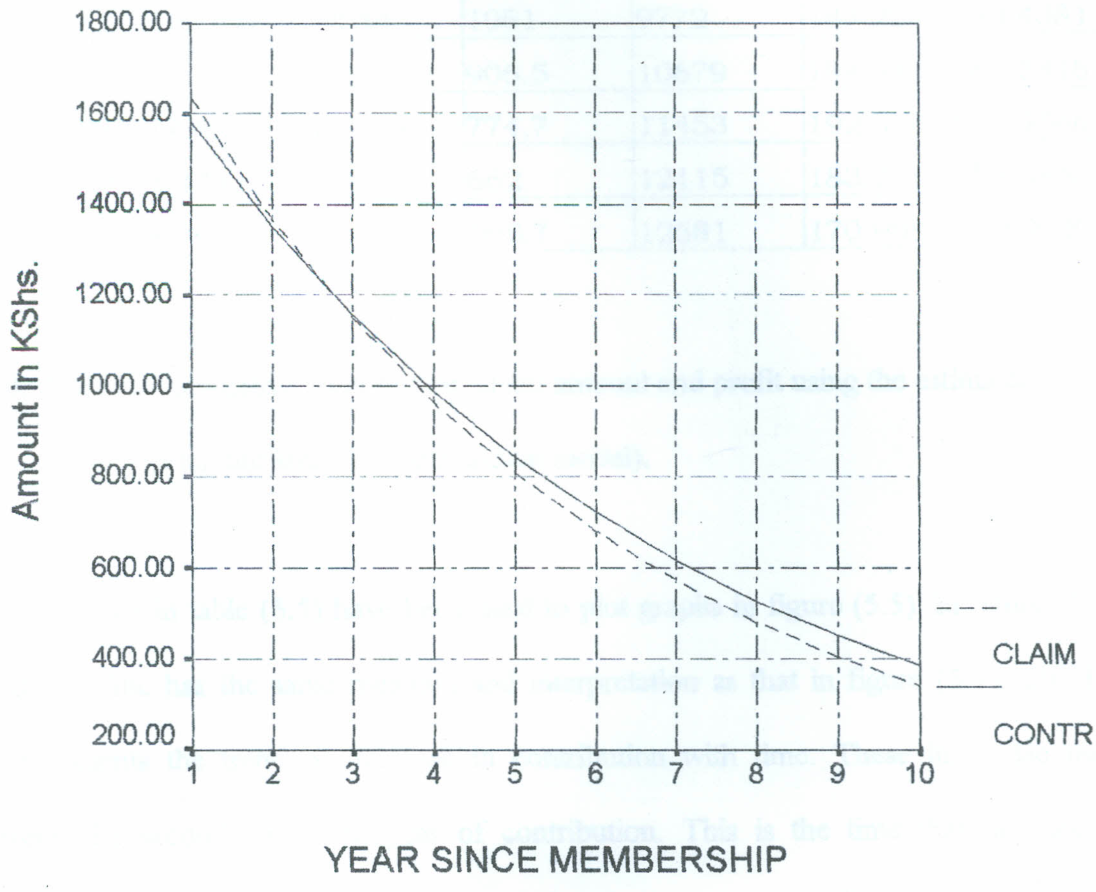


Figure 5.4: A graph showing growth in contribution, claim amount, and profits for the one-dependant model (using estimated premium size).

5.3.2. Two-dependant model.

Year n	*Expected claim amount*	*Discounted claim amount*	*cumulative claim amount*	*Expected contribution (discounted)*	*Discounted cumulative contribution*	*Profit*	*Growth in profit*
1	2850	2591	2591	2328	2328	-263.131	-263
2	2497.11	2064	4655	1989	4317	-74.5344	-337.534
3	2199.09	1652	6307	1700	6017	47.6496	-289.885
4	1946.21	1329	7636	1453	7470	123.3231	-166.562
5	1730.57	1075	8711	1241	8711	166.7751	0.213485
6	1545.77	872.6	9584	1061	9772	188.2246	188.4381
7	1386.59	711.6	10295	906.5	10679	194.9435	383.3816
8	1248.77	582.6	10878	774.7	11453	192.075	575.4566
9	1128.83	478.8	11357	662	12115	183.228	758.6846
10	1023.94	394.8	11751	565.7	12681	170.9083	929.5929

Table 5.5: Cumulative contribution, claim amount and profit using the estimated premium size (Two-dependant model).

Values in table (5.5) have been used to plot graphs in figure (5.5). In figure (5.5), the dotted line has the same meaning and interpretation as that in figure (5.4). The full line represents the trend of decrease in contribution with time. These lines also meet between the second and third year of contribution. This is the time that the rate of contribution equals that of claims. We note that the two are more dispersed than was the case in figure (5.4). This implies that the amounts diverge more as the number of dependants increase.

ESTIMATED PREMIUM SIZE

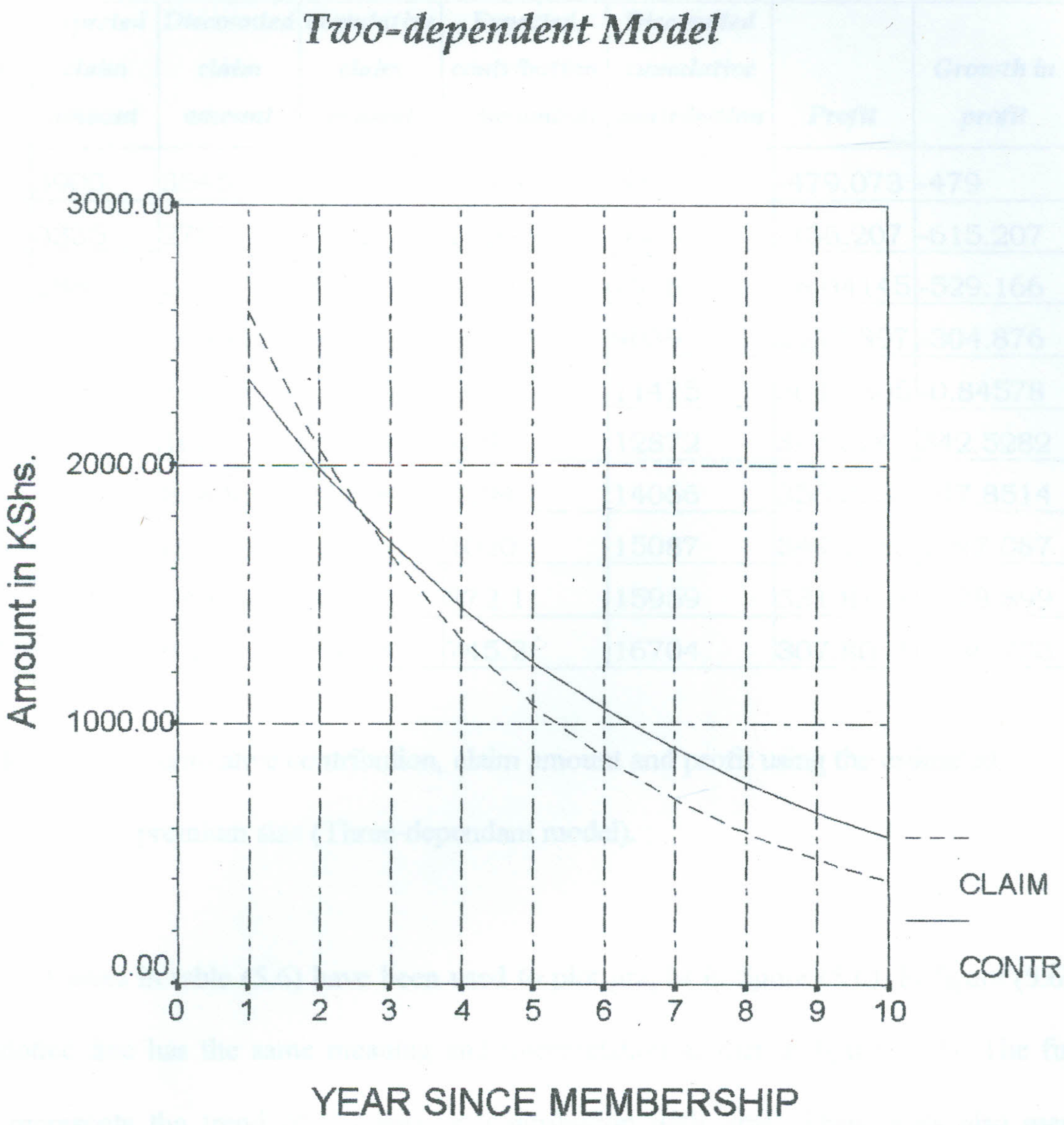


Figure 5.5: A graph showing growth in contribution and claim amount, for the two-dependant model (using estimated premium size).

5.3.3. Three -dependant Model.

Year n	*Expected claim amount*	*Discounted claim amount*	*cumulative claim amount*	*Expected contribution (discounted)*	*Discounted cumulative contribution*	*Profit*	*Growth in profit*
1	3900	3545	3545	3066	3066	-479.073	-479
2	3335	2757	6302	2620	5686	-136.207	-615.207
3	2866	2153	8455	2239	7926	86.04145	-529.166
4	2473	1689	10144	1914	9839	224.2897	-304.876
5	2144	1331	11475	1635	11475	304.0305	-0.84578
6	1867	1054	12529	1397	12872	343.374	342.5282
7	1635	838.9	13368	1194	14066	355.3232	697.8514
8	1439	671.3	14040	1020	15087	349.2353	1047.087
9	1274	540.2	14580	872.1	15959	331.8121	1378.899
10	1134	437.4	15017	745.2	16704	307.8041	1686.703

Table 5.6: Cumulative contribution, claim amount and profit using the estimated premium size (Three-dependant model).

Values in table (5.6) have been used to plot graphs in figure (5.6). In figure (5.6), the dotted line has the same meaning and interpretation as that in figure (5.5). The full line represents the trend of decrease in contribution with time. These lines also meet between the second and third year of contribution. This is the time that the rate of contribution equals that of claims. We note that the two are more dispersed than was the case in figure (5.4) and (5.5). This implies that the amounts diverge more as the number of dependants increase.

ESTIMATED PREMIUM SIZE

Three-dependant Model.

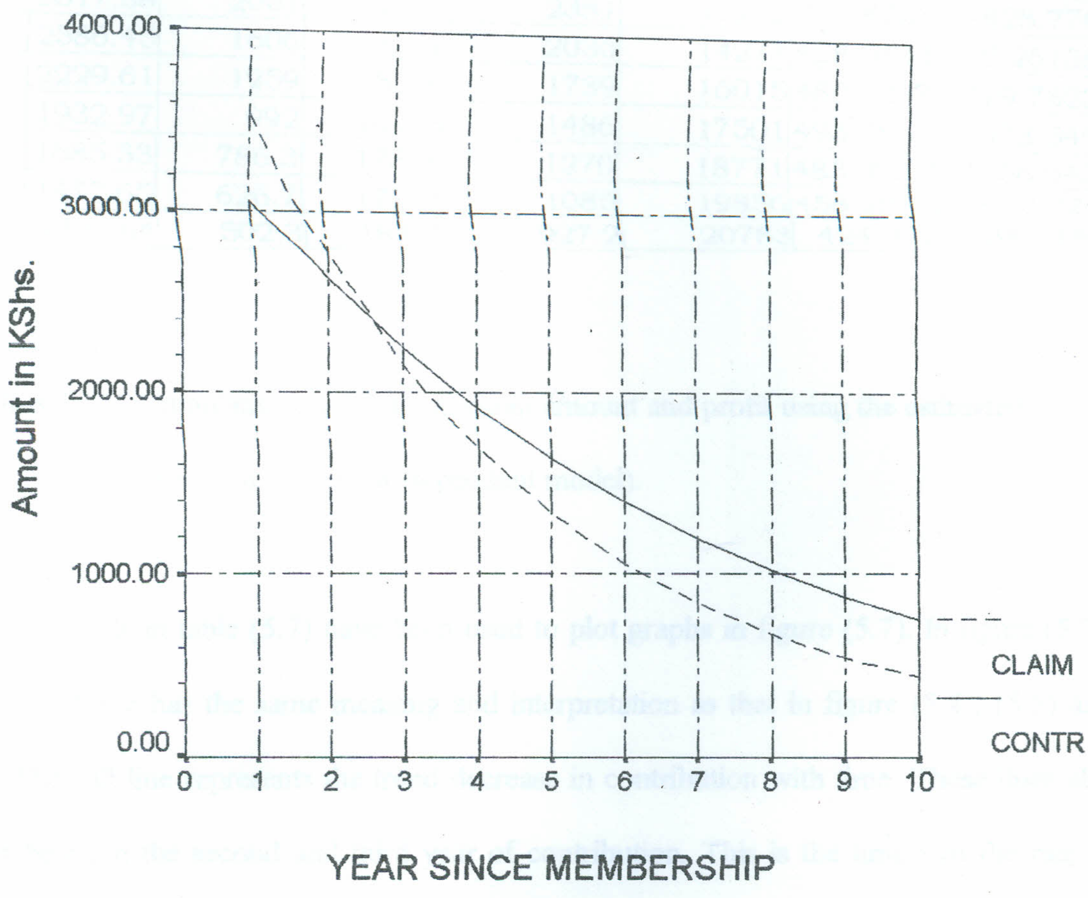


Figure 5.6: A graph showing growth in contribution and claim amount, for the three-dependant model (using estimated premium size).

5.3.4. Four-dependant Model

Year *n*	*Expected claim amount*	*Discounted claim amount*	*cumulative claim amount*	*Expected contribution (discounted)*	*Discounted cumulative contribution*	*Profit*	*Growth in profit*
1	4950	4500	4500	3815	3815	-684.856	-685
2	4175.01	3450	7950	3260	7075	-190.205	-875.205
3	3539.73	2660	10610	2786	9861	126.5585	-748.647
4	3017.38	2061	12671	2381	12242	319.8706	-428.776
5	2586.45	1606	14277	2035	14277	428.5148	-0.26138
6	2229.61	1259	15536	1739	16015	480.0237	479.7623
7	1932.97	992	16528	1486	17501	493.7867	973.549
8	1685.33	786.3	17314	1270	18771	483.3935	1456.943
9	1477.65	626.7	17941	1085	19856	458.2819	1915.224
10	1302.64	502.3	18443	927.2	20783	424.922	2340.146

Table 5.7: Cumulative contribution, claim amount and profit using the estimated premium size (Four-dependant model).

Values in table (5.7) have been used to plot graphs in figure (5.7). In figure (5.7), the dotted line has the same meaning and interpretation as that in figure (5.4), (5.5) and 5.6. The full line represents the trend decrease in contribution with time. These lines also meet between the second and third year of contribution. This is the time that the rate of contribution equals that of claims. Note that the two are more dispersed than was the case in figure (5.6). This confirms that the amounts diverge more as the number of dependants increase.

ESTIMATED PREMIUM SIZE

Four-dependant Model.

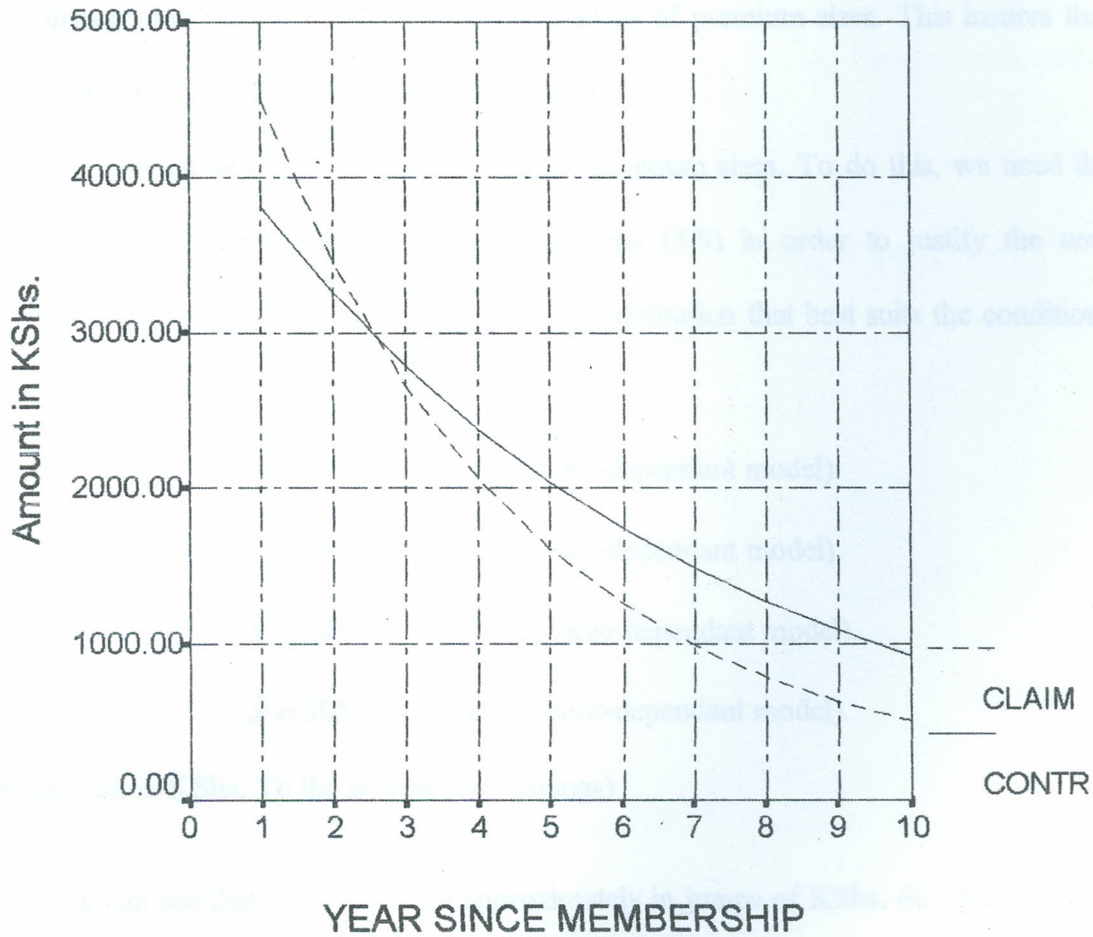


Figure 5.7: **A graph showing growth in contribution and claim amount, for the four-dependant model (using estimated premium size).**

5.3.5. Observation.

It comes out clearly that the more the dependants, the more disperse the two lines of each of the four grids. This implies that there will be increased profit to the insurer as a contributor has more dependants. This is the reverse of the uncorrected situation where the more the dependants the larger the loss the insurer expects.

In order to precisely have a balance between the contribution and profit expected by the insurer, the insurer ought to set several levels of premium sizes. This insures that he achieves both his goals and those of the investor.

We now study the profit due to the new premium sizes. To do this, we need the data in table (5.8) and graphs in figure (5.8) and (5.9) in order to justify the new modification. We suggest the mode of premium contribution that best suits the conditions assumed (that is, y=10) to be as follows.

$\hat{x} = 125$ (For the one-dependant model).

$\hat{x} = 185$ (For the two-dependant model).

$\hat{x} = 245$ (For the three-dependant model).

$\hat{x} = 305$ (For the four-dependant model).

(Amounts are in KShs. To the nearest five shillings)

We can see that the figures are approximately in jumps of KShs. 60. We therefore suggest that a member of the Benevolent Scheme contributes 125 shillings per month if he has only the spouse as the dependant. In addition to that, he ought to give Ksh.60 for every other dependant (Child) he wishes to register as a beneficiary to the Scheme. This is definitely non-arbitrary since the values have been clearly derived using our m-dependant model of the Benevolent Scheme. The profit trends are illustrated next.

Year n	*Profit 1-dep. model*	*Profit 2-dep. model*	*Profit 3-dep. model*	*Profit 4-dep. model*
1	-52.526	-263.131	-479.073	-684.856
2	-16.919	-74.5344	-136.207	-190.205
3	8.00755	47.6496	86.04145	126.5585
4	24.8887	123.3231	224.2897	319.8706
5	35.7652	166.7751	304.0305	428.5148
6	42.2078	188.2246	343.374	480.0237
7	45.4227	194.9435	355.3232	493.7867
8	46.3299	192.075	349.2353	483.3935
9	45.6271	183.228	331.8121	458.2819
10	43.8391	170.9083	307.8041	424.922

Table 5.8. A table of profit for the estimated premium sizes in the one, two, three, and four-dependant models.

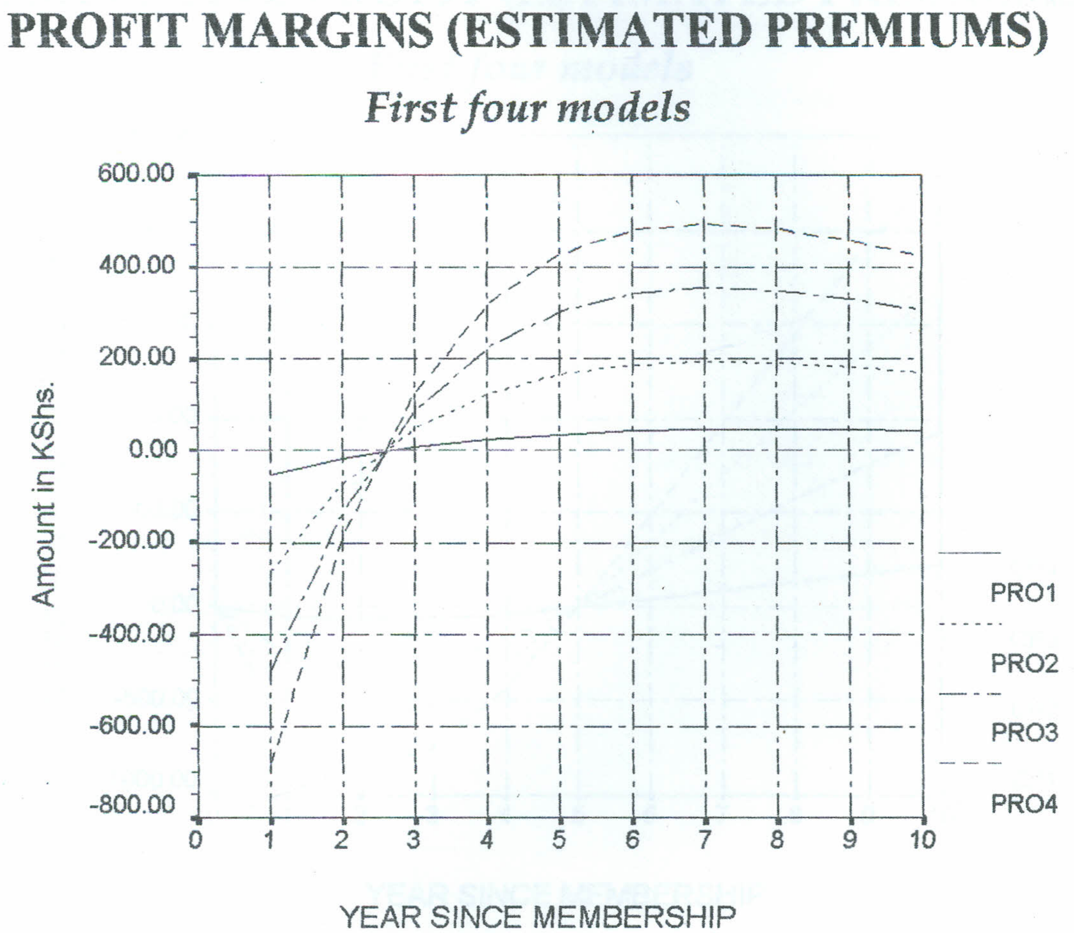


Figure 5.8. A graph showing the new profit margins on using the estimated premium sizes. Cases are the one, two, three, and four-dependant models.

Year n	Cumulative Profit 1-dep. model	Cumulative Profit 2-dep. model	Cumulative Profit 3-dep. model	Cumulative Profit 4-dep. model
1	-52	-263	-479	-685
2	-68.9193	-337.534	-615.207	-875.205
3	-60.9118	-289.885	-529.166	-748.647
4	-36.0231	-166.562	-304.876	-428.776
5	-0.25777	0.213485	-0.84578	-0.26138
6	41.95003	188.4381	342.5282	479.7623
7	87.37274	383.3816	697.8514	973.549
8	133.7027	575.4566	1047.087	1456.943
9	179.3298	758.6846	1378.899	1915.224
10	223.169	929.5929	1686.703	2340.146

Table 5.9. A table of Cumulative profit for the estimated premium sizes in the one, two, three, and four-dependant models.

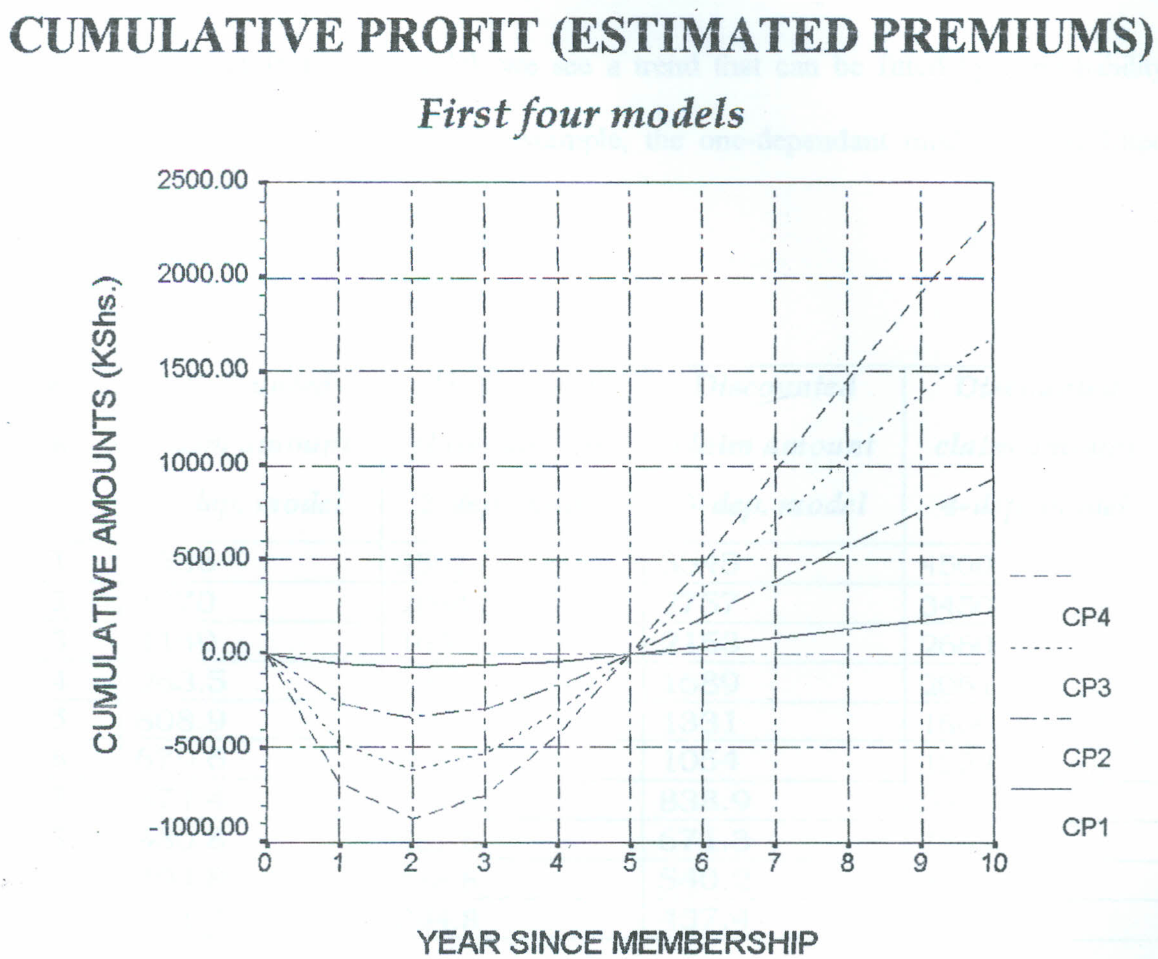


Figure 5.9. A graph showing the new cumulative profit (estimated premium sizes). Cases are the one, two, three, and four-dependant models.

KEY:

PRO1→ Cumulative profit for the one-dependant model.

PRO2→ Cumulative profit for the two-dependant model.

PRO3→ Cumulative profit for the three-dependant model.

PRO4→ Cumulative profit for the four-dependant model.

CP1→ Cumulative profit for the one-dependant model.

CP2→ Cumulative profit for the two-dependant model.

CP3→ Cumulative profit for the three-dependant model.

CP4→ Cumulative profit for the four-dependant model.

5.4. SUGGESTED APPROXIMATE MODEL FOR THE SCHEME

As eminent from our model, we see a trend that can be fitted by a probability distribution or a model function. For example, the one-dependant model can be fitted with a CUBIC function as follows.

Year *n*	*Discounted claim amount 1-dep. model*	*Discounted claim amount 2-dep. model*	*Discounted claim amount 3-dep. model*	*Discounted claim amount 4-dep. model*
1	1636	2591	3545	4500
2	1370	2064	2757	3450
3	1149	1652	2153	2660
4	963.5	1329	1689	2061
5	808.9	1075	1331	1606
6	679.6	872.6	1054	1259
7	571.4	711.6	838.9	992
8	480.8	582.6	671.3	786.3
9	404.8	478.8	540.2	626.7
10	341.1	394.8	437.4	502.3

Table 5.10. A table of expected claim amount for the first four models.

Curve estimation

Using SPSS Application package, the regression equations are as follows:

MODEL: MOD_1. (THE CUBIC MODEL)

Independent: YEAR

Dependent	Mth	Rsq	Sigf	b0	b1	b2	b3
CLAIM1	CUB	1.000	.000	1940.71	-329.93	24.3241	-.7341
CLAIM2	CUB	1.000	.000	3205.91	-680.82	61.3786	-2.1465
CLAIM3	CUB	1.000	.000	4469.93	-1030.1	97.4510	-3.4863
CLAIM4	CUB	1.000	.000	5733.23	-1380.6	135.130	-4.9524

In the graphs that follow, dotted lines represent the expected amounts while the continuous lines represent the fitted model.

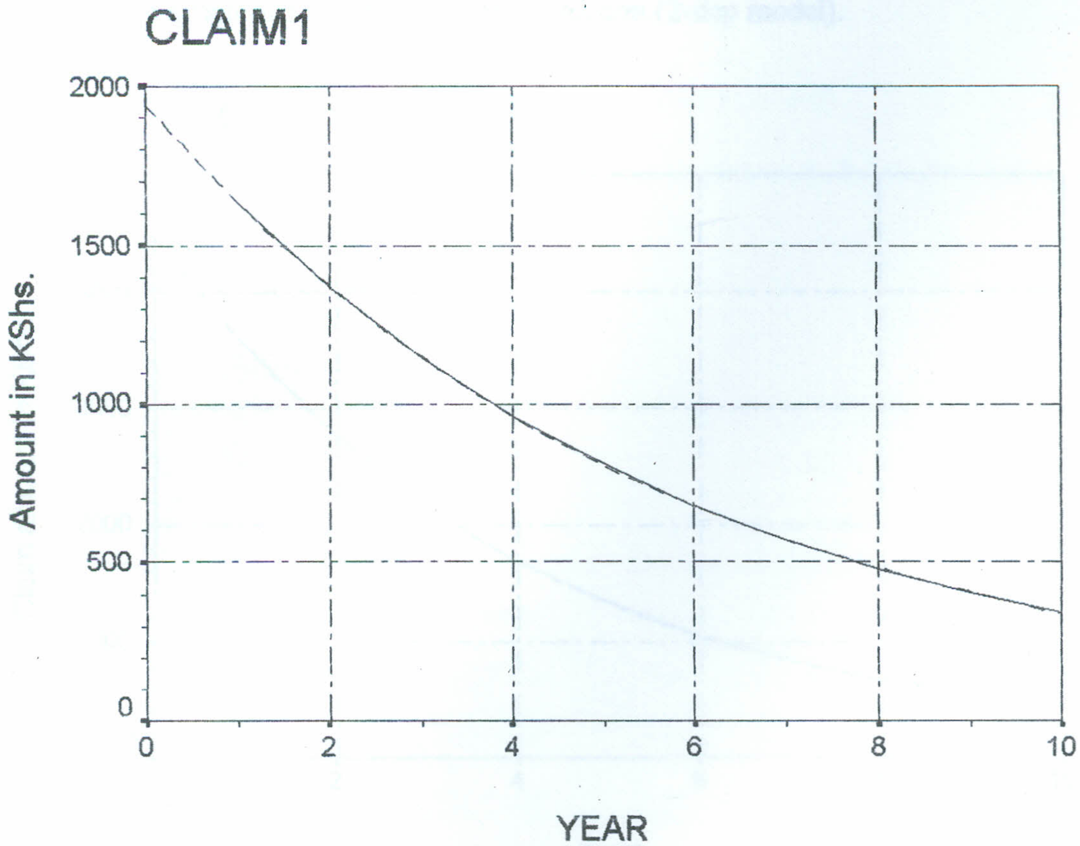


Figure 5.10a. A graph of fit with the CUBIC function (one-dependant model).

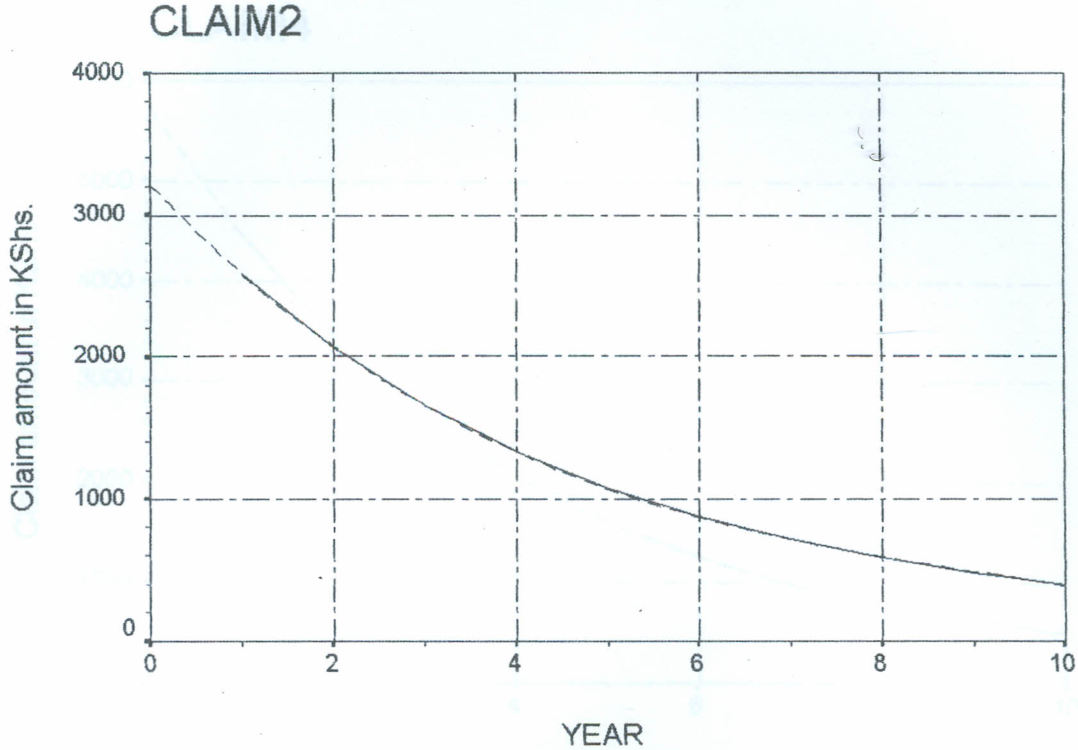


Figure 5.10b. A graph of fit with the CUBIC function (2-dep model).

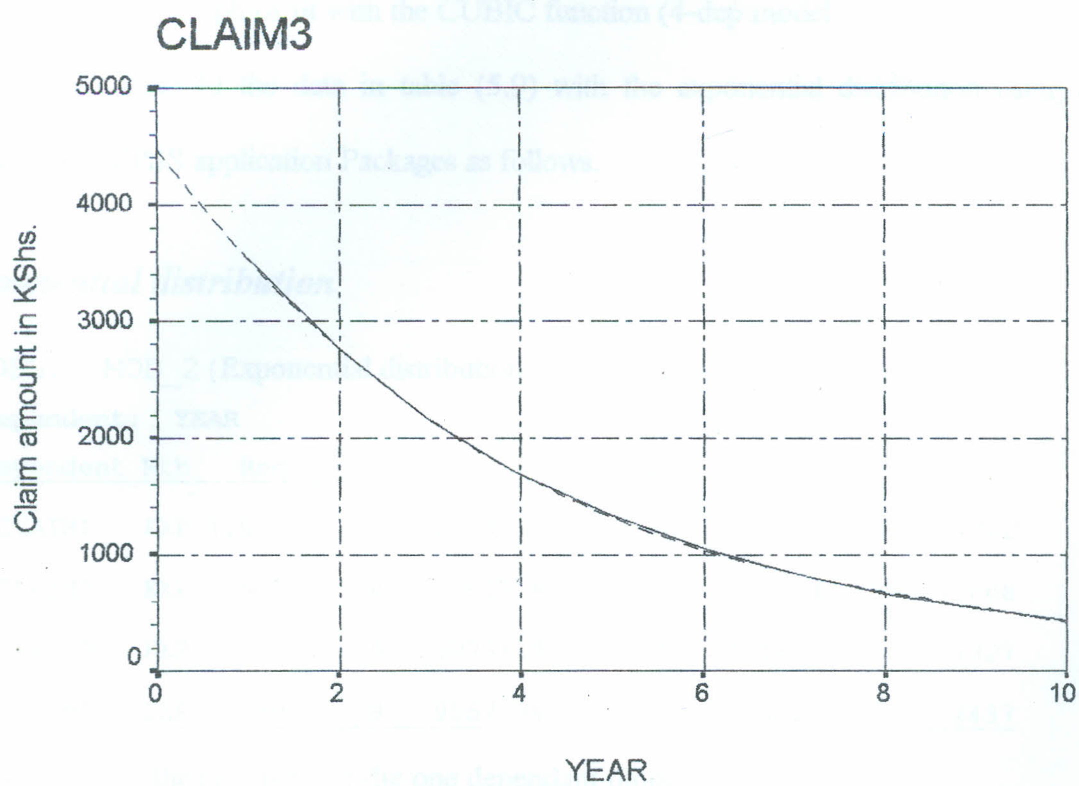


Figure 5.10c. A graph of fit with the CUBIC function (3-dep model).

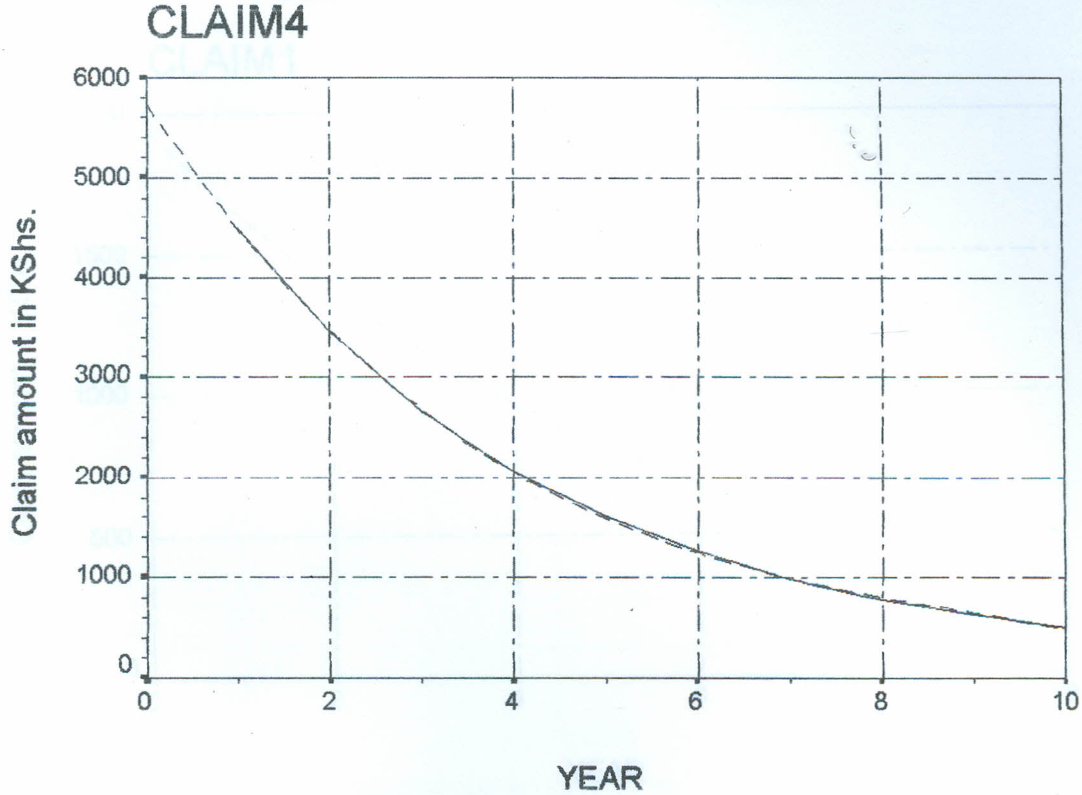


Figure 5.10d. A graph of fit with the CUBIC function (4-dep model).

Next is to fit the data in table (5.9) with the exponential distribution using the Minitab and SPSS application Packages as follows.

Exponential distribution.

MODEL: MOD_2 (Exponential distribution) .

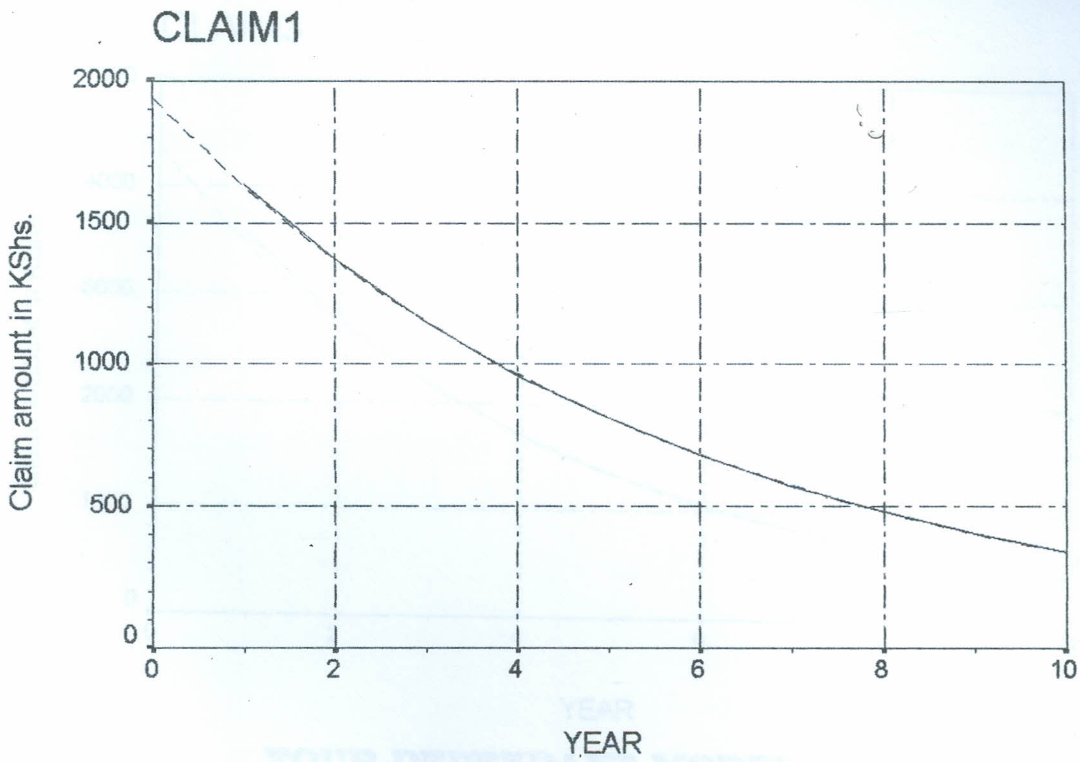
Independent: YEAR

Dependent	Mth	Rsq	d.f.	F	Sigf	b0	b1
CLAIM1	EXP	1.000	8	273411	.000	1938.53	-.1742
CLAIM2	EXP	.999	8	11507.6	.000	3107.32	-.2088
CLAIM3	EXP	.999	8	10794.8	.000	4346.09	-.2327
CLAIM4	EXP	.999	8	9557.04	.000	5553.97	-.2437

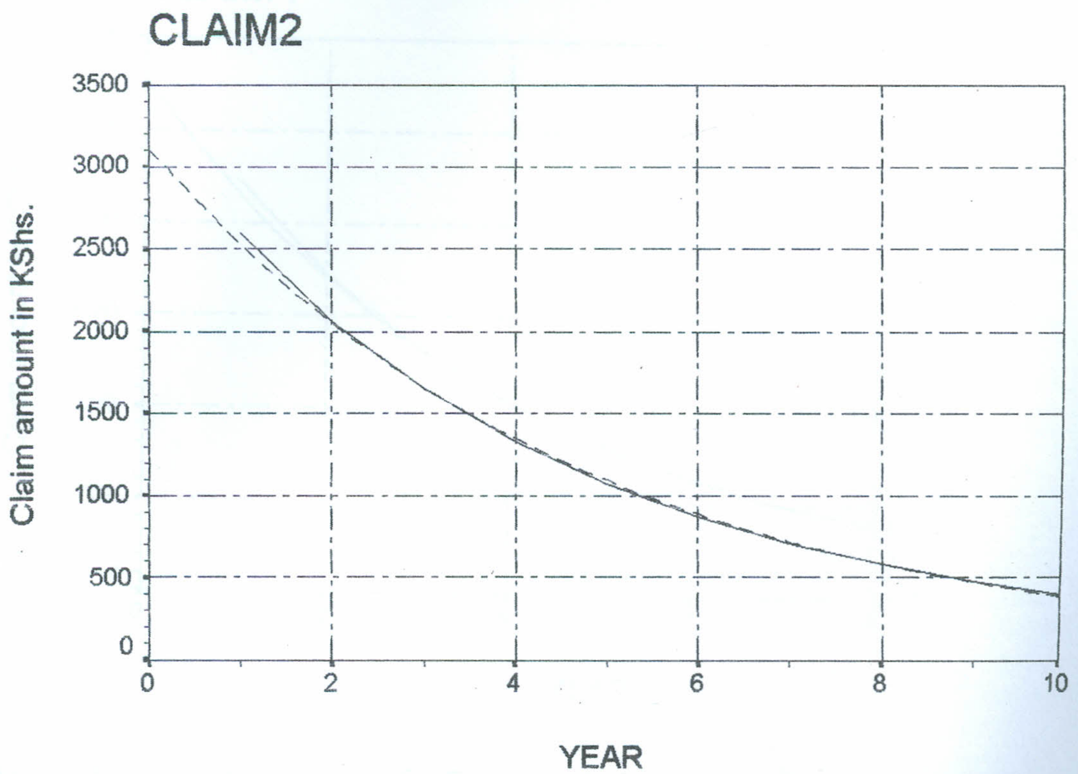
As an example, the equation for the one dependant model is:

$$\textbf{CLAIM1} = \textbf{1938.53*exp (-0.1743*year)}$$

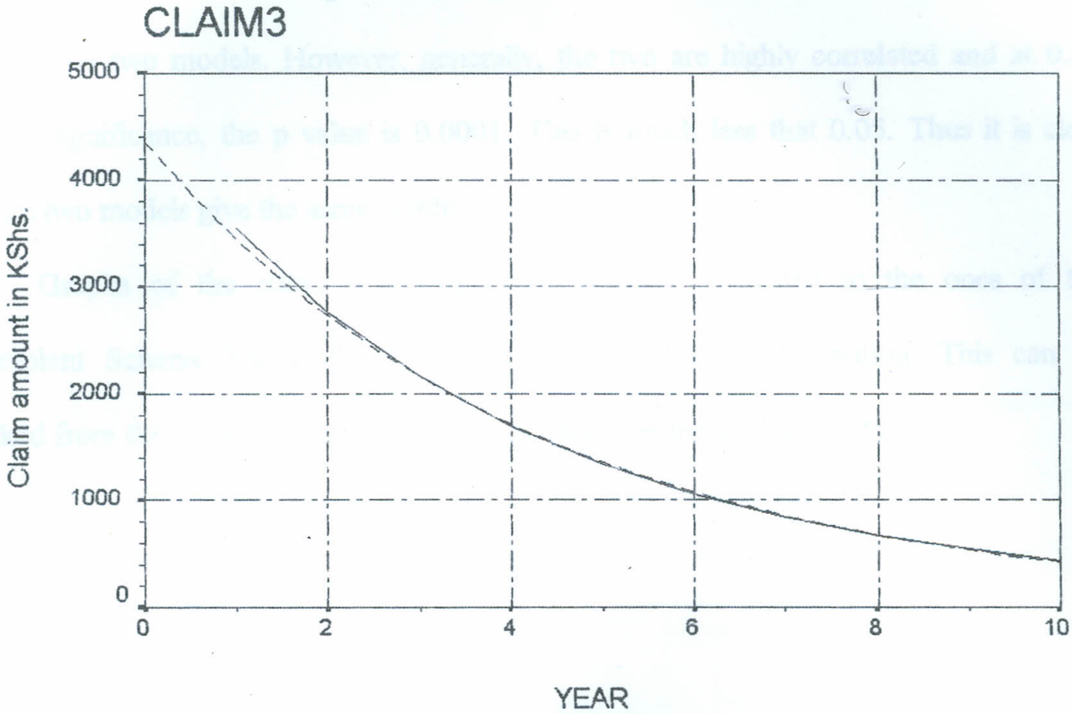
ONE-DEPENDANT MODEL



TWO-DEPENDANT MODEL



THREE-DEPENDANT MODEL



FOUR-DEPENDANT MODEL

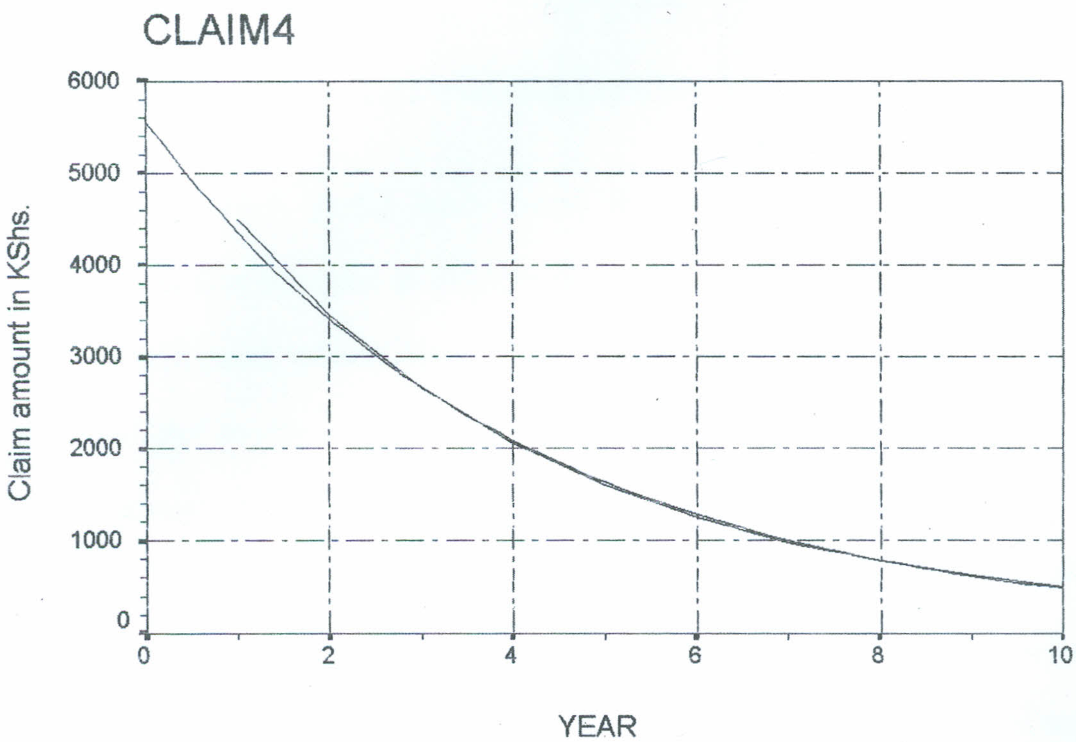


Figure 5.11. Graphs of fit of the expected Claim with the Exponential distribution.

We note that the larger the number of dependants, the lower the correlation between the two models. However, generally, the two are highly correlated and at 0.01 level of significance, the p value is 0.0001. This is much less that 0.05. Thus it is clear that the two models give the same results.

Graphs of the cubic model are more closely correlated to the ones of the Benevolent Scheme Model than those of the Exponential distribution. This can be checked from the Square of regression column (Rsq.) in the analysis tables.

CHAPTER SIX.

6: CONCLUSION

In this chapter, we summarize the results that have been arrived at throughout the whole research. First we compare the Branching probabilistic model with the Markov model for the Benevolent Scheme. Other results as the estimated premium sizes for different numbers of dependants will also be recorded. Suggestions for further modifications that can be made to the models are given. Finally, a conclusion in line with stated objectives will be made.

6.1: COMPARISON OF THE TWO METHODS.

The Branching (probabilistic) model for the benevolent scheme has been developed from first principles in chapters two and three. The calculations involved are seen to be junky and tedious although they are straightforward. However the markov model formulated in chapter four has proved to be short and easier to apply despite the fact that it is more technical.

All in all, it is interesting to note that both models give the same result. That is, the formula for calculation of expected claim amount, arrived at in both cases, is the same. It is also plausible that both methods predict the same about the future of a member of the scheme. Essentially, as time $n \rightarrow \infty$ the probability of survival and hence the membership diminishes to zero. In the Markov model, such a phenomenon is called

Steady state or the stationary distribution of the of the markov process. In reality, no one can stay alive forever. After some time, the contributor is bound to quit due to death, dismissal or retirement.

6.2: SUMMARY OF RESULTS.

The following were our objectives of the study that we now wish to claim to have fulfilled.

i. Formulating a Statistical model that will be used in calculating the claim amount and profit the insurer expects to earn from the benevolent scheme.

ii. Estimate the appropriate premium size to be contributed in order for the insurer to realize modest profit.

iii. Applying the model by use of data from the Maseno University SACCO in order to forecast the financial status of the cooperative (as an example).

After going through all necessary resources and personal inputs, we came up with the following.

i. The one dependant model for the benevolent scheme was derived as

$$_1\xi_n = p_1^{n-1}\{p_2^{n-1}\Phi_1 + (1-p_2^{n-1})\Phi_{1\backslash 2}\}.$$

The two-dependant model is

$$_2\xi_n = p_1^{n-1}\{p_2^{n-1}p_3^{n-1}\Phi_2 + p_2^{n-1}(1-p_3^{n-1})\Phi_{2\backslash 3} + p_3^{n-1}(1-p_2^{n-1})\Phi_{2\backslash 2} + (1-p_2^{n-1})*(1-p_3^{n-1})\Phi_{2\backslash 2,3}\}.$$

The three-dependant model is

$$
{}_3\xi_n = p_1^{n-1}\{p_2^{n-1}p_3^{n-1}p_4^{n-1}\Phi_3 + p_2^{n-1}p_3^{n-1}\left(1-p_4^{n-1}\right)\Phi_{3\backslash 4} + p_2^{n-1}p_4^{n-1}\left(1-p_3^{n-1}\right)\Phi_{3\backslash 3} +
$$

$$
p_4^{n-1}p_3^{n-1}\left(1-p_2^{n-1}\right)\Phi_{3\backslash 2} + p_2^{n-1}(1-p_3^{n-1})\left(1-p_4^{n-1}\right)\Phi_{3\backslash 3,4} + (1-p_2^{n-1})p_3^{n-1}\left(1-p_4^{n-1}\right)\Phi_{3\backslash 2,4}
$$

$$
p_4^{n-1}(1-p_3^{n-1})\left(1-p_2^{n-1}\right)\Phi_{3\backslash 2,3} + (1-p_4^{n-1})(1-p_3^{n-1})\left(1-p_2^{n-1}\right)\Phi_{3\backslash 2,3,4}\}.
$$

The four-dependant model is

$$
{}_4\xi_n = p_1^{n-1}\{p_2^{n-1}p_3^{3(n-1)}\Phi_4 + p_2^{n-1}\Big(3p_3^{2(n-1)}(1-p_3^{n-1})\Phi_{4\backslash 3} +
$$

$$
3p_3^{(n-1)}(1-p_3^{n-1})^2\Phi_{4\backslash 3,3} + (1-p_3^{n-1})^3\Phi_{4\backslash 3,3,3}\Big) +
$$

$$
p_3^{(n-1)}(1-p_2^{n-1})\Big(p_3^{2(n-1)}\Phi_{4\backslash 2} + 3p_3^{(n-1)}(1-p_3^{n-1})\Phi_{4\backslash 2,3} +
$$

$$
3(1-p_3^{n-1})^2\Phi_{4\backslash 2,3,3}\Big) + (1-p_2^{n-1})(1-p_2^{n-1})^3\Phi_{4\backslash 2,3,3,3}\}.
$$

and more generally, the m-dependant model: The expected amount to be spent in compensation to a contributor with m dependants is

$$
{}_m\xi_n = p_1^{n-1}\left\{\prod_{k=2}^{m+1} p_k^{n-1}\Phi_m + \Psi_{m,n} + \Gamma_{m,n}\right\}
$$

Where

$$
\Gamma_{m,n} = \left(\prod_{k=2}^{m+1}\left(1-p_k^{n-1}\right)\right)\Phi_{m\backslash 2,3,\ldots,m+1}
$$

$$\Psi_{m,n} = \sum_{g=1}^{m-1} \left\{ \sum_{f_1=2}^{3} \sum_{f_2}^{4} \cdots \sum_{f_3}^{m} \sum_{f_g=m}^{m+1} \left[\left(\prod_{k=2}^{m+1} p_{k \setminus \bigcup_{i=1}^{g} f_i} \right)^{n-1} \left(\prod_{i=1}^{g} \left[1 - p_{f_i}^{n-1} \right] \right) \right] \Phi_{m \setminus \bigcup_{i=1}^{g} f_i} \right\}$$

and

$$\Phi_{m \setminus a,b,\ldots,J} = \sum_{j=1}^{m+1} q_j D_j - \left(\sum_{\forall i \in \{a,b,\ldots,J\}} (q_i D_i) \right)$$

The expected profit is

$${}_m\Pi_n = P\left(1 + \frac{I}{100}\right) \sum_{k=1}^{n} V^{k-1} p_1^k - \sum_{k=1}^{n} V^k {}_m\xi_k .$$

$$= \sum_{k=1}^{n} V^k \left\{ \frac{P\left(1 + \frac{I}{100}\right)}{V} p_1^k - {}_m\xi_k \right\}.$$

These results have been successfully arrived at by both the Branching method and the Markov approach.

ii. Using collected data, the average number of years a contributor is expected to stay as a member of the scheme (of Maseno University SACCO in this case) was ten years. This together with the assumption that the insurer ought to start earning profit midway the stay of the contributor, led to our estimate for premium size given as

$$\hat{P}(m) = \frac{\sum_{k=1}^{\left(\frac{y}{2}\right)} V^{k} {}_{m}\xi_{k}}{\left(1 + \frac{I}{100}\right)\sum_{k=1}^{\left(\frac{y}{2}\right)} V^{k-1} p_{1}^{k}}$$

Per annum.

iii. From collected data and suggested formulae, we found that $p_1=p_2=0.94$ (The probability of survival of the contributor and spouse) $p_3=p_4 \ldots =0.85$ (The probability of survival of the contributor's siblings) and $\bar{c} = 3$ is the average number of children per contributor. The trend as $n \rightarrow \infty$ was that the claim amount, ${}_{m}\xi_{n} \rightarrow 0 \ \forall \ m$ where m=1, 2, 3, …, 30.

The Exponential distribution is found to be the most suitable probability distribution function that best matches the Benevolent Scheme Model.

In chapter two, the one-dependant model was formulated. Data from the Maseno University Burial and Benevolent Fund (BBF), which is a sub-sector of the SACCO, was applied and resultant tables and graphs made. From the results, expected cooperative claim amount was found to be decreasing with time. It was found that, under the current system of contribution, and compensation, a contributor with only one dependant (the spouse) or two dependants, is at a disadvantage in terms of benefits from the scheme.

Under the three-dependant model, the insurer could only expect profit from the twenty-fourth year, while in the case of four dependants, the insurer could not expect any profit at all, within the first thirty years of contribution. It is natural that most of the members to the scheme could be having more than three dependants. This implies that the insurer is expected to run out of funds to manage the program if there were no other source other than member contributions.

Using our formula for the estimated premium size, we got results that were proved to be better than if current values were charged. We therefore strongly suggest that the premium size shouldn't be constant for all family sizes. The estimated premium size for a contributor with one dependant should be KShs.125. Any additional beneficiary should be accommodated by an increase on the premium size by KShs. 60 for each. With this implemented, we expect favorable insurer-insured co-existence. That is, the application of the estimated premium size will lead to improvement of services to members.

6.3: SUGGESTIONS AND CONCLUSSION.

With the summary in section 6.2, we can claim to have fully succeeded in satisfying our objectives. However, there are various assumptions that led to our results. This to some extent could impact negatively on the application of our model. So, more work can be done to eliminate some of the assumptions.

We have suggested the various premium sizes to be paid by contributors with different numbers of dependants. This may complicate the system of contribution and compensation due to the diverse properties of each contributor. Therefore there is need to estimate a general premium size for each member regardless the number of dependants.

We have dealt with a case where a pure death process is involved. New in-births registrations, departures (not by death) and other irregular occurrences have not been taken care of. This leaves room for further research and modifications to the model.

This statistical model that we have formulated may not only be applied by the financial institutions which incorporate the Benevolent Scheme, but also other sectors under the field of insurance. With some modification, the application can be extended to engineering where reliability of systems of machines and repair at some cost is required.

REFERENCES

Aitkin M. and Clayton D (1980) "The fitting of exponential, Weibull and extreme value Distributions in complex censored survival data using GLIM" *Applied statistics.* 29 156-163.

Biger, N and Y. Kahane, (1978), "Risk consideration in insurance ratemaking", *Journal of Risk and Insurance, 45: 121-132.*

Clements, R.R., (1982), "The development of methodologies of mathematical modelling", *Teaching mathematics applications, 1,125.*

Cooper, R.W., (1974), "Investment Return and Property-Liability Insurance Ratemaking." *Philadelphia: S.S. Heubner foundation, University of Pennsylvania.*

Cox, D.R., (1972b) "Regression Models in life tables" *(with discussion) Journal of Royal Statistical Society,* Series B 34 187-220.

Cummins, J.D. and L. Chang (1983) "An analysis of the New Jersey formula for including investment income in Property liability insurance Ratemaking." *Journal of Insurance Regulation 1, 555-573.*

Davis, M.H.A. (1993), "Markov Models and Optimisation." Chapman and Hall, London.

Fairley, W., (1979), "Investment Income and Profit Margins in Property Liability Insurance." *Bell Journal of Economics, 10: 192-210.*

Feller W. (1968), "An introduction to Probability theory and its Applications" vol.1, John Wiley and sons inc. USA.

Ferrari, J.R., (1968), "A note on the basic relationship of underwriting, investments, leverage and exposure of total return on owners' equity." *Proceedings of the casualty actuarial Society, 55: 295-302.*

Gnedenko B. and I. Ushakov (1995) "Probabilistic reliability engineering", John Wiley and sons inc. USA.

Grimmet G.R. and D.R. Stirzaker (1992) " Probability and Random Processes" Claredon, England.

Gross A.J. and Clark V.A., (1975) "Survival distributions": *Reliability applications in the biomedical Sciences.* Wiley.

Henri, L. (1991), "Risk information and insurance." *Essay in memory of Karl H. Borch.* Kluwer Academic Publishers, U.S.A.

Hill, R., (1979), "Profit Regulation in Property Liability Insurance." *Bell Journal of Economics, 10: 172-191.*

Kalbfleisch , J.D., and Prentice R .L., (1980), *The Statistical analysis of failure time data.* Wiley.

Murray, A. et al. (1990), "Statistical Modelling in GLIM." *Oxford Science Publications.* OUP Inc. New York.

Myers, S. and R. Cohn, (1987), "Insurance Rate Regulation and the Capital Asset Prising Model" *Fair Rate of Return in Property liability Insurance.* Kluwer Academic Publishers, U.S.A.

National Council On Compensation Insurance (NCCI), (1987), "The impact of the Tax Reform Act of 1986 on Property liability Insurance Profits." *The Council,* New York.

Pratt, J.W., (1964), "Risk aversion in the small and in the large." *Econometrica, 32: (1-2): 122-136.*

Stirzaker D.R. (1994), "Elementary probability " Cambridge University Press, *Cambridge.*