(p,q)-SUMMING MULTIPLIERS.

By

OGIK, WYCLIFE RAO

A thesis submitted in partial fulfillment of the requirements for the degree of

Master of science

In

Pure Mathematics

Department of Mathematics
Maseno University
Kenya

September 2003

ACKNOWLEDGEMENT

I would like to acknowledge the invaluable support I have received from my supervisors, Dr. Shem Aywa and Prof. John Ogonji. They have been able supervisors and always available whenever I needed them. Their constant encouragement through the course work and research kept me moving on.

Thanks to Prof. Omolo Ongati for his invaluable pieces of advice concerning this research.

I thank the ministry of education, which through the Teachers Service Commission facilitated my studies by granting me study leave. Thanks to the HELB for the loan they awarded me. This enabled me to finance my postgraduate programme.

Next, I would like to express my gratitude to my wife Betty, for the contribution she has made towards the success of this work. Her encouragement, occasional financial hardship both her and the children had to go through and a conducive family environment they provided made it possible for me to plough through the rigour of postgraduate studies.

Finally, glory be to God who by His grace, favour and wisdom has enabled me go through this work.

ABSTRACT

Summing multipliers is an important class of operators in the geometric theory of general Banach spaces. They are particularly useful in the study of the structure of the classical spaces. The work done by Grothendieck and Pietsch provides a good basis for the study of this class of operators. The topic of this study is (p, q)-summing multipliers. These are sequences of bounded linear operators mapping weakly p-summable sequences into strongly q-summable sequences. This study is concerned with using the concepts of absolute and p-summing multipliers to characterize the space of all (p, q)-summing multipliers. In particular we show that the space of all (p, q)-summing multipliers is complete. This is accomplished through a detailed study of the concepts of the summing operators and absolute and p-summing multipliers.

CHAPTER 1

INTRODUCTION

Summing multipliers is an important class of operators in the geometric theory of general Banach spaces. They are particularly useful in the study of the structure of the classical spaces. The work done by Grothendieck and Pietsch provides a good basis for the study of this class of operators.

In this study we present concepts and results that are used in our work in chapter two. Some acquaintance with linear algebra and basic concepts of functional analysis is assumed. In chapter three we present a discussion on summing operators and summing multipliers. This chapter also contains our proposition on the (p,q)-summing multipliers with a critical analysis of the same. Chapter four is a summary outlining our contribution.

1.1 LITERATURE REVIEW

Up to the late 1960's, the available knowledge in the area of vector lattices of linear operators between Banach lattices was still fragmentary and incoherent. This knowledge was therefore of little importance to the mainstream of operator theory.

The roots of the theory of p-summing operators lie in the work undertaken by Alexandre Grothendieck from the 1950s. However, it was only in 1967 that Albrecht Pietsch clearly isolated this class of operators and established many of their fundamental theorems. Actually the classes of 1-summing and 2-summing operators were studied before in Grothendieck's Resume [8]. Whereas he was in possession of the 1-summing norm in the first case, the norm, which he attributed to the 2-summing operators, is only equivalent to 2-summing norm. This came to light through Pietsch's Domination theorem. Indeed the depth of Pietsch's contribution comes in large part from his isolation of the simple finitary defining inequality governing an operator's inclusion in the class of p-summing operators [12]. Within a year, Pietsch's work gained recognition as was evident by the appearance of seminal paper by Joram Lindenstrauss and Aleksander Pelczynski [10].

Fourie [6] introduced the concept of absolutely summing multiplier of a Banach space as follows. A sequence $(u_j) \in B(X)$ is called an absolutely summing multiplier of X if $(u_j x_j)$ is absolutely summable in X whenever (x_j) is weakly absolutely summable in X; hence, in notation $(u_j x_j) \in l_1(X)$ for all $(x_j) \in l_1^w(X)$. Aywa [1] has shown that if X is an infinite dimensional Banach space, then the space of all absolutely summing multipliers on X, $l_{\Pi_1}(X)$, is an l_p-space.

1.2 STATEMENT OF THE PROBLEM

From the overview, it is clear that a lot of work has been done on summing operators. Diestel, Jarchow and Tonge [3] have studied Absolutely Summing Operators. Fourie and Aywa [7] have done work on Absolutely p-summing multipliers and their applications. In this study we look at many of the meaningful generalizations of the concepts of p-summing multipliers and apply them to characterize (p,q)-summing multipliers. In particular we attempt to show that the space of all (p,q)-summing multipliers is complete.

1.3 OBJECTIVES OF THE STUDY

The purpose of this study is to accomplish the following:

(i) Study the concepts of p-summing multipliers and apply these generalizations to characterize (p,q)-summing multipliers.

(ii) Investigate whether the space of all (p,q)-summing multipliers is a Banach space under the operator norm of the (p,q)-summing multiplier norm.

1.4 SIGNIFICANCE OF THE STUDY

It is hoped that with the accomplishment of the above objectives, more avenues will be paved for other mathematicians to pursue further the study of (p,q)-summing multipliers from where we would have stopped.

CHAPTER 2

BASIC CONCEPTS

In this chapter we present definitions and concepts that are used in our work in chapter three. These concepts are found in most functional analysis texts, however where a particular approach is preferred, the source is indicated. Concepts of convergence of sequences including Cauchy sequences and convergence of sequences in a Banach space are presented in section 2.1. In section 2.2 we give examples of some of the sequence spaces that are used in our work. In section 2.3 we present the concepts of summability of sequences. The idea of the space of bounded linear operators, which is shown to be Banach, is the content of section 2.4. The concept of the dual of l_p spaces is presented in section 2.5.

2.1 CONVERGENCE OF SEQUENCES

2.1.1 Definition: Convergence of a sequence

A sequence (x_n) in a normed space X is said to converge or to be convergent if there is an $x \in X$ such that

$$\lim_{n \to \infty} \|x_n - x\| = 0.$$

x is called the limit of (x_n) and we write

$$\lim_{n \to \infty} x_n = x$$

2.1.2 Definition: Cauchy sequence

A sequence (x_n) in a normed space X is said to be Cauchy (or fundamental) if for every $\varepsilon > 0$, there is an $N = N(\varepsilon)$ such that

$$\|x_n - x_m\| < \varepsilon \qquad \text{for every } m, n > N$$

The space X is said to be complete if every Cauchy sequence in X converges in X. A complete normed space is called a Banach space.

Consider any normed space X and define a metric on it by $d(x, y) = \|x - y\|$ for all $x, y \in X$. We shall denote the norm topology of X by $\tau(X)$ which is to say that $\tau(X)$ is a class of all open sets of X as given by the metric d i.e. $G \in \tau(X)$ implies that for all $a \in G$ there exists an $r > 0$ such that

$$S(a, r) = \{x \in X : d(a, x) < r\} \subset G.$$

We say that a sequence (x_n) in a normed space X is strongly convergent to an element x if and only if $x_n \to x$ as $n \to \infty$ in the normed topology $\tau(X)$. That is to say that

$$\lim_{n \to \infty} \|x_n - x\| = 0.$$

This is the convergence defined above (definition 2.1.1) and in this case we say that x is a strong limit of the sequence (x_n). Strong convergence can also be written as $x_n \xrightarrow{s} x$ as $n \to \infty$.

2.1.3 Definition

Let X be a normed space. Then the set of all bounded linear functionals on X constitutes a normed space with the norm defined by

$$\|x^*\| = \sup\left\{\frac{|x^*(x)|}{\|x\|} : x \in X, x \neq 0\right\}$$

$$\|x^*\| = \sup\left\{|x^*(x)| : x \in X, \|x\| = 1\right\}$$

which is called the dual space of X and is denoted by X^*.

For a given normed space *X*, we introduce a new topology called weak topology on *X* determined by X^*. For any $a \in X$ and functionals $x_1^*, x_2^*, \ldots, x_n^* \in X^*$ and $\varepsilon > 0$, define the following subset of *X*;

$$U(a, x_1^*, x_2^*, \ldots, x_n^*, \varepsilon) = \left\{x \in X : \sup_{1 \le k \le n} |x_k^*(x-a)| < \varepsilon\right\}.$$

Note that $a \in U(a, x_1^*, x_2^*, \ldots, x_n^*, \varepsilon)$ so that in some sense *U* is a neighbourhood of *a*. We refer to *U* as a weak neighbourhood of *a*.

Let *G* be a nonempty subset of *X*. Then we say that *G* is weakly open if and only if for every $a \in G$, there exist $x_1^*, x_2^*, \ldots, x_n^* \in X^*$ and $\varepsilon > 0$ such that

$$U(a, x_1^*, x_2^*, \ldots, x_n^*, \varepsilon) \subset G.$$

By $\sigma(X, X^*)$ we denote the class of all weakly open sets together with the empty set. Then $\sigma(X, X^*)$ is a topology on *X* and we call it the weak topology on *X* determined by X^*. Weak convergence is defined in terms of linear functionals on *X* as follows.

2.1.4 Definition: Weak convergence

A sequence (x_n) in a normed space X is said to be weakly convergent if there is an $x \in X$ such that for every $x^* \in X^*$,

$$\lim_{n\to\infty} x^*(x_n) = x^*(x).$$

This is written

$$x_n \xrightarrow{w} x.$$

The element x is called the weak limit of (x_n) and we say that (x_n) converges weakly to x.

Weak convergence means convergence of the sequence of numbers $a_n = x^*(x_n)$ for every $x^* \in X^*$.

2.1.5 Definition

A directed set A is a partially ordered set having the property that for every pair α, and β in A there exists $\gamma \in A$ such that $\gamma \geq \alpha$ and $\gamma \geq \beta$. A net is a function $\alpha \to \lambda_\alpha$ on a directed set. If the λ_α all lie in a topological space X, then the net is said to converge to λ in X if for each neighbourhood U of λ there exists α_U in A such that λ_α is in U for $\alpha \geq \alpha_U$. Two topologies on a space X coincide if they have the same convergent nets.

2.1.6 Definition

A net $\{f_\alpha\}_{\alpha\in A}$ in a Banach space X is said to be a Cauchy net if for every $\varepsilon > 0$, there exists α_o in A such that $\alpha_1, \alpha_2 \geq \alpha_o$ implies that $\|f_{\alpha_1} - f_{\alpha_2}\| < \varepsilon$.

2.1.7 Proposition
In a Banach space each Cauchy net is convergent.

Proof
Let $\{f_\alpha\}_{\alpha\in A}$ be a Cauchy net in the Banach space *X*. Choose α_1 such that $\alpha \geq \alpha_1$ implies $\|f_\alpha - f_{\alpha_1}\| < 1$. Also choose $\{\alpha_k\}_{k=1}^n$ in *A* and $\alpha_{n+1} \geq \alpha_n$ such that $\alpha \geq \alpha_{n+1}$ implies

$$\|f_\alpha - f_{\alpha_{n+1}}\| < \frac{1}{n+1}.$$

The sequence $\{f_{\alpha_n}\}_{n=1}^\infty$ is clearly Cauchy and, since *X* is complete, there exists *f* in *X* such that $\lim_{n\to\infty} f_{\alpha_n} = f$.

It is clear that $\lim_{\alpha\in A} f_\alpha = f$ since given ε >0, we can choose *n* such that 1/n < ε / 2 and $\|f_{\alpha_n} - f\| < \varepsilon/2$. Then for $\alpha \geq \alpha_n$ we have

$$\|f_\alpha - f\| \leq \|f_\alpha - f_{\alpha_n}\| + \|f_{\alpha_n} - f\| < 1/n + \varepsilon/2 < \varepsilon.$$

□

2.1.8 Definition
Let $\{f_\alpha\}_{\alpha\in A}$ *be a set of vectors in the Banach space X. and let* $\mathcal{F}$ *={F⊂A: F finite}.*

If we define $F_1 \leq F_2$ *for* $F_1 \subset F_2$*, then* $\mathcal{F}$ *is a directed set. For each F in* $\mathcal{F}$ *let* $g_F = \sum_{\alpha\in F} f_\alpha$. *If the net* $\{g_F\}_{F\in\mathcal{F}}$ *converges to some g in X.,, then the sum* $\sum_{\alpha\in A} f_\alpha$ *is said to converge and we write g=*$\sum_{\alpha\in A} f_\alpha$

2.1.9 Proposition

If $\{f_\alpha\}_{\alpha\in A}$ is a set of vectors in the Banach space X such that $\sum_{\alpha\in A}\|f_\alpha\|$ converges in **R**, *then $\sum_{\alpha\in A} f_\alpha$ converges in X*

Proof

It suffices to show, in the notation of definition 2.1.8, that the net $\{g_F\}_{F\in \mathrm{F}}$ is Cauchy. Since $\sum_{\alpha\in A}\|f_\alpha\|$ converges, for $\varepsilon > 0$, there exists F_o in $\mathcal{F}$ such that $F \geq F_0$ implies

$$\left|\sum_{\alpha\in F}\|f_\alpha\| - \sum_{\alpha\in F_o}\|f_\alpha\|\right| < \varepsilon .$$

Thus for $F_1, F_2 \geq F_o$ we have

$$\|g_{F_1} - g_{F_2}\| = \left\|\sum_{\alpha\in F_1} f_\alpha - \sum_{\alpha\in F_2} f_\alpha\right\|$$

$$= \left\|\sum_{\alpha\in F_1\setminus F_2} f_\alpha - \sum_{\alpha\in F_2\setminus F_1} f_\alpha\right\|$$

$$\leq \sum_{\alpha\in F_1\setminus F_2}\|f_\alpha\| + \sum_{\alpha\in F_2\setminus F_1}\|f_\alpha\|$$

$$\leq \sum_{\alpha\in F_1\cup F_2}\|f_\alpha\| - \sum_{\alpha\in F_o}\|f_\alpha\| < \varepsilon$$

Therefore, $\{g_F\}_{F\in \mathrm{F}}$ is Cauchy and $\sum_{\alpha\in A} f_\alpha$ converges by definition 2.1.8.

□

2.2 SEQUENCE SPACES

2.2.1 Sequence space l_∞

This is the set of all bounded sequences of complex numbers; that is, every element of l_∞ is a complex sequence

$$x=(x_1, x_{2\ldots}) \qquad \text{briefly } x=(x_j)$$

such that for all $j=1, 2, \ldots,$ we have

$$|x_j| \le c_x$$

where c_x is a real number, which may depend on x, but does not depend on j. This space is a Banach space i.e. complete normed space[9] with the norm defined by

$$\|x\|_\infty = \sup_{j\in \mathrm{N}} |x_j| < \infty$$

2.2.2 Sequence space c

The space c consists of all convergent sequences $x=(x_j)$ of complex numbers with the norm induced from the space l_∞. Where the limit is zero, we denote the space by c_0 and call it the null convergent sequence space. This space being a closed subspace of the complete normed space l_∞, is also complete.

2.2.3 Sequence space l_p

Let $p\geq 1$ be a fixed real number. By definition each element in the space l_p is a sequence $x=(x_j)=(x_1, x_2, \ldots.)$ of numbers such that

$$\sum_{j=1}^{\infty} |x_j|^p < \infty \qquad (p \ge 1, \textit{fixed})$$

and the norm is defined by

$$\|x\|_p = \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p}$$

This sequence space is also a complete normed space[9]. In the case *p* = *2*, we have the famous Hilbert sequence space which is also a complete normed space under the norm defined by

$$\|x\|_2^2 = \left(\sum_{j=1}^{\infty} |x_j|^2 \right)^{1/2}$$

The above norm certainly satisfies the parallelogram equality as seen below

$$\|x + y\|_2 = \left(\sum_{j=1}^{\infty} |x_j + y_j|^2 \right)^{1/2} \Rightarrow \|x + y\|_2^2 = \sum_{j=1}^{\infty} |x_j + y_j|^2$$

$$\|x - y\|_2 = \left(\sum_{j=1}^{\infty} |x_j - y_j|^2 \right)^{1/2} \Rightarrow \|x - y\|_2^2 = \sum_{j=1}^{\infty} |x_j - y_j|^2$$

$$\begin{aligned}
\|x + y\|_2^2 + \|x - y\|_2^2 &= \sum_{j=1}^{\infty} |x_j + y_j|^2 + \sum_{j=1}^{\infty} |x_j - y_j|^2 \\
&= \sum_{j=1}^{\infty} \left\{ |x_j + y_j|^2 + |x_j - y_j|^2 \right\} \\
&= \sum_{j=1}^{\infty} 2|x_j|^2 + 2|y_j|^2 \\
&= 2\left(\sum_{j=1}^{\infty} |x_j|^2 + \sum_{j=1}^{\infty} |y_j|^2 \right) \\
&= 2\left(\|x\|_2^2 + \|y\|_2^2 \right).
\end{aligned}$$

2.3 ABSOLUTE SUMMABILITY

A sequence (x_n) in a normed space is absolutely summable if $\sum_n \|x_n\| < \infty$ and is unconditionally summable if $\sum_n x_{\sigma(n)}$ converges, regardless of the permutation σ of the indices. It is conventional to say that the series $\sum_n x_n$ is absolutely convergent if the sequence (x_n) is absolutely summable. Similarly the series $\sum_n x_n$ is unconditionally convergent if the sequence (x_n) is unconditionally summable.

2.3.1 Theorem

A sequence (x_n) in a Banach space is unconditionally summable if and only if it is sign summable, that is, $\Sigma_n \varepsilon_n x_n$ converges for all signs $\varepsilon_n = \pm 1$.

The proof of this theorem is found in [3].

2.3.2 Vector Valued Sequences

Now we introduce the concept of vector valued sequences. We shall work with the index $1 \leq p < \infty$ and a Banach space X. The vector sequence (x_n) in X is strongly p-summable (alternatively, a strong l_p-sequence) if the corresponding scalar sequence $(\|x_n\|)$ is in l_p. We denote by

$$l_p\,(X)$$

the set of all such sequences in X. This is a vector space under pointwise operations, and a natural norm is given by

$$\|(x_n)\|_p := \left(\sum_n \|x_n\|^p\right)^{1/p}.$$

$l_p(X)$ is a Banach space [3]. Strong p-summability makes reference to the strong (or norm) topology on X.

The vector sequence (x_n) in X is weakly p-summable (alternatively, a weak l_p-sequence) if the scalar sequences $(\langle x^*, x_n \rangle)$ are in l_p for every $x^* \in X^*$. We denote by

$$l_p^w(X)$$

the set of all such sequences in X. This is also a vector space under pointwise operations.

2.3.3 Lemma

$$\|(x_n)\|_p^w := \sup\left\{ \left(\sum_n |\langle x^*, x_n \rangle|^p \right)^{1/p} : x^* \in B_{X^*} \right\}.$$

is a norm on $l_p^w(X)$.

2.3.4 Theorem

$l_p^w(X)$ *is a complete normed space.*

Proof

We begin by showing that it is a normed space. For any $x, y \in l_p^w(X)$ and any scalar α we have that

$$\textbf{N1. } \|x\|_p^w = \|(x_n)\|_p^w = \sup_{x^* \in B_{X^*}} \left\{ \left(\sum_n |\langle x^*, x_n \rangle|^p \right)^{1/p} \right\} \geq 0$$

$$\textbf{N2. } \text{Let } \|x\|_p^w = 0 \text{ then } \|(x_n)\|_p^w = \sup_{x^* \in B_{X^*}} \left\{ \left(\sum_n |\langle x^*, x_n \rangle|^p \right)^{1/p} \right\} = 0$$

$$\Rightarrow |\langle x^*, x_n \rangle| = 0 \text{ for all } x^* \in B_{X^*}, \forall\, n$$

$$\Rightarrow \langle x^*, x_n \rangle = 0 \text{ for all } x^* \in B_{X^*}, \forall\, n$$

$$\Rightarrow x = 0$$

Let $x = 0$. Then $\langle x^*, x_n \rangle = 0$ for all $x^* \in B_{X^*}$, $\forall\, n$ and clearly $\|x\| = 0$

$$\therefore \|x\|_p^w = 0 \Leftrightarrow x = 0$$

N3. $\|\alpha x\|_p^w = \|\alpha(x_n)\|_p^w = \|(\alpha x_n)\|_p^w = \sup\limits_{x^* \in B_{X^*}} \left\{ \left(\sum_n |\langle x^*, \alpha x_n \rangle|^p \right)^{1/p} \right\}$

$$= \sup_{x^* \in B_{X^*}} \left\{ \left(\sum_n |\alpha \langle x^*, x_n \rangle|^p \right)^{1/p} \right\}$$

$$= \sup_{x^* \in B_{X^*}} \left\{ \left(\sum_n |\alpha|^p |\langle x^*, x_n \rangle|^p \right)^{1/p} \right\}$$

$$= |\alpha| \sup_{x^* \in B_{X^*}} \left\{ \left(\sum_n |\langle x^*, x_n \rangle|^p \right)^{1/p} \right\}$$

$$= |\alpha| \|x\|_p^w$$

N4. $\|x + y\|_p^w = \|(x_n) + (y_n)\|_p^w = \|(x_n + y_n)\|_p^w = \sup\limits_{x^* \in B_{X^*}} \left\{ \left(\sum |\langle x^*, x_n + y_n \rangle|^p \right)^{1/p} \right\}$

$$= \sup_{x^* \in B_{X^*}} \left\{ \left(\sum_n |\langle x^*, x_n \rangle + \langle x^*, y_n \rangle|^p \right)^{1/p} \right\}$$

$$\leq \sup_{x^* \in B_{X^*}} \left\{ \left(\sum_n |\langle x^*, x_n \rangle|^p \right)^{1/p} + \left(\sum_n |\langle x^*, y_n \rangle|^p \right)^{1/p} \right\} \text{ by Minkowski inequality}$$

$$\leq \sup_{x^* \in B_{X^*}} \left\{ \left(\sum_n \left| \langle x^*, x_n \rangle \right|^p \right)^{1/p} \right\} + \sup_{x^* \in B_{X^*}} \left\{ \left(\sum_n \left| \langle x^*, y_n \rangle \right|^p \right)^{1/p} \right\}$$

$$= \|x\|_p^w + \|y\|_p^w .$$

Next we show that $l_p^w(X)$ is complete.

Take a Cauchy sequence $x^{(k)} = \left(x_n^{(k)}\right)_n$ in $l_p^w(X)$. Thus given $\varepsilon > 0$, there is a natural number N such that for $k, r \geq N$ we have

$$\left\|x_n^{(k)} - x_n^{(r)}\right\|_p^w = \sup\left\{ \left(\sum_n \left| \langle x^*, x_n^{(k)} - x_n^{(r)} \rangle \right|^p \right)^{1/p} : x^* \in B_{X^*} \right\} < \varepsilon$$

$$\Rightarrow \quad \sum_n \left| \langle x^*, x_n^{(k)} \rangle - \langle x^*, x_n^{(r)} \rangle \right|^p \leq \varepsilon^p \qquad (1)$$

for each $x^* \in B_{X^*}$. Thus for each $x^* \in B_{X^*}$ and every n, we have

$$\left| \langle x^*, x_n^{(k)} \rangle - \langle x^*, x_n^{(r)} \rangle \right| < \varepsilon .$$

This tells us that the sequence $\left(\langle x^*, x_n^{(k)} \rangle\right)$ is Cauchy in **C**, and so fixing n and letting $k \to \infty$ the sequence $\left(\langle x^*, x_n^{(k)} \rangle\right)$ converges to $\left(\langle x^*, x_n \rangle\right)$ for every $x^* \in B_{X^*}$. Thus we have $x = (x_n)$ as the weak limit of $(x^{(k)})_k$. We are then to show that $x \in l_p^w(X)$.

Letting $r \to \infty$ and for $k \geq N$ the inequality (1) implies

$$\sum_{n=1}^{j} \left| \langle x^*, x_n^{(k)} \rangle - \langle x^*, x_n \rangle \right|^p \leq \varepsilon^p \qquad \text{for every } x^* \in B_{X^*}.$$

Thus letting $j \to \infty$ we get

$$\left(\sum_{n=1}^{\infty}\left|\left\langle x^{*}, x_{n}-x_{n}^{(k)}\right\rangle\right|^{p}\right)^{1/p} \leq \varepsilon \text{ for every } x^{*} \in B_{X^{*}}.$$

This implies that $x - x^{(k)}$ belong to $l_p^w(X)$. Given $\left(x_n^{(k)}\right)_n \in l_p^w(X)$ then $x = x_n^{(k)} - \left(x_n^{(k)} - x\right) \in l_p^w$, implying that x belongs to $l_p^w(X)$ with $\left\|x - x^{(k)}\right\|_p^w \leq \varepsilon$ for $k \geq N$. Since $\varepsilon > 0$ was arbitrary, $\left(x_n^{(k)}\right)_n$ converges to $x = (x_n)$ in the norm $\|\cdot\|_p^w$. Thus $l_p^w(X)$ is a Banach space. □

For $p = \infty$, if (x_n) is a bounded sequence in the Banach space X, then

$$\sup_n \left\|x_n\right\| = \sup_{x^{*} \in B_{X^{*}}} \sup_n \left|\left\langle x^{*}, x_n\right\rangle\right|.$$

In other words the spaces $l_\infty(X)$ and $l_\infty^w(X)$ are identical and $\left\|(x_n)\right\|_\infty = \left\|(x_n)\right\|_\infty^w$.

Henceforth we shall refer to them simply as

$$l_\infty(X)$$

and use $\left\|(x_n)\right\|_\infty$ for the norm.

We write

$$c_0^w(X)$$

for the closed subspace of $l_\infty(X)$ consisting of all sequences (x_n) in X with $\lim_{n\to\infty}\left\langle x^{*}, x_n\right\rangle = 0$ for all $x^{*} \in X^{*}$. This in turn has as a closed subspace the collection

$$c_0(X)$$

of all sequences (x_n) in X with $\lim_{n\to\infty}\|x_n\| = 0$. The members of $c_0^w(X)$ and $c_0(X)$ are called respectively the weak null sequences and the strong null sequences in the Banach space X.

In the following theorem we see that when $1 \leq p < \infty$, we have $l_p^w(X) = l_p(X)$ if and only if X is finite dimensional.

2.3.5 Weak Dvoretzky-Rogers Theorem

Let $1 \leq p < \infty$. Every infinite dimensional Banach space X contains a weakly p-summable sequence, which fails to be strongly p-summable.

Proof

If not, id_X would be p-summing. But $id_X = (id_X)^2$, so id_X would be compact, which is only possible if X is finite dimensional.

□

Thus $l_p(X)$ is a vector (linear) subspace of $l_p^w(X)$ with the inclusion being strict unless X is finite dimensional. We also have natural isometric isomorphism $l_p^w(X) \cong B(l_{p'}, X)$ $(1 < p \leq \infty)$ and $l_1^w(X) \cong B(c_0, X)$ by associating to each operator u the sequence $(x_j) \subset X$ given by the image of the canonical basis $x_j = u(e_j)$. In this sense, the space $K(c_0, X)$ of compact operators corresponds with the sequence (x_j) such that the series $\sum x_j$ converges unconditionally.

2.4 NORMED SPACE OF OPERATORS

2.4.1 Definition: Bounded linear operator

Let X and Y be normed spaces and u: D(u) $\rightarrow$ Y a linear operator, where D(u) $\subset$ X. The operator u is said to be bounded if there is a real number c such that for all $x \in D(u)$,

$$\|ux\| \leq c\|x\|. \qquad (2)$$

In (2) the norm on the left is that on *Y*, and the norm on the right is that on *X*. This formula shows that a bounded linear operator maps bounded sets in *D(u)* onto bounded sets in *Y*.

From (2) we have

$$\frac{\|ux\|}{\|x\|} \leq c \qquad (x \neq 0)$$

and this shows that the least *c* for which the above inequality holds is the supremum of the expression on the left taken over *D(u) - {0}*. This quantity is denoted by $\|u\|$; thus

$$\|u\| = \sup\left\{\frac{\|ux\|}{\|x\|} : x \in D(u), x \neq 0\right\}. \qquad (3)$$

$\|u\|$ is called the norm of the operator *u*.

2.4.2 Theorem

An alternative formula for $\|u\|$ is

$$\|u\| = \sup\{\|ux\| : x \in D(u), \|x\| = 1\}.$$

Proof

We write $\|x\| = a$ and set $y = \left(\frac{1}{a}\right)x$, where $x \neq 0$. Then $\|y\| = \frac{\|x\|}{a} = 1$, and since u is linear, (3) gives

$$\|u\| = \sup\left\{\frac{1}{a}\|ux\| : x \in D(u), x \neq 0\right\}$$

$$= \sup\left\{\left\|u\left(\frac{1}{a}x\right)\right\| : x \in D(u), x \neq 0\right\}$$

$$= \sup\{\|uy\| : y \in D(u), \|y\| = 1\}$$

Writing x for y in the last expression we have

$$\|u\| = \sup\{\|ux\| : x \in D(u), \|x\| = 1\}$$

□

Given any two normed spaces *X* and *Y* (both real or complex), we consider the set

B (X, Y)

consisting of all bounded linear operators from X into Y, that is, each such operator is defined on all of *X* and its range lies in *Y*. From the proof that $\|u\|$ satisfies the four axioms of a norm, it follows that *B (X, Y)* is a normed space which we state in the following theorem.

2.4.3 Theorem

The vector space B (X, Y) of all bounded linear operators from a normed space X into a normed space Y is itself a normed space with norm defined by

$$\|u\| = \sup\left\{\frac{\|ux\|}{\|x\|} : x \in D(u), x \neq 0\right\}$$

$$= \sup\{\|ux\| : x \in D(u), \|x\| = 1\}.$$

Proof

For any $u, v \in B(X, Y)$ and $\alpha \in \mathbf{R}$, we have that

N1. $\|u\| = \sup\{\|ux\| : x \in D(u), \|x\| = 1\} \geq 0$

N2. Let $\|u\| = 0$, then $\sup\{\|ux\| : x \in D(u), \|x\| = 1\} = 0$ for all $x \neq 0$

$$\Rightarrow \|ux\| = 0 \text{ for all } x \neq 0$$

$$\Rightarrow u = 0$$

Let $u = 0$. Then clearly $\|u\| = 0$

$\therefore \|u\| = 0$ if and only if $u = 0$.

N3. $\|\alpha u\| = \sup\{\|(\alpha u)x\| : x \in D(u), \|x\| = 1\}$

$$= \sup\{\|\alpha ux\| : x \in D(u), \|x\| = 1\}$$

$$= |\alpha| \sup\{\|ux\| : x \in D(u), \|x\| = 1\}$$

$$= |\alpha| \, \|u\|$$

N4. $\|u + v\| = \sup\{\|(u + v)x\| : x \in D(u) \cup D(v), \|x\| = 1\}$

$$= \sup\{\|ux + vx\| : x \in D(u) \cup D(v), \|x\| = 1\}$$

$$\leq \sup\{\|ux\| : x \in D(u), \|x\| = 1\} + \sup\{\|vx\| : x \in D(v), \|x\| = 1\}$$

$$= \|u\| + \|v\|$$

□

The following theorem shows us the condition under which the space *B (X, Y)* becomes a Banach space. It is worth noting that the condition does not involve *X*; that is *X* may or may not be complete.

2.4.3 Theorem: Completeness of *B (X, Y)*

If Y is a Banach space then B (X, Y) is a Banach space.

Proof

We consider an arbitrary Cauchy sequence (u_n) in $B(X, Y)$ and show that (u_n) converges to an operator $u \in B(X, Y)$. Since (u_n) is Cauchy, for every $\varepsilon > 0$ there is an N such that

$$\|u_n - u_m\| < \varepsilon \qquad (m, n > N).$$

For all $x \in X$ and $m, n > N$ we thus obtain

$$\|u_n x - u_m x\| = \|(u_n - u_m)x\| \leq \|u_n - u_m\|\|x\| < \varepsilon\|x\| \qquad (4)$$

Now for any fixed x and $\tilde{\varepsilon}$ we may choose $\varepsilon = \varepsilon_x$ so that $\varepsilon_x\|x\| < \tilde{\varepsilon}$. Then from (4) we have $\|u_n x - u_m x\| < \tilde{\varepsilon}$ and see that $(u_n x)$ is Cauchy in Y. Since Y is complete, $(u_n x)$ converges, say, $u_n x \to y$. Clearly, the limit $y \in Y$ depends on the choice of $x \in X$. This defines an operator $u: X \to Y$, where $y = ux$. The operator u is linear since

$$\lim u_n(\alpha x + \beta z) = \lim(\alpha u_n x + \beta u_n z) = \alpha \lim u_n x + \beta \lim u_n z.$$

We prove that u is bounded and $u_n \to u$, that is $\|u_n - u\| \to 0$.

Since (4) holds for every $m > N$ and $u_n x \to ux$, we may let $m \to \infty$. Using the continuity of the norm, we thus obtain from (4) for every $n > N$ and all $x \in X$

$$\|u_n x - ux\| = \left\|u_n x - \lim_{m\to\infty} u_m x\right\| = \lim\|u_n x - u_m x\| \leq \varepsilon\|x\|. \qquad (5)$$

This shows that $(u_n - u)$ with $n > N$ is a bounded linear operator. Since u_n is bounded, $u = u_n - (u_n - u)$ is bounded, that is $u \in B(X, Y)$. Furthermore, if in (5) we take the supremum over all x of norm 1, we obtain

$$\|u_n - u\| \leq \varepsilon \qquad (n > N)$$

Hence

$$\|u_n - u\| \to 0. \qquad \square$$

2.5 THE DUAL OF l_p SPACES

Since a linear functional on X maps X into **R** or **C** (the scalar field of X), and since **R** or **C** taken with the usual metric is complete, we see that X^* is $B(X, Y)$ with the complete space Y = **R** or **C**. Thus the dual space X^* of a normed space X is a Banach space (whether or not X is). Below are examples of some of the spaces that appear in our work and how their dual spaces look like. We recall that an isomorphism of a normed space X onto a normed space $\tilde{X}$ is a bijective linear operator $u: X \to \tilde{X}$, which preserves the norm, that is, for all $x \in X$,

$$\|ux\| = \|x\|$$

X then is called isomorphic with $\widetilde{X}$ and from an abstract point of view, X and $\widetilde{X}$ are identical, that is, they differ at most by the nature of their points but are similar as far as their structure (in this case, the norm) is concerned.

1. Space l_1: *The dual of l_1 is l_∞*

Proof

A Schauder basis for l_1 is (e_k), where $e_k = (\delta_{kj})$ has 1 in the k^{th} place and zeros otherwise. Then every $x \in l_1$ has a unique representation

$$x = \sum_{k=1}^{\infty} x_k e_k .$$

We consider any $x^* \in (l_1)^*$, where $(l_1)^*$ is the usual dual space of l_1. Since x^* is linear and bounded,

$$x^*(x) = \sum_{k=1}^{\infty} x_k \gamma_k \qquad\qquad \gamma_k = x^*(e_k) \qquad\qquad (6)$$

where the numbers $\gamma_k = x^*(e_k)$ are uniquely determined by x^*. Also $\|e_k\| = 1$ and

$$|\gamma_k| = |x^*(e_k)| \le \|x^*\|\|e_k\| = \|x^*\| \qquad\qquad \Rightarrow \sup_k |\gamma_k| \le \|x^*\| \qquad\qquad (7)$$

Hence $(\gamma_k) \in l_\infty$

On the other hand, for every $b = (\beta_k) \in l_\infty$ we can obtain a corresponding bounded linear functional y^* on l_1. In fact, we may define y^* on l_1 by

$$y^*(x) = \sum_{k=1}^{\infty} x_k \beta_k$$

where $x = (x_k) \in l_1$. Then y^* is linear and boundedness follows from

$$\left|y^*(x)\right| = \left|\sum x_k\beta_k\right| \le \sum\left|x_k\beta_k\right| \le \sup_j\left|\beta_j\right|\sum\left|x_k\right| = \|x\|\sup_j\left|\beta_j\right|$$

Hence $y^* \in (l_1)^*$.

We finally show that the norm of x^* is the norm on the space l_∞. From (6) we have

$$\left|x^*(x)\right| = \left|\sum x_k\gamma_k\right| \le \sup_j\left|\gamma_j\right|\sum\left|x_k\right| = \|x\|\sup_j\left|\gamma_j\right|.$$

Taking supremum over all x of norm 1, we see that

$$\left\|x^*\right\| \le \sup_j\left|\gamma_j\right|.$$

From this and (7)

$$\left\|x^*\right\| = \sup_j\left|\gamma_j\right|$$

which is the norm on l_∞. Hence this formula can be written $\left\|x^*\right\| = \|c\|_\infty$, where $c = (\gamma_j) \in l_\infty$. It shows that the bijective linear mapping of l_1^* onto l_∞ defined by $x^* \mapsto c = (\gamma_j)$ is an isomorphism.

□

2. Space l_p: *The dual space of l_p is l_q; here $1 < p < \infty$ and q is the conjugate of p, that is* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof

A Schauder basis of l_p is (e_k), where $e_k = (\delta_{kj})$ as in the above example. Every $x \in l_p$ has a unique representation,

$$x = \sum_{k=1}^{\infty} x_k e_k .$$

We consider any $x^* \in l_p^*$, where l_p^* is the dual space of l_p. Since x^* is linear and bounded,

$$x^*(x) = \sum_{k=1}^{\infty} x_k \gamma_k \qquad \gamma_k = x^*(e_k) \qquad (8).$$

Let q be the conjugate of p and consider $x_n = \left(x_k^{(n)}\right)$ with

$$x_k^{(n)} = \begin{cases} \dfrac{|\gamma_k|^q}{\gamma_k} & \text{if } k \le n \text{ and } \gamma_k \neq 0 \\ 0 & \text{if } k > n \text{ or } \gamma_k = 0 \end{cases} \qquad (9)$$

By substituting this into (8) we obtain

$$x^*(x_n) = \sum_{k=1}^{\infty} x_k^{(n)} \gamma_k = \sum_{k=1}^{n} |\gamma_k|^q .$$

We also have, using (9) and $(q - 1)\, p = q$

$$x^*(x_n) \le \|x^*\| \|x_n\| = \|x^*\| \left(\sum \left|x_k^{(n)}\right|^p \right)^{1/p}$$

$$= \|x^*\| \left(\sum |\gamma_k|^{(q-1)p} \right)^{1/p}$$

$$= \|x^*\| \left(\sum_{k=1}^{n} |\gamma_k|^q \right)^{1/p} .$$

Together

$$x^*(x_n) = \sum |\gamma_k|^q \le \|x^*\| \left(\sum |\gamma_k|^q \right)^{1/p}.$$

Dividing by the last factor and using $1 - \frac{1}{p} = \frac{1}{q}$, we get

$$\left(\sum_{k=1}^{n} |\gamma_k|^q \right)^{1-\frac{1}{p}} = \left(\sum_{k=1}^{n} |\gamma_k|^q \right)^{1/q} \le \|x^*\|.$$

Since n is arbitrary, letting $n \to \infty$, we obtain

$$\left(\sum_{k=1}^{\infty} |\gamma_k|^q \right)^{1/q} \le \|x^*\| \qquad (10).$$

Hence $(\gamma_k) \in l_q$.

Conversely, for any $b = (\beta_k) \in l_q$ we can get a corresponding bounded linear functional y^* on l_p. In fact, we may define y^* on l_p by setting

$$y^*(x) = \sum_{k=1}^{\infty} x_k \beta_k$$

where $x = (x_k) \in l_p$. Then y^* is linear and boundedness follows from

$$|y^*(x)| = \left| \sum x_k \beta_k \right| \le \left(\sum |x_k|^p \right)^{1/p} \left(\sum |\beta_k|^q \right)^{1/q} \qquad \text{Holder's inequality}$$

$$= \|x\| \left(\sum |\beta_k|^q \right)^{1/q}.$$

Hence $y^* \in l_p^*$.

We finally prove that the norm of x^* is the norm on the space l_q. From (8) and Holder inequality we have

$$\left|x^*(x)\right| = \left|\sum x_k \gamma_k\right| \le \left(\sum |x_k|^p\right)^{1/p} \left(\sum |\gamma_k|^q\right)^{1/q} = \|x\| \left(\sum |\gamma_k|^q\right)^{1/q}$$

hence by taking supremum over all x of norm 1 we obtain

$$\|x^*\| \le \left(\sum |\gamma_k|^q\right)^{1/q}.$$

From this and (10) we see that the equality sign must hold, that is,

$$\|x^*\| = \left(\sum |\gamma_k|^q\right)^{1/q} \qquad (11).$$

This can be written $\|x\| = \|c\|_q$, where $c = (\gamma_j) \in l_q$ and $\gamma_k = x^* (e_k)$. The mapping of l_p^* onto l_q defined by $x^* \mapsto c\|x\| = \|c\|_q$ is linear and bijective and from (11) we see that it is norm preserving, so that it is an isomorphism.

□

CHAPTER 3

DISCUSSION AND RESULTS

This chapter contains two sections, the first, section 3.1, is a presentation of the concept of summing operators from which the concept of summing multipliers was abstracted. In section 3.2 we present the concepts of absolutely and p-summing multipliers. Using the definition of (p,q)-summing multipliers and the ideas developed from the concepts of summing operators and absolutely and p-summing multipliers, we show that the space of all (p,q)-summing multipliers is a Banach space.

3.1 SUMMING OPERATORS

3.1.1 Definition: Absolutely summing operator

Let X and Y be Banach spaces. An operator $u \in B(X,Y)$ is absolutely summing if for every unconditionally convergent series $\sum x_j$ in X, it holds that $\sum ux_j$ is absolutely convergent in Y.

As was stated earlier, the root of the study of this class of operators is traced to the work undertaken by Grothendieck. His theorem is a consequence of a matrix inequality called Grothendieck's inequality.

3.1.2 Grothendieck's Inequality

There is a universal constant k_G for which, given any Hilbert space H, any

$n \in N$, any n x n scalar matrix (a_{ij}) and any vectors $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ in B_H, we have

$$\left|\sum_{i,j} a_{ij}\left(x_i \middle| y_j\right)\right| \leq k_G \cdot \max\left\{\left|\sum_{i,j} a_{ij}\varepsilon_i\varepsilon_j\right| : \varepsilon_i, \varepsilon_j = \pm 1\right\}.$$

The possible k_G-currently unknown-is generally called *Grothendieck's constant*; it depends on the chosen scalar field. The above inequality is called Grothendieck's inequality.

3.1.2 Grothendieck's Theorem [3]

Every continuous linear operator $u: l_1 \to l_2$ is absolutely summing.

Proof

We assume that $\|u\| < 1$ and restrict ourselves to real scalars. Let (x_n) be an unconditionally summable sequence in l_1, *then* $\Sigma_n \varepsilon_n x_n$ converges for any sequence of signs ε_n, and we have

$$\left\|\sum_n \varepsilon_n x_n\right\| \leq \sup_{x^* \in B_{l_\infty}} \sum_n \left|\langle x^*, x_n\rangle\right| = \|v\|,$$

where v is the operator from $l_\infty = l_1^*$ to l_1 *i.e.* $v: l_\infty \to l_1$.

We need to show that $\sum_n \|ux_n\|_2 < \infty$. First we reduce to finite dimensions so that we can apply Grothendieck's inequality.

Let $m \in N$ and $\delta > 0$ be given. Choose $n \geq m$, and vectors $y_1, \ldots, y_m$ in $l_1^n \subset l_1$ so that $\|x_i - y_i\| \leq \delta/2^i$ for $1 \leq i \leq m$. If n happens to be strictly greater than m, set $y_{m+1} = \ldots = y_n = 0$ as well. For each i, write $y_i = \sum_{j=1}^{n} a_{ij} e_j$ for the expansion of y_i with respect to the unit coordinate vectors in l_1^*. This gives us a matrix $a = (a_{ij})$ for use in Grothendieck's inequality.

Assuming absolute summability in l_2 we have

$$\sum_{i=1}^{n} \|uy_i\|_2 = \sum_{i=1}^{n} \left\| \sum_{j=1}^{n} a_{ij} u e_j \right\|_2 = \sum_{i,j} a_{ij} \left(z_i | u e_j \right)$$

for appropriate $z_1, \ldots, z_n \in B_{l_2^n}$. This is well suited for insertion into the left hand side of Grothendiek's inequality.

Now going to unconditional summability in l_1 which is interpreted as sign summability, we have that given $\varepsilon_1, \ldots, \varepsilon_n = \pm 1$,

$$\left\| \sum_{i=1}^{n} \varepsilon_i y_i \right\|_1 = \left\| \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ij} \varepsilon_i \right) e_j \right\|_1 = \sum_j \left| \sum_i a_{ij} \varepsilon_i \right|$$

$$= \max \left\{ \left| \sum_{i,j} a_{ij} \varepsilon_i \varepsilon_j \right| : \varepsilon_j = \pm 1 \right\}.$$

This is well suited for insertion into the right hand side of Grothendieck's inequality. Thus in terms of x_i's

$$\sum_{1 \le n} \|ux_i\|_2 \le \sum_{i \le n} \|uy_i\|_2 + \delta \le \sup \left| \sum_{i,j} a_{ij} \left(x_i | y_j \right) \right| + \delta$$

$$\le k_G \cdot \max \left\{ \left| \sum_{i,j} a_{ij} \varepsilon_i \varepsilon_j \right| : \varepsilon_i, \varepsilon_j = \pm 1 \right\} + \delta$$

$$= k_G \cdot \|a\| + \delta = k_G \cdot \max \left\{ \left\| \sum_{i \le n} \varepsilon_i y_i \right\|_1 : \varepsilon_i = \pm 1 \right\} + \delta$$

$$\le k_G \cdot \max \left\{ \left\| \sum_{i \le n} \varepsilon_i x_i \right\|_1 : \varepsilon_i = \pm 1 \right\} + (1 + k_G)$$

$$\le k_G \cdot \|v\| + (1 + k_G) \cdot \delta$$

Letting $\delta \to 0$, we get

$$\sum_n \|ux_n\|_2 \le k_G \cdot \|u\| \cdot \sup_{\varepsilon_i = \pm 1} \left\| \sum \varepsilon_i x_i \right\| < \infty$$

for any choice of (x_i) in l_1.

□

This formulation of Grothendieck's theorem and the meaning of k_G were given by Pietsch and Lindenstrauss and Pelczynski, by changing from the original tensor norms context to the Pietsch's operator ideals setting. This particular proof of the Grothendieck's theorem is taken from [8].

3.1.4 Definition: p-summing Operator

Suppose that $1 \le p < \infty$ and that $u: X \to Y$ is a linear operator between Banach spaces. We say that u is p-summing if there is a constant $c \ge 0$ such that regardless of the natural number n and regardless of the choice of $x_1, \ldots, x_n$ in X we have

$$\left(\sum_{i=1}^{n} \|ux_i\|^p \right)^{1/p} \le c \cdot \sup \left\{ \left(\sum_{i=1}^{n} \left| \langle x^*, x_j \rangle \right|^p \right)^{1/p} : x^* \in B_{X^*} \right\} \qquad (1)$$

The least c for which the inequality (1) holds is denoted by

$\pi_p(u)$.

We shall write

$\prod_p (X, Y)$

for the set of all p-summing operators from X to Y. $\prod_p (X, Y)$ is a linear subspace of $B (X, Y)$, the space of all bounded linear operators from X into Y, and that $\pi_p (u)$ defines a norm on $\prod_p (X, Y)$ with

$$\|u\| \leq \pi_p (u)$$

for all $u \in \prod_p (X, Y)$. It has been shown [3] that $\prod_p (X, Y)$ is a Banach space.

3.1.5 Theorem
$\Pi_p (X, Y)$ is a Banach space under the norm π_p.

Proof
Let (u_n) be a π_p-Cauchy sequence in $\Pi_p (X, Y)$. Since $\|\cdot\| \leq \pi_p(\cdot)$, (u_n) is Cauchy in $B (X, Y)$ and so converges to some $u \in B (X, Y)$ in the uniform norm. u gives rise to the operator $\hat{u} : l_p^w(X) \to l_p^w(Y) : (x_k) \mapsto (ux_k)$. For each n, $\hat{u}_n : l_p^w(X) \to l_p(Y) : (x_k) \mapsto (u_n x_k)$ is well defined, and this implies that $(\hat{u}_n)$ is a Cauchy sequence in $(l_p^w(X), l_p(Y))$. This is a Banach space since $l_p(Y)$ is complete, and so $(\hat{u}_n)$ converges. Its limit is a map with values in $l_p^w(Y)$ and so must be $\hat{u}$. Thus u is p-summing and $\lim_{n\to\infty} \pi_p(u - u_n) = 0$. Hence $\Pi_p (X, Y)$ is complete. □

3.1.6 Definition: (p, q)-summing operators
Given $1 \leq p \leq q < \infty$, the space $\prod_{p,q} (X, Y)$ of (p,q)-summing operators is formed by those operators that map sequences in $l_p^w(X)$ into sequences in $l_q (Y)$. In other words if there exists a constant c such that

$$\left\|\left(ux_j\right)\right\|_{l_q(Y)} \leq c\left\|\left(x_j\right)\right\|_{l_p^w(X)}$$

for any finite family of vectors x_j in X.

The least of such c is the (p, q)-summing norm of u. It is denoted $\pi_{p,q}(u)$.

Note: (p, p)-summing is just the same as p-summing.

3.2 SUMMING MULTIPLIERS

Let X and Y be two real or complex Banach spaces and $E(X)$ and $F(Y)$ be two Banach spaces whose elements are defined by sequences of vectors in X and Y.

3.2.1 Definition: Multiplier sequence

A sequence of operators $(u_j) \in B(X,Y)$ is called a multiplier sequence from E (X) to F (Y) if there exists a constant $c > 0$ such that

$$\left\|\left(u_j x_j\right)_{j=1}^n\right\|_{F(Y)} \leq c\left\|\left(x_j\right)_{j=1}^n\right\|_{E(X)}$$

for all finite families $x_1, \ldots, x_n$ in X.

Notes:

(1) The set of all multiplier sequences is denoted by $(E(X), F(Y))$. In this study, we consider the case of classical sequence spaces $E(X) = l_p^w(x)$ and $F(Y) = l_q(Y)$.

(2) If $u \in \Pi_{p,q}(X, Y)$ then the constant sequence $u_j = u$ for all $j \in$ **N**, belonging to $\left(l_p^w(X), l_q(Y)\right)$.

(3) If we set $u_j = u\lambda_j$ then $(u_j) \in \left(l_p^w(X), l_q(Y)\right)$ for all $(\lambda_j) \in l_{p'}$ where $\frac{1}{p}+\frac{1}{p'}=1$ if and only if $u \in \Pi_{p,q}(X, Y)$.

These facts suggest the use of notation $l_{\Pi_{p,q}}(X,Y)$ when considering the sequence (u_j) in $\left(l_p^w(X), l_q(Y)\right)$ and $l_{\Pi_p}(X,Y)$ for $p = q$.

3.2.2 Definition: Absolutely summing multiplier

A sequence $(u_j) \in B(X)$ is called absolutely summing multiplier of X if $(u_j x_j)$ is absolutely summable in X whenever (x_j) is weakly absolutely summable in X; hence $(u_j x_j) \in l_1(X)$ for all $(x_j) \in l_1^w(X)$.

The space of all absolutely summing multipliers of X is denoted by $l_{\Pi_1}(X)$. It is a vector subspace of $l_\infty(B(X))$. The space $l_{\Pi_1}(X)$ is a complete normed space with respect to the operator norm

$$\left\|(u_j)\right\|_{1,1} := \sup\left\{\left\|(u_j x_j)\right\| : (x_j) \in l_1^w(X), \left\|(x_j)\right\| = 1\right\}.$$

As we saw in chapter 2, $l_1^w(X) = l_1(X)$ if and only if X has finite dimension, so that $l_{\Pi_1}(X) = l_\infty(B(X))$.

Using the Dvoretzky-Rogers theorem, it is proved in [7] that for any infinite dimensional Banach space X we have

$$l_1 \subseteq l_{\Pi_1}(X) \subseteq l_2$$

3.2.3 Theorem: Dvoretzky-Roger theorem

Let X be an infinite dimensional Banach space. Then no matter how we choose $(\lambda_n) \in l_2$ there is always an unconditional summable sequence (x_n) in X with $\|x_n\| = |\lambda_n|$ for all n.

The proof of this theorem uses the following lemma

3.2.4 Lemma

Let X be a 2n-dimentional Banach space. There exists n vectors $x_1,\ldots,x_n \in B_X$, each of norm $\geq \frac{1}{2}$, such that regardless of the scalars $\lambda_1,\ldots,\lambda_n$ we have

$$\left\|\sum_{j\leq n}\lambda_j x_j\right\| \leq \left(\sum_{j\leq n}|\lambda_j|^2\right)^{\frac{1}{2}}.$$

The proof of this lemma is found in [3].

Proof of the Dvoretzky-Rogers Theorem.

Fix $(\lambda_n) \in l_2$ and choose positive integers $n_1 < n_2 < \ldots$, such that, for each $k \in \mathbf{N}$,

$$\sum_{n\geq n_k}|\lambda_n|^2 \leq 2^{-2k}.$$

Since X is infinite dimensional, we apply the above lemma and find a sequence of vectors (y_n) in B_X, each of norm $\geq \frac{1}{2}$, such that for every scalar sequence (α_n) and any k we have

$$\left\|\sum_{n=n_k}^{N}\alpha_n y_n\right\| \leq \left(\sum_{n=n_k}^{N}|\alpha_n|^2\right)^{\frac{1}{2}}$$

no matter how we select $n_k \leq N \leq n_{k+1}$. We set $x_j = \dfrac{\lambda_j y_j}{\|y_j\|}$ and note that regardless of the sign $\varepsilon_n = \pm 1$ and regardless of $n_k \leq N \leq n_{k+1}$ we have

$$\left\| \sum_{n=n_k}^{N} \varepsilon_n x_n \right\| \leq \left(\sum_{n=n_k}^{N} \frac{|\lambda_n|^2}{\|y_n\|^2} \right)^{1/2} \leq 2^{-k+1}.$$

It follows that the partial sums of $(\varepsilon_n x_n)$ are Cauchy. Hence (x_n) is sign summable, and so unconditionally summable. The setting of x_j above ensures that $\|x_n\| = |\lambda_n|$ for all n.

□

3.2.5 Theorem

Let X be an infinite dimensional Banach space, then

$$l_1 \subseteq l_{\Pi_1}(X) \subseteq l_2..$$

Proof

Let $(\alpha_n) \in l_1$. Since each $(x_n) \in l_1^w(X)$ is norm bounded in X, it follows that $\sum_{n=1}^{\infty} \|\alpha_n x_n\| < \infty$.

Hence $(\alpha_n) \in l_{\Pi_1}(X)$.

Conversely, let $(\alpha_n) \in l_{\Pi_1}(X)$. For $(\beta_i) \in l_2$ there is by the Dvoretzky-Rogers theorem a sequence $(x_i) \in l_1^w(X)$ such that $|\beta_j| = \|x_j\|$ for $i=1,2,\ldots$.Then $(\alpha_j x_i) \in l_1(X)$. Hence it follows that $\sum_{i=1}^{\infty} |\alpha_i \beta_i| < \infty$. Since $(\beta_i) \in l_2$ was arbitrary, then $(\alpha_i) \in (l_2)^* = l_2$

$$\Rightarrow l_1 \subseteq l_{\Pi_1}(X) \subseteq l_2..$$

□

3.2.6 Definition: p-summing multiplier

Let $1\leq p < \infty$. A sequence (u_j): $X\rightarrow Y$ of operators is called a p-summing multiplier if

$\sum_{j=1}^{\infty}\|u_j x_j\|^p < \infty$ *in Y for all sequences* $(x_j)\in l_p^w(X)$. *Put*

$$l_{\Pi_p}(X,Y)=\left\{(u_j)\in B(X,Y):\sum_{j=1}^{\infty}\|u_j x_j\|^p<\infty,\forall(x_j)\in l_p^w(X)\right\}.$$

On the vector space $l_{\Pi_p}(X,Y)$, we define a norm

$$\|(u_j)\|_{p,p} := \sup\left\{\left(\sum_{j=1}^{\infty}\|u_j x_j\|^p\right)^{1/p} : (x_j)\in l_p^w(X), \|(x_j)\|=1\right\},$$

which is well defined because for each $(u_j)\in l_{\Pi_p}(X,Y)$, this is the operator norm of the bounded linear operator.

$T_u : l_p^w(X)\rightarrow l_p(Y):(x_j)\alpha\ (u_j x_j).$

3.2.7 Definition: (p, q)-summing multiplier

A sequence $(u_j)_{j\in N}$ of operators in B (X, Y) is a (p, q)-summing multiplier, in short $(u_j)\in l_{\Pi_{p,q}}(X,Y)$, if there exists a constant $c>0$ such that, for any finite collection of vectors $x_1,\ldots,x_n$ in X, it holds that

$$\left(\sum_{j=1}^{n}\|u_j x_j\|^q\right)^{1/q}\leq c\cdot\sup\left\{\left(\sum_{j=1}^{n}|\langle x^*,x_j\rangle|^p\right)^{1/p} : x^*\in B_{X^*}\right\}.$$

The least c for which the above inequality holds is the (p, q)-summing norm of (u_j) and is denoted by $\pi_{p,q}(u_j)$. The space of all (p, q)-summing operators is denoted by $l_{\Pi_{p,q}}(X,Y)$.

3.2.8 Proposition

Let X and Y be Banach spaces and $1 \leq p \leq q < \infty$. $(l_{\Pi_{p,q}}(X,Y), \pi_{p,q}(u_j))$ is a Banach space.

Proof

From the definition of a (p, q)-summing multiplier we have that

$$\left\|(u_j x_j)\right\|_{l_q(Y)} \leq c \cdot \left\|(x_j)\right\|_{l_p^w(X)}$$

where

$$\left\|(u_j x_j)\right\| = \left(\sum_{j=1}^{\infty} \left\|u_j x_j\right\|^q\right)^{1/q}$$ is the norm in $l_q(Y)$,

$$\left\|(x_j)\right\| = \sup\left\{\left(\sum_{j=1}^{\infty} \left|\langle x^*, x_j\rangle\right|^p\right)^{1/p} : x^* \in B_{X^*}\right\}$$ is norm in $l_p^w(X)$ and the least c for which the above inequality holds is called the (p,q)-summing norm of $(u_j)$and is denoted by $\pi_{p,q}(u_j)$. Thus we define

$$\pi_{p,q}(u_j) = \sup\left\{\left\|(u_j x_j)\right\| : (x_j) \in l_p^w(X), \left\|(x_j)\right\| = 1\right\}.$$

We now show that $\pi_{p,q}(u_j)$ defines a norm on $l_{\Pi_{p,q}}(X,Y)$. Given $(u_j), (v_j) \in l_{\Pi_{p,q}}(X,Y)$ and $\alpha \in \boldsymbol{R}$ we have

N1. $\pi_{p,q}(u_j) = \sup\left\{\left\|(u_j x_j)\right\| : (x_j) \in l_p^w(X), \left\|(x_j)\right\| = 1\right\} \geq 0$ since $\left\|(u_j x_j)\right\| \geq 0 \quad \forall j, \forall x_j \neq 0$

N2. Let $\pi_{p,q}(u_j) = 0$, then $\pi_{p,q}(u_j) = \sup\left\{\left\|(u_j x_j)\right\| : (x_j) \in l_p^w(X), \left\|(x_j)\right\| = 1\right\} = 0$

$\Rightarrow \left\|(u_j x_j)\right\| = 0 \iff u_j x_j = 0 \quad \forall j, \forall x_j \neq 0$ thus $(u_j) = 0, \forall j$

Let $(u_j) = 0$. Then $\pi_{p,q}(u_j) = 0$

$\Rightarrow \pi_{p,q}(u_j) = 0$ if and only if $(u_j) = 0$

N3. $\pi_{p,q}(\alpha u_j) = \sup\{\|((\alpha u_j)x_j)\| : (x_j) \in l_p^w(X), \|(x_j)\| = 1\}$

$= \sup\{\|(\alpha u_j x_j)\| : (x_j) \in l_p^w(X), \|(x_j)\| = 1\}$

$= \sup\{\|\alpha(u_j x_j)\| : (x_j) \in l_p^w(X), \|(x_j)\| = 1\}$

$= \sup\{|\alpha| \|(u_j x_j)\| : (x_j) \in l_p^w(X), \|(x_j)\| = 1\}$

$= |\alpha| \sup\{\|(u_j x_j)\| : (x_j) \in l_p^w(X), \|(x_j)\| = 1\}$

$= |\alpha| \pi_{p,q}(u_j)$

N4. $\pi_{p,q}(u_j + v_j) = \sup\{\|((u_j + v_j)x_j)\| : (x_j) \in l_p^w(X), \|(x_j)\| = 1\}$

$= \sup\{\|(u_j x_j + v_j x_j)\| : (x_j) \in l_p^w(X), \|(x_j)\| = 1\}$

$= \sup\{\|(u_j x_j) + (v_j x_j)\| : (x_j) \in l_p^w(X), \|(x_j)\| = 1\}$

$\leq \sup\{\|(u_j x_j)\| : (x_j) \in l_p^w(X), \|(x_j)\| = 1\} + \sup\{\|(v_j x_j)\| : (x_j) \in l_p^w(X), \|(x_j)\| = 1\}$

$= \pi_{p,q}(u_j) + \pi_{p,q}(v_j)$.

Next we show that $l_{\Pi_{p,q}}(X,Y)$ is a complete normed space. We take a Cauchy sequence $(u^{(n)})$ in $l_{\Pi_{p,q}}(X,Y)$, where $(u^{(n)}) = (u_j^{(n)})$. Therefore given $\varepsilon > 0$ there is a natural number N such that for $m, n > N$ we have

$$\pi_{p,q}(u_j^{(n)} - u_j^{(m)}) = \sup(\|(u_j^{(n)} - u_j^{(m)})x_j\| : (x_j) \in l_p^w(X), \|(x_j)\| = 1)$$

$$= \sup(\|(u_j^{(n)} x_j - u_j^{(m)} x_j)\| : (x_j) \in l_p^w(X), \|(x_j)\| = 1) < \varepsilon \quad (2)$$

Therefore for every j and for $m, n > N$, we have

$$\left\|\left(u_j^{(n)}x_j - u_j^{(m)}x_j\right)\right\| = \left(\sum_{j=1}^{\infty}\left\|u_j^{(n)}x_j - u_j^{(m)}x_j\right\|^q\right)^{1/q} < \varepsilon$$

$$\Rightarrow \left\|u_j^{(n)}x_j - u_j^{(m)}x_j\right\| < \varepsilon \qquad \text{for every } j \text{ and for } m, n > N.$$

Thus $\left(u_j^{(n)}x_j\right)$ is Cauchy in $l_q(Y)$. Given $l_q(Y)$ is complete, $\left(u_j^{(n)}x_j\right)$ converges in it, and so fixing j and letting $n \to \infty$ we have that $\left(u_j^{(n)}x_j\right) \to \left(u_jx_j\right) \in l_q(Y)$

Letting $n \to \infty$ in (2) and for $n > N$, we have

$$\pi_{p,q}\left(u_j^{(n)} - u_j\right) = \sup\left(\left\|\left(u_j^{(n)}x_j - u_jx_j\right)\right\| : \left(x_j\right) \in l_p^w(X), \left\|\left(x_j\right)\right\| = 1\right) < \varepsilon$$

This implies that $\left(u_j^{(n)} - u_j\right) \in l_{\Pi_{p,q}}(X,Y)$ and that $\left(u_j^{(n)}\right)$ converges to (u_j). But

$$u_j = u_j^{(n)} - \left(u_j^{(n)} - u_j\right) \in l_{\Pi_{p,q}}(X,Y).$$

Hence

$$(u_j) \in l_{\Pi_{p,q}}(X,Y)$$

and thus $l_{\Pi_{p,q}}(X,Y)$ is Banach. □

CHAPTER 4

CONCLUSION

The concept of Banach spaces is basic to the study functional analysis, hence the significance of identifying that a given space is Banach.

In this research, we have shown that the space of all (p, q)-summing multipliers, $l_{\Pi_{p,q}}(X,Y)$, between Banach spaces X and Y, is itself a Banach space under the operator norm denoted

$$\pi_{p,q}\,(u_j) = \sup\{\|(u_j x_j)\| : (x_j) \in l_p^w(X), \|(x_j)\| = 1\}.$$

It is worth noting that the context of our work has been a finite family of vectors $x_1,\ldots,x_n$ in X. This is in keeping with the Grothendieck theorem and the work already done in summing operators.

REFERENCE

[1] AYWA, S.[1999] *Compact Operators on Sequences and Function Spaces: Characterizations and Duality,* PhD Thesis, Potchefsroom University

[2] CONWAY, J. B. [1985] *A Course in Functional Analysis,* Spinger-Verlag N.Y.

[3] DIESTEL, J., JARCHOW, H. AND TONGE, A. [1995] *Absolutely Summing Operators*. Cambridge Studies in Advanced Mathematics 43, Cambridge University Press.

[4] DOUGLAS, R. G. [1972] *Banach Algebra Techniques in Operator Theory*. Academic Press New York

[5] DVORETZKY, A AND ROGERS, C. A [1950] Absolute and Unconditional Convergence in Normed Linear Spaces. Proc. Nat. Acad. USA **36** 192-197.

[6] FOURIE, J. H. [1991] Operator Summing Multipliers And Topologies On L(X,Y). Quaestiones Math., **14**(1):51-64

[7] FOURIE, J H. AND AYWA, S. [2001] On Summing Multiplier and Their Applications. Journal of Mathematics Analysis and Application 253, 166-186.

[8] GROTHENDIECK, A [1953a] Resume de la Theorie des Produits Tensoriels Topologiques. Bol. Soc. Mat. Sao Paulo **8** (1953/1956) 1-79.

[9] KREYSZIG, E. [1978] *Introductory Functional Analysis with Applications* John Wiley & Sons. New York.

[10] LINDENSTRAUSS,J AND PELCZYNSKI, A [1968] Absolutely Summing Operators in L_p-spaces and their Application. Studia Math. **29,** 275-326.

[11] MADDOX, I. J. [1988] *Elements of Functional Analysis*, 2nd Edition, Cambridge University Press, Cambridge

[12] PIETSCH, A [1967], Absolut p-simmierende Abbildungen in Normierten Raumen. Studia Math. **28,** 333-353.