On Joint Essential Numerical Ranges

by

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Abstract

The concept of numerical range on a Hilbert space was first introduced by O. Toeplitz in 1918 for matrices. This notion was independently extended by G. Lumer and F. Bauer in the sixties on finite dimensional Banach spaces. J. G. Stampfli introduced the maximal numerical range, proved its convexity and used it to derive an identity for the norm of derivation in 1970. In 1972, J. G. Stampfli and J. P. Williams defined and studied the essential numerical range of an operator. In our work, we looked at the joint essential numerical ranges. In particular, this study has shown that the properties of numerical ranges such as compactness, nonemptiness and convexity do hold for the joint essential numerical range. The study has also shown that the closure of the joint numerical range of an operator is star-shaped with elements in the joint essential numerical range of the operator as star centers. Further, we have shown that the joint essential spectrum is contained in the joint essential numerical range by looking at the boundary of the joint spectrum. Convexity, nonemptiness and compactness of the joint essential numerical range were shown by first proving the equivalent definitions of the joint essential numerical range. Basing on the convexity of the joint essential numerical range, other results were obtained. The results of this study are helpful in the development of the research on numerical ranges and may also be applied by mathematicians in solving several problems in operator theory.

Chapter 1

Introduction

1.1 Background Information

Denote by $B(X)$ the algebra of bounded linear operators acting on complex Hilbert space X. The numerical range of $T \in B(X)$ is defined as

$$W(T) = \{\langle Tx, x \rangle : x \in X, \langle x, x \rangle = 1\}$$

which is useful in studying operators, see [2, 5, 13, 14, 28, 29, 30]. Let $\mathcal{A}$ be a complex normed algebra with unit e, let $a = (a_1, ..., a_m) \in \mathcal{A}^m$. The joint algebra numerical range $V_m(a, \mathcal{A})$ is defined by

$$V_m(a, \mathcal{A}) = \{(f(a_1), ..., f(a_m)) : f \in \mathcal{A}^*, \| f \| = 1 = f(e)\}$$

where $\mathcal{A}^*$ is the set of all bounded linear functionals on $\mathcal{A}$. It was shown in [13] that $V_m(a, \mathcal{A})$ is always a compact convex subset of $\mathbb{C}^m$.

Denote the set of self-adjoint operators in $B(X)$ by $S(X)$. Since every $T \in B(X)$ admits a decomposition $T = T_1 + iT_2$ with $T_1, T_2 \in S(X)$,

$W(T)$ can be identified with

$$\{(\langle T_1x,x\rangle,\langle T_2x,x\rangle):x\in X,\langle x,x\rangle=1\}.$$

This leads to the joint Numerical range of $T=(T_1,...,T_m)\in S(X)^m$,

$$W_m(T)=\{(\langle T_1x,x\rangle,...,\langle T_mx,x\rangle):x\in X,\langle x,x\rangle=1\},$$

which has been studied by many researchers in order to understand the joint behaviour of several operarors $T_1,...,T_m$, see [1, 8, 16, 21, 22, 24, 31] .

Let $\mathcal{K}(X)$ be the ideal of all compact operators in $B(X)$. Researchers, while studying finite rank or compact perturbations of operators, considered the joint essential numerical range of $T\in S(X)^m$ as

$$W_{e_m}(T)=\bigcap\{\overline{W_m(T+K)}:K=(K_1,...,K_m)\in\mathcal{K}(X)\}.$$

When $m=2$, $W_{e_m}(T)$ is identified with essential numerical range of $T=T_1+iT_2\in B(X)$ defined by

$$W_e(T)=\bigcap\{\overline{W(T+K)}:K\in\mathcal{K}(X)\},$$

see [4, 10, 17, 19, 26, 27, 43]. The essential numerical range for $m=1$ was introduced and studied in [27]. The joint essential numerical range was studied, for example, in [10, 38] among others.

Convexity of $W_m(T)$ and $\overline{W_m(T+K)}$ has been studied by various researchers and concluded that $\overline{W_m(T+K)}$ is usually non-convex while $W_m(T)$ is convex for $m=1$ and is not convex in general for $m\geq 2$, see [

8, 13, 14, 19, 27, 32]. It is thus unexpected for the set $W_{e_m}(T)$ to be convex since it is an intersection of non-convex sets. This could be the reason why convexity of $W_{e_m}(T)$ is rarely discussed for $m > 3$. Many properties of $W_{e_m}(T)$ have been studied by some researchers under the assumption that $W_{e_m}(T)$ is convex. Other researchers studied $W_{e_m}(T)$ without discussing its convexity. This study has shown that $W_{e_m}(T)$ is always convex by first establishing several equivalent formulations of the joint essential numerical range for $T \in S(X)^m$ then showing that $\overline{W_m(T)}$ is star-shaped with the elements in $W_{e_m}(T)$ as star centers. We have also shown that $W_{e_m}(T)$ contains the joint essential spectrum $\sigma_{e_m}(T)$ by looking at the boundary of the joint spectrum. Here, the joint essential spectrum $\sigma_{e_m}(T)$ is defined as the joint spectrum $\sigma_m(T)$, where $\sigma_m(T) = \sigma^l_m(T) \bigcup \sigma^r_m(T)$ while the left (right) joint spectrum $\sigma^l_m(T)(\sigma^r_m(T))$ is the set of all $\lambda = (\lambda_1, ..., \lambda_m) \in \mathbb{C}^m$ such that $\{T_i - \lambda_i\}_{i=1}^m$ generates a proper left(right) ideal in the Calkin algebra of X.

1.2 Literature Review

The joint essential numerical range has lots of application in mathematics, mostly in solving problems in operator theory. For the purpose of this study it was important to have a good background in functional analysis, operator theory and general topology. Knowledge of numerical ranges, algebra numerical ranges and the joint essential spectrum was most important for this study.

Numerical ranges of a single operator T has been greatly studied and its properties determined. In [13, 28, 29, 32, 33, 39], it was established that this numerical range of a single operator T is convex and its closure contains the spectrum of the operator T. For a normal operator T, they established that the closure of the numerical range is the convex hull of the spectrum of T. It was also shown that the numerical radius is norm equivalent to the operator norm $\| T \|$ which satisfies $\frac{1}{2} \| T \| \leq w(T) \leq \| T \|$.

Much work has also been done on the algebra numerical range and its properties. In [13] for instance, F. F. Bonsall and J. Duncan showed that the algebra numerical range is a compact convex subset of a complex plane.

It is worth noting that much has also been done on the essential numerical ranges [10, 27, 34]. The essential numerical range for $m = 1$ was introduced and studied in [27] by P. A. Fillmore, J. G. Stampfli and J. P. Williams. It is therefore known that the essential numerical range of T is contained in the closure of the numerical range of T. This knowledge and much more about the essential numerical range was vital for this study. The joint numerical range of T has too been studied by several researchers. Convexity of the joint numerical range and that of the closure of the joint numerical range was studied by several researchers and concluded that the closure of the joint numerical range is usually non-convex while the joint numerical range is convex for $m = 1$ and is not convex in general for $m \geq 2$, [8, 13, 14, 19, 27, 32].

The study of the Joint essential numerical ranges has captured great interest especially in [17, 38] in which it was defined and several of its properties examined. It was thus unexpected for the set of the joint essential numerical range to be convex since it is an intersection of non-convex sets. This could be the reason why convexity of the joint essential numerical range is rarely discussed for $m > 3$. Many properties of $W_{e_m}(T)$ have been studied by some researchers under the assumption that the joint essential numerical range is convex. Other researchers studied the joint essential numerical range without discussing its convexity. This study, being an extension of the study of the numerical ranges, has determined that the properties of the numerical ranges also hold for the joint essential numerical ranges. In particular, the study has shown that the joint essential numerical range is always a convex set.

It must be noted that much has also been done on the joint essential spectrum. In the case of a single operator, the boundary points of the numerical range of an (self-adjoint) operator belong to its spectrum; consequently in the finite-dimensional case, they are in fact eigenvalues. Abramov [1], Buoni and Wadhwa [16] have investigated the relation between the joint spectrum and the joint numerical range. Abramov [1]

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showed that the conical point of the closure of the joint numerical range of an m-tuple of operators $T = (T_1, ..., T_m)$ belongs to the joint approximate point spectrum of T. Dash [23] studied the relationship between the joint essential spectrum and the joint spectrum and showed that the two are equal. This fact was useful in the sequel.

In this study, we also showed that the joint essential spectrum is contained in the joint essential numerical range. This study relied heavily on [27].

1.3 Statement of the Problem

The properties of $W_{e_m}(T)$ have not been exhaustively studied. It was not clear whether the general properties of the numerical ranges hold for $W_{e_m}(T)$. This study therefore investigated whether the properties of the classical numerical ranges also hold for the joint essential numerical range.

1.4 Objectives of the Study

This study was aimed at:

1. Determining whether $W_{e_m}(T)$ is a compact convex set.
2. Investigating whether $\overline{W_m(T)}$ is always star-shaped with elements in $W_{e_m}(T)$ as star centers.
3. Investigating whether $\sigma_{e_m}(T) \subset W_{e_m}(T)$

1.5 Significance of the Study

The study of the joint essential numerical ranges is of great interest to mathematicians since its knowledge plays an important role in solving several problems in operator theory. This study was an extension of the study of numerical ranges. We sincerely hope that the results obtained will be vital in the development of the research on numerical ranges and may also be applied by mathematicians in solving several problems in operator theory.

1.6 Research Methodology

To obtain the results, we first proved many equivalent definitions of the joint essential numerical range. This study then showed that the closure of the joint numerical range is star-shaped with the elements in the joint essential numerical range as star centers. To determine whether the joint essential spectrum is contained in the joint essential numerical range, this study looked at the boundary of the spectrum.

Chapter 2

Basic Concepts

2.1 Introduction

In this chapter, definitions essential to the study are simplified and given.

2.2 Normed and Banach Spaces

Definition 2.2.1

A real valued function

$$\| \, . \, \|: V \rightarrow \mathbb{R}$$

is called a norm on a vector space V if it satisfies the following properties:

(i) $\| x \| \geq 0 \; \forall \, x \in V$

(ii) $\| x \| = 0 \iff x = 0$

(iii) $\| \alpha x \| = | \alpha | \| x \| \;\; \forall \, x \in V$ and $\alpha \in \mathbb{K}$

(iv) $\| x + y \| \leq \| x \| + \| y \| \;\; \forall \, x, y \in V$ (Triangle inequality)

$| . |$ denotes the usual absolute value. If $\| . \|$ is a function with properties (iii) and (iv) only it is called a semi-norm.

Definition 2.2.2

A normed space X is a vector space with a norm defined on it. We denote it by $(X, \| . \|)$. If $\mathbb{K} = \mathbb{R}$ it is a real normed space. If $\mathbb{K} = \mathbb{C}$ it is a complex normed space.

Definition 2.2.3

A sequence (x_n) is a Cauchy sequence (or fundamental) if $\forall \, \epsilon > 0 \, \exists \, N = N(\epsilon)$ such that for all $n, m > N \Longrightarrow \| x_n - x_m \| < \epsilon$.

Definition 2.2.4

A normed linear space X is said to be complete if all Cauchy sequences in X are convergent.

Definition 2.2.5

A Banach space is a complete normed space.

Definition 2.2.6

An inner product space is a linear space X together with an inner product $\langle , \rangle : X \times X \longrightarrow \mathbb{K}$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ such that,

(i) $\langle \, x, y \, \rangle = \langle \, \overline{y, x} \, \rangle$

(ii) $\langle \, \lambda \, x + \mu \, y, z \, \rangle = \lambda \, \langle \, x, z \, \rangle + \mu \, \langle \, y, z \, \rangle$

(iii) $\langle \, x, x \, \rangle \geq 0$ with equality only for $x = 0$, the zero vector.

The function $\langle , \rangle$ is called an inner or scalar product. We denote the inner product space (or sometimes a pre-Hilbert space) by $(\, X, \langle , \rangle \,)$.

Definition 2.2.7

The norm $\| . \|$ in X given by $\| \, x \, \| = \sqrt{\langle x, x \rangle}$ is called the norm defined by the inner product $\langle , \rangle$ in the inner product space X.

Definition 2.2.8

A Hilbert space $(X, \langle , \rangle)$ is a strongly complete inner product space.

Definition 2.2.9

A subset S of vectors of X is said to be orthonormal if

$$\text{(i) } \| \ x \ \| = 1 \, \forall \, x \, \in \, S$$

$$\text{(ii) } \langle x, y \rangle \ = \ 0 \text{ if } x \ \neq \ y \text{ and } x, y \ \in \ S$$

Definition 2.2.10

An operator is a mapping from a vector space X to a vector space Y over the same field that preserves the algebraic properties of the vector spaces.

Definition 2.2.11

An operator T is linear if:

(i) The domain $\mathcal{D}(T)$ is a vector space and the range lies in the vector space over the same field.

(ii) For all $x, y \ \in \mathcal{D}(T)$ and scalar $\lambda, \ \alpha \in \mathbb{K}$,

$$T(\lambda \, x \ + \ \alpha \, y) \ = \ \lambda T x \ + \ \alpha T y.$$

Definition 2.2.12

A linear operator $T : D(T) \longrightarrow Y$, where $D(T) \subset X$, is said to be bounded if there is a real number $m \ > \ 0$ such that

$$\| \, Tx \, \| \leq \ m \, \| \, x \, \| \quad \forall \, x \, \in D(T) \qquad \textbf{(2.1)}.$$

Here, the norm on the left is that of Y, and the norm on the right is that of X. The norm of the bounded linear operator T is

$$\| T \| = \sup_{x \in D(T)\,:\, x \neq 0} \frac{\| Tx \|}{\| x \|}.$$

With $m = \| T \|$, the above formula, (**2.1**) becomes

$$\| Tx \| \leq \| T \| \, \| x \| .$$

Definition 2.2.13

The adjoint of a linear operator $T \in B(X)$ is a linear operator $T^* \in B(X)$ defined by the relation $\langle Tx, y\rangle = \langle x, T^*y\rangle \ \forall\, y, x \in X$.

Definition 2.2.14

A bounded linear operator $T : X \longrightarrow X$ on a Hilbert space X is said to be

- Self-adjoint or Hermitian if $T^* = T$,
- Normal if $TT^* = T^*T$,
- Unitary if T is bijective and $T^* = T^{-1}$,
- Hyponormal if $T^*T - TT^* \geq 0$ or equivalently, $\| Tx \| \geq \| T^*x \|$ $\forall\, x \in X$.

Definition 2.2.15

If $\mathcal{M}$ is a closed linear subspace of the Hilbert space X then X is the orthogonal direct sum of $\mathcal{M}$ and $\mathcal{M}^{\perp}$, and we write $X = \mathcal{M} \oplus \mathcal{M}^{\perp}$. If every $x \in X$ is written uniquely in the form $x = y + z \, : y \in \mathcal{M}, \ z \in \mathcal{M}^{\perp}$, we write $x = y \oplus z$.

The operator $P = P\mathcal{M}$ defined by $Px = y$ is the projection (or orthogonal projection) on $\mathcal{M}$. The space $\mathcal{M}$ is called the subspace of the projection P.

Definition 2.2.16

For a Hilbert space X, $P \in B(X)$ is called an idempotent if $P^2 = P$. If P is an idempotent, then so is $1 - P$.

Definition 2.2.17

A functional is a mapping from a Normed space X into the scalar field $\mathbb{R}$ or $\mathbb{C}$.

A functional $f : D(f) \longrightarrow \mathbb{K}$ is linear if it's domain is in a vector space X and the range is in the scalar field $\mathbb{K}$ of X.

Definition 2.2.18

A linear functional $f : X \longrightarrow \mathbb{R}$ or $\mathbb{C}$ is said to be bounded if there exists a real number $m > 0$ such that

$$| f(x) | \leq m \| x \| \; \forall x \in X \quad \textbf{(2.2)}.$$

Furthermore, the norm of f is

$$\| f \| = \sup_{x \in D(f) \,:\, |x\|=1} | f(x) | \,.$$

With $m = \| f \|$, formula **(2.2)** now implies

$$| f(x) | \leq \| f \| \| x \| \,.$$

We shall denote the set of bounded linear functional by $\mathcal{A}^*$.

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Definition 2.2.19

X^*, the set all linear functionals on a vector space X, is the dual space of X.

2.3 Compact Operators

Definition 2.3.1

A subset S of a normed linear space is compact if and only if every sequence (s_m) in S has a subsequence $(s_{m_j}) = (s_{m_1}, s_{m_2}, ...)$ that converges in S.

Definition 2.3.2

A pre-compact subset in a normed linear space is one whose closure is compact.

A linear map $T : X \rightarrow Y$, where X and Y are a pre-Hilbert space and a Hilbert space respectively, is compact if it maps the unit ball in X to a pre-compact set in Y.

Equivalently, T is compact if and only if it maps bounded sequences in X to sequences in Y with convergent subsequences. That is, $T \in B(X)$ is compact if for every bounded sequence $(x_m) \in X$, the sequence $(Tx_m) \in Y$ has convergent subsequence.

Compact operators on X will be denoted by $\mathcal{K}(X)$.

A compact linear operator is continuous, whereas the converse is not always true.

Definition 2.3.3

The rank of an operator is the dimension of its range. An operator with finite dimensional range is therefore of finite rank.

The identity operator on a Hilbert space X is compact if and only if X is finite dimensional.

2.4 Algebras

Definition 2.4.1

An algebra over a field $\mathbb{K}$ is a vector space $\mathcal{A}$ such that for each ordered pair of elements $x, y \in \mathcal{A}$ a unique product $xy \in \mathcal{A}$ is defined with the properties:

$$\begin{aligned} (xy)z &= x(yz) \\ x(y+z) &= xy + xz \\ (x+y)z &= xz + yz \\ \alpha(xy) &= (\alpha x)y = x\,(\alpha\, y)\ \forall\, x,\ y\ \in X,\ \alpha\ \in\ \mathbb{K},\ \ \mathbb{K} = \mathbb{R}\ or\ \mathbb{C} \end{aligned}$$

A normed Algebra is a Normed space $\mathcal{A}$ which is an Algebra such that $\forall\, x,\ y\ \in \mathcal{A}$

i) $\parallel x\, y \parallel \leq \parallel x \parallel \parallel y \parallel$

ii) and if $\mathcal{A}$ has an identity e, then $\parallel e \parallel = 1$

A Banach algebra is a complete Normed algebra.

Definition 2.4.2

A subalgebra of an algebra $\mathcal{A}$ is a vector subspace $\mathcal{M}$ such that $\forall\, x, x^{'} \in \mathcal{M}$, we have $xx^{'} \in \mathcal{M}$.

Definition 2.4.3

An involution on an algebra $\mathcal{A}$ is a mapping $^* : \mathcal{A} \longrightarrow \mathcal{A}$ defined by $x \longmapsto x^*$ such that $\forall\, x, y \in \mathcal{A}$, and $\lambda \in \mathbb{C}$, the following conditions are satisfied:

$$\text{(i) } (x+y)^* = x^* + y^*$$

$$\text{(ii) } (\lambda x)^* = \bar{\lambda} x^*$$

$$\text{(iii) } (xy)^* = y^* x^*$$

$$\text{(iv) } (x^*)^* = x$$

*-algebra or an involutive algebra is an algebra $\mathcal{A}$ with an involution.

Definition 2.4.4

A Banach *-algebra is a Banach algebra $\mathcal{A}$ with an involution satisfying the property

$$\| x \| = \| x^* \|, \forall\, x \in \mathcal{A}.$$

Definition 2.4.5

A *-algebra (with identity) is called symmetric if $e + x^*x$ has an inverse for every $x \in \mathcal{A}$.

Definition 2.4.6

C^* - algebra is a symmetric Banach * - algebra $\mathcal{A}$ such that $\| xx^* \| = \| x \|^2 \ \forall\, x \in \mathcal{A}$.

Definition 2.4.7

A positive linear functional is a linear functional f on a Banach algebra $\mathcal{A}$ with an involution that satisfies

$$f(xx^*) \geq 0 \,\forall\, x \in \mathcal{A}.$$

Definition 2.4.8

A left (or right) ideal in an algebra $\mathcal{A}$ is a vector subspace $\mathcal{M} \subset \mathcal{A}$ such that for all $x \in \mathcal{A}$ and $y \in \mathcal{M}$, $xy \in \mathcal{M}$ (or $yx \in \mathcal{M}$).

An ideal in $\mathcal{A}$ is a vector subspace that is both a left and right ideal in $\mathcal{A}$.

2.5 Convex and Star-shaped Sets.

Definition 2.5.1

A subset C of a linear space M is convex if $\forall\ x, y \in C$ the segment joining x and y is contained in C, that is, $tx + (1-t)y \in C\ \forall\ t \in [\,0, 1\,]$.

A set S is starshaped if $\exists\ y \in S$ such that $\forall\ x \in S$ the segment joining x and y is contained in S, that is $\lambda x + (1-\lambda)y \in S\ \ \forall\ \lambda \in [\,0, 1\,]$.

A point $y \in S$ is a star center of S if there is a point $x \in S$ such that the segment joining x and y is contained in S.

Starshapedness is related to convexity in that a convex set is starshaped with all its points being star centers. A starshaped set is not necessarily convex.

Recall that a subset M of $\mathbb{C}$ is connected if it cannot be split into two nonempty open sets.

Definition 2.5.2

A norm $\|.\|$ is strictly convex if $\| x \| = 1$, $\| y \| = 1$, $\| x + y \| = 2$ together imply that $x = y$.

Strict convexity is automatic for Hilbert spaces.

Definition 2.5.3

If M is a subset of a linear space X, then a convex hull M, represented by $conv(M)$ is the smallest convex subset of X containing M and is thus the intersection of all convex subsets of X that contain M.

2.6 Numerical Ranges and the Spectrum

Definition 2.6.1

The numerical range of a bounded single linear operator T on a Hilbert space X is subset of the complex numbers given by

$$W(T) = \{\langle Tx, x\rangle \ : \ x \in X, \ \| x \| = 1 \}.$$

Theorem 2.6.2

For any operator T on X, the following properties hold:

(i) $W(\alpha I + \beta T) \ = \ \alpha + \beta W(T) \ \forall \, \alpha, \beta \in \mathbb{C}$.

(ii) $W(T^*) \ = \ \{\overline{\lambda} \ : \ \lambda \in W(T)\}$.

(iii) $W(U^*TU) \ = \ W(T)$ *for all unitary operators* U *(i.e.* $U^*U \ = \ I \ = \ UU^*$*).*

P. Halmos book [29] has a detailed account of this subject.

Definition 2.6.3

The spectrum of an operator T is defined as

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible } \}.$$

Definition 2.6.4

The resolvent set $\rho(T)$ of the operator T is defined as

$$\rho(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is invertible } \}.$$

Lemma 2.6.5

(Ellipse Lemma). Let T be an operator on a two dimensional space. Then $W(T)$ is an ellipse whose foci are the eigenvalues of T.

See Gustafson and Rao [28] for the proof.

Theorem 2.6.6

(Toeplitz-Hausdorff). The numerical range of an operator is convex.

See Gustafson and Rao [28] for the proof.

Theorem 2.6.7

The spectrum of an operator is contained in the closure of the numerical range.

This was proved by Gustafson and Rao [28] by looking at the boundary of the spectrum which is included in the approximate point spectrum.

Definition 2.6.8

An operator $T \in B(X)$ is convexoid if

$$\overline{W(T)} = \operatorname{conv} \sigma(T).$$

Here conv $\sigma(T)$ is the convex hull of the spectrum of T.

Definition 2.6.9

The numerical radius $w(T)$ and the spectral radius $r(T)$ of the operator T are defined as follows:

$$w(T) = \sup\{|\lambda| : \lambda \in W(T)\} = \sup\{|\langle Tx, x\rangle|, \| x \| = 1\}$$

and

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

Definition 2.6.10

An operator $T \in B(X)$ is

(i) spectraloid if $w(T) = r(T)$

and

(ii) normaloid if $\| T \| = r(T)$

or equivalently

$$\| T \|^m = \| T^m \| \quad (m = 1, 2, ...)$$

P. Halmos [29] showed that the classes of normaloids and convexoids are both contained in the class of spectraloids.

Definition 2.6.11

A point $\lambda = (\lambda_1, ..., \lambda_m) \in W_m(T)$ is an extreme point if λ is not in any open line segment contained in $W_m(T)$.

It is known that a set of extreme points is contained in a set of boundary points.

For a bounded normal operator T on a Hilbert space X, the extreme points of the closure of the numerical range are in the spectrum of T. This is from the fact that the convex hull of the spectrum is the closure of the numerical range and because the extreme points of the convex hull of a compact set are in the compact set.

See Berberian [11] for this and more.

Theorem 2.6.12

Every extreme point of the numerical range $W(T)$ is an eigenvalue of the spectrum $\sigma(T)$.

See J.G. Stampfli [39] for the proof.

Definition 2.6.13

An operator $T \in B(X)$ is paranormal if

$$\| Tx \|^2 \leq \| T^2 x \| \| x \| \quad \forall\, x \in X.$$

Definition 2.6.14

The algebra numerical range of an arbitrary element $a \in \mathcal{A}$ is defined by

$$V(a) = \{f(a) : f \in \mathcal{A}^*,\ f(I) = 1 = \| f \|\}$$

where $\mathcal{A}$ is a complex Normed algebra with unit and $\mathcal{A}^*$ the set of all bounded linear functionals on $\mathcal{A}$.

Definition 2.6.15

Let $\mathcal{K}(X)$ be the ideal of all compact operators acting on complex Banach space X, and let π be the canonical projection from $B(X)$ onto the Calkin algebra $B(X)/\mathcal{K}(X)$. Denote also by $\| \,.\, \|_e$ the essential norm $\| T \|_e = \inf \{\| T + K \| : K \in \mathcal{K}(X)\}$.

The essential numerical range $W_e(T)$ of T is defined by

$$W_e(T) = V(\pi(T), B(X)/\mathcal{K}(X), \|\cdot\|_e)$$

for $T \in B(X)$ and X is infinite dimensional space.

Definition 2.6.16

Let $B(X)$ denote the set of bounded linear operators acting on the complex Banach space X. An operator $T \in B(X)$ is said to be Fredholm if it has closed range with finite dimensional null space and its range of finite co-dimension.

We shall denote the null space and range of T by $\mathcal{N}(T)$ and $\mathcal{R}(T)$ respectively.

The index of a Fredholm operator $T \in B(X)$ is given by

$$i(T) = \alpha(T) - \beta(T)$$

where $\alpha(T) = \dim\ (\mathcal{N}(T))$, and $\beta(T) = \text{codim}\ (\mathcal{R}(T))$.

Definition 2.6.17

An operator $T \in B(X)$ is called Weyl if it is Fredholm of index zero.

An operator $T \in B(X)$ is called Browder if it is Fredholm and $T - \lambda I$ is invertible for $\lambda \neq 0 \in \mathbb{C}$.

Definition 2.6.18

The essential spectrum $\sigma_e(T)$, the Weyl's spectrum $\sigma_w(T)$ and the Browder's spectrum $\sigma_b(T)$ are defined by

$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm }\}$

$\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl }\};$

$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder }\}.$

Definition 2.6.19

The joint spectrum $\sigma_m(a)$ of a commuting m-tuple of elements $a = (a_1, ..., a_m) \in X$ is defined as

$$\sigma_m(a) \; = \; \sigma_m^l(a) \; \cup \; \sigma_m^r(a)$$

where the left (right) joint spectrum $\sigma_m^l(a)(\sigma_m^r(a))$ is the set of all $\lambda = (\lambda_1, ..., \lambda_m) \in \mathbb{C}^m$ such that $\{b_i - \lambda_i\}_{i=1}^m$ generates a proper left (right) ideal in the Calkin algebra and $b_i = \pi(T_i)$ is the coset containing $T_i \;\; \forall\, i \in [1, m]$ and π the canonical homomorphism from $B(X)$ to the Calkin algebra $B(X)/\mathcal{K}(X)$.

Consult F. F. Bonsall and J. Duncan [13] for the notion of the joint spectrum.

Definition 2.6.20

The joint essential spectrum of an m-tuple of operators $T = (T_1, ..., T_m)$ denoted by $\sigma_{e_m}(T)$ is the joint spectrum $\sigma_m(a)$ of $a = (a_1, ..., a_m) \in X$.

The set $\sigma_m^l(T)(\sigma_m^r(T))$ is known as the left (right) joint essential spectrum and denoted by $\sigma_{e_m}^l(T)(\sigma_{e_m}^r(T))$. The $\sigma_{e_m}(T)$ is a nonempty compact subset of $\mathbb{C}^m$ for an m-tuple of essentially commuting (commuting modulo the compact) operators $T = (T_1, ..., T_m)$.

According to A. T. Dash [23], the joint essential spectrum $\sigma_{e_m}(T)$ of $T = (T_1, ..., T_m)$ is equivalently defined as

$$\sigma_{e_m}(T) = \sigma^l_{e_m}(T) \cup \sigma^r_{e_m}(T)$$

where

$\sigma^l_{e_m}(T) = \{\lambda = (\lambda_1, ..., \lambda_m) \; : \; B_1(T_1 - \lambda_1 I) \; + ... + \; B_m(T_m - \lambda_m I)$ is not
a Fredholm operator for all operators $B = (B_1, ..., B_m)$ on X $\}$

and

$\sigma^r_{e_m}(T) = \{\lambda = (\lambda_1, ..., \lambda_m) \; : \; (T_1 - \lambda_1 I)B_1 \; + ... + \; (T_m - \lambda_m I)B_m$ is not
a Fredholm operator for all operators $B = (B_1, ..., B_m)$ on X $\}$.

Definition 2.6.21

A point $\lambda = (\lambda_1, ..., \lambda_m) \in C^m$ is the joint approximate compression spectrum $\sigma_c(T)$ of T if and only if there exists a sequence $\{x_n\}$ of unit vectors in X such that

$$\| (\lambda_i - T_i)^* x_n \| \longrightarrow 0 \; (n \longrightarrow \infty), \; i = 1, ..., m.$$

Definition 2.6.22

A point $\lambda = (\lambda_1, ..., \lambda_m) \in W_m(T)$ is a bare point of $W_m(T)$ if there is a spherical surface through λ such that no points of $W_m(T)$ lie outside this spherical surface.

The set of the bare points of $W_m(T)$ is included in the set of extreme points of K and dense in it. See S. K. Berberian [11].

Definition 2.6.23

A joint approximate point spectrum $\sigma_\pi(T)$ of an operator $T = (T_1, ..., T_m)$ is a point $\lambda = (\lambda_i, ..., \lambda_m) \in \mathbb{C}^m$ such that for a sequence $\{x_n\}$ of unit vectors in X we have

$$\| (\lambda_i - T_i)x_n \| \longrightarrow 0 \quad (n \longrightarrow \infty), \ \ i = 1, ..., m.$$

A joint eigenvalue (point spectrum) , $\sigma_p(T)$, of an operator $T = (T_1, ..., T_m)$ is a point $\lambda = (\lambda_1, ..., \lambda_m)$ such that for a nonzero eigenvector x there is

$$T_i x \ = \ \lambda_i x, \ i = 1, ..., m.$$

The joint residual spectrum $\sigma_r(T)$ of T is a point $\lambda = (\lambda_1, ..., \lambda_m)$ such that for a nonzero vector x there is

$$T_i^* x \ = \ \overline{\lambda}_i x, \ i = 1, ..., m$$

where $\overline{\lambda}_i$ is the complex conjugate of λ_i and T_i^* is the adjoint of the operator T_i.

Definition 2.6.24

If we denote by F, the set of all Fredholm operators, then if for $F \subset \mathbb{C}^m$ and $\lambda = (\lambda_1, ..., \lambda_m) \in F$ there exists a closed convex cone K with vertex $(0, ..., 0)$ such that $F \subset K - \lambda$, we shall call the point λ a conical point of F. Here, a closed subset $K \subset \mathbb{C}^m$ satisfies the following properties:

(1) $K + K \subset K$,

(2) $\alpha K \subset K$ for all $\alpha \ \geq \ 0$,

(3) $K \cap (-K) = \{(0, ..., 0)\}$.

Definition 2.6.25

For any $m-$ tuple $T = (T_1, ..., T_m)$ of operators, the joint operator norm, joint numerical radius, joint spectral radius and joint approximate point spectral radius respectively, of T are defined by:

$$\| T \| = \sup \{(\| T_1 x \|^2 + ... + \| T_m x \|^2)^{1/2} : \| x \| = 1\},$$

$$w_m(T) = \sup \{(| \langle T_1 x, x \rangle |^2 + ... + | \langle T_m x, x \rangle |^2)^{1/2} : \| x \| = 1\},$$

$$r_m(T) = \sup \{(| \lambda_1 |^2 + ... + | \lambda_m |^2)^{1/2} : \lambda \in \sigma(T)\},$$

$$r_\pi(T) = \sup \{(| \lambda_1 |^2 + ... + | \lambda_m |^2)^{1/2} : \lambda \in \sigma_\pi(T)\}.$$

Chapter 3

Numerical Ranges

3.1 Introduction

The concept of numerical range or the classical field of values on a Hilbert space was first introduced by Toeplitz in 1918 for matrices. In 1962, Bauer introduced the notion of numerical range on finite dimensional Banach spaces. The subject of numerical range and numerical radius has connections and applications to various areas such as C^*-algebras, iterations methods, Krein space operators, factorisation of matrix polynomials, dilation theory and unitary similarity which all constitute an active field of research in operator theory [5, 7, 9, 12]. For a bounded linear operator T on a Hilbert space X, the numerical range $W(T)$ has been studied by various writers and its properties given. It is therefore known that $W(T)$ is bounded (not necessarily closed) convex set whose closure $\overline{W(T)}$ contains the spectrum $\sigma(T)$ of T. If T is normal, $\overline{W(T)} = \text{conv } \sigma(T)$ and T is said to be convexoid. Furthermore, the extreme points of $W(T)$ are eigenvalues of T. See [16, 18, 19, 28] for these and more details.

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The notion of the joint numerical range $W_m(T)$ was investigated by Halmos [29] and Dash [22] among others. They sought to find out how much of the knowledge about the numerical range in the single operator case carried over to the analogous situation in the case of an m-tuple of operators. Studying convexity of $W_m(T)$, researchers concluded that $W_m(T)$ was convex for $m = 1$ and not convex in general for $m \geq 2$.

This first section of this Chapter focuses on the numerical range of two linear operators on a Hilbert space. The study has in this first section showed that the properties of the numerical range of a single operator also hold for the numerical range of two operators. The second section of this chapter will focuss on the joint algebra numerical range.

3.2 Numerical Range for two Linear Operators.

Definition 3.2.1

The joint numerical range $W(T, A)$ of two linear operators $T, A \in B(X)$ where X is a Hilbert space is defined as

$$W(T, A) = \{\langle Tx, x\rangle, \langle Ax, x\rangle : x \in X \ \langle x, x\rangle = 1\}.$$

The numerical radius $w(T, A)$ is defined by

$$w(T, A) = \sup \{|\lambda| : \lambda \in W(T, A)\}.$$

The spectrum $\sigma(T, A)$ of two linear operators T and A is defined as

$$\sigma(T, A) = \{\lambda \in \mathbb{C} : ((T - \lambda I)\ (A - \lambda I)) \text{ is not invertible } \}.$$

The spectral radius $r(T, A)$ is defined as

$$r(T, A) = \sup \{|\lambda| : \lambda \in \sigma(T, A)\}.$$

The joint eigenvalue (point spectrum), $\sigma_p(T, A)$, of the operators $T, A \in B(X)$ is a point $\lambda \in \mathbb{C}$ such that for a nonzero eigenvector x there is

$$(Tx - \lambda x)(Ax - \lambda x) = 0.$$

The joint approximate point spectrum $\sigma_\pi(T, A)$ of the operators $T, A \in B(X)$ is a point $\lambda \in \mathbb{C}^m$ such that for a sequence $\{x_m\}$ of unit vectors in X with $\| x_m \| = 1$ we have

$$\| (T - \lambda)x_m \| \, \| (A - \lambda)x_m \| \longrightarrow 0 \quad (m \longrightarrow \infty).$$

The following theorem, whose proof was adopted from Charles Amelin [3], shows that the numerical range of two linear operators is convex.

Theorem 3.2.2

$W(T, A)$ is a convex set.

PROOF. Let $(\gamma_1, \gamma_2) \in W(T, A)$. Let also $(y_1, y_2) \in X$ such that $\langle Ty_1, y_1 \rangle = \gamma_1$, $\langle Ty_2, y_2 \rangle = \gamma_2$, $\langle Ay_1, y_1 \rangle = 1$ and $\langle Ay_2, y_2 \rangle = 1$. Consider the binary forms $C_i : C \times C \to C$, $i = 1, 2$ defined by

$$C_1(\alpha_1, \alpha_2) = \langle T(\alpha_1 y_1 + \alpha_2 y_2), (\alpha_1 y_1 + \alpha_2 y_2) \rangle$$

$$= \gamma_1\alpha_1\overline{\alpha}_1 + \overline{\alpha}_1\alpha_2 \langle Ty_2, y_1 \rangle + \alpha_1\overline{\alpha}_2 \langle Ty_1, y_2 \rangle + \gamma_2\alpha_2\overline{\alpha}_2,$$

$$C_2(\alpha_1, \alpha_2) = \langle A(\alpha_1 y_1 + \alpha_2 y_2), (\alpha_1 y_1 + \alpha_2 y_2) \rangle$$

$$= \alpha_1\overline{\alpha}_1 + \overline{\alpha}_1\alpha_2 \langle Ay_2, y_1 \rangle + \alpha_1\overline{\alpha}_2 \langle Ay_1, y_2 \rangle + \alpha_2\overline{\alpha}_2.$$

It is to be shown that $C_2 = 1$ while C_1 assumes every value on the line segment that joins γ_1 and γ_2. To do this, for

$$(\gamma_1 - \gamma_2)\alpha_{12} = \langle Ty_2, y_1 \rangle - \gamma_2 \langle Ay_2, y_1 \rangle$$

and

$$(\gamma_1 - \gamma_2)\alpha_{21} = \langle Ty_1, y_2 \rangle - \gamma_2 \langle Ay_1, y_2 \rangle,$$

assume that

$$C(\alpha_1, \alpha_2) = \frac{C_1 - \gamma_2 C_2}{\gamma_1 - \gamma_2} = \alpha_1\overline{\alpha}_1 + \alpha_{12}\overline{\alpha}_1\alpha_2 + \alpha_{21}\alpha_1\overline{\alpha}_2.$$

Then, for C_1 to exhibit the above behaviour, it must be shown that while $C_2 = 1$, C takes on every real value from 0 to 1 inclusive. Choose ω of modulus 1 such that

$$Re\langle Ay_1, y_2 \rangle \overline{\omega} \geq 0 \text{ and}$$
$$(\alpha_{12} - \overline{\alpha}_{21})\, \omega \text{ is real.}$$

For real variables $\mathfrak{m}$ and $\mathfrak{n}$, let $\alpha_1 = \mathfrak{m}$ and $\alpha_2 = \omega\mathfrak{n}$. Let $\mathfrak{t} = \omega\alpha_{12} + \overline{\omega}\alpha_{21}$ be real and $z = Re\,(\omega \langle Ay_2, y_1 \rangle) \geq 0$. Thus $C = \mathfrak{m}^2 + \mathfrak{t}\mathfrak{m}\mathfrak{n}$ and $C_2 = \mathfrak{m}^2 + 2z\mathfrak{m}\mathfrak{n} + \mathfrak{n}^2$. Solving for $C_2(\mathfrak{m}, \omega\mathfrak{n}) = 1$, we obtain

$$\mathfrak{n} = -z\mathfrak{m} \pm \sqrt{1 + (z^2 - 1)\mathfrak{m}^2}$$

which is real valued for $\mathfrak{m} \in [-1, +1]$. Now $\mathfrak{n}$ is a function of $\mathfrak{m}$. With the above value of $\mathfrak{n}$, the form C becomes

$$C = \mathfrak{m}^2(1 - \mathfrak{t}z) \pm \mathfrak{t}\mathfrak{m}\sqrt{1 + (z^2 - 1)\mathfrak{m}^2}.$$

We can let $C_o(\mathfrak{m}) = C(\mathfrak{m}, \omega\mathfrak{n})$. For $\mathfrak{m} \in [0, 1], C_o(\mathfrak{m})$ is a continuous real valued function with $C_o(0) = 0$, $C_o(1) = 1$ so that $C_o(\mathfrak{m})$ assumes all values between 0 and 1 as we had desired. □

We now abbreviate the essential numerical range of two linear operators $T, A \in B(X)$ by $W_e(T, A)$.

Definition 3.2.3

The essential numerical range, $W_e(T, A)$ of two linear operators T and A is defined as

$$W_e(T, A) = \bigcap\{\overline{W((T, A) + K)} : K \in \mathcal{K}(X)\}.$$

Theorem 3.2.4

Let T be nonnegative, self-adjoint operator and $TA = AT$. Then

$$W_e(T, A) \subseteq W_e(T)\, W_e(A)$$

Proof. Suppose $\mu \in W_e(T, A)$, we must show that $\mu \in W_e(T)\, W_e(A)$. There is a sequence of unit vectors $(x_m) \in X$ converging weakly to $0 \in X$ such that

$$(\langle Tx_m, x_m\rangle\langle Ax_m, x_m\rangle) \longrightarrow \mu.$$

Let $e_m = T^{\frac{1}{2}}x_m$. There is $e_{m_k} = 0$ for some subsequence such that $0 \in W_e(T, A)$ and $0 \in W_e(T)\, W_e(A)$. If not, let $e_m \neq 0\ \forall\ m$. Let $y_m = \frac{e_m}{\|e_m\|}$.

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Thus $(y_m) \in X$ is sequence of unit vectors converging weakly to $0 \in X$ such that

$$(\langle Ay_m, y_m\rangle \ \langle Tx_m, x_m\rangle) \longrightarrow \mu.$$

Since $\langle Ay_m, y_m\rangle \in W_e(A)$, $\mu \in W_e(T)\, W_e(A)$ □

Theorem 3.2.5

The joint approximate point spectrum $\sigma_\pi(T, A)$ is contained in $\overline{W(T, A)}$

PROOF. Suppose $\lambda \in \sigma_\pi(T, A)$. There is a sequence $x_m \in X \ : \ \langle\ Ax_m, x_m\ \rangle =$ 1 and $\|\ (T-\lambda)x_m\ \|\ \|\ (A-\lambda)x_m\ \| \longrightarrow 0 \quad (m \longrightarrow \infty)$.

Then, by Schwarz inequality,

$$\begin{aligned} |(\langle\ Tx_m, x_m\ \rangle\ \langle\ Ax_m, x_m\ \rangle) - \lambda| &= |(\langle\ (T-\lambda)x_m, x_m\rangle\ \langle A-\lambda)x_m, x_m\ \rangle)| \\ &\leq \|\ (T-\lambda)x_m\ \|\ \|\ (A-\lambda)x_m\ \| \end{aligned}$$

Thus $(\langle\ Tx_m, x_m\ \rangle\ \langle\ Ax_m, x_m\ \rangle) \rightarrow \lambda$ as $m \rightarrow \infty$.

Therefore, $\lambda \in \overline{W(T, A)}$

and

$\sigma_\pi(T, A) \subset \overline{W(T, A)}$.

□

The immediate consequence of the above theorem is the next corollary which we state without proof.

Corollary 3.2.6

Conv $\sigma_\pi(T, A) \subseteq \overline{W(T, A)}$.

Here Conv $\sigma_\pi(T, A)$ denotes the convex hull of the joint approximate point spectrum of the two operators T and A.

Recall that the point spectrum $\sigma_p(T, A)$ is contained in the spectrum $\sigma(T, A)$.

Recall also that the essential spectrum is a subset of the spectrum and its complement is called the discrete spectrum.

Theorem 3.2.7

Spectrum $\sigma(T, A) \subset \overline{W(T, A)}$

PROOF. We look at the boundary of $\sigma(T, A)$. Let $\lambda \in \partial\sigma(T, A)$ where $\partial\sigma(T, A)$ is the boundary of $\sigma(T, A)$. We need to show that $\lambda \in \overline{W(T, A)}$. From Halmos, [29], $\lambda \in \partial\sigma(T, A)$ is contained in $\sigma_\pi(T, A)$. Since $W(T, A)$ is convex by Theorem 3.2.2, it suffices to show that $\sigma_\pi(T, A)$ is contained in $\overline{W(T, A)}$. Theorem 3.2.5 completes our proof. □

3.3 The Joint Algebra Numerical Range

The study of the algebra numerical range of an element $a \in \mathcal{A}$ where $\mathcal{A}$ is a complex normed algebra with unit has drawn the attention of many researchers. For instance, F. F. Bonsall [13] showed that this algebra numerical range is a compact convex subset of the complex plane and that it contains the spectrum $\sigma(a)$ of a. In this section, the properties of the joint algebra numerical range of an arbitrary element a were examined and results obtained.

Definition 3.3.1

Let $\mathcal{A}$ be a C^* -algebra with identity 1. A linear functional f is a state if $f(x^*x) \geq 0 \ \forall \ x \in \mathcal{A}$ and $\| f \| = f(I)$. If $\| f \| = 1$ then f is called a normalized state. The set of all states of $\mathcal{A}$ is denoted as $\mathcal{P}(\mathcal{A})$ or $\mathcal{P}$.

States separate points of $\mathcal{A}$.

Recall that $\mathcal{P}$ is nonvoid, since by Hahn-Banach theorem, there is $f \in \mathcal{A}^* : f(e) = 1 = \| f \|$ ($\mathcal{A}^*$ is a set of all bounded linear functionals on $\mathcal{A}$).

Also, $\mathcal{P}$ is a convex set for the w^*-topology. See [25] for this and more.

An element a of a unital Banach algebra $\mathcal{A}$ is hermitian if $V_m(a) \subset \mathbb{R}$. Also a is strongly hermitian if a and a^2 are both hermitian.

If $a = h + ik$ where h and k are (strongly) hermitian and $hk = kh$, then a is (strongly) normal.

See K. Mattila [36] for these and more.

Theorem 3.3.2

$V_m(a)$ is a nonempty convex and compact set.

PROOF. We first show that $V_m(a)$ is nonempty. By Hahn-Banach theorem, $\mathcal{P} \neq \emptyset$ since there is $f \in \mathcal{A}^* : f(a) \neq 0$ and $\| f \| = 1$ for $a = (a_1, ..., a_m)$. Thus $f(a) \in V_m(a)$ and hence $V_m(a) \neq \emptyset$.

To show convexity, let $f_1, f_2 \in \mathcal{P}, \lambda \in [0, 1]$. Let also, $f = \lambda f_1 + (1 - \lambda) f_2$. But $\mathcal{P}$ is convex and $f \in \mathcal{P}$. Also, f is linear, positive and $\| f \| = 1$ giving $f(a) = \lambda f_1(a) + (1 - \lambda) f_2(a) \in V_m(a)$.

For compactness, since the map $a^\Lambda : \{f \longmapsto f(a), (\mathcal{P}, w^* - topology) \longmapsto \mathbb{C}\}$ is continuous, $\mathcal{P} = V_m(a)$ is compact.

$\square$

REMARK 3.3.3

The joint essential spectrum, $\sigma_{e_m}(T)$, of an operator $T \in B(X)$ is contained in the spectrum $\sigma(T)$.

If there is no confusion, we will in what follows, use the arbitrary element T instead of the element a when dealing with the algebra numerical range. It should therefore not be construed for the operator T.

Proposition 3.3.4

$\sigma(T) \subset V_m(T)$.

PROOF. From the definition $\sigma(T) = \{\lambda \in \mathbb{C} : (T-\lambda) \text{ is not invertible}\}$. It must therefore be shown that $\lambda \in V_m(T)$. Now, $(T-\lambda)$ is not left invertible implies $1 \notin \mathcal{A}(T-\lambda)$. For an arbitrary $x \in \mathcal{A}$ not left invertible, we have $x(T-\lambda)$ and $(x^{'}x(T-\lambda) = (x^{'}x)(T-\lambda)$ is impossible). By Neumann series $(1-(1-y)^{-1})$, y is invertible if $y \in \mathcal{A}$ and $\| 1-y \| < 1$. Thus, $\forall\, x \in \mathcal{A},\ \| 1 - x(T-\lambda) \| \geq 1$. We construct $f \in \mathcal{P} \,:\, \lambda = f(T)$. Choosing $f \in \mathcal{A}^*$, by Hahn Banach theorem, we obtain

$$\forall\, x \in \mathcal{A},\ f(x(T-\lambda)) = 0,\ f(I) = 1.$$

Given $x \in \mathcal{A}, \eta \in \mathbb{C}$, let $y = x(T-\lambda) + \eta$ such that $f(y) = f(\eta) = \eta$. Suppose $\eta \neq 0$, then $\| y \| = \| \eta \| \| 1 + \eta^{-1}x(T-\lambda) \| \geq \| \eta \|$ implying that $|f(y)| \leq \| y \|$. Hence $\| f \| = 1,\ f \in \mathcal{P}$. If $x = 1$, then, $f(T-\lambda) = f(T) - \lambda = 0$ implying that $\lambda = f(T) \in V_m(T)$. Thus $\sigma(T) \subset V_m(T)$. $\square$

Theorem 3.3.5

$\sigma_{e_m}(T) \subset V_m(T)$

PROOF. From remark 3.3.3 and Proposition 3.3.4, it is clear that $\sigma_{e_m}(T) \subset V_m(T)$. □

Theorem 3.3.6

For an operator $T = B(X)$ where X is a complex Hilbert space and $a = T \in B(X)$, the joint algebra numerical range $V_m(T)$ equals the closure of the joint numerical range $W_m(T)$, i.e $V_m(T) = \overline{W_m(T)}$

PROOF. To prove the above theorem, the following result by S. Mecheri [37] is needed.

Proposition 3.3.7

$$\sup\{Re\mu, \mu \in V_m(a)\} = \inf_{t>0}(\frac{1}{t}(\| 1+ta \| -1)) = \lim_{t\to 0^+}(\frac{1}{t}(\| 1+ta \| -1)).$$

PROOF. Let $h = \sup\{Re\mu, \mu \in V_m(a)\} = \sup\{Ref(a), f \in \mathcal{P}\}$. Then,

$$\forall\, y > 0,\ 1 + tRef(a) \leq |1 + tf(a)| = |f(1+ta)| \leq \| 1 + ta \| .$$

Thus

$$h \leq \inf_{t>0}\{(\frac{1}{t}(\| 1+ta \| -1))\}.$$

Let $m = \inf_{t>0}\{(\frac{1}{t}(\| 1+ta \| -1))\}$. Therefore, $h \leq m$.

Conversely, let $x \in \mathcal{A}$ be such that $\| x \| = 1$, $f \in \mathcal{A}^*$ and $f(x) = \| f \| = 1$. Consider the map $g : \mathcal{A} \mapsto \mathbb{C}$ defined by $g(y) = f(yx)$. Then $g \in \mathcal{P}$ and g is linear and $|g(y)| \leq \| y \|$, $g(1) = 1$.

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Thus,

$$\| (1-ta)x \| \geq f((1-ta)x) \geq 1 - tRef(ax) = 1 - tReg(a) \geq 1 - th.$$

Therefore,

$$\| (1-ta)x \| \geq (1-th) \| x \|, \forall\, x \in X.$$

Let $x = 1 + ta$, then,

$$(1-th) \| (1+ta) \| \leq \| (1-ta)(1+ta) \| \leq (1+t^2) \| a \|^2 .$$

Now, let $t \leq h^{-1}$.

Then,

$$m \leq t^{-1}(\| 1+ta \| -1) \leq \frac{h + t \| a \|^2}{1-th}$$

and

$$m \leq \inf_{t \to 0^+} \frac{h+t \| a \|^2}{1-th} = h = \inf_{t \to 0^+} \frac{h+t \| a \|^2}{1-th}.$$

Thus, there is $\lim_{t\to 0^+}(\frac{1}{t}(\| 1+ta \| -1))$. □

To prove Theorem 3.3.6, proposition 3.3.7 is applied to show that,

$$\sup\{Re\mu, \mu \in V_m(T)\} = \lim_{t\to 0^+}(\frac{1}{t}(\| 1+tT \| -1)).$$

By the same way as the first part of the proof of proposition 3.3.7, the map $x^\wedge : B \mapsto (Bx, x), B(X) \mapsto \mathbb{C}$ is an element of $\mathcal{P}$. Letting

$$h_1 = \sup\{Re\mu, \mu \in V_m(T)\}, \; m = \lim_{t\to 0^+}(\frac{1}{t}(\| 1 + tT \| -1)),$$

then $h_1 \leq m$.

Conversely,

$$\| (1 - tT)x \| \| x \| \geq Re((1 - tT)x, x) \geq (1 - th_1) \| x \|^2 .$$

Choose t small enough to get $1 - th_1 > 0$. Let $x = (1 + tT)y$ with $\| y \| = 1$. Then

$$\| (1 - t^2T^2)y \| \geq (1 - th_1) \| (1 + tT)y \| .$$

Thus

$$(1 - th_1) \| 1 + tT \| \leq 1 + t^2 \| T \|^2 .$$

Just as in the proof of proposition 3.3.7, this last inequality implies that $m \leq h_1$ □

Theorem 3.3.8

Let T be an $m-$ tuple self-adjoint operator. Then $W_{e_m}(T) = \overline{W_m(T)}$ if and only if $Ext(W_m(T)) \subseteq W_{e_m}(T)$ where $Ext(W_m(T))$ denotes the set of extreme points on $W_m(T)$.

Proof. Let $Ext(W_m(T)) \subseteq W_{e_m}(T)$, then

$Ext(\overline{W_m(T)}) \subseteq W_{e_m}(T)$. Therefore,

$$\overline{W_m(T)} \subseteq conv(Ext(\overline{W_m(T)}) \subseteq conv(W_{e_m}(T)) = W_{e_m}(T).$$

But $W_{e_m}(T) \subseteq \overline{W_m(T)}$,

thus $W_{e_m}(T) = \overline{W_m(T)}$

Conversely,

If $W_{e_m}(T) = \overline{W_m(T)}$, then

$Ext(W_m(T)) \subseteq W_m(T) \subseteq W_{e_m}(T)$.

Thus we have $Ext(W_m(T)) \subseteq W_{e_m}(T)$.

□

Theorem 3.3.9

$W_{e_m}(T) = V_m(T)$

PROOF. This is clear from Theorems 3.3.6 and 3.3.8 □

Chapter 4

Joint Essential Numerical Range

4.1 Introduction

The essential numerical range $W_e(T)$ of a single operator T is defined as the algebra numerical range of the coset $T + \mathcal{K}(X)$ in the Calkin agebra $B(X)/\mathcal{K}(X)$ where $\mathcal{K}(X)$ is the ideal of all compact operators on X. In [13], Bonsall and Duncan proved that $W_e(T)$ is nonempty, compact and satisfies $W_e(T+\beta) = W_e(T) + \beta$ for any scalar β. Further, they showed that $0 \in W_e(T)$ if and only if T is compact.

The joint essential numerical range $W_{e_m}(T)$ of an m-tuple of operator $T = (T_1, ..., T_m) \in B(X)$ was studied in [17, 38] among others. While discussing the properties of $W_{e_m}(T)$, researchers did not show whether the general properties of numerical ranges hold for this set i.e whether it is compact, convex and contains the joint essential spectrum of the operator T. The first section of this chapter has shown that $W_{e_m}(T) \subset W_m(T)$ and that $W_{e_m}(T)$ is a compact convex set. To prove convexity, this study

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came up with several equivalent definitions of $W_{e_m}(T)$ using the work done in [27] on a single operator $T \in B(X)$. Since $W_m(T)$ is not convex for $m \geq 3$ and $W_{e_m}(T)$ is the intersection of non convex sets, it was showed that $\overline{W_m(T)}$ is star shaped and each element of $W_{e_m}(T)$ is a star center of $\overline{W_m(T)}$. This, together with the above equivalent definitions led to the main result of the research that $W_{e_m}(T)$ is a compact convex set.

4.2 Joint Essential Numerical Range

The following theorem shows the relation between the joint numerical range $W_m(T)$ of T, its closure $\overline{W_m(T)}$ and the joint essential numerical range $W_{e_m}(T)$. One consequence of the theorem is that $W_m(T)$ is closed if and only if $W_{e_m}(T) \subset W_m(T)$.

Theorem 4.2.1

$\overline{W_m(T)} = conv\ (W_m(T) \cup W_{e_m}(T))$.

Here conv denotes the convex hull.

Proof. From the properties of the joint numerical range, it is linear and so $W_m(\beta T + \alpha) = \beta W_m(T) + \alpha \ \ \forall\, \beta, \alpha \in \mathbb{C}$. Also, an arbitrary element $T = (T_1, ..., T_m)$ of a unital algebra $\mathcal{A}$ has a joint algebra numerical range $V_m(T)$ defined as

$$V_m(T) = \{f(T_1), ..., f(T_m) : f \in \mathcal{A}^*, f(I) = 1 = \| f \|\}$$

where a linear functional f is a state and $\mathcal{A}^*$ is a set of all bounded linear functionals on $\mathcal{A}$. By Theorem 3.3.6, $V_m(T) = \overline{W_m(T)}$. The joint essential numerical range of an operator $T = (T_1, ..., T_m) \in B(X)$ is the joint algebra numerical range $V_m(\pi(T))$ of the coset in $B(X)/\mathcal{K}(X)$ that contains T. Here, π is the canonical projection from $B(X)$ to the Calkin algebra $B(X)/\mathcal{K}(X)$ and $\mathcal{K}(X)$ is the ideal of all compact operators on X. This joint essential numerical range is denoted as

$$W_{e_m}(T) = \bigcap\{\overline{W_m(T+K)} : K = (K_1, ..., K_m) \in \mathcal{K}(X)\}.$$

$\overline{W_m(T)}$ and $W_{e_m}(T)$ are therefore obtained by evaluation of all states of $B(X)$ (that vanish on $\mathcal{K}(X)$) at the operator T. From Dixmier [25], every state f on $B(X)$ is convex with the form

$$f = \lambda f_0 + (1-\lambda) f_A$$

where $\lambda \in [0,1]$, f_0 is a state that annihilates $\mathcal{K}(X)$ and f_A is a state induced by the nonnegative trace class operator A : $f_A(Y) = \text{trace}(YA)$ for $Y \in B(X)$.

Therefore $\overline{W_m(T)}$ is the convex hull of $W_{e_m}(T)$ and the trace class numerical range consisting of the numbers $f_A(T)$. We then show that the trace class numerical range consisting of the numbers $f_A(T)$ is just $W_m(T)$. The operator A has a spectral decomposition $A = \Sigma\lambda_m(\cdot, x_m)x_m$ where the x_m is an orthonormal set, $\lambda_m \geq 0$, and $\Sigma\lambda_m = 1$. Consequently

$$f_A(T) = \Sigma\lambda_i\langle Tx_m, x_m\rangle : i = 1, ..., m$$

belongs to $W_m(T)$ because any convex subset of the plane (or $\mathbb{R}^m$) contains convex combinations of its countable subsets. $\square$

Corollary 4.2.2

$W_m(T)$ is closed if and only if $W_{e_m}(T) \subset W_m(T)$.

PROOF. Suppose $W_m(T)$ is closed,

then $W_{e_m}(T) \subset \overline{W_m(T)} = \text{conv } W_m(T) = W_m(T)$.

Conversely, if $W_{e_m}(T) \subset W_m(T)$, then $\overline{W_m(T)} = \text{conv} W_m(T) = W_m(T)$.

$\square$

Theorem 4.2.3

Let X be an infinite dimensional complex Hilbert space and $T = (T_1, ..., T_m) \in B(X)$. Let $r = (r_1, ..., r_m) \in \mathbb{C}^m$. The following properties are equivalent:

(1) $r \in W_{e_m}(T) = \bigcap \{\overline{W_m(T+K)} : K = (K_1, ..., K_m) \in \mathcal{K}(X)\}$.

(2) $r \in \bigcap \{\overline{W_m(T+F)} : F = (F_1, ..., F_m) \in \mathcal{F}(X)\}$. *Here, $\mathcal{F}(X)$ is a set of finite rank operators in $B(X)$.*

(3) *There exists an orthonormal sequence of vectors $\{x_n\}_{n=1}^{\infty} \in X$ such that*

$$\langle Tx_n, x_n \rangle \longrightarrow r.$$

(4) *There exists a sequence of unit vectors $\{x_n\}_{n=1}^{\infty} \in X$ converging weakly to $0 \in X$ such that*

$$\langle Tx_n, x_n \rangle \longrightarrow r.$$

(5) *There exists an infinite-dimensional projection P such that*

$$P(T_j - r_jI)P \in \mathcal{K}(X) \text{ for } j = 1, ..., k.$$

PROOF. First, we show that (1) implies (3). Let $r \in W_{e_m}(T)$, then there is a sequence $\{x_n\}$ of vectors such that

$$\langle Tx_n, x_n \rangle \longrightarrow r, \; \| x_n \| = 1, \; x_n \longrightarrow 0 \text{ weakly}.$$

Suppose we have chosen the set $\{x_1, ..., x_n\}$ satisfying $|\langle Tx_n, x_n \rangle - r| < \frac{1}{i}, \quad \forall\, i$. Let $\mathcal{M}$ be the subspace spanned by $x_1, ..., x_n$ and P be the projection onto $\mathcal{M}$. Then we have $\| Px_n \| \longrightarrow 0$ as $n \longrightarrow \infty$. Let

$$z_n = \| (I - P)x_n \|^{-1} ((I - P)x_n).$$

We have

$$Tz_n = \| (I - P)x_n \|^{-1} (T(I - P)x_n).$$

This gives

$$\begin{aligned}\langle Tz_n, z_n\rangle &= \langle \| (I - P)x_n \|^{-1} (T(I - P)x_n), \| (I - P)x_n \|^{-1} (T(I - P)x_n)\rangle \\ &= \| (I - P)x_n \|^{-2} \{\langle Tx_n, x_n\rangle - \langle Tx_n, Px_n\rangle - \langle TPx_n, x_n\rangle \\ &\quad +\langle TPx_n, Px_n\rangle\} \to r.\end{aligned}$$

We choose n large enough such that

$$|\langle Tz_n, z_n \rangle - r| < \frac{1}{n+1}.$$

If we let $z_n = x_{n+1}$ we get

$$|\langle Tx_{n+1}, x_{n+1} \rangle - r| < \frac{1}{n+1}$$

To prove that (3) implies (5)

Let $\{x_n\}$ be an orthonormal sequence with $\langle Tx_n, x_n \rangle \longrightarrow r$. By passing to a subsequence we can assume that

$$\sum_{n=1}^{\infty} |\langle Tx_n, x_n \rangle|^2 < \infty \tag{4.3}$$

Let $n_1 = 1$. Then

$$\sum_{n=1}^{\infty} |\langle Tx_{n_1}, x_n \rangle|^2 \leq \| Tx_{n_1} \|^2$$

and

$$\sum_{n=1}^{\infty} |\langle Tx_n, x_{n_1} \rangle|^2 \leq \| T^* x_{n_1} \|^2 .$$

Thus, by Bessel's inequality, there is an integer $n_2 > n_1$ such that

$$\sum_{n=n_2}^{\infty} |\langle Tx_{n_1}, x_n \rangle|^2 < \frac{1}{2}$$

and

$$\sum_{n=n_2}^{\infty} |\langle Tx_n, x_{n_1} \rangle|^2 < \frac{1}{2}.$$

If this procedure is repeated, a strictly increasing sequence $\{n_t\}_{t=1}^{\infty}$ of

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positive integers is obtained such that we have

$$\sum_{n=n_{t+1}}^{\infty} |\langle Tx_{n_t}, x_n \rangle|^2 < \frac{1}{2^t}$$

and

$$\sum_{n=n_{t+1}}^{\infty} |\langle Tx_n, x_{n_t} \rangle|^2 < \frac{1}{2^t} \qquad (4.4)$$

(3.3) and (3.4) imply that

$$\sum_{t,l=1}^{\infty} |\langle Tx_{n_t}, x_{n_l} \rangle|^2 < \infty \qquad (4.5)$$

If P is an orthogonal projection onto the subspace $\mathcal{M}$ spanned by $x_{n_1}, x_{n_2}, ...,$ then

$$\sum_{t,l=1}^{\infty} |\langle PTPx_{n_t}, x_{n_l} \rangle|^2 = \sum_{t,l=1}^{\infty} |\langle Tx_{n_t}, x_{n_l} \rangle|^2 < \infty \ (by\ (4.5)),$$

hence PTP is a Hilbert - Schmidt operator and therefore $PTP \in \mathcal{K}(X)$.

We now show that (2) implies (3).

Let $\{x_n\}$ be a sequence of unit vectors with $x_n \longrightarrow 0$ weakly such that

$$\langle Tx_n, x_n \rangle \longrightarrow r.$$

Suppose we have an orthonormal set of vectors $\{x_1, ..., x_n\}$ such that $|\langle Tx_n, x_n \rangle| < \frac{1}{2^n}$. Let P be the orthogonal projection onto subspace $\mathcal{M}$ spanned by $\{x_1, ..., x_n\}$. In order to exhibit a unit vector $x_{n+1} \in \mathcal{M}^\perp$ with $|\langle Tx_{n+1}, x_{n+1} \rangle| < \frac{1}{2^{n+1}}$, we must show that $0 \in \overline{W_m(I-P)T_i(I-P)|_{\mathcal{M}^\perp}}$: $i = 1, ..., m$. To do this, let $\mu = (\mu_1, ..., \mu_m) \in W_m((I-P)T_i(I-P)|_{\mathcal{M}^\perp}) : i = 1, ..., m$ and $\mu I_{\mathcal{M}} = (\mu_1 I_{\mathcal{M}}, ..., \mu_m I_{\mathcal{M}})$. Also, let

$B = ((I-P)T_1(I-P)|_{\mathcal{M}^\perp}, ..., (I-P)T_m(I-P)|_{\mathcal{M}^\perp}))$

and

$F = \mu P - PTP - (I-P)TP - PT(I-P)$. Here, F is of finite rank.

For $\overline{P} = I - P$, we obtain,

$$\mu I_{\mathcal{M}} \oplus B = (\mu_1 P + \overline{P}T_1\overline{P}, ..., \mu_m P + \overline{P}T_m\overline{P}) = T_i + F_i : i \in [1, m],\ F \in \mathcal{F}(X).$$

From Magajna [35], $W_m(\mu I_{\mathcal{M}} \oplus B) = conv(W_m(\mu I_{\mathcal{M}}) \cup W_m(B))$. Therefore, since $W_m((I-P)T_i(I-P)|_{\mathcal{M}^\perp})$ is convex and contains μ, it follows that

$$W_m(T_i + F_i) = W_m((I-P)T_i(I-P)|_{\mathcal{M}^\perp}).$$

Hence $0 \in \overline{W_m(I-P)T_i(I-P)|_{\mathcal{M}^\perp}}$ as required.

We then show that (3) implies (4).

Let $\{x_n\}$ be an orthonormal sequence with $\langle Tx_n, x_n \rangle \longrightarrow r$. But every orthonormal sequence $\{x_n\}$ converges weakly to zero and $\| x_n \| = 1$.

To show that (4) implies (1).

Suppose that for a point $r \in \mathbb{C}^m$ there is a sequence $\{x_n\} \in X$ such that

$$\langle Tx_n, x_n \rangle \to r.$$

Since every sequence $\{x_n\} \to 0$ weakly, and $\| x_n \| = 1$, we have $r \in W_{e_m}(T)$.

To prove that (5) implies (1), let $P \in B(X)$ be an infinite dimensional projection such that $(PT_jP - r_jP) \in \mathcal{K}(X)$, $j \in [1, m]$. There is thus an orthonormal sequence $\{x_n\} \in X$ such that $Px_n = x_n \ \forall \ n$. Let $K = (K_1, ..., K_m) \in \mathcal{K}(X)$. For any $K_j : j \in [1, m]$, $PT_jP - r_jP = K_j$ and thus $\langle (PT_jP - r_jP)x_n, x_n \rangle = \langle K_jx_n, x_n \rangle$ implying $\langle T_jx_n, x_n \rangle = r_j + \langle K_jx_n, x_n \rangle$. From the orthonormality of sequence $\{x_n\}$, we get K_jx_n converging weakly to 0 in norm as $n \to \infty$, $j \in [1, m]$. Therefore, $\langle T_jx_n, x_n \rangle \longrightarrow r_j$ as $n \to \infty$ implying $r \in W_{e_m}(T)$.

To prove that (1) implies (2), let $r \in W_{e_m}(T)$, then there is a sequence $\{x_n\}$ of vectors such that

$$\langle Tx_n, x_n \rangle \longrightarrow r, \ \| x_n \| = 1, \ x_n \longrightarrow 0 \text{ weakly.}$$

But $W_{e_m}(T) = \bigcap \{\overline{W_m(T + F)} : F = (F_1, ..., F_m) \in \mathcal{F}(X)\}$, where, $\mathcal{F}(X)$ is a set of finite rank operators in $B(X)$. This implies that $r \in \bigcap \{\overline{W_m(T + F)} : F = (F_1, ..., F_m) \in \mathcal{F}(X)\}$.

□

Theorem 4.2.4

Suppose $T = (T_1, ..., T_m) \in B(X)$.

(i) $W_{e_m}(T)$ *is nonempty, compact and convex.*

(ii) *Each element* $r \in W_{e_m}(T)$ *is a star center of* $\overline{W_m(T)}$.

Proof. To prove this, for the self-adjoint operators $T = (T_1, ..., T_m) \in B(X)$, let $W_{e_m}(T)$ fulfil condition 3 of Theorem 4.2.3. We first prove that $W_{e_m}(T)$ is nonempty. For an orthonormal sequence $\{x_n\}_{n=1}^{\infty}$ of vectors in

X, the sequence $\{\langle Tx_n, x_n\rangle\}_{n=1}^{\infty}$ is bounded. Choose a subsequence and assume that $\langle Tx_n, x_n\rangle$ converges. Then $W_{e_m}(T)$ is nonempty.

The compactness of $W_{e_m}(T)$ can be seen right from its definition. The joint essential numerical range, $W_{e_m}(T)$ is defined as the intersection of all sets of the form

$$\overline{W_m(T+K)}\ : K\ \in \mathcal{K}(X)$$

where $\mathcal{K}(X)$ denote the sets of compact operators in $B(X)$. Being an intersection of compact sets, the joint essential numerical range is also compact.

To prove that each element $r \in W_{e_m}(T)$ is a star center of $\overline{W_m(T)}$, it should be shown that $(1-\lambda)\,p + \lambda r \in \overline{W_m(T)}\ : \lambda \in [0,1]$ where $r \in W_{e_m}(T)$ and $p \in \overline{(W_m(T)}$. Assume without loss of generality that $\| T \| = 1$. Suppose $s\ \in \overline{(W_m(T)}$ so that $s\ =\ \lambda r\ +\ (1-\lambda)p$. Let $\{x_n\}$ and $\{e_n\}$ be orthonormal sequences in X such that

$$r = \langle\, Tx_n, x_n\rangle,\ p\ =\ \langle\, Te_n, e_n\,\rangle$$

and

$$\| x_n \| = \| e_n \| =\ 1.$$

Then,

$$\begin{aligned} s\ &=\ \lambda\langle\, Tx_n, x_n\,\rangle\ +\ (1-\lambda)\langle\, Te_n, e_n\,\rangle \\ &=\ \langle\, T\sqrt{\lambda}x_n, \sqrt{\lambda}x_n\,\rangle\ +\ \langle\, T\sqrt{1-\lambda}e_n, \sqrt{1-\lambda}e_n\,\rangle \end{aligned}$$

$$= \langle\, (T\sqrt{\lambda}x_n \;+\; T\sqrt{1-\lambda}e_n\,), (\sqrt{\lambda}x_n \;+\; \sqrt{1-\lambda}e_n\,)\,\rangle$$

$$\begin{aligned} \|\, \sqrt{\lambda}x_n \;+\; \sqrt{1-\lambda}e_n \,\|^2 &= (\|\, \sqrt{\lambda}x_n \,\|^2 \;+\; \|\, \sqrt{1-\lambda}e_n \,\|^2) \\ &= \lambda \,\|\, x_n \,\|^2 \;+ (1-\lambda) \,\|\, e_n \,\|^2 \\ &= \lambda \;+\; (1-\lambda) \;=\; 1 \end{aligned}$$

Thus, $(1-\lambda)\, r + \lambda p \in \overline{W_m(T)}$.

Convexity of $W_{e_m}(T)$ is proved by showing that for $r, p \in W_{e_m}(T)$ and $\lambda \in [0,1]$, $\lambda r + (1-\lambda)p \in W_{e_m}(T)$. Now, $r \in W_{e_m}(T) \;=\; W_{e_m}(T+F)$ for every $F \in \mathcal{F}(X)$ and $p \in W_{e_m}(T) \subseteq \overline{W_m(T+F)}$. From Theorem 4.2.3 above, $\lambda r + (1-\lambda)p \in \overline{W_m(T+F)}$. Thus,
$\lambda r + (1-\lambda)p \in \bigcap\{\overline{W_m(T+F)} : F \in \mathcal{F}(X)\} \;=\; W_{e_m}(T)$.
Hence $W_{e_m}(T)$ is convex.

□

4.3 Joint Essential Spectrum

The concept of joint spectrum for a family of operators was first introduced by R. Arens and A. P. Calderón [6]. Since then, some researchers have asserted its definition and properties which have been generalized to the joint essential spectrum and in some instances to joint Browder spectrum. The successful definitions among them have been carried out by J. L. Taylor [41] and A. T. Dash [23]. This study has used the definition by A. T. Dash. However, it must be noted that it was shown by M. Chō and M. Takagunchi [20] that Taylor's joint spectrum coincides with

Dash's joint spectrum in the case of commuting normal operators.

Many authors showed the relation between the joint numerical range and the joint spectrum. For instance, Abramov [1] has shown that the conical point of the closure of the joint numerical range of an m-tuple operator $T = (T_1, ..., T_m)$ belongs to the joint approximate point spectrum of T of a family of self-adjoint operators. Studying on the boundary points of the joint numerical range, M. Chō and M. Takagunchi [19] proved that the extreme points of the closure of the joint numerical range of an m-tuple operator belong to the joint approximate point spectrum of the operator. This study has in this section showed the relation between the joint essential spectrum and the joint essential numerical range of an operator $T = (T_1, ..., T_m)$. In particular, the study has proved among others that the joint essential spectrum of the operator T is contained in the joint essential numerical range of T for $T = (T_1, ..., T_m)$. To do this, the study has made good use of the available literature above. In addition to the above literature, A. T. Dash's proof that the joint spectrum equals the joint essential spectrum was quite useful in the sequel. Also important to the study was the result by M. Chō [18] that the joint spectrum for a strongly commuting m-tuple of operators on a Banach space equals the joint approximate point spectrum for it.

Lemma 4.3.1

(A. T. Dash [23])

Let $d = (d_1, ..., d_m)$ be an m-tuple of elements in a unital C^*-algebra of X. Then:

(a) $(\lambda_1, ..., \lambda_m) \in \sigma^l_m(d_1, ..., d_m)$ *if and only if*

$$0 \in \sigma_m(\sum_{i=1}^{m}(d_i - \lambda_i)^*(d_i - \lambda_i))$$

(b) $(\lambda_1, ..., \lambda_m) \in \sigma^r_m(d_1, ..., d_m)$ *if and only if*

$$0 \in \sigma_m(\sum_{i=1}^{m}(d_i - \lambda_i)(d_i - \lambda_i)^*).$$

See A. T. Dash [23] for the proof.

The above result by A. T. Dash was important for the proof of the main result in this section.

Corollary 4.3.2

For an m-tuple of operators $T = (T_1, ..., T_m)$ on X;

(a) $(\lambda_1, ..., \lambda_m) \in \sigma^l_{e_m}(T)$ *if and only if*

$$0 \in \sigma_{e_m}(\sum_{i=1}^{m}(T_i - \lambda_i)^*(T_i - \lambda_i))$$

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(b) *$(\lambda_1, ..., \lambda_m) \in \sigma^r_{em}(T)$ if and only if*

$$0 \in \sigma_{em}(\sum_{i=1}^{m}(T_i - \lambda_i)(T_i - \lambda_i)^*).$$

See A. T. Dash [23] for the proof.

The following proof was then used by A. T. Dash to show the relationship between the joint spectrum and the joint essential spectrum of an m-tuple of operators.

Theorem 4.3.3

(A. T. Dash [23])

Let $T = (T_1, ..., T_m)$ be an $m-$tuple of operators on X. Then:

(a) $\sigma^l_m(T) = \sigma^l_{em}(T) \cup \sigma_p(T)$

(b) $\sigma^r_m(T) = \sigma^r_{em}(T) \cup \sigma_p(T^*)^*$, *and hence we have*

(c) $\sigma_m(T) = \sigma_{em}(T) \cup \sigma_p(T^*)^*$, *where $T^* = (T_1^*, ..., T_m^*)$ and star on the right represents complex conjugates.*

It is well known that $\sigma^l_m(T) = \sigma_\pi(T)$ and $\sigma^r_m(T) = \sigma_\delta(T)$. Clearly, $\lambda = (\lambda_1, ..., \lambda_n) \in \sigma_\pi(T)$ if and only if

$$0 \in \sigma_\pi(\sum_{i=1}^{m}(T_j - \lambda_j)^*(T_j - \lambda_j))$$

and $\lambda \in \sigma_\delta(T)$ if and only if $\lambda^* \in \sigma_\pi(T^*)$ if and only if

$$0 \in \sigma_\pi(\sum_{i=1}^{m}(T_j - \lambda_j)(T_j - \lambda_j)^*).$$

See A. T. Dash [23] for the proof.

The following Theorem by Huang Danrun and Zhang Dianzhow proves that $\sigma_m(T) = \sigma_\pi(T)$

Theorem 4.3.4

Let $T = T_1, ..., T_m$ be a commuting $m-$ tuple of normal operators. Then $\sigma_m(T) = \sigma_\pi(T)$.

See [31] for the proof.

Theorem 4.3.5

Let $T = (T_1, ..., T_m) \in B(X)$ be an m- tuple of operators. Then

$$conv\ \sigma_\pi(T) \subseteq \overline{W_m(T)}.$$

See Wrobel Volker [44] for the proof.

It was proved that $\sigma_\pi(T)$ is a nonempty compact subset of $\mathbb{C}^m$ by J. Bunce [15] while Berberian [11] showed that every compact set contains the extreme points of its closed convex hull.

Theorem 4.3.6

Let $T = (T_1, ..., T_m) \in B(X)$ be an m- tuple of bounded linear operators on a Hilbert space X.

$$\sigma_\pi(T) \subset \overline{W_m(T)}.$$

PROOF. Suppose $\lambda = (\lambda_1, ..., \lambda_m) \in \sigma_\pi(T)$. There is a sequence $x_m \in X$ of unit vectors such that $\langle x_m, x_m \rangle = \| x_m \|^2 = 1$ and

$$\| T_i x_m - \lambda_i x_m \| \longrightarrow 0 : (m \longrightarrow \infty),\ i = 1, ..., m.$$

By Schwarz inequality,

$$\| (T_i - \lambda_i I)x_m \| \geq |\langle (T_i - \lambda_i I)x_m, x_m \rangle| = |\langle T_i x_m, x_m \rangle - \lambda_i|.$$

Therefore,

$$\langle Tx_m, x_m \rangle \longrightarrow \lambda \implies \lambda = (\lambda_1, ..., \lambda_m) \in \overline{W_m(T)}.$$

□

Recall that the boundary points of the joint spectrum are contained in the joint approximate point spectrum. Consequently, the joint spectrum is contained in the closure of the joint numerical range.

The following theorem by A. T. Dash is important for the next theorem. See [23] for its proof.

Theorem 4.3.7

Let $T = (T_1, ..., T_m)$ be a commuting m-tuple on X. Then

1. *$\lambda = (\lambda_1, ..., \lambda_m) \in \sigma^l_{em}(T)$ if and only if there exists a sequence $\{x_m\}$ of unit vectors in X with $x_m \longrightarrow 0$ weakly such that*

$$\| (T_i - \lambda_i)x_m \| \longrightarrow 0 \text{ as } m \longrightarrow \infty, \text{ for each } i,\ 1 \leq i \leq m.$$

2. *$\lambda = (\lambda_1, ..., \lambda_m) \in \sigma^r_{em}(T)$ if and only if there exists a sequence $\{x_m\}$ of unit vectors in X with $x_m \longrightarrow 0$ weakly such that*

$$\| (T_i^* - \lambda_i^*)x_m \| \longrightarrow 0 \text{ as } m \longrightarrow \infty, \text{ for each } i,\ 1 \leq i \leq m.$$

Moreover, the sequence $\{x_m\}$ can be chosen orthonormal.

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Theorem 4.3.8

The joint essential spectrum $\sigma_{e_m}(T)$ of the operator $T = (T_1, ..., T_m)$ is contained in the joint essential numerical range $W_{e_m}(T)$ of the operator $T = (T_1, ..., T_m)$.

PROOF. By letting $\lambda = (\lambda_1, ..., \lambda_m) \in \sigma_{e_m}(T)$, it should be shown that $\lambda = (\lambda_1, ..., \lambda_m) \in W_{e_m}(T)$. Dash defines the joint essential spectrum as

$$\sigma_{e_m}(T) = \sigma^l_{e_m}(T) \cup \sigma^r_{e_m}(T).$$

Suppose $\lambda = (\lambda_1, ..., \lambda_m) \in \sigma^l_{e_m}(T)$ then from Theorem 4.3.7 there is a sequence (x_m) of unit vectors in X such that $\| (T_i - \lambda_i I)x_m \| \longrightarrow 0 \; \forall i = 1, ..., m$ as $x_m \longrightarrow 0$ weakly.

Now, by Schwarz inequality,

$| \langle (T_i - \lambda_i I)x_m, x_m \rangle | \leq \| (T_i - \lambda_i I)x_m \| \longrightarrow 0 \; \forall i = 1, ..., m.$

Therefore, $\langle T_i x_m, x_m \rangle \longrightarrow \lambda_i \; \forall i = 1, ..., m.$

Hence $\lambda = \lambda_1, ..., \lambda_m \in W_{e_m}(T)$.

Likewise, let $\lambda = (\lambda_1, ..., \lambda_m) \in \sigma^r_{e_m}(T)$ then $\lambda^* = (\lambda_1^*, ..., \lambda_m^*) \in \sigma^l_{e_m}(T^*)$. This gives $\lambda = (\lambda_1, ..., \lambda_m) \in W_{e_m}(T^*) = [W_{e_m}(T)]^*$ (the complex conjugate of $W_{e_m}(T)$) and again $\lambda = (\lambda_1, ..., \lambda_m) \in W_{e_m}(T)$. □

Chapter 5

Summary and Recommendation

5.1 Summary

In this section the main results of the study are highlighted basing on the objectives of the study. Our objectives were to determine whether $W_{e_m}(T)$ is a compact convex set; to investigate whether $\overline{W_m(T)}$ is always star-shaped with elements in $W_{e_m}(T)$ as star centers and finally to investigate whether $\sigma_{e_m}(T) \subset W_{e_m}(T)$. We therefore state that these objectives were achieved. It was shown in section 4.2 that the closure of the joint numerical range is star-shaped with the elements of the joint essential numerical range as star centers of this closure. We also, in this same section, showed that the joint essential numerical range is nonempty, compact and convex. Section 4.3 contains the other main result that the joint essential spectrum of an operator is contained in the joint essential numerical range of the operator. This was proved by looking at the boundary of the joint spectrum of T.

5.2 Recommendation.

This study has clearly shown that the study of the joint essential numerical ranges is still an interesting and active area to be researched on in pure mathematics. Although we have investigated several properties of the numerical ranges for the joint essential numerical range, there is still a lot that needs be investigated. We therefore invite researchers to investigate several other properties of the joint essential numerical range. There is also need for investigation on whether the knowledge of the joint essential numerical range can be applied to various areas such as C^*-algebras, iterations methods, Krein space operators, factorisation of matrix polynomials and dilation theory which all constitute an active field of research in operator theory.