ON COMPLETELY BOUNDED OPERATORS.

BY

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Abstract

Calculating norms of matrices when the entries are not constants is the first problem tackled in this thesis. We have considered the space of matrices with entries from the algebra of bounded linear operators and have managed to approximate norm in this space. The basic idea has been to identify this space with the space of bounded operators from $\mathcal{H}^n$ (where $\mathcal{H}^n$ is the orthogonal sum of n-copies of $\mathcal{H}$) to $\mathcal{H}^m$ and calculating the norm of an operator on $\mathcal{H}^n$. This forms the content of chapter two. The notion of completely bounded operators is a fairly new and developing area in Mathematics. It started its life in the early 1980's following Stinespring's and Arveson's work on completely positive operators. It later gave rise to operator spaces, a new branch in operator algebra. Progress in this new area of Mathematics has been rapid and it is difficult to say which results motivated others. We have investigated the norm of completely bounded operators and have shown that they form an increasing sequence. The idea was to apply Hilbert-Schmidt norm to the definition of these operators. We have also given examples of these operators for illustration, something which is missing in the available literature. We have also investigated operator spaces, especially their algebraic tensor product. Specific interest has been in the matricial tensor product.

Chapter 1

Introduction

Norms of matrices are induced from the vector norms,(see [17]). Calculating these norms is not easy especially when the matrix entries are not constants. In our study, we have considered the space $M_{m,n}(B(\mathcal{H}))$ where the entries are bounded linear operators. We have managed to approximate the norms from this space. This was possible since the space $M_{m,n}(B(\mathcal{H}))$ has been identified with the space of bounded operators from $\mathcal{H}^n$ to $\mathcal{H}^m$. All of these are covered in chapter two. Chapter one basically covers the basic concepts that are vital in the understanding of this thesis.

The motivation of the study of operator spaces ties up with the notion of quantisation. In fact, this notion started its life with the 'matrix mechanics' of Heisenberg (see [7]). Influenced by this work of Heisenberg, Von Neumann suggested that one should seek quantised analogues of Mathematics, in the sense of replacing functions by operators. Murray and Von Neumann put this into practice by producing the operator (quantised) version of integration. This gave birth to the whole field of operator algebras. Similarly, one can seek for the notion of 'quantisation of Banach spaces' which turns out to be operator spaces. On the other

hand, the study of operator spaces is related to the study of complete boundedness of operators or mappings (i.e. morphisms between operator spaces) which was found to be useful in the study of operator algebras long before the operator space theory was axiomatised, (see [9]).
As noted in the above, operator spaces basically means spaces of bounded operators on some Hilbert spaces.

The theory of completely bounded maps is the basis for operator space theory. It emerged in the early 1980′s through the works of Haagerup, Wittstock and Paulsen, who proved independently, a fundamental factorisation and extension theorem for completely bounded maps, (see [9]). This factorisation theorem is a generalisation of an earlier important work by Stinespring and Arveson (see [1]), who proved a factorisation/extension theorem for completely positive maps. Completely bounded operators developed from 1980 onwards with the basic linear results complete by 1984. Progress was rapid and it is difficult to explain which results motivated others. In our study, we have investigated the norms of these operators, given some examples of completely bounded operators and finally, we have investigated the tensor norm of operator spaces. All of these form the content of chapter three.

1.1 Literature review

Let $\mathcal{H}$ be a Hilbert space, $B(\mathcal{H})$ be the algebra of bounded linear operators and $M_{m,n}(B(\mathcal{H}))$ be the space of $m \times n$ matrices with entries from $B(\mathcal{H})$. When speaking about the norm of a matrix $\mathbf{T} \in M_{m,n}(B(\mathcal{H}))$, we will always mean its norm as an operator from $\mathcal{H}^n$ to $\mathcal{H}^m$, (see [7]). Let

$M_{m,n}(B(\mathcal{H}))$ be the algebra of $m \times n$ matrix of operators acting on n-dimensional complex Hilbert space $\mathcal{H}^n$ and if $\mathbf{T} \in B(\mathcal{H}^n; \mathcal{H}^m)$, then the norm of this operator is given by

$$||\mathbf{T}|| = \sup\{||\mathbf{T}h|| : h \in \mathcal{H}^n; ||h|| = 1\}.$$

The following theorem provides formulae that can be used to calculate the three commonly used matrix norms

Theorem 1.1.1. *If the elements of an* $n \times n$ *matrix* $\boldsymbol{A}$ *are* a_{ij}, *then*

i) $||\boldsymbol{A}||_1 = \max\{\sum_{i=1}^{n} |a_{ij}| : 1 \leq j \leq n\}$ *the matrix 1-norm.*

ii) $||\boldsymbol{A}||_\infty = \max\{\sum_{j=1}^{n} |a_{ij}| : 1 \leq i \leq n\}$ *the* ∞*-norm.*

iii) $||\boldsymbol{A}||_2 = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2}$ *the Frobenius or Hilbert-Schmidt norm*

Proof. See [17] for the proof of this theorem. □

The notion of completely bounded maps first appeared in the early 1980's following Stinespring's pioneering work and Arveson's fundamental results on completely positive maps, (see [1]). Stinespring showed that completely positive maps have a representation of the form $\pi_2[\phi(\mathcal{A}_2)] = V^*\pi_1(\mathcal{A}_1)V$, where π_1 and π_2 are representations of the algebras $\mathcal{A}_1$, $\mathcal{A}_2$, ϕ is completely positive operator and V is a bounded operator.

Theorem 1.1.2. ***(Stinespring's representation theorem).*** *Let* $\mathcal{A}$ *be a unital* C^**-algebra and let* $\phi : \mathcal{A} \to B(\mathcal{H})$ *be completely positive map, then there exists a Hilbert space* $\mathcal{K}$*, a bounded operator* $V : \mathcal{H} \to \mathcal{K}$ *and a unital *-homomorphism,* $\pi : \mathcal{A} \to B(K)$ *such that* $\phi(a) = V^*\pi(a)V$*, for every* $a \in \mathcal{A}$.

For the proof see [14].

Arveson showed that, if $S \subseteq \mathcal{A}$ is an operator system and $\phi : S \to B(\mathcal{H})$ is completely positive, then, there exists a completely positive map $\psi : \mathcal{A} \to B(\mathcal{H})$ that extends ϕ such that $\psi(a) = \phi(a)$ for every $a \in S$.

Theorem 1.1.3. *(**Arveson's Extension Theorem**) Let $S \subseteq \mathcal{A}$ be an operator system and let $\phi : S \to B(\mathcal{H})$ be completely positive, then there exists a completely positive map $\psi : \mathcal{A} \to B(\mathcal{H})$ that extends ϕ, such that $\psi(a) = \phi(a)$ for every $a \in S$.*

For the proof see [14].

This result by Arveson yielded another result due to Wittstock, who worked on operator spaces instead of operator systems and showed that if $\mathcal{M} \subseteq \mathcal{A}$ is an operator space and $\phi : \mathcal{M} \to B(\mathcal{H})$ is completely bounded, then there exists a completely bounded map $\psi : \mathcal{A} \to B(\mathcal{H})$ that extends ϕ and satisfies $||\psi||_{cb} = ||\phi||_{cb}$.

Theorem 1.1.4. *(**Wittstock's Extension Theorem.**) Let $\mathcal{M} \subseteq \mathcal{A}$ be an operator space and let $\phi : \mathcal{M} \to B(\mathcal{H})$ be completely bounded, then there exists a completely bounded map $\psi : \mathcal{A} \to B(\mathcal{H})$ that extends ϕ and satisfies $||\psi||_{cb} = ||\phi||_{cb}$.*

For the proof see [14].

Wittstock and Paulsen proved that the span of the completely positive maps from a C^*-algebra into an injective C^*-algebra is identical with the set of completely bounded maps. Hadwin showed that a bounded unital homomorphism from a C^*-algebra into $L(\mathcal{H})$ (algebra of linear operators on a Hilbert space) is similar to a *-homomorphism if and only if the homomorphism belongs to the span of the completely positive maps. Together, these two results by Wittstock and Paulsen and Hadwin, prove that a bounded unital homomorphism from a C^*-algebra into $L(\mathcal{H})$ is similar to a *-homomorphism if and only if it is completely bounded. Recently, Paulsen [[14], chapter 10] proved that a bounded linear operator on a Hilbert space is similar to a contraction if and only if it is completely polynomially bounded.

Haagerup, Paulsen and Wittstock proved independently the fundamental factorisation theorem for completely bounded operators, (see [9]).

Theorem 1.1.5. *(Fundamental Factorisation/Extension theorem.) Consider a completely bounded map $\phi : B(\mathcal{H}) \supset E \to F \subset B(K)$. Then there is a Hilbert space $\hat{\mathcal{H}}$, a representation $\pi : B(\mathcal{H}) \to B(\hat{\mathcal{H}})$ and operators $V_1 : K \to \hat{\mathcal{H}}$, $V_2 : \hat{\mathcal{H}} \to K$ such that $||V_1||\,||V_2|| = ||\phi||_{cb}$ and*

$$\forall x \in E \qquad \phi(x) = V_2\pi(x)V_1 \tag{1.1.0.1}$$

Conversely, if (1.1.0.1) holds then ϕ is completely bounded and $||\phi||_{cb} < ||V_1||||V_2||$. Moreover, ϕ admits a completely bounded extension $\tilde{\phi} : B(\mathcal{H}) \to B(K)$ such that $||\tilde{\phi}||_{cb} = ||\phi||$.

See [14] for the proof of this theorem.

We shall continue with the analysis of these completely bounded maps as has been done by Paulsen, Wittstock and other Mathematicians. In our case, we have investigated their norms especially the norm of the multiplicity maps, given examples of completely bounded operators and finally, have investigated the algebraic tensor product of operator spaces.

1.2 Statement of the problem

Let $\mathcal{H}$ be a complex Hilbert space. Consider the space $B(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$. Then clearly, $B(\mathcal{H})$ is a C^*-algebra.

Let $M_{m,n}(B(\mathcal{H}))$ be the space of $m \times n$ matrices with entries in $B(\mathcal{H})$. Let also $\mathcal{H}^n = \mathcal{H} \oplus \ldots \oplus \mathcal{H}$ (n copies) and $\mathcal{H}^m = \mathcal{H} \oplus \ldots \oplus \mathcal{H}$ (m copies) be the orthogonal sum of n, m copies of $\mathcal{H}$ respectively and $B(\mathcal{H}^n, \mathcal{H}^m)$ be the space of all bounded linear operators from $\mathcal{H}^n$ to $\mathcal{H}^m$. Then, we shall calculate norms on $M_{m,n}(B(\mathcal{H}))$ using the fact that the space $M_{m,n}(B(\mathcal{H}))$ have been identified with $B(\mathcal{H}^n, \mathcal{H}^m)$. We shall also investigate norms of the multiplicity maps and the algebraic tensor product of operator spaces with respect to the matrix tensor norms.

1.3 Mathematical background

In this section we give definitions of some of the terms that are very fundamental in understanding this study. We have also shown that $\mathcal{B}(\mathcal{H})$ is indeed a C*-algebra. Finally, we have given a brief account of the Gelfand-Naimark-Segal construction.

1.3.1 Vector spaces

5such a way that

Definition 1.3.1. Free vector space- Given any nonempty set X, let $\mathbb{K}$ be a field. F_X is a vector space over $\mathbb{K}$ with X as basis and
$F_X = \{\sum_{i=1}^{n} r_i x_i : x_i \in X,\ r_i \in \mathbb{K}\}$,
where the operations are as expected -i.e. combine like terms using the rules
$rx + sx = (r + s)x$
$r(sx) = (rs)x.$
The vector space F_X is called the free vector space, (see[13]).

Definition 1.3.2. Tensor product- Let U and V be vector spaces over $\mathbb{K}$, and let I be the subspace of the free vector space $F_{U\times V}$ generated by all vectors of the form
$r(u, v) + s(u', v) - (ru + su', v)$ and
$r(u, v) + s(u, v') - (u, rv + sv')$ for all $r, s \in \mathbb{K},\ u, u' \in U$ and $v, v' \in V$.
The quotient space $F_{U\times V}/I$ is called the tensor product of U and V and

is denoted by $U \otimes V$. An element of $U \otimes V$ has the form

$$\sum_{i=1}^{n} r_i(u_i, v_i) + I.$$

It is customary to denote the coset $(u, v) + I$ by $u \otimes v$ hence any element of $U \otimes V$ has the form

$$\sum_{i=1}^{n} u_i \otimes v_i,$$

where $r(u \otimes v) + s(u' \otimes v) = (ru + su') \otimes v$ and

$r(u \otimes v) + s(u \otimes v') = u \otimes (rv + sv')$.

Definition 1.3.3. Operator/Functional- Functionals are mappings from vector space X to the field of scalars $\mathbb{K}$, while operators are mappings from one vector space X to another vector space Y or to the same vector space X.

Definition 1.3.4. Linear Operator- Let X and Y be vector spaces over the same field $\mathbb{K}$. An operator $T : X \to Y$ is a linear map if

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2), \ \forall \ x_1, x_2 \in X, \ \alpha_1, \ \alpha_2 \in \mathbb{K}.$$

The vector space of linear operators from X to Y is denoted by $\mathcal{L}(X, Y)$.

Definition 1.3.5. Norm- A real valued function $||.|| : V \to \mathbb{R}$, where V is a vector space over the field $\mathbb{K}$ is called a **norm** if it satisfies the following conditions:

(i) $||x|| \geq 0, \ \forall \, x \in V$;

(ii) $||x|| = 0$ if and only if $x = 0, \ \forall\, x \in V$;

(iii) $||cx|| = |c|\,||x||, \ \forall\, x \in V, \ c \in \mathbb{K}$;

(iv) $||x + y|| \leq ||x|| + ||y||, \ \forall\, x, \ y \in V$.

Definition 1.3.6. Bounded operator- A linear operator $T \in \mathcal{L}(X, Y)$ is bounded if there is a constant $N > 0$ such that

$$||Tx|| \leq N||x||, \ \forall\, x \in X.$$

We shall write $B(X, Y)$ for the set of bounded linear operators from X to Y.

Definition 1.3.7. Operator Norm- Let $B(X, Y)$ be the set of bounded linear operators from X to Y. Let $T \in B(X, Y)$ then the norm of T is defined as

$$||T|| = \sup_{x \in X}\{||Tx|| \ : \ ||x|| = 1\}.$$

Definition 1.3.8. Inner product- Let X be a vector space over $\mathbb{C}$. An inner product is a map $\langle ., . \rangle : X \times X \to \mathbb{C}$ satisfying, for $x, y, z \in X$ and scalars $\alpha \in \mathbb{C}$,

i) $\langle x, y \rangle = \overline{\langle y, x \rangle}$,

ii) $\langle x, x \rangle \geq 0$, with $\langle x, x \rangle = 0 \iff x = 0$,

iii) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$,

iv) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$.

The pair $(X, \langle , \rangle)$ is called an inner product space or pre-Hilbert space.

Definition 1.3.9. Hilbert space- A complex Hilbert space $\mathcal{H}$ is a vector space over $\mathbb{C}$ with an inner product such that $\mathcal{H}$ is complete in the metric $d(x, y) = ||x - y|| = \langle x - y, x - y\rangle^{1/2}$.

1.3.2 Algebra

Definition 1.3.10. Algebra- An algebra $\mathcal{A}$ over $\mathbb{K}$ is a vector space $\mathcal{A}$ over $\mathbb{K}$ that also has a multiplication defined on it making $\mathcal{A}$ into a ring such that for $\alpha \in \mathbb{K}$ and $a, b \in \mathcal{A}$,

$$\alpha(ab) = (\alpha a)b = a(\alpha b).$$

Example 1.3.11. Let $\mathcal{H}$ be a Hilbert space , $B(\mathcal{H})$ the set of all bounded linear operators on the Hilbert space $\mathcal{H}$. Then $B(\mathcal{H})$ is an algebra when multiplication is defined pointwise.

Example 1.3.12. If S is a set , $\ell^{\infty}(S)$ the set of all bounded complex-valued functions on the set S is an algebra where the operations are defined as follows:

$$(f + g)(x) = f(x) + g(x)$$

$$(fg)(x) = f(x)g(x)$$

$$(\lambda f)(x) = \lambda f(x)$$

$\forall x \in S$,

Definition 1.3.13. Sub-algebra- A sub-algebra of an algebra $\mathcal{A}$ is a vector space M such that for all $u, u' \in M$ we have $uu' \in M$.

Definition 1.3.14. Unital Algebra- If an algebra $\mathcal{A}$ admits a unit 1 such that $a1 = 1a = a$, $\forall a \in \mathcal{A}$, then we say that $\mathcal{A}$ is a unital algebra, otherwise it is non-unital.

Definition 1.3.15. Involution- If $\mathcal{A}$ is an algebra, an involution is a map $a \mapsto a^*$ of $\mathcal{A}$ into itself such that $\forall\, a, b \in \mathcal{A}$, and $\alpha \in \mathbb{C}$ the following conditions hold:

(i) $(a+b)^* = a^* + b^*$;

(ii) $(\alpha a)^* = \overline{\alpha} a^*$;

(iii) $(ab)^* = b^* a^*$;

(iv) $(a^*)^* = a$.

An algebra $\mathcal{A}$ with an involution $a \mapsto a^*$ is called a ***-algebra** or an involutive algebra.

Definition 1.3.16. Banach algebra- A Banach algebra is an algebra $\mathcal{A}$ over $\mathbb{K}$ that has a norm $||.||$ relative to which $\mathcal{A}$ is a Banach space and such that for all a, b in $\mathcal{A}$,

$$||ab|| \leq ||a||||b||.$$

A **Banach *-algebra** is a Banach algebra $\mathcal{A}$ with involution satisfying the property $||a|| = ||a^*||$, $\forall a \in \mathcal{A}$.

Definition 1.3.17. Spectrum of an operator- If $\mathcal{A}$ is a Banach algebra with identity and $a \in \mathcal{A}$, the spectrum of a, denoted by $\sigma(a)$, is defined by

$$\sigma(a) = \{\lambda \in \mathbb{K} : a - \lambda \text{ is not invertible}\}.$$

Definition 1.3.18. Spectral radius- If $\mathcal{A}$ is a Banach algebra with identity and $a \in \mathcal{A}$, the spectral radius of a, $r(a)$, is defined by

$$r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}.$$

Definition 1.3.19. Algebra norm- Let $\mathcal{A}$ be an algebra. An algebra norm on $\mathcal{A}$ is a map $||.|| : \mathcal{A} \to \mathbb{R}^+$ such that $(\mathcal{A}, ||.||)$ is a normed space and , further:

$$||ab|| \leq ||a|| ||b|| \quad a, b \in \mathcal{A}$$

The normed algebra $(\mathcal{A}, ||.||)$ is a Banach algebra if $||.||$ is a complete norm i.e. if every Cauchy sequence converges.

Definition 1.3.20. Abelian algebra- An algebra $\mathcal{A}$ which is commutative, that is $ab = ba$, $\forall\, a, b \in \mathcal{A}$ is referred to as an Abelian algebra. If the product is non-commutative, it is known as a non-Abelian algebra.

Example 1.3.21. Let $C(\Omega)$ be the Banach space of all complex continuous functions on a non-empty compact Hausdorff space Ω, with the supremum norm. Define multiplication in the usual way:

$$(fg)(x) = f(x)g(x) \qquad x \in \Omega.$$

This makes $C(\Omega)$ into a commutative or Abelian Banach algebra, where the constant function 1 is the unit element.

Definition 1.3.22. Adjoint of an operator- If $T \in B(\mathcal{H}, K)$, where $\mathcal{H}$ and K are Hilbert spaces, then the linear operator $T^* \in B(K, \mathcal{H})$ satisfying $\langle Tx, y\rangle = \langle x, T^*y\rangle$ $\forall x \in \mathcal{H}$ and $y \in K$ is called the adjoint of T.

Definition 1.3.23. C*-algebra- A Banach *-algebra $\mathcal{A}$ such that

$$||aa^*|| = ||a||^2 \qquad \forall a \in \mathcal{A}$$

is called a C^*-algebra.

Example 1.3.24. The algebra of all bounded linear operators $B(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ is a C*-algebra with the usual adjoint operation as involution. To show this, we note first that this follows from the well known identity

$$\begin{aligned} ||T^*T|| &= \sup_{||x||=1} |\langle T^*Tx, x\rangle| \\ &= \sup_{||x||=1} |\langle Tx, Tx\rangle| \\ &= ||T||^2. \end{aligned}$$

$B(\mathcal{H})$ is a vector space over $\mathbb{C}$. In fact it is an algebra if multiplication is defined pointwise (see [2]) i.e. for $S, T \in B(\mathcal{H})$ where $S, T : \mathcal{H} \to \mathcal{H}$ then

$$STx = S(Tx) \;\; \forall\, S, T \in B(H) \text{ and } x \in H.$$

Since $B(\mathcal{H})$ is complete and $\forall\, T \in B(\mathcal{H})$,

$$||T|| = \sup\{||Tx|| \;:\; ||x|| = 1\}.$$

is a norm on it. This norm is submultiplicative i.e.

$$\begin{aligned} ||STx|| &= ||S(Tx)|| \\ &\leq ||S|| ||Tx|| \\ &\leq ||S|| ||T|| ||x||. \end{aligned} \tag{1.3.2.1}$$

and satisfies the C*-condition

$||T||^2 = ||T^*T||$ and hence the assertion follows.

Definition 1.3.25. Self-adjoint operator- A bounded operator $T \in B(\mathcal{H})$ is said to be self-adjoint if $T^* = T$.

Definition 1.3.26. Unitary operator- A unitary operator is a bounded linear operator U on a Hilbert space $\mathcal{H}$ satisfying $U^*U = UU^* = I$, where U^* is the adjoint of U and I is the identity operator.

1.3.3 Positive linear functional

Definition 1.3.27. positive linear functional- A positive linear functional is a linear functional on a Banach *-algebra $\mathcal{A}$ with involution such that $f(aa^*) \geq 0$. $\forall a \in \mathcal{A}$.

Definition 1.3.28. State- Let $\mathcal{A}$ be an involutive algebra. Then the linear functional f is called a state on $\mathcal{A}$ if f is positive and $||f|| = f(e) = 1$ where e is a unit element in $\mathcal{A}$.

Theorem 1.3.29. *If f is a positive linear functional on a C^*-algebra $\mathcal{A}$, then it is bounded.*

Proof. If f is not bounded, then the $\sup_{a\in S} f(a) = \infty$ where S is the set of all positive elements of $\mathcal{A}$ of norm not greater than one. Hence , there is a sequence $(a_n) \subset S$ such that
$2^n \leq f(a_n)$ for all $n \in \mathbb{N}$.
Set $a = \sum_{n=0}^{\infty} a_n/2^n$, so $a \in \mathcal{A}^+$
Now $1 \leq f(a_n/2^n)$ and therefore

$$N \leq \sum_{n=0}^{N-1} f(a_n/2^n) = f(\sum_{n=0}^{N-1} a_n/2^n) \leq f(a)$$

Hence $f(a)$ is an upper bound for the set N , which is impossible. This shows that f is bounded. □

This proof was obtained from [13].

Proposition 1.3.30. *Every positive linear functional f on a Banach *-algebra $\mathcal{A}$ has the following properties :*

i) $f(x^*) = \overline{f(x)}$,

ii) $|f(xy^*)|^2 \leq f(xx^*)f(yy^*)$,

iii) $|f(x)|^2 \leq f(1)f(xx^*)$ *or* $|f(x)|^2 \leq ||f||f(xx^*)$ *since* $||f|| = f(1)$.

Proof. See [3] for the proof. □

1.3.4 Representations of C*-algebras.

In this section, we shall develop the basic properties of representations of C*-algebras. This will culminate in a proof of the fundamental theorem of

Gelfand and Naimark that every C*-algebra is isomorphic to a C*-algebra of operators.

Definition 1.3.31. *-homomorphism- Suppose $\mathcal{A}$ and $\mathcal{B}$ are C*-algebras. A mapping $\pi : \mathcal{A} \to \mathcal{B}$ is said to be a C*-homomorphism or simply a *-homomorphism , if for any $\alpha, \beta \in \mathbb{C}$ and $a, b \in \mathcal{A}$, the following conditions are satisfied

(a) $\pi(\alpha a + \beta b) = \alpha\pi(a) + \beta\pi(b)$.

(b) $\pi(ab) = \pi(a)\pi(b)$

(c) $\pi(a^*) = (\pi(a))^*$

(d) π maps a unit in $\mathcal{A}$ to a unit in $\mathcal{B}$.

Definition 1.3.32. Representation- A representation of a C^*-algebra $\mathcal{A}$ is a pair $(\mathcal{H}, \pi)$ where $\mathcal{H}$ is a Hilbert space and $\pi : \mathcal{A} \to B(\mathcal{H})$ is a *-homomorphism. We say $(\mathcal{H}, \pi)$ is faithful if π is injective. If $\mathcal{A}$ is non-zero, we define its universal representation to be the direct sum of all the representations $(\mathcal{H}_f, \pi_f)$ where f ranges over $S(\mathcal{A})$

Definition 1.3.33. *-representation- A *-representation of a C*-algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a mapping $\pi : \mathcal{A} \to B(\mathcal{H})$ such that

i) π is a ring homomorphism which carries involution on $\mathcal{A}$ into involution on operators.

ii) π is non-degenerate i.e. the space of vectors $\pi(a)x$ is dense as a ranges through $\mathcal{A}$ and x ranges through $\mathcal{H}$.

For a representation π of a C*-algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, an element x is called a cyclic vector if the set of vectors $\{\pi(a)x : a \in \mathcal{A}\}$ is norm dense in $\mathcal{H}$, in which case π is called a cyclic representation.

Definition 1.3.34. Ideal- A subset I of a commutative complex algebra $\mathcal{A}$ is said to be an ideal if

- I is a subspace of $\mathcal{A}$ (in the vector sense) and
- xy and yx are in I whenever $x \in \mathcal{A}$ and $y \in I$.

Associated to any positive linear functional is a positive semi-definite sesquilinear form on $\mathcal{A}$ given by

$$\langle a, b\rangle = f(b^*a).$$

that is $\langle ., .\rangle$ is linear in the first variable and conjugate linear in the second.

The key to representing a C*-algebra on a Hilbert space is to build representations from states. This important procedure is called the GNS construction named after Gelfand, Naimark and Segal, (see [12, 11]).

Theorem 1.3.35. *If f is a state of a C^*-algebra $\mathcal{A}$, there is a cyclic representation π_f of $\mathcal{A}$ on a Hilbert space H_f, and a unit cyclic vector x_f for π_f, such that $f = \omega_{x_f} \circ \pi_f$ that is*

$$f(a) = \langle \pi_f(a)x_f, x_f\rangle, \qquad \forall\ a \in \mathcal{A}.$$

ω_{x_f} is a positive linear functional on $B(\mathcal{H})$.

Proof. Let $\mathcal{N} = \{a \in \mathcal{A} : f(a^*a) = 0\}$. Then

$\mathcal{N}$ is a closed left ideal of $\mathcal{A}$. That is: If $a \in \mathcal{N}$ and $b \in \mathcal{A}$ then by proposition (1.3.30), we have

$|f(b^*a)|^2 \leq f(b^*b)f(a^*a) = 0$, but $f(b^*a) \geq 0$

so $f(b^*a) = 0$.

Upon replacing b by b^*ba, it follows that

$f((b^*ba)^*a) = f((ba)^*ba) = 0$ by definition (1.3.27). So $ba \in \mathcal{N}$ whenever $a \in \mathcal{N}$ and $b \in \mathcal{A}$. Hence $\mathcal{N}$ is a left ideal of $\mathcal{A}$.

To show that $\mathcal{N}$ is closed, let $(x_n) \subset \mathcal{N}$ such that $x_n \xrightarrow{s} x$.

Since f is positive linear functional, it is bounded by theorem (1.3.29) and therefore continuous. Hence $f(x_n^*x_n) \xrightarrow{s} f(x^*x)$

but $(x_n) \subset \mathcal{N}, \ \forall n \in \mathbb{N}$.

$\Rightarrow \ f(x_n^*x_n) \to 0$, so $f(x^*x) = 0$ implying that $x \in \mathcal{N}$ and hence $\mathcal{N}$ is closed. Now, define a positive definite inner product on $\mathcal{A}/\mathcal{N}$ by

$$\langle x + \mathcal{N}, y + \mathcal{N} \rangle = f(y^*x).$$

This is well defined because if

$N_1, N_2 \in \mathcal{N}$, then

$$\begin{aligned}
f((y + N_2)^*(x + N_1)) &= f((y^* + N_2^*)(x + N_1)) \\
&= f(y^*x + (y + N_2)^*N_1 + N_2^*x) \\
&= f(y^*x) + f((y + N_2)^*N_1) + f(N_2^*x) \\
&= f(y^*x) + f((y + N_2)^*N_1) + \overline{f(xN_2^*)} \\
&= f(y^*x) + 0 + 0 \\
&= f(y^*x.)
\end{aligned}$$

$$\begin{aligned}\pi_f(ab)(c+\mathcal{N}) &= abc+\mathcal{N} \\ &= \pi_f(a)(bc+\mathcal{N}) \\ &= \pi_f(a)\pi_f(b)(c+\mathcal{N}), \end{aligned} \tag{1.3.4.2}$$

$$\begin{aligned}\langle b+\mathcal{N}, \pi_f(a)^*(c+\mathcal{N})\rangle = \langle \pi_f(a)(b+\mathcal{N}), c+\mathcal{N}\rangle &= \langle ab+\mathcal{N}, c+\mathcal{N}\rangle \\ &= f(c^*ab) \\ &= f((a^*c)^*b) \\ &= \langle b+\mathcal{N}, a^*c+\mathcal{N}\rangle \\ &= \langle b+\mathcal{N}, \pi_f(a^*)c+\mathcal{N}\rangle. \end{aligned} \tag{1.3.4.3}$$

From (1.3.4.1), (1.3.4.2) , (1.3.4.3) and since $\mathcal{A}/\mathcal{N}$ is everywhere dense in $\mathcal{H}_f$, it follows that

$$\pi_f(\alpha a+\beta b) = \alpha\pi_f(a)+\beta\pi_f(b),$$

$$\pi_f(ab) = \pi_f(a)\pi_f(b),$$

$$\pi_f(a)^* = \pi_f(a^*).$$

Thus π_f is a *-homomorphism of $\mathcal{A}$ into $B(\mathcal{H}_f)$. Accordingly, π_f is a representation of $\mathcal{A}$ on $\mathcal{H}_f$ with x_f the vector $I+\mathcal{N}$ in $\mathcal{A}/\mathcal{N}$.
Now $\pi_f(a)x_f = \pi_f(a)(I+\mathcal{N}) = a+\mathcal{N} \quad \forall\, a \in \mathcal{A}$. Hence $\pi_f(\mathcal{A})x_f$ is the everywhere-dense subset $\mathcal{A}/\mathcal{N}$ of $\mathcal{H}_f$, and x_f is a cyclic vector for π_f.
Moreover,

$$\langle \pi_f(a)x_f, x_f\rangle = \langle a+\mathcal{N}, I+\mathcal{N}\rangle = f(a), \qquad \forall\, a \in \mathcal{A};$$

i.e. $f(a) = \langle \pi_f(a)x_f, x_f \rangle$. □

For more details see [11, 13, 21] .

Theorem 1.3.36. *(Gelfand-Naimark) If $\mathcal{A}$ is a C^*-algebra , then it has a faithful representation. Specifically, its universal representation is faithful.*

Proof. For the proof of this theorem, see [13]. □

The Gelfand-Naimark theorem (1.3.37) is one of those results that are used all the time and in this thesis it has been used in the proof of theorem (2.2.4)

Theorem 1.3.37. *(Gelfand-Naimark) A C^*-algebra $\mathcal{A}$ is isomorphic to an algebra of bounded operators in a Hilbert space.*

Proof. See [19] for the proof of this theorem. □

Chapter 2

Calculating Norm on $M_{m,n}(B(\mathcal{H}))$

In this chapter, we shall first identify the space $M_{m,n}(B(\mathcal{H}))$ of $m \times n$ matrices with entries from $B(\mathcal{H})$ with $B(\mathcal{H}^n, \mathcal{H}^m)$, the space of bounded linear operators from $\mathcal{H}^n$ to $\mathcal{H}^m$ and then use this identification to determine the norm of any element $[T_{i,j}] \in M_{m,n}(B(\mathcal{H}))$.

Given a Hilbert space $\mathcal{H}$, and operators, $T_{i,j} \in B(\mathcal{H}), 1 \leq i \leq m,\ 1 \leq j \leq n$, we identify the $m \times n$ matrix of operators, $[T_{i,j}]$ with an operator from $\mathcal{H}^{(n)} = \mathcal{H} \oplus \ldots \oplus \mathcal{H}$ (n copies) to $\mathcal{H}^{(m)} = \mathcal{H} \oplus \ldots \oplus \mathcal{H}$ (m copies) by regarding vectors in these spaces as columns and performing usual matrix multiplication. This endows $M_{m,n}(B(\mathcal{H}))$ with a norm and this collection of norms on $B(\mathcal{H})$ are often referred to as the **matrix norms** on $B(\mathcal{H})$, (see [14, 15]). When speaking about the norm of a matrix $[T_{i,j}] \in M_{m,n}(B(\mathcal{H}))$, we will always mean its norm as an operator from $\mathcal{H}^n$ to $\mathcal{H}^m$. That is, $M_{m,n}(B(\mathcal{H}))$ will be the space of $m \times n$ matrix of operators acting on an n-dimensional complex Hilbert space $\mathcal{H}^n$ to $\mathcal{H}^m$. We note that when $m = n$, then $M_{m,n}(B(\mathcal{H})) = M_{n,n}(B(\mathcal{H}))$ and $B(\mathcal{H}^n, \mathcal{H}^m)$

is $B(\mathcal{H}^n)$. If $\mathbf{T} \in B(\mathcal{H}^n, \mathcal{H}^m)$, then the norm of $\mathbf{T}$ is given by

$$||\mathbf{T}|| = \sup\{||\mathbf{T}h|| : h \in \mathcal{H}^n; ||h|| = 1\}.$$

2.1 Matrix inequalities

In this section, we prove some important matrix inequalities which we shall use later in the sequel.

Proposition 2.1.1. *Let $M_n(B(\mathcal{H}))$ be an $n \times n$ matrix with entries from $B(\mathcal{H})$. Let $[T_{i,j}] \in M_n(B(\mathcal{H}))$, then we have that*

$$||T_{i,j}|| \leq ||[T_{i,j}]|| \leq \sum_{i=1}^{n} \sum_{j=1}^{n} ||T_{i,j}|| \tag{2.1.0.1}$$

Proof. By the matrix 1-norm, we have

$$||[T_{i,j}]|| = \max_{j} \sum_{i=1}^{n} ||T_{i,j}|| \geq ||T_{i,j}||; \quad j = 1, \ldots, n.$$

This implies that

$$||[T_{i,j}]|| \geq ||T_{i,j}||. \tag{2.1.0.2}$$

Also, we have

$$||[T_{i,j}]|| = \max_{j} \sum_{i=1}^{n} ||T_{i,j}|| \leq \sum_{j=1}^{n} \sum_{i=1}^{n} ||T_{i,j}||.$$

is $B(\mathcal{H}^n)$. If $\mathbf{T} \in B(\mathcal{H}^n, \mathcal{H}^m)$, then the norm of $\mathbf{T}$ is given by

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$$||T_{i,j}|| \leq ||[T_{i,j}]|| \leq \sum_{i=1}^{n}\sum_{j=1}^{n} ||T_{i,j}|| \qquad (2.1.0.1)$$

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This implies that

$$||[T_{i,j}]|| \geq ||T_{i,j}||. \qquad (2.1.0.2)$$

Also, we have

$$||[T_{i,j}]|| = \max_{1\leq j\leq n} \sum_{i=1}^{n} ||T_{i,j}|| \leq \sum_{j=1}^{n}\sum_{i=1}^{n} ||T_{i,j}||.$$

Implying that

$$||[T_{i,j}]|| \leq \sum_{j=1}^{n}\sum_{i=1}^{n}||T_{i,j}||. \qquad (2.1.0.3)$$

Hence, from (2.1.0.2) and (2.1.0.3) we obtain

$$||T_{i,j}|| \leq ||[T_{i,j}]|| \leq \sum_{i=1}^{n}\sum_{j=1}^{n}||T_{i,j}||, \quad (i,j = 1,\ldots,n.)$$

□

Although these inequalities do not determine the matrix norms, they provide important constraints on their properties. In particular, any two such norms must be equivalent on $M_n(B(\mathcal{H}))$ and a sequence say $\mathbf{T}(k)$ ($k \in \mathbb{N}$) in $M_n(B(\mathcal{H}))$ converges if and only if the entries $T(k)_{i,j}$ converges. It is also apparent from these inequalities that $B(\mathcal{H})$ is complete if and only if each of the normed spaces $M_n(B(\mathcal{H}))$ is complete.

2.2 Identification of $M_{m,n}(B(\mathcal{H}))$ with $B(\mathcal{H}^n, \mathcal{H}^m)$

If we can identify the space $M_n(B(\mathcal{H}))$ with $B(\mathcal{H}^n)$ then we can do the same to $M_{m,n}(B(\mathcal{H}))$ with $B(\mathcal{H}^n, \mathcal{H}^m)$, after which we can equate norm on $M_{m,n}(B(\mathcal{H}))$ to norm on $B(\mathcal{H}^n, \mathcal{H}^m)$. This norm is defined by letting the space $M_n(B(\mathcal{H}))$ act on a Hilbert space H^n. We shall denote a typical element of $M_n(B(\mathcal{H}))$ by $[T_{i,j}]$.

In fact the space $M_n(B(\mathcal{H}))$ is an involutive algebra if we define multiplication as

$$[T_{i,k}].[S_{k,j}] = (\sum_{k=1}^{n} T_{i,k}S_{k,j}) \text{ for any } [T_{i,j}],\ [S_{i,j}] \in M_n(B(\mathcal{H})),$$

and define involution as

$$[T_{i,j}]^* = [T_{j,i}^*].$$

It is not obvious that the *-algebra $M_{m,n}(B(\mathcal{H}))$ is a C*-algebra with this identification. We shall prove in Proposition (2.2.2) that indeed $M_{m,n}(B(\mathcal{H}))$ is a C*-algebra.

Now, let $\mathcal{H}^n$ denote the direct sum of n copies of $\mathcal{H}$. Then we shall define an inner product on $\mathcal{H}^n$ by: $\forall\ h, f \in \mathcal{H}^n$ where $h = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$ and

$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

$$\langle h, f\rangle = \left\langle \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \right\rangle_{\mathcal{H}^n} = \langle h_1, f_1\rangle_{\mathcal{H}} + \ldots + \langle h_n, f_n\rangle_{\mathcal{H}},$$

and also a norm by:

$$||h||^2 = \left\| \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \right\|^2 = ||h_1||^2 + \ldots + ||h_n||^2$$

So $\mathcal{H}^n$ is a normed space. In fact it is a Hilbert space.

Proposition 2.2.1. *Let $M_n(B(\mathcal{H}))$ and $B(\mathcal{H}^n)$ be *-algebras. Then there exists a linear mapping $\pi : M_n(B(\mathcal{H})) \rightarrow B(\mathcal{H}^n)$ such that π is a *-isomorphism.*

Proof. Let $\pi : M_n(B(\mathcal{H})) \rightarrow B(\mathcal{H}^n)$ be a mapping between these two *-algebras. If $[T_{i,j}] \in M_n(B(\mathcal{H}))$, we define $\pi([T_{i,j}]) \in B(\mathcal{H}^n)$ by setting

$$\pi([T_{i,j}]) \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = \pi \begin{pmatrix} T_{11} & \dots & T_{1n} \\ \vdots & & \vdots \\ T_{n1} & \dots & T_{nn} \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} T_{1,j} h_j \\ \vdots \\ \sum_{j=1}^{n} T_{n,j} h_j \end{pmatrix}$$
$$= \begin{pmatrix} T_{11} & \dots & T_{1n} \\ \vdots & & \vdots \\ T_{n1} & \dots & T_{nn} \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$$

for all $\begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \in \mathcal{H}^n$.

From the above development, the map π is just the ordinary matrix multiplication. We need to show that this map $\pi : M_n(B(\mathcal{H})) \rightarrow B(\mathcal{H}^n)$ is a *-isomorphism, whence it would follow that $M_n(B(\mathcal{H})) \equiv B(\mathcal{H}^n)$.

To do this, it suffices to show that π is a *-homomorphism and is bijective.

Let $\alpha, \beta \in \mathbb{C}$, $[T_{i,j}], [S_{i,j}] \in M_n(B(\mathcal{H}))$ and $h \in \mathcal{H}^n$, then

$$\begin{aligned}
\pi(\alpha[T_{i,j}] + \beta[S_{i,j}])(h) &= \begin{pmatrix} \sum_{j=1}^n (\alpha T_{1,j} + \beta S_{1,j})(h_j) \\ \vdots \\ \sum_{j=1}^n (\alpha T_{n,j} + \beta S_{n,j})(h_j) \end{pmatrix} \\
&= \begin{pmatrix} \sum_{j=1}^n (\alpha T_{1,j}(h_j) + \beta S_{1,j}(h_j)) \\ \vdots \\ \sum_{j=1}^n (\alpha T_{n,j}(h_j) + \beta S_{n,j}(h_j)) \end{pmatrix} \\
&= \begin{pmatrix} \sum_{j=1}^n \alpha T_{1,j}(h_j) + \sum_{j=1}^n \beta S_{1,j}(h_j) \\ \vdots \\ \sum_{j=1}^n \alpha T_{n,j}(h_j) + \sum_{j=1}^n \beta S_{n,j}(h_j) \end{pmatrix} \\
&= \alpha \begin{pmatrix} \sum_{j=1}^n T_{1,j}(h_j) \\ \vdots \\ \sum_{j=1}^n T_{n,j}(h_j) \end{pmatrix} + \beta \begin{pmatrix} \sum_{j=1}^n S_{1,j}(h_j) \\ \vdots \\ \sum_{j=1}^n S_{n,j}(h_j) \end{pmatrix} \\
&= \alpha\pi([T_{i,j}])(h) + \beta\pi([S_{i,j}])(h) \\
&= (\alpha\pi([T_{i,j}]) + \beta\pi([S_{i,j}]))(h)
\end{aligned}$$

Thus π is linear.

Next, we show that $\pi((S_{i,k})(T_{k,j})) = \pi((S_{i,k}))\pi((T_{k,j}))$. Let $[R_{i,j}] = [S_{i,k}][T_{k,j}]$. Then

$$\pi([R_{i,j}])(h) = \begin{pmatrix} \sum_{j=1}^{n} R_{1,j}h_j \\ \vdots \\ \sum_{j=1}^{n} R_{n,j}h_j \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{n}\sum_{j=1}^{n} S_{1,k}T_{k,j}h_j \\ \vdots \\ \sum_{k=1}^{n}\sum_{j=1}^{n} S_{n,k}T_{k,j}h_j \end{pmatrix} = \begin{pmatrix} \sum_{k,j=1}^{n} S_{1,k}T_{k,j}h_j \\ \vdots \\ \sum_{k,j=1}^{n} S_{n,k}T_{k,j}h_j \end{pmatrix} \quad (2.2.0.4)$$

Also

$$(\pi([S_{i,k}])\pi([T_{k,j}]))(h) = \pi([S_{i,k}])\begin{pmatrix} \sum_{j=1}^{n} T_{1,j}h_j \\ \vdots \\ \sum_{j=1}^{n} T_{n,j}h_j \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{n} S_{1,k}\left(\sum_{j=1}^{n} T_{k,j}h_j\right) \\ \vdots \\ \sum_{k=1}^{n} S_{n,k}\left(\sum_{j=1}^{n} T_{k,j}h_j\right) \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{n}\sum_{j=1}^{n} S_{1,k}T_{k,j}h_j \\ \vdots \\ \sum_{k=1}^{n}\sum_{j=1}^{n} S_{n,k}T_{k,j}h_j \end{pmatrix} = \begin{pmatrix} \sum_{k,j=1}^{n} S_{1,k}T_{k,j}h_j \\ \vdots \\ \sum_{k,j=1}^{n} S_{n,k}T_{k,j}h_j \end{pmatrix} \quad (2.2.0.5)$$

From (2.2.0.4) and (2.2.0.5) it follows that $\pi([S_{i,k}][T_{i,k}]) = \pi([S_{i,k}])\pi([T_{k,j}])$.
Hence π is a homomorphism.

We then show that, for this π, $\pi([T_{i,j}])^* = \pi([T^*_{j,i}])$.

Let $\begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \in \mathcal{H}^n$

Then

$$\left\langle \pi([T_{i,j}]) \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, \pi([T_{i,j}])^* \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \right\rangle \tag{2.2.0.6}$$

Also

$$
\begin{aligned}
\left\langle \pi([T_{i,j}]) \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \right\rangle &= \left\langle \begin{pmatrix} \sum_{j=1}^n T_{1,j}h_j \\ \vdots \\ \sum_{j=1}^n T_{n,j}h_j \end{pmatrix}, \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \right\rangle \\
&= \left\langle \sum_{j=1}^n T_{1,j}h_j, f_1 \right\rangle + \ldots + \left\langle \sum_{j=1}^n T_{n,j}h_j, f_n \right\rangle \\
&= \sum_{i=1}^n \sum_{j=1}^n \langle T_{i,j}h_j, f_i \rangle \\
&= \sum_{i,j=1}^n \langle T_{i,j}h_j, f_i \rangle \\
&= \sum_{i,j=1}^n \langle h_j, T_{j,i}^* f_i \rangle \\
&= \left\langle h_1, \sum_{i=1}^n T_{1,i}^* f_i \right\rangle + \ldots + \left\langle h_n, \sum_{i=1}^n T_{n,i}^* f_i \right\rangle \\
&= \left\langle \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, \begin{pmatrix} \sum_{i=1}^n T_{1,i}^* f_i \\ \vdots \\ \sum_{i=1}^n T_{n,i}^* f_i \end{pmatrix} \right\rangle \\
&= \left\langle \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, \pi([T_{j,i}^*]) \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \right\rangle \qquad (2.2.0.7)
\end{aligned}
$$

From (2.2.0.6) and (2.2.0.7) and since $\mathcal{H}$ is a complex Hilbert space, it follows that

$$\pi([T_{i,j}])^* = \pi([T_{j,i}^*]).$$

Thus π is a *-homomorphism.

It now remains to show that π is bijective.

Let

$$E_j : \mathcal{H} \to \mathcal{H}^n \tag{2.2.0.8}$$

be a map defined by

$E_k(h) =$ vector that has h for its k-th entry and is 0 elsewhere

Now, suppose $\pi([T_{i,j}]) = 0$, then

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \pi([T_{i,j}]E_k(h)) = \begin{pmatrix} T_{1,k}h \\ \vdots \\ T_{n,k}h \end{pmatrix}; \quad k = \{1, \ldots, n\}$$

This implies that $T_{i,k}h = 0$ for all $h \in \mathcal{H}$ and for all $i, k = \{1, \ldots, n\}$. Hence $[T_{i,j}] = 0$. Therefore, π is injective. We next show that π is onto. To do this , we define a map

$$E_k^* : \mathcal{H}^n \to \mathcal{H} \tag{2.2.0.9}$$

We shall first show that, this map sends a vector in $\mathcal{H}^n$ to its j-th component. Note that from (2.2.0.8), $\sum_{k=1}^{n} E_k h_k = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$.

To show that 2.2.0.9 is the required map, let $h \in \mathcal{H}$, $\begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \in \mathcal{H}^n$.

Then by the definition of adjoints,

$\left\langle E_j^* \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, h \right\rangle = \left\langle \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, E_j h \right\rangle = \langle h_j, h \rangle$. Thus E_j^* is the map that sends $\begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$ to h_j as required. We then proceed to show that π is onto. To prove this, it is enough to show that $\pi([T_{i,j}]) = T$ for any $T \in B(\mathcal{H}^n)$.

To show this, let $T_{i,j} = E_i^* T E_j$, $\begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \in \mathcal{H}^n$.

Then

$$\begin{aligned}
\left\langle \pi([T_{i,j}]) \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}, \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \right\rangle &= \left\langle \begin{pmatrix} \sum_{j=1}^n T_{1,j} h_j \\ \vdots \\ \sum_{j=1}^n T_{n,j} h_j \end{pmatrix}, \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \right\rangle \\
&= \sum_{i,j=1}^n \langle T_{i,j} h_j, f_i \rangle \\
&= \sum_{i,j=1}^n \langle E_i^* T E_j h_j, f_i \rangle \\
&= \sum_{i,j=1}^n \langle T E_j h_j, E_i f_i \rangle \\
&= \left\langle \sum_{j=1}^n T E_j h_j, \sum_{i=1}^n E_i f_i \right\rangle \\
&= \left\langle T \left(\sum_{j=1}^n E_j h_j \right), \sum_{i=1}^n E_i f_i \right\rangle \\
&= \left\langle T \left(\begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \right), \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \right\rangle
\end{aligned} \tag{2.2.0.10}$$

Thus, from (2.2.0.10), and since $\mathcal{H}$ is a complex Hilbert space, it follows that $\pi([T_{i,j}]) = T$. Hence π is onto and therefore is a *-isomorphism.

□

Moreover, this π is a representation of $M_n(B(\mathcal{H}))$ on the Hilbert space $\mathcal{H}^n$. We call π the canonical *-isomorphism of $M_n(B(\mathcal{H}))$ onto $B(\mathcal{H}^n)$. Therefore, we can identify $M_{m,n}(B(\mathcal{H}))$ with $B(\mathcal{H}^n, \mathcal{H}^m)$. This identification gives us a norm that makes the *-algebra $M_{m,n}(B(\mathcal{H}))$ a C*-algebra as evident in the following proposition.

Proposition 2.2.2. *Let $\pi : M_{m,n}(B(\mathcal{H})) \longrightarrow B(\mathcal{H}^n, \mathcal{H}^m)$ be a *-isomorphism. Then the norm defined by*

$$\|[T_{i,j}]\| = \|\pi([T_{i,j}])\| \qquad (2.2.0.11)$$

*makes the *-algebra $M_{m,n}(B(\mathcal{H}))$ a C*-algebra.*

Proof. It is clear that $\|[T_{i,j}]\| = \|\pi([T_{i,j}])\|$ is a norm. In fact it is submultiplicative. It now remains to show that equation (2.2.0.11), satisfies the condition $\|[T_{i,j}]^*[T_{i,j}]\| = \|[T_{i,j}]\|^2$.

$$\begin{aligned}
\|[T_{i,j}]^*[T_{i,j}]\| &= \|\pi([T_{i,j}]^*[T_{i,j}])\| \\
&= \|\pi([T_{i,j}]^*)\pi([T_{i,j}])\| \\
&\leq \|\pi([T_{i,j}]^*)\|\|\pi([T_{i,j}])\| \\
&\leq \|\pi\|\|[T_{i,j}]^*\|\|\pi\|\|[T_{i,j}]\| \\
&\leq \|[T_{i,j}]^*\|\|[T_{i,j}]\| \\
&= \|[T_{i,j}]\|^2 \text{ since } \|[T_{i,j}]\| = \|[T_{i,j}]^*\| \\
& \text{ because } M_{m,n}(B(\mathcal{H})) \text{ is a *-algebra.}
\end{aligned}$$

Hence

$$\|[T_{i,j}]^*[T_{i,j}]\| \leq \|[T_{i,j}]\|^2. \tag{2.2.0.12}$$

On the other hand,

$$\begin{aligned}
0 \leq \|[T_{i,j}]\|^2 &= \|\pi([T_{i,j}])\|^2 \\
&= \langle \pi([T_{i,j}])h, \pi([T_{i,j}])h\rangle \text{ for } h \in H^n \text{ and of unit length.} \\
&= \langle \pi([T_{i,j}])^*\pi([T_{i,j}])h, h\rangle \\
&= \langle \pi([T_{i,j}]^*[T_{i,j}])h, h\rangle \\
&= |\langle \pi([T_{i,j}]^*[T_{i,j}])h, h\rangle| \\
&\leq \|\pi([T_{i,j}]^*[T_{i,j}])h\|\|h\| \text{ by C.B.S inequality.} \\
&\leq \|\pi([T_{i,j}]^*[T_{i,j}])\|\|h\|^2 \\
&\leq \|\pi\|\|[T_{i,j}]^*[T_{i,j}]\|\|h\|^2 \\
&\leq \|[T_{i,j}]^*[T_{i,j}]\|.
\end{aligned}$$

Hence

$$\|[T_{i,j}]\|^2 \leq \|[T_{i,j}]^*[T_{i,j}]\| \tag{2.2.0.13}$$

From (2.2.0.12) and (2.2.0.13), it follows that $\|[T_{i,j}]^*[T_{i,j}]\| = \|[T_{i,j}]\|^2$ implying (2.2.0.11) satisfies the C*-condition, and therefore $M_{m,n}(B(\mathcal{H}))$ is a C*-algebra. □

The norm defined above is unique. Before showing this, we shall require the following result.

Proposition 2.2.3. *If* $\|.\|$ *is a complete* C^**-norm on a* **-algebra* $\mathcal{A}$*, then it is given by the expression*

$$\|a\| = r(a^*a)^{\frac{1}{2}}, \quad \forall a \in \mathcal{A}, \tag{2.2.0.14}$$

where $r(a)$ is the spectral radius of a. Hence a C^*-norm on a $*$-algebra is unique if it exists.

Proof. See [22, 13] for the proof. □

Theorem 2.2.4. *If $B(\mathcal{H})$ is a C^*-algebra, then there is a unique norm on $M_n(B(\mathcal{H}))$ making it a C^*-algebra.*

Proof. $M_n(B(\mathcal{H}))$ being a C*-algebra has been shown in Proposition (2.2.2). It now remains to show that the C*- norm defined in Proposition (2.2.2) is unique. The uniqueness of this C*-norm follows from expression of equation (2.2.0.14). For, if $||.||_1$ and $||.||_2$ are norms on the *-algebra $M_n(B(\mathcal{H}))$ making it a C*-algebra, then

$$||[T_{i,j}]||_1^2 = ||[T_{i,j}]^*[T_{i,j}]||_1 = r([T_{i,j}]^*[T_{i,j}]) = \sup_{\lambda \in \sigma([T_{i,j}]^*[T_{i,j}])} |\lambda|.$$

and

$$||[T_{i,j}]||_2^2 = ||[T_{i,j}]^*[T_{i,j}]||_2 = r([T_{i,j}]^*[T_{i,j}]) = \sup_{\lambda \in \sigma([T_{i,j}]^*[T_{i,j}])} |\lambda|$$

which implies that $||[T_{i,j}]||_1 = ||[T_{i,j}]||_2$, i.e. the two norms are equal. □

2.3 Norm on $M_{m,n}(B(\mathcal{H}))$

There is a natural way to regard an element of $M_{m,n}(B(\mathcal{H}))$ as a linear map on $\mathcal{H}^n$, by using the usual rules for matrix products (see [7]).

Let $[T_{i,j}] \in M_{m,n}(B(\mathcal{H}))$ and $h \in \mathcal{H}^n$, then

$$[T_{i,j}](h) = [T_{i,j}] \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} T_{1,j} h_j \\ \vdots \\ \sum_{j=1}^{n} T_{m,j} h_j \end{pmatrix}.$$

We shall check if this element of $M_{m,n}(B(\mathcal{H}))$, is a bounded linear operator on $\mathcal{H}^n$, i.e. if $\exists\, c \in \mathbb{R} : ||[T_{i,j}]h|| \leq c||h||$.

$$
\begin{aligned}
\left\| [T_{i,j}] \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \right\|^2 &= \left\| \begin{pmatrix} T_{11} & \dots & T_{1n} \\ \vdots & & \vdots \\ T_{m1} & \dots & T_{mn} \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \right\|^2 \\
&= \left\| \begin{pmatrix} \sum_{j=1}^{n} T_{1,j} h_j \\ \vdots \\ \sum_{j=1}^{n} T_{m,j} h_j \end{pmatrix} \right\|^2 \\
&= \left\| \sum_{j=1}^{n} T_{1,j} h_j \right\|^2 + \ldots + \left\| \sum_{j=1}^{n} T_{m,j} h_j \right\|^2 \\
&= \sum_{i=1}^{m} \left\| \sum_{j=1}^{n} T_{i,j} h_j \right\|^2 \\
&\leq \sum_{i=1}^{m} \sum_{j=1}^{n} \|T_{i,j} h_j\|^2 \qquad \text{by triangle inequality.} \\
&\leq \sum_{i=1}^{m} \sum_{j=1}^{n} \|T_{i,j}\|^2 \|h_j\|^2 \\
&\leq \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \|T_{i,j}\|^2 \right) \sum_{j=1}^{n} \|h_j\|^2 \\
&= c \sum_{j=1}^{n} \|h_j\|^2 \text{ where } c = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \|T_{i,j}\|^2 \right)
\end{aligned}
$$

and the assertion follows.

Putting $\sum_{j=1}^{n} \|h_j\|^2 = 1$ and then taking the square root on both sides we obtain

$$
\left\| [T_{i,j}] \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} \right\| \leq \sqrt{c}.
$$

$\Rightarrow \quad \|[T_{i,j}]\| \quad \leq \quad \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} \|T_{i,j}\|^2}$. Since $[T_{i,j}]$ was picked arbitrarily, we have that the norm of $[T_{i,j}] \in M_{m,n}(B(\mathcal{H}))$ can be approximated by $\|[T_{i,j}]\| \leq \sqrt{\left(\sum_{i=1}^{m} \sum_{j=1}^{n} \|T_{i,j}\|^2\right)}$.

Chapter 3

Operator Spaces and Completely Bounded Operators

3.1 Completely bounded operators.

This chapter contains the formal definition of operator spaces, complete boundedness for linear operators between C*-algebras, the main representation theory for completely bounded operators, Wittstock's decomposition theorem for c.b. operators, algebraic tensor product of operator spaces and a discussion of other basic related results.

Definition 3.1.1. Let $\mathcal{H}$ be a Hilbert space, $B(\mathcal{H})$ a set of bounded linear operators on $\mathcal{H}$ and let $\mathcal{M} \subseteq B(\mathcal{H})$ be a subspace. Let $M_{n,m}(B(\mathcal{H}))$ be a $n \times m$ matrix algebra with entries from $B(\mathcal{H})$. Then the inclusion, $M_{n,m}(\mathcal{M}) \subseteq M_{n,m}(B(\mathcal{H}))$ endows this vector space with a collection of matrix norms and we call $\mathcal{M}$ together with this collection of matrix norms on $M_{n,m}(\mathcal{M})$ an operator space. When $m = n$, then $M_{n,m}(\mathcal{M})=M_{n,n}(\mathcal{M}) = M_n(\mathcal{M})$.

Alternatively, an operator space is a closed subspace of $B(\mathcal{H})$.

The study of operator spaces is related to the study of completely bounded operators (i.e. the morphisms between operator spaces).

Definition 3.1.2. Given a C*-algebra $\mathcal{A}$, an operator space $\mathcal{M} \subseteq \mathcal{A}$, and a linear map $\phi : \mathcal{M} \to B(\mathcal{H})$, we define a linear operator $\phi_n : M_n(\mathcal{M}) \to M_n(B(\mathcal{H}))$ by

$$\phi_n([a_{ij}]) = [\phi(a_{ij})]$$

for $[a_{ij}] \in M_n(\mathcal{M})$. We call ϕ completely bounded, if $||\phi||_{cb} = \sup_{n\in\mathbb{N}} ||\phi_n||$, is finite

The space of completely bounded operators from $\mathcal{A}$ to $\mathcal{B}$ with this norm is denoted by $\mathcal{CB}(\mathcal{A},\mathcal{B})$. We shall show that $||.||_{cb}$ is a norm on $\mathcal{CB}(\mathcal{A},\mathcal{B})$.

Proposition 3.1.3. *$||.||_{cb}$ is a norm on the linear space $\mathcal{CB}(\mathcal{A},\mathcal{B})$.*

Proof. Clearly $||\phi||_{cb}$ is non-negative and is zero if and only if ϕ_n is zero for every $n \in \mathbb{N}$.
Now, let $\lambda \in \mathbb{C}$, then

$$\begin{aligned} ||\lambda\phi||_{cb} &= \sup_n ||\lambda\phi_n|| \\ &= \sup_n |\lambda|.||\phi_n|| \\ &= |\lambda| \sup_n ||\phi_n|| \\ &= |\lambda|.||\phi||_{cb} \end{aligned}$$

Let ψ, ϕ be completely bounded linear operators, then it's clear that $\psi + \phi$ is also completely bounded. Therefore

$$\begin{aligned} ||\psi + \phi||_{cb} &= \sup_n ||\psi_n + \phi_n|| \\ &\leq \sup_n \{||\psi_n|| + ||\phi_n||\} \\ &\leq \sup_n ||\psi_n|| + \sup_n ||\phi_n|| \\ &= ||\psi||_{cb} + ||\phi||_{cb} \end{aligned}$$

Thus $||\phi||_{cb} = \sup_n ||\phi_n||$. is indeed a norm. □

Proposition 3.1.4. *Let $\mathcal{M}$, $\mathcal{M}^*$ be operator spaces in the C^*-algebra $\mathcal{A}$ such that $\mathcal{M}^* = \{a^* : a \in \mathcal{M}\}$. Let $B(\mathcal{H})$ be another C^*-algebra such that $\phi : \mathcal{M} \rightarrow B(\mathcal{H})$ is a linear map. Then the map $\phi^* : \mathcal{M}^* \rightarrow B(\mathcal{H})$ defined by*

$$\phi^*(a) = \phi(a^*)^*$$

is also linear map and $||\phi_n|| = ||\phi_n^*||$.

Proof. Let $\mathcal{M}$ and $\mathcal{M}^*$ be operator spaces. Let also $\alpha, \beta \in \mathbb{C}$ and $a, b \in \mathcal{M}$, then by definition, $\alpha a + \beta b \in \mathcal{M}$ so that $(\alpha a + \beta b)^* \in \mathcal{M}^*$.
Now

$$\begin{aligned} \phi^*(\alpha a + \beta b) &= \phi((\alpha a + \beta b)^*)^*. \\ &= \phi(\overline{\alpha} a^* + \overline{\beta} b^*)^* \\ &= [\overline{\alpha}\phi(a^*) + \overline{\beta}\phi(b^*)]^* \text{ since } \phi \text{ is linear} \\ &= \overline{\overline{\alpha}}\phi(a^*)^* + \overline{\overline{\beta}}\phi(b^*)^* \\ &= \alpha\phi^*(a) + \beta\phi^*(b) \end{aligned}$$

Hence ϕ^* is a linear map.

With these linear maps: $\phi : \mathcal{M} \to B(\mathcal{H})$, $\phi^* : \mathcal{M}^* \to B(\mathcal{H})$, we can define their corresponding linear maps:

$$\phi_n : M_n(\mathcal{M}) \to M_n(B(\mathcal{H}))$$

via $\phi_n([a_{i,j}]) = [\phi(a_{i,j})]$ where $[a_{i,j}] \in M_n(\mathcal{M})$ and

$$\phi_n^* : M_n(\mathcal{M}^*) \to M_n(B(\mathcal{H}))$$

via $\phi_n^*([a_{i,j}]) = [\phi^*(a_{i,j})] = [\phi(a_{i,j})^*]^*$ for all $i, j = 1, \ldots, n$.

For the case when $n = 2$, we have

$$\begin{aligned}
\phi_2^*([a_{i,j}]) &= [\phi((a_{i,j})^*)]^* \\
&= [\phi(a_{j,i}^*)]^* \\
&= \begin{bmatrix} \phi(a_{1,1}^*) & \phi(a_{2,1}^*) \\ \phi(a_{1,2}^*) & \phi(a_{2,2}^*) \end{bmatrix}^* \\
&= \begin{bmatrix} \phi(a_{1,1}^{**}) & \phi(a_{1,2}^{**}) \\ \phi(a_{2,1}^{**}) & \phi(a_{2,2}^{**}) \end{bmatrix} \\
&= \begin{bmatrix} \phi(a_{1,1}) & \phi(a_{1,2}) \\ \phi(a_{2,1}) & \phi(a_{2,2}) \end{bmatrix} \\
&= \phi_2([a_{i,j}])
\end{aligned}$$

From the above calculation, we have, for any $n \in \mathbb{N}$ that

$$\begin{aligned}
\phi_n^*([a_{i,j}]) = [\phi^*(a_{i,j})] &= [\phi(a_{i,j})^*]^* \\
&= [\phi(a_{j,i}^*)]^* \\
&= [\phi(a_{i,j}^{**})] \\
&= [\phi(a_{i,j})] \\
&= \phi_n([a_{i,j}])
\end{aligned}$$

Hence $||\phi_n^*([a_{i,j}])|| = ||\phi_n([a_{i,j}])|| \leq ||\phi_n||||[a_{i,j}]||$
Thus $||\phi_n^*([a_{i,j}])|| \leq ||\phi_n||.||[a_{i,j}]||$.
Taking the supremum norm on both sides with $||[a_{i,j}]|| = 1$, one thus obtains $||\phi_n^*|| \leq ||\phi_n||$
Similarly
$||\phi_n([a_{i,j}])|| = ||\phi_n^*([a_{i,j}])|| \leq ||\phi_n^*||.||[a_{i,j}]||$
Thus $||\phi_n([a_{i,j}])|| \leq ||\phi_n^*||.||[a_{i,j}]||$.
Again taking the supremum norm on both sides with $||[a_{i,j}]|| = 1$, one thus obtains $||\phi_n|| \leq ||\phi_n^*||$. Hence $||\phi_n|| = ||\phi_n^*||$. Therefore

$$||\phi||_{cb} = \sup_n ||\phi_n|| = \sup_n ||\phi_n^*|| = ||\phi^*||_{cb}.$$

□

Proposition 3.1.5. *Given a C^*-algebra $\mathcal{A}$, an operator space $\mathcal{M} \subseteq \mathcal{A}$, and a linear map, $\phi : \mathcal{M} \to B(\mathcal{H})$, The norms $||\phi_n||_{n\in\mathbb{N}}$ form an increasing sequence*

$$||\phi|| \leq ||\phi_2|| \leq \ldots \leq ||\phi_n|| \leq \ldots \leq ||\phi||_{cb}.$$

and

$$||\phi_n|| \leq n||\phi||.$$

Proof. We shall first prove the first inequality. Notice that when $n = 1$, then by the definition of ϕ_n, ϕ_1 coincides with ϕ and hence $||\phi|| = ||\phi_1||$. Now let's consider cases when $n = 2, 3$.

Let $[a_{i,j}] \in M_2(\mathcal{M})$ $\quad i,\ j = 1, 2$, then for

$\phi_2 : M_2(\mathcal{M}) \longrightarrow M_2(B(\mathcal{H}))$, we have

$$\phi_2\left(\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}\right) = \begin{pmatrix} \phi(a_{1,1}) & \phi(a_{1,2}) \\ \phi(a_{2,1}) & \phi(a_{2,2}) \end{pmatrix}$$

and

$$\begin{aligned} \left\|\phi_2\left(\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}\right)\right\| &= \left\|\begin{pmatrix} \phi(a_{1,1}) & \phi(a_{1,2}) \\ \phi(a_{2,1}) & \phi(a_{2,2}) \end{pmatrix}\right\| \\ &= \sqrt{\sum_{i=1}^{2}\sum_{j=1}^{2}||\phi(a_{i,j})||^2} \qquad \text{by Hilbert-Schmidt norm} \\ &= \sqrt{||\phi(a_{1,1})||^2 + ||\phi(a_{1,2})||^2 + ||\phi(a_{2,1})||^2 + ||\phi(a_{2,2})||^2} \\ &\geq \sqrt{||\phi(a_{1,1})||^2} = ||\phi(a_{1,1})|| = ||\phi_1(a_{1,1})|| \end{aligned}$$

Thus $||\phi_2|| = \sup_{[a_{i,j}]\in M_2(\mathcal{M})} ||\phi_2([a_{i,j}])|| \geq \sup ||\phi_1(a_{1,1})|| = ||\phi_1||$. Hence $||\phi_2|| \geq ||\phi_1||$.

For $n = 3$, we have

$$\phi_3\left(\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}\right) = \begin{pmatrix} \phi(a_{1,1}) & \phi(a_{1,2}) & \phi(a_{1,3}) \\ \phi(a_{2,1}) & \phi(a_{2,2}) & \phi(a_{2,3}) \\ \phi(a_{3,1}) & \phi(a_{3,2}) & \phi(a_{3,3}) \end{pmatrix}$$

So

$$\begin{aligned}
\left\|\phi_3\left(\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}\right)\right\| &= \left\|\begin{pmatrix} \phi(a_{1,1}) & \phi(a_{1,2}) & \phi(a_{1,3}) \\ \phi(a_{2,1}) & \phi(a_{2,2}) & \phi(a_{2,3}) \\ \phi(a_{3,1}) & \phi(a_{3,2}) & \phi(a_{3,3}) \end{pmatrix}\right\| \\
&= \sqrt{\sum_{i=1}^{3}\sum_{j=1}^{3}||\phi(a_{i,j})||^2} \\
&= (\|\phi(a_{1,1})\|^2 + \|\phi(a_{1,2})\|^2 + \|\phi(a_{1,3})\|^2 + \|\phi(a_{2,1})\|^2 + \\
&\quad \|\phi(a_{2,2})\|^2 + \|\phi(a_{2,3})\|^2 + \|\phi(a_{3,1})\|^2 + \|\phi(a_{3,2})\|^2 + \\
&\quad \|\phi(a_{3,3})\|^2)^{1/2} \\
&\geq \sqrt{\|\phi(a_{1,1})\|^2 + \|\phi(a_{1,2})\|^2 + \|\phi(a_{2,1})\|^2 + \|\phi(a_{2,2})\|^2} \\
&= \sqrt{\sum_{i=1}^{2}\sum_{j=1}^{2}||\phi(a_{i,j})||^2} \\
&= ||\phi_2([a_{i,j}])||
\end{aligned}$$

Thus $||\phi_3|| = \sup_{[a_{i,j}]\in M_3(\mathcal{M})} ||\phi_3([a_{i,j}])|| \geq \sup_{[a_{i,j}]\in M_2(\mathcal{M})} ||\phi_2([a_{i,j}])|| = ||\phi_2||$. Hence $||\phi_3|| \geq ||\phi_2||$.

Therefore in general, let's consider

$$\phi_{n+1} : M_{n+1}(\mathcal{M}) \rightarrow M_{n+1}(B(\mathcal{H}))$$

defined by $\phi_{n+1}([a_{i,j}]) = [\phi(a_{i,j})]$ for all $i, j = 1, \ldots, n+1$. By the above calculations we have

$$\begin{aligned} \|\phi_{n+1}([a_{i,j}])\| &= \|[\phi(a_{i,j})]\| \\ &= \sqrt{\sum_{i=1}^{n+1}\sum_{j=1}^{n+1} \|\phi(a_{i,j})\|^2} \\ &\geq \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \|\phi(a_{i,j})\|^2} \\ &= \|\phi_n([a_{i,j}])\| \end{aligned}$$

Taking supremum on both sides we thus obtain $\|\phi_{n+1}\| \geq \|\phi_n\|$.
The completely bounded norm of ϕ is given by

$$||\phi||_{cb} = \sup_n ||\phi_n||.$$

$\Rightarrow \|\phi\|_{cb} \geq \|\phi_n\| \;\; \forall n \in \mathbb{N}$.

Thus $||\phi|| \leq ||\phi_2|| \leq \ldots \leq ||\phi_n|| \leq \ldots \leq ||\phi||_{cb}$

To show that $||\phi_n|| \leq n||\phi||$, let $||a_{i,j}|| \leq 1 \ \forall \ i, j$, then

$$\begin{aligned}
||\phi_n([a_{i,j}])|| &= ||[\phi(a_{i,j})]|| \\
&= \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} ||\phi(a_{i,j})||^2} \\
&\leq \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} ||\phi||^2||(a_{i,j})||^2} \\
&= ||\phi||\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} ||(a_{i,j})||^2} \\
&= ||\phi||(||a_{1,1}||^2 + \cdots + ||a_{1,n}||^2 + ||a_{2,1}||^2 + \cdots + ||a_{2,n}||^2 + \ldots + \\
&\quad ||a_{n,1}||^2 + \ldots + ||a_{n,n}||^2)^{1/2} \\
&\leq ||\phi||\sqrt{(n+n+n+n+\ldots+n)} \text{ i.e. add n n-times since } ||a_{i,j}|| \leq 1 \\
&= ||\phi||\sqrt{n^2} \\
&= n||\phi||
\end{aligned}$$

Thus $||\phi_n|| \leq n||\phi||$.

□

3.2 Representation of completely bounded operators.

Each C*-algebra $\mathcal{A}$ has an isometric *-representation in the algebra $B(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$, where $\mathcal{H}$ is a Hilbert space (see [4]). This representation also identifies $M_n(\mathcal{A})$ with a subalgebra of $M_n(B(\mathcal{H})) = B(\mathcal{H}^{(n)})$. In discussing completely bounded linear operators from a C*-algebra $\mathcal{A}$ into another, initially it will be assumed that the range algebra is $B(\mathcal{H})$. This leads to the Haagerup-Paulsen-Wittstock representation theorem for a completely bounded operator into $B(\mathcal{H})$, (see [14]).

Theorem 3.2.1. ***(Representation theorem).*** *Let $\mathcal{A}$ be a unital C*-algebra, $\phi : \mathcal{A} \to B(\mathcal{H})$ completely bounded. Then there exists a Hilbert space K, a unital *-homomorphism $\pi : \mathcal{A} \to B(K)$ and bounded linear operators $V_1, V_2 : \mathcal{H} \to K$ with $||V_1||.||V_2|| = ||\phi||_{cb}$ such that for all $a \in \mathcal{A}$*

$$\phi(a) = V_1^* \pi(a) V_2.$$

Moreover, if $||\phi||_{cb} = 1$ then V_1 and V_2 may be taken to be isometries.

Proof. For the proof of this theorem, see ([14], theorem 8.4.) □

Definition 3.2.2. Let $\mathcal{A}$, $\mathcal{B}$ be C*-algebras and $\phi : \mathcal{A} \to \mathcal{B}$ be a linear map, then ϕ is called a positive map provided that it maps positive elements of $\mathcal{A}$ to positive elements of $\mathcal{B}$.

Definition 3.2.3. Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras, $M_n(\mathcal{A})$ and $M_n(\mathcal{B})$ be $n \times n$ matrices with entries in $\mathcal{A}$ and $\mathcal{B}$ respectively. For each linear map $\phi : \mathcal{A} \to \mathcal{B}$, we define a linear map $\phi_n : M_n(\mathcal{A}) \to M_n(\mathcal{B})$ by

$$\phi_n[a_{i,j}] = [\phi(a_{i,j})].$$

If ϕ_n is positive, then ϕ is said to be n**-positive**. If ϕ is n-positive for all n, then ϕ is said to be **completely positive**.

Definition 3.2.4. If $\mathcal{A}$ is a unital C^*-algebra, then a subspace $S \subseteq \mathcal{A}$ with $I \in S$ and $S^* = S$ is called an **operator system.**

Completely positive maps are all completely bounded but it is worth noting here that not all positive maps are completely positive and not all bounded maps are completely bounded (see [9]).

Proposition 3.2.5. *Let $\mathcal{S} \subseteq \mathcal{A}$ be an operator system, let $\mathcal{B}$ be a C^*-algebra, and $\phi : \mathcal{S} \to \mathcal{B}$ be completely positive. Then ϕ is completely bounded and* $||\phi(I)|| = ||\phi|| = ||\phi||_{cb}$

Proof. By theorem (3.2.1) ϕ is a completely bounded operator. It now remains to show that $||\phi||_{cb.} = ||\phi||_. = ||\phi(I)||$. Clearly, we have that $||\phi(I)|| \leq ||\phi|| \leq ||\phi||_{cb}$, so it is sufficient to show that $||\phi||_{cb} \leq ||\phi(I)||$. Let $\mathbf{A} = [a_{i,j}] \in M_n(\mathcal{S})$ with $||\mathbf{A}|| \leq 1$. Let I_n be the unit of $M_n(S)$. Since

$$\begin{pmatrix} I_n & \mathbf{A} \\ \mathbf{A}^* & I_n \end{pmatrix}$$

is positive, we have that

$$\phi_{2n}\left(\begin{pmatrix} I_n & \mathbf{A} \\ \mathbf{A}^* & I_n \end{pmatrix}\right) = \begin{pmatrix} \phi_n(I_n) & \phi_n(\mathbf{A}) \\ \phi_n(\mathbf{A})^* & \phi_n(I_n) \end{pmatrix}$$

is positive, since $\mathcal{S}$ is an operator system.

Thus we have

$||\phi_n(\mathbf{A})||^2 = ||\phi_n(\mathbf{A})\phi_n(\mathbf{A}^*)|| \leq ||\phi_n(I_n)\phi_n(I_n)|| \leq ||\phi_n(I_n)||.||\phi_n(I_n)|| =$
$||\phi_n(I_n)||^2 = ||\phi(I)||^2$.

Since $\mathcal{S}$ is an operator system, and $\mathbf{A} \in \mathcal{S}$ we have

$||\phi_n(\mathbf{A})||^2 \leq ||\phi_n(I_n)||^2 = ||\phi(I)||^2$.

Taking square root on both sides we obtain

$||\phi_n(\mathbf{A})|| \leq ||\phi_n(I_n)|| = ||\phi(I)||$. So

$||\phi||_{cb} = \sup_n ||\phi_n(\mathbf{A})|| \leq ||\phi(I)||$ which completes the proof.
An alternative proof can be obtained from [14]. □

Define $Re\phi = \frac{1}{2}[\phi + \phi^*]$ and $Im\phi = \frac{1}{2i}[\phi - \phi^*]$ so that $Re\phi$, $Im\phi$ are self-adjoint.

Completely bounded operators can be decomposed into at most four completely positive maps, (see [9]). The following decomposition theorem is by Wittstock:

Theorem 3.2.6. *Let $\mathcal{A}$ be a unital C^*-algebra, $\phi : \mathcal{A} \longrightarrow B(\mathcal{H})$ completely bounded. Then there exists a completely positive map $\psi : \mathcal{A} \longrightarrow B(\mathcal{H})$ such that $||\psi||_{cb} \leq ||\phi||_{cb}$ and that $\psi \pm Re\phi$, $\psi \pm Im\phi$ are completely positive. Consequently, every completely bounded operator is a linear combination of at most four completely positive maps.*

Proof. We shall use the Haagerup-Paulsen-Wittstock representation theorem (3.2.1). Write $\phi(a) = V_1^*\pi(a)V_2$ for all $a \in \mathcal{A}$ with

$$||V_1||^2 = ||V_2||^2 = ||\phi||_{cb}$$

Define $\psi(a) = \frac{1}{2}[V_1^*\pi(a)V_1 + V_2^*\pi(a)V_2]$, then ψ is completely positive according to ([21],theorem 3.6) and $||\psi||_{cb} = ||\psi(I)||$ since $\mathcal{A}$ is unital. Therefore

$$||\psi(I)|| = ||\psi||_{cb} \leq ||\phi||_{cb}.$$

Notice that

$$
\begin{aligned}
\phi^*(a) &= (V_1^*\pi(a)V_2)^* \\
&= V_2^*\pi(a)(V_1^*)^* \\
&= V_2^*\pi(a)V_1
\end{aligned}
$$

and the real and imaginary parts of ϕ are given as:

$Re\phi = \frac{1}{2}(\phi + \phi^*)$

$Im\phi = \frac{1}{2i}(\phi - \phi^*)$.

So

$Re\phi(a) = \frac{1}{2}(V_1^*\pi(a)V_2 + V_2^*\pi(a)V_1)$

$Im\phi(a) = \frac{1}{2i}(V_1^*\pi(a)V_2 - V_2^*\pi(a)V_1) = \frac{i}{2}(V_2^*\pi(a)V_1 - V_1^*\pi(a)V_2)$

Now,

$$
\begin{aligned}
2[\psi(a) + Re\phi(a)] &= V_1^*\pi(a)V_1 + V_2^*\pi(a)V_2 + V_1^*\pi(a)V_2 + V_2^*\pi(a)V_1 \\
&= V_1^*\pi(a)V_1 + V_2^*\pi(a)V_1 + V_1^*\pi(a)V_2 + V_2^*\pi(a)V_2 \\
&= (V_1^* + V_2^*)\pi(a)V_1 + (V_1^* + V_2^*)\pi(a)V_2 \\
&= (V_1^* + V_2^*)\pi(a)(V_1 + V_2) \\
&= (V_1 + V_2)^*\pi(a)(V_1 + V_2)
\end{aligned}
$$

This shows that $2[\psi(a) + Re\phi(a)]$ is completely positive.

Similarly, $2[\psi(a) - Re\phi(a)] = (V_1 - V_2)^*\pi(a)(V_1 - V_2)$ which is also completely positive.

$$\begin{aligned}
2[\psi(a) + Im\phi(a)] &= V_1^*\pi(a)V_1 + V_2^*\pi(a)V_2 + i[V_2^*\pi(a)V_1 - V_1^*\pi(a)V_2] \\
&= V_1^*\pi(a)V_1 + iV_2^*\pi(a)V_1 + V_2^*\pi(a)V_2 - iV_1^*\pi(a)V_2 \\
&= (V_1^* + iV_2^*)\pi(a)V_1 - i(V_1^* + iV_2)\pi(a)V_2 \\
&= (V_1^* + iV_2^*)[\pi(a)V_1 - i\pi(a)V_2] \\
&= (V_1^* + iV_2^*)\pi(a)(V_1 - iV_2) \\
&= (V_1 - iV_2)^*\pi(a)(V_1 - iV_2)
\end{aligned}$$

This again shows that $2[\psi(a) + Im\phi(a)]$ is completely positive. Similarly, $2[\psi(a) - Im\phi(a)] = (V_1 + iV_2)^*\pi(a)(V_1 + iV_2)$. It therefore follows that $\psi(a) + Re\phi(a)$, $\psi(a) - Re\phi(a)$, $\psi(a) + Im\phi(a)$ and $\psi(a) - Im\phi(a)$ are all completely positive. So

$$\begin{aligned}
&\psi(a) + Re\phi(a) - (\psi(a) - Re\phi(a)) + \\
&i[(\psi(a) + Im\phi(a)) - (\psi(a) - Im\phi(a))] = \psi(a) + Re\phi(a) - \psi(a) + Re\phi(a)) + \\
&\qquad i[\psi(a) + Im\phi(a)) - \psi(a) + Im\phi(a)] \\
&= 2Re\phi(a + 2iIm\phi(a)) \\
&= 2[Re\phi(a) + iIm\phi(a)] \\
&= 2\phi(a)
\end{aligned}$$

Hence

$$\phi = \frac{1}{2}[\psi + Re\phi - (\psi - Re\phi) + i(\psi + Im\phi - (\psi - Im\phi))]$$

Taking

$$\begin{aligned} u_1 &= 2^{-1}(\psi + Re\phi), \\ u_2 &= 2^{-1}(\psi - Re\phi), \\ u_3 &= 2^{-1}(\psi + Im\phi) \text{ and} \\ u_4 &= 2^{-1}(\psi - Im\phi), \end{aligned}$$

we have

$$\phi = u_1 - u_2 + i(u_3 - u_4),$$

where u_1, u_2, u_3 and u_4 are all completely positive maps.
An alternative proof can be obtained from [9, 14]. □

Lemma 3.2.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras, and let $\pi : \mathcal{A} \rightarrow \mathcal{B}$ be a *-homomorphism with $\pi(I_{\mathcal{A}}) = I_{\mathcal{B}}$. Then π is completely positive and completely bounded and that $||\pi|| = ||\pi_n|| = ||\pi||_{cb} = 1$.*

Proof. If π is a *-homomorphism with $\pi(I_{\mathcal{A}}) = I_{\mathcal{B}}$, then π maps invertible elements in $\mathcal{A}$ into invertible elements in $\mathcal{B}$. So

$$\sigma(\pi(a)) \subseteq \sigma(a) \text{ for any } a \in \mathcal{A}.$$

It follows that

$$\begin{aligned} ||\pi(a)||^2 &= ||\pi(a)^*\pi(a)|| \\ &= ||\pi(a^*a)|| \\ &= r(\pi(a^*a)) \\ &\leq r(a^*a) \\ &= ||a^*a|| = ||a||^2 \end{aligned}$$

where $r(a^*a)$ denotes the spectral radius of a^*a.

Thus $||\pi|| \leq 1$ which implies that π is bounded and moreover π is contractive.

Since $\pi(I_{\mathcal{A}}) = I_{\mathcal{B}}$,

$$||\pi(I_{\mathcal{A}})|| = ||I_{\mathcal{B}}|| = 1.$$

Hence $||\pi|| = 1$.

If a is a positive element of $\mathcal{A}$, then there exists an $x \in \mathcal{A}$ such that $a = x^*x$. Therefore

$$\pi(a) = \pi(x^*x) = \pi(x^*)\pi(x) = \pi(x)^*\pi(x) \geq 0.$$

This shows that π is positive.

Define π_n by

$$\pi_n([a_{i,j}]) = [\pi(a_{i,j})].$$

Since π is positive, $\pi(a_{i,j}) \geq 0 \ \forall \ i, j$. Hence

$[\pi(a_{i,j})] \geq 0$ implying that

$\pi_n([a_{i,j}]) = [\pi(a_{i,j})] \geq 0$ for every $n \in \mathbb{N}$. This shows that π_n is positive for every $n \in \mathbb{N}$. Hence π is completely positive.

In $\pi_n([a_{i,j}]) = [\pi(a_{i,j})]$, take $[a_{i,j}] = I_n$, the $n \times n$ identity matrix.

But

$$\pi_n([I_n]) \stackrel{def}{=} \pi([I_n]),$$

thus $||\pi_n([I_n])|| = ||\pi([I_n])|| = ||[I_n]|| = 1$.

Hence $||\pi_n|| = 1$ for every $n \in \mathbb{N}$.

Therefore π_n is bounded for all $n \geq 1$.

So

$$||\pi||_{cb} = \sup_n ||\pi_n|| = 1 = ||\pi||.$$

This shows that π is completely bounded and

$||\pi|| = ||\pi_n|| = ||\pi||_{cb} = 1$. □

Corollary 3.2.8. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be C^*-algebras and let $\varphi : \mathcal{A} \to \mathcal{B}$ and $\psi : \mathcal{B} \to \mathcal{C}$ be completely positive maps. Then $\psi \circ \varphi$ is completely positive.*

Proof. Given that φ and ψ are completely positive, it suffices to show that $(\psi \circ \varphi)_n = \psi_n \circ \varphi_n$ for every $n \in \mathbb{N}$, whence it would follow that $\psi \circ \varphi$ is completely positive.

$$\begin{aligned} (\varphi \circ \psi)_n([a_{i,j}]) &= [\varphi \circ \psi(a_{i,j})] \\ &= [\varphi.\psi(a_{i,j})] \\ &= [\varphi.\psi_n([a_{i,j}])] \end{aligned}$$

Put $\psi_n([a_{i,j}]) = b_n$.

$\Rightarrow$

$$
\begin{aligned}
(\varphi \circ \psi)_n([a_{i,j}]) &= [\varphi(b_n)] \\
&= \varphi_n([b_n]) \\
&= \varphi_n . \psi_n([a_{i,j}]) \\
&= \varphi_n \circ \psi_n([a_{i,j}])
\end{aligned}
$$

Hence $(\varphi \circ \psi)_n = \varphi_n \circ \psi_n$. □

The same proof holds when φ and ψ are completely bounded.

3.3 Examples of completely bounded operators.

The following examples are given purposely for illustration.

i) Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras. Let $\pi : \mathcal{A} \rightarrow \mathcal{B}$ be a *-homomorphism, then π is completely positive and completely contractive, see [9, 19]. Each map $\pi_n : M_n(\mathcal{A}) \rightarrow M_n(\mathcal{B})$ is a *-homomorphism. *-homomorphism is both positive and contractive. Thus $\sup_n ||\pi_n|| \leq 1$. Hence $||\pi||_{cb} = \sup_n ||\pi_n|| \leq 1 \Rightarrow$ π is a completely bounded operator.

ii) Let $\mathcal{A}$ and $\mathcal{B}$ be as in the example above. Fix $x, y \in \mathcal{A}$ where x and y are diagonal elements.

Define $\varphi : \mathcal{A} \to \mathcal{A}$ by

$$\varphi(a) = xay \ \ \forall\, a \in \mathcal{A}.$$

If $[a_{i,j}] \in M_n(\mathcal{A})$ then

$||\varphi_n([a_{i,j}])|| = ||x[a_{i,j}]y|| =$

$$= \left\| \begin{pmatrix} x_{1,1} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & x_{m,n} \end{pmatrix} \cdot \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{pmatrix} \cdot \begin{pmatrix} y_{1,1} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & y_{n,m} \end{pmatrix} \right\|$$

$\leq ||x||\,||[a_{i,j}]||\,||y|| \Rightarrow ||\varphi_n|| \leq ||x||\,||y||.$

Hence $\sup_n ||\varphi_n|| \leq ||x||\,||y||$ for all $n \in \mathbb{N}$.

Thus φ is completely bounded and $||\varphi||_{cb} \leq ||x||\,||y||$.

iii) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, $V_i : \mathcal{H}_1 \to \mathcal{H}_2$ $i = 1, 2$ be bounded operators, and let $\pi : \mathcal{A} \to B(\mathcal{H}_2)$ be a *-homomorphism. Define a map $\varphi : \mathcal{A} \to B(\mathcal{H}_1)$ by

$$\varphi(a) = V_2^* \pi(a) V_1, \quad \forall\, a \in \mathcal{A}.$$

We show that φ is completely bounded and that

$$||\varphi||_{cb} \leq ||V_1||\,||V_2||.$$

Let ξ, η in $\mathcal{H}_1$ be of unit lengths, then

$$\begin{aligned}
|\langle \varphi_n(a)\xi, \eta\rangle| &= |\langle V_2^* \otimes I_n \pi(a) V_1 \otimes I_n \xi, \eta\rangle| \\
&= |\langle \pi(a) V_1 \otimes I_n \xi, V_2 \otimes I_n \eta\rangle| \\
&\leq ||\pi(a) V_1 \otimes I_n \xi||\, ||V_2 \otimes I_n \eta|| \text{ by C.B.S inequality.} \\
&\leq ||\pi(a)||\, ||V_1 \otimes I_n||\, ||\xi||\, ||V_2 \otimes I_n||\, ||\eta|| \\
&\leq ||\pi||\, ||(a)||\, ||V_1||\, ||\xi||\, ||V_2||\, ||\eta|| \\
&\leq ||(a)||\, ||V_1||\, ||\xi||\, ||V_2||\, ||\eta|| \text{ since } ||\pi|| \leq 1. \\
So\, ||\varphi_n|| &\leq ||V_1||\, ||V_2||.
\end{aligned}$$

Thus $\sup_n ||\varphi_n|| \leq ||V_1||\, ||V_2||$ which is finite since V_1 and V_2 are bounded operators.

Hence

$$||\varphi||_{cb} = \sup_n ||\varphi_n|| \leq ||V_1||\, ||V_2|| < \infty$$

Therefore φ is completely bounded and $||\varphi||_{cb} \leq ||V_1||\, ||V_2||$.

iv) In this example, we shall consider the transpose mapping. Let $\{E_{i,j}\}_{i,j=1}^2$ denote the system of matrix units for $M_2(\mathbb{C})$. That is $E_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{1,2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$E_{2,1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $E_{2,2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Let $\phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be the transpose mapping so that $\phi(E_{i,j}) = E_{j,i}$. It is easy to show that the transpose of a positive matrix is positive and that the norm of the transpose of a matrix is the same as the norm of the matrix. Clearly $||\phi|| = 1$. This is true for any $n \in \mathbb{N}$

and hence ϕ is bounded.

Now let's consider

$\phi_2 : M_2(M_2(\mathbb{C})) \to M_2(M_2(\mathbb{C}))$ and

let $\mathbf{A} = (E_{j,i})_{i,j=1}^{2} \in M_2(M_2(\mathbb{C}))$, then

$$\begin{aligned}\mathbf{A} &= \begin{pmatrix} E_{1,1} & E_{2,1} \\ E_{1,2} & E_{2,2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\end{aligned}$$

So

$$\begin{aligned}\phi_2(\mathbf{A}) &= [\phi(\mathbf{A})] \\ &= \begin{bmatrix} \phi(E_{1,1}) & \phi(E_{2,1}) \\ \phi(E_{1,2}) & \phi(E_{2,2}) \end{bmatrix} \\ &= \begin{bmatrix} E_{1,1} & E_{1,2} \\ E_{2,1} & E_{2,2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}\end{aligned}$$

Therefore

$$||\phi_2([\mathbf{A}])|| = \left\|\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}\right\|$$

$$= 2$$

and $\sup ||\phi_2|| = 2 < \infty$. Thus ϕ is completely bounded.

The following example is a counterexample to show that not all bounded operators are completely bounded.

v) When the underlying space is infinitely-dimensional, then ϕ turns out not to be completely bounded, see [4, 14]. To see this, Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space with a countable, orthonormal basis, $\{e_n\}_{n=1}^{\infty}$. Every bounded linear operator say T on $\mathcal{H}$ can be thought of as an infinite matrix whose $(i,j)th$ entry is the inner product $\langle Te_i, e_i \rangle$.

Define a map $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ by the transpose mapping. Let $\{E_{i,j}\}_{i,j=1}^{\infty}$ be matrix units. For a fixed $n \in \mathbb{N}$, let $\mathbf{A} = (E_{j,i})$, be an element of $M_n(\mathbb{C})$ whose $(i,j)th$ entry is $E_{j,i}$, i.e.

$$\mathbf{A} = \begin{pmatrix} E_{1,1} & E_{2,1} & \dots \\ E_{1,2} & E_{2,2} & \dots \\ \vdots & \vdots & \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Each column entry of matrix $\mathbf{A}$ has only one element 1 in each column (and in each row). Thus, taking the matrix 1-norm, we obtain that $||\mathbf{A}|| = 1$, but $||\phi_n([\mathbf{A}])|| = |||[\phi(\mathbf{A})]||| = n$. So $\sup_n ||\phi_n|| = \infty$. So ϕ is not completely bounded.

3.4 Tensor product of operator spaces.

In this section, we give the basics of tensor products, tensor norms, tensor product of operator spaces and other related results.

Let $\mathcal{H}$ and K be vector spaces and $\mathcal{H} \times K = \{(x,y) : x \in \mathcal{H},\ y \in K\}$ be the cartesian product of $\mathcal{H}$ and K.

Definition 3.4.1. Let $\mathcal{H}$, K and Z be linear spaces over the same scalar field, say $\mathbb{C}$. A function $\varphi : \mathcal{H} \times K \to Z$ is bilinear if $\varphi(x,.) : K \to Z$ is linear for each $x \in \mathcal{H}$ and $\varphi(.,y) : \mathcal{H} \to Z$ is linear for each $y \in K$.

The algebraic tensor product of vector spaces $\mathcal{H}$ and K denoted by $\mathcal{H} \otimes K$ is the linear span of the collection of elementary tensors, $\{x \otimes y : x \in \mathcal{H}, y \in K\}$. So a typical $u \in \mathcal{H} \otimes K$ has the form

$$u = \sum_{i=1}^{n} \lambda_i x_i \otimes y_i$$

where $\lambda_1, \ldots, \lambda_n$ are scalars, $x_1, \ldots, x_n \in \mathcal{H}$ and $y_1, \ldots, y_n \in K$ and $n \in \mathbb{N}$ is arbitrary.

Theorem 3.4.2. *Let $\tau : \mathcal{H} \times K \to \mathcal{H} \otimes K$, $\tau(x, y) = x \otimes y$ be a mapping from a cross product to tensor product space. Then τ is a bilinear map.*

Proof. Let $x, x_1, x_2 \in \mathcal{H}$, $y, y_1, y_2 \in K$. and $\alpha, \beta \in \mathbb{C}$. To show that τ is bilinear, it suffices to show that τ is linear in both the vector spaces $\mathcal{H}$ and K.

We show linearity in $\mathcal{H}$.

Since

$$\tau(x, y) = x \otimes y,$$

$$\begin{aligned} \tau(\alpha x_1 + \beta x_2, y) &= (\alpha x_1 + \beta x_2) \otimes y \\ &= (\alpha x_1 \otimes y) + (\beta x_2 \otimes y) \\ &= \alpha(x_1 \otimes y) + \beta(x_2 \otimes y) \\ &= \alpha\tau(x_1, y) + \beta\tau(x_2, y). \end{aligned}$$

Hence τ is linear in $\mathcal{H}$.

Next, we show linearity in K.

$$\begin{aligned}\tau(x, \alpha y_1 + \beta y_2) &= x \otimes (\alpha y_1 + \beta y_2) \\ &= (x \otimes \alpha y_1) + (x \otimes \beta y_2) \\ &= \alpha(x \otimes y_1) + \beta(x \otimes y_2) \\ &= \alpha\tau(x, y_1) + \beta\tau(x, y_2).\end{aligned}$$

Hence τ is linear in K and therefore τ is a bilinear map. □

If $\mathcal{H}$ and K are Hilbert spaces, then the algebraic tensor product $\mathcal{H} \otimes K$ is a pre-Hilbert space with the inner product determined by

$$\langle x \otimes y, x_1 \otimes y_1 \rangle = \langle x, x_1 \rangle \langle y, y_1 \rangle.$$

This leads to the following theorem.

Theorem 3.4.3. *Let $\mathcal{H}$ and K be Hilbert spaces. Then there is a unique inner product $\langle , \rangle$ on $\mathcal{H} \otimes K$ such that*

$$\langle x \otimes y, x_1 \otimes y_1 \rangle = \langle x, x_1 \rangle \langle y, y_1 \rangle \quad x, x_1 \in \mathcal{H} \;\; y, y_1 \in K.$$

For the proof of this theorem see [13].

Remark 3.4.4. We note that in the theorem above, the tensor product space $\mathcal{H} \otimes K$ with the above inner product forms a pre-Hilbert space. The completion of this pre-Hilbert space $\mathcal{H} \otimes K$ which is denoted by $\mathcal{H} \hat{\otimes} K$ makes it a Hilbert space.

Lemma 3.4.5. *Let $\mathcal{H}$ and K be Hilbert spaces. Then if $E_1 = \{e_i : i \in I\}$ is an orthonormal basis for $\mathcal{H}$ and $E_2 = \{e'_j : j \in J\}$ is an orthonormal* *basis for K, then the set*

$$E_1 \otimes E_2 = \{e_i \otimes e'_j : i \in I, j \in J\}, \text{ where } I \text{ and } J \text{ are index sets.}$$

is an orthonormal basis for $\mathcal{H}\hat{\otimes}K$.

Proof. We need to show that the elements of $E_1 \otimes E_2$ are linearly independent and that $E_1 \otimes E_2$ spans $\mathcal{H}\hat{\otimes}K$.

Now suppose that

$$\sum_{i,j} \lambda_{i,j}(e_i \otimes e'_j) = \sum_i e_i \otimes (\sum_j \lambda_{i,j} e'_j) = 0.$$

Since e_i are linearly independent for all $i \in I$, it implies that

$$\sum_i e_i \neq 0$$

and hence

$$\sum_j \lambda_{i,j} e'_j = 0, \quad \forall\, j \in J$$

But $e'_{j's}$ are also linearly independent, meaning $e'_j \neq 0, \quad \forall\, j \in J$

Therefore, $\lambda_{i,j} = 0, \quad \forall\, i \in I,\ j \in J.$

Hence $e_i \otimes e'_j$ are linearly independent.

Let $x \otimes y \in \mathcal{H}\hat{\otimes}K$ such that $x = \sum_i^n \lambda_i e_i$ and $y = \sum_j^m \alpha_j e'_j$, with $\lambda_i,\ \alpha_j \in \mathbb{K},\ \forall\, i, j$

Then

$$
\begin{aligned}
x \otimes y &= \sum_{i}^{n} \lambda_i e_i \otimes \sum_{j}^{m} \alpha_j e'_j \\
&= \sum_{j}^{m} \alpha_j (\sum_{i}^{n} \lambda_i e_i \otimes e'_j) \\
&= \sum_{j}^{m} \alpha_j \sum_{i}^{n} \lambda_i (e_i \otimes e'_j) \\
&= \sum_{i,j}^{k} \lambda_i \alpha_j (e_i \otimes e'_j),
\end{aligned}
$$

where k is taken to be the $\max\{n, m\}$.

Since $x \otimes y$ was picked arbitrarily in $\mathcal{H}\hat{\otimes}K$, any vector in $\mathcal{H}\hat{\otimes}K$ can be expressed as a linear combination of the vectors $e_i \otimes e'_j$. Hence $E_1 \otimes E_2$ spans $\mathcal{H}\hat{\otimes}K$ and is therefore an orthonormal basis for it. □

Proposition 3.4.6. *Let $\mathcal{H}$ and K be Hilbert spaces. We denote by $\mathcal{H} \otimes K$ the tensor product space between $\mathcal{H}$ and K. The elements of $\mathcal{H} \otimes K$ are denoted by $x \otimes y$ where $x \in \mathcal{H}$ and $y \in K$. Then*

$$||x \otimes y|| = ||x|| ||y||$$

defines a norm.

Proof. It is clear that $||x \otimes y|| \geq 0$ and

$||x \otimes y|| = 0 \quad \Leftrightarrow x \otimes y = 0.$

Next, we show that for any $\alpha \in \mathbb{C}$ we must have $||\alpha(x \otimes y)|| = |\alpha| ||x \otimes y||$.

Recall that

$$\begin{aligned}||x \otimes y||^2 &= \langle x \otimes y, x \otimes y \rangle \\ &= \langle x, x \rangle \langle y, y \rangle \\ &= ||x||^2 ||y||^2\end{aligned}$$

and from the algebraic properties of tensor products, for any $\alpha \in \mathbb{C}$ we have $\alpha(x \otimes y) = (\alpha x \otimes y) = (x \otimes \alpha y)$.

Therefore

$$\begin{aligned}||\alpha(x \otimes y)||^2 &= \langle \alpha x \otimes y, \alpha x \otimes y \rangle \\ &= \langle x \otimes \alpha y, x \otimes \alpha y \rangle \\ &= \langle x, x \rangle \langle \alpha y, \alpha y \rangle \\ &= ||x||^2 |\alpha|^2 ||y||^2 \\ &= |\alpha|^2 ||x||^2 ||y||^2 \\ &= |\alpha|^2 ||x \otimes y||^2.\end{aligned}$$

Taking square roots on both sides, the assertion follows.

Finally, we show the triangular inequality.

Let $x_1, x_2 \in \mathcal{H}$ and $y_1, y_2 \in K$ then

$$\begin{aligned}
||(x_1 \otimes y_1) + (x_2 \otimes y_2)||^2 &= \langle (x_1 \otimes y_1) + (x_2 \otimes y_2), (x_1 \otimes y_1) + (x_2 \otimes y_2) \rangle \\
&= \langle x_1 \otimes y_1, x_1 \otimes y_1 \rangle + \langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle + \langle x_2 \otimes y_2, x_1 \otimes y_1 \rangle + \\
&\quad \langle x_2 \otimes y_2, x_2 \otimes y_2 \rangle \\
&= \langle x_1, x_1 \rangle \langle y_1, y_1 \rangle + \langle x_1, x_2 \rangle \langle y_1, y_2 \rangle + \langle x_2, x_1 \rangle \langle y_2, y_1 \rangle + \\
&\quad \langle x_2, x_2 \rangle \langle y_2, y_2 \rangle \\
&= ||x_1||^2 ||y_1||^2 + ||x_2||^2 ||y_2||^2 + \langle x_1, x_2 \rangle \langle y_1, y_2 \rangle + \overline{\langle x_1, x_2 \rangle}\, \overline{\langle y_1, y_2 \rangle} \\
&= ||x_1||^2 ||y_1||^2 + ||x_2||^2 ||y_2||^2 + 2Re\langle x_1, x_2 \rangle \langle y_1, y_2 \rangle \\
&\leq ||x_1||^2 ||y_1||^2 + ||x_2||^2 ||y_2||^2 + 2||x_1|| ||x_2|| ||y_1|| ||y_2|| \text{ by the} \\
&\quad \text{C.B.S inequality} \\
&= (||x_1|| ||y_1|| + ||x_2|| ||y_2||)^2 \\
&= (||x_1 \otimes y_1|| + ||x_2 \otimes y_2||)^2.
\end{aligned}$$

Taking square roots on both sides, we obtain

$$||(x_1 \otimes y_1) + (x_2 \otimes y_2)|| \leq ||x_1 \otimes y_1|| + ||x_2 \otimes y_2||.$$

□

Theorem 3.4.7. *Suppose $\mathcal{H}$ and K are Hilbert spaces and $B(\mathcal{H})$ and $B(K)$ Banach spaces of bounded linear operators of $\mathcal{H}$ and K respectively. Let $u \in B(\mathcal{H})$ and $v \in B(K)$. Then the operator defined as*

$$(u \widehat{\otimes} v)(x \otimes y) = u(x) \otimes v(y) \quad x \in \mathcal{H}, y \in K,$$

is a bounded linear operator on $B(\mathcal{H}\widehat{\otimes}K)$. *Moreover,*

$$||u\widehat{\otimes} v|| = ||u||||v||.$$

Proof. See [13] for the proof of this theorem. □

Remark 3.4.8. In case $u : X_1 \to X_2$ and $v : Y_1 \to Y_2$ are completely bounded maps between operator spaces, then there is a map $u \otimes v$ on the algebraic tensor product $X_1 \otimes Y_1$ into $X_2 \otimes Y_2$.
Now if ω is a norm on $X \otimes Y$, then we denote by $X \otimes_\omega Y$ the completion of the algebraic tensor product $X \otimes Y$ for the norm ω. Recalling that $X \otimes_\omega Y$ is defined for all pairs of operator spaces, it may be the case that $u \otimes v : X_1 \otimes_\omega Y_1 \to X_2 \otimes_\omega Y_2$ is completely bounded and satisfies $||u \otimes v||_{cb} = ||u||_{cb}||v||_{cb}$.

The most important result on operator spaces is the Ruan's theorem, [9] which gives the abstract characterisation of operator spaces:

Theorem 3.4.9. *(**Ruan**) A vector space* V *with a sequence of norms* $||.||_n$ *on* $M_n(V)$ $(n \in \mathbb{N})$ *is an operator space if and only if the following two conditions are satisfied:*
For all $m, n \in \mathbb{N}$,

i) $||a \oplus b||_{n+m} = \max\{||a||_n, ||b||_m\}$ *for all* $a \in M_n(V), b \in M_m(V)$ *where* $a \oplus b$ *denotes the matrix*

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in M_{n+m}(V).$$

ii) $||\alpha a\beta||_n \leq ||\alpha||||a||||\beta||$ *for all* $\alpha \in M_n(\mathbb{C}), a \in M_n(V), \beta \in M_n(\mathbb{C})$ *(the matrix multiplication being the natural one).*

Proof. Let $\mathcal{H}_n$ and K_n $(n \in \mathbb{N})$ be Hilbert spaces and $t_n : \mathcal{H}_n \to K_n$ be operators such that $\sup_{n\in\mathbb{N}}\{||t_n||\} < \infty$. Then $\mathcal{H} = \oplus_{n\in\mathbb{N}}\mathcal{H}_n$ and $K = \oplus_{n\in\mathbb{N}}K_n$ are Hilbert spaces and the n-tuple $(t_n)_{n\in\mathbb{N}}$ determines a bounded operator from $\mathcal{H}$ to K which we shall denote by t.
Now, the norm of t is given by

$$\begin{aligned} ||t|| &= \sup\{||t_1||, ||t_2||, \ldots, ||t_n|| : n \in \mathbb{N}\} \\ &= \max\{||t_1||, ||t_2||, \ldots, ||t_n|| : n \in \mathbb{N}\} \qquad (3.4.0.1) \end{aligned}$$

With the identification, $M_n(B(\mathcal{H})) \equiv B(\mathcal{H}^n)$, let $a \in M_n(B(\mathcal{H}))$ and $b \in M_m(B(\mathcal{H}))$ and using 3.4.0.1 we have that $||a \oplus b|| = \max\{||a||_n, ||b||_m\}$. Note that, we can identify $B(\mathcal{H}^n)$ with $B(\mathbb{C}^n \otimes \mathcal{H})$. Therefore, for any $\alpha \in M_{m,n}(\mathbb{C})$, $\beta \in M_{n,m}(\mathbb{C})$ and $a \in M_n(B(\mathcal{H}))$, we have

$$\begin{aligned} ||\alpha a\beta|| &= ||\alpha \otimes I_n a\beta \otimes I_n|| \\ &\leq ||\alpha \otimes I_n||||a||||\beta \otimes I_n|| \\ &\leq ||\alpha||||I_n||||a||||\beta||||I_n|| \\ &= ||\alpha||||a||||\beta||. \end{aligned}$$

which completes the proof. □

norms on $M_n(X)$, induced by regarding $M_n(X)$ as a subspace of $M_n(B(\mathcal{H}))$ and identifying this algebra with $B(\mathcal{H}^n)$.
Each rectangular space $M_{p,q}(X)$ embeds into a larger square one, and so

we also obtain a norm on $M_{p,q}(X)$ for any pair of integers p and q.

3.5 The matricial tensor product.

Given operator spaces V, W and elements $v \in M_{p,q}(V), w \in M_{r,s}(W)$, we define

$$v \otimes w \in M_{pr,qs}(V \otimes W)$$

by

$$(v \otimes w)_{ij} = v_{i_1 j_1} \otimes w_{i_2 j_2}$$

See [6] for more details.

Also given integers p, q, r and $v \in M_{p,q}(V), w \in M_{q,r}(W)$, we define

$$v \odot w \in M_{p,r}(V \otimes W)$$

by

$$(v \odot w)_{ij} = \sum v_{ik} \otimes w_{kj}$$

For any operator spaces V and W, we say that an operator space matrix norm $||.||_\mu$ on $V \otimes W$ is subcross matrix norm if

$$||v \otimes w||_\mu \leq ||v||||w||$$

for all $v \in M_p(V)$ and $w \in M_q(W)$. If in addition,

$$||v \otimes w||_\mu = ||v||||w||,$$

then we say that $||.||_\mu$ is a cross matrix norm on $V \otimes W$.

Given an element $u \in M_n(V \otimes W)$, we define the projective tensor norm by

$$||u||_\wedge = \inf\{||\alpha||||v||||w||||\beta|| : u = \alpha(v \otimes w)\beta\}$$

where the infimum is taken over arbitrary decompositions of u with $v \in M_p(V)$, $w \in M_q(W)$, $\alpha \in M_{n,p\times q}$ and $\beta \in M_{p\times q,n}$ and $p, q \in \mathbb{N}$ arbitrary. We shall show that $V \otimes_\wedge W = (V \otimes W, ||.||_\wedge)$ is again an operator space, and define the operator space projective tensor product $V\widehat{\otimes}W$ to be the completion of this space i.e. $||.||_\wedge$ is the projective tensor norm.

Theorem 3.5.1. *Given operator spaces V and W, $V \otimes_\wedge W$ is again an operator space.*

Proof. Given $u_1 \in M_m(V \otimes W), u_2 \in M_n(V \otimes W)$ and $\epsilon > 0$, we may find decompositions $u_k = \alpha_k(v_k \otimes w_k)\beta_k$, $(k = 1, 2)$ where $||\alpha_k||||v_k||||w_k||||\beta_k|| \leq ||u_k||_\wedge + \epsilon$.

We may assume $||v_k|| = ||w_k|| = 1$, and that

$||\alpha_k|| = ||\beta_k|| \leq (||u_k||_\wedge + \epsilon)^{1/2}$.

If $v = v_1 \oplus v_2$ and $w = w_1 \oplus w_2$, then we have the natural identification

$$\begin{aligned} v \otimes w &= \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} \otimes \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix} \\ &= \begin{pmatrix} v_1 \otimes w_1 & 0 & 0 & 0 \\ 0 & v_1 \otimes w_2 & 0 & 0 \\ 0 & 0 & v_2 \otimes w_1 & 0 \\ 0 & 0 & 0 & v_2 \otimes w_2 \end{pmatrix}. \end{aligned}$$

and thus

$$\begin{aligned} u_1 \oplus u_2 &= \alpha_1(v_1 \otimes w_1)\beta_1 \oplus \alpha_2(v_2 \otimes w_2)\beta_2 \\ &= (\alpha_1 v_1 \oplus \alpha_2 v_2) \otimes (w_1\beta_1 \oplus w_2\beta_2) \\ &= \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}\begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} \otimes \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix}\begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix} \\ &= \alpha(v \otimes w)\beta \end{aligned}$$

where $\alpha = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 \end{pmatrix}$, $\beta = \begin{pmatrix} \beta_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \beta_2 \end{pmatrix}$ since $v \otimes w$ is a 4×4 matrix and multiplication here is the natural one. α and β above are scalar matrices. So

$$\begin{aligned}
||u_1 \oplus u_2|| &= \inf\{||\alpha(v \otimes w)\beta|| : u_k = \alpha_k(v_k \otimes w_k)\beta_k\} \\
&\leq ||\alpha||||v \otimes w||||\beta|| \\
&= ||\alpha||||v||||w||||\beta|| \\
&= ||\alpha||||\beta|| \text{ since } ||v|| = ||w|| = 1 \\
&= ||\alpha\alpha^*||^{1/2}||\beta^*\beta||^{1/2}
\end{aligned}$$

Note that

$$\begin{aligned}
\alpha\alpha^* &= \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \alpha_2 \end{pmatrix} \\
&= \begin{pmatrix} \alpha_1\alpha_1 & 0 \\ 0 & \alpha_2\alpha_2 \end{pmatrix}
\end{aligned}$$

So

$$\begin{aligned}
||\alpha\alpha^*|| = ||\begin{pmatrix} \alpha_1\alpha_1 & 0 \\ 0 & \alpha_2\alpha_2 \end{pmatrix}|| &= \max\{||\alpha_1\alpha_1||, ||\alpha_2\alpha_2||\} \\
&= \max\{||\alpha_i\alpha_i|| : i = 1, 2\} \\
&\leq \max\{||\alpha_i||||\alpha_i|| : i = 1, 2\} \\
&= \max\{||\alpha_i||^2 : i = 1, 2\}
\end{aligned}$$

Similarly, $||\beta^*\beta|| \leq \max\{||\beta_j||^2 : j = 1, 2\}$

Therefore

$$\begin{aligned}
||u_1 \oplus u_2|| &\leq ||\alpha\alpha^*||^{1/2}||\beta^*\beta||^{1/2} \\
&\leq (\max\{||\alpha_i||^2\})^{1/2}(\max\{||\beta_j||^2\})^{1/2} \\
&\leq (\max\{||u_i||_\wedge\} + \epsilon)^{1/2}(\max\{||u_i||_\wedge\} + \epsilon)^{1/2} \\
&= \max\{||u_i||\} + \epsilon
\end{aligned}$$

and since $\epsilon > 0$ is arbitrary, we have

$$||u_1 \oplus u_2|| \leq \max\{||u_1||_\wedge, ||u_2||_\wedge\}. \qquad (3.5.0.2)$$

Given scalars $\gamma \in M_{p,m}(\mathbb{C})$ and $\delta \in M_{m,p}(\mathbb{C})$, then $\gamma u_1 \delta = \gamma\alpha_1(v_1 \otimes w)\beta_1\delta$ and thus

$$\begin{aligned}
||\gamma u_1 \delta||_\wedge &= ||\gamma\alpha_1(v_1 \otimes w)\beta_1\delta|| \\
&\leq ||\gamma\alpha_1||||v_1||||w||||\beta_1\delta|| \\
&\leq ||\gamma||(||u_1||_\wedge + \epsilon)||\delta||.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have

$$||\gamma u_1 \delta||_\wedge \leq ||\gamma||||u_1||||\delta||. \qquad (3.5.0.3)$$

It follows from (3.5.0.2) and (3.5.0.3) that $V \otimes_\wedge W$ is an operator space. An alternative proof can be obtained from [6]. □

Definition 3.5.2. Injective(spatial) tensor product

Given operator spaces V and W, we define the injective matrix norm $||.||_\vee$

on $V \otimes W$ by

$$||u||_\vee = \sup\{||f \otimes g_n(u)|| : f \in M_p(V^*),\ g \in M_q(W^*),\ ||f||, ||g|| \leq 1\}$$

for each matrix $u \in M_n(V \otimes W)$.

Definition 3.5.3. The Haagerup tensor product

Given operator spaces V and W and an element $u \in M_n(V \otimes W)$, we define the Haagerup tensor product matrix norm $||.||_h$ by

$$||u||_h = \inf\{||v||||w|| : u = v \odot w, v \in M_{n,r}(V), w \in M_{r,n}(W), r \in \mathbb{N}\}.$$

The Haagerup norm is not a C*-norm, but if the definition is repeated for $n \in \mathbb{N}$ and $u \in M_n(V \otimes W)$ for operator spaces V and W, it turns out that the Haagerup norm is an operator cross-norm (see [16]). Note that the Haagerup tensor product (i.e.$V \otimes W$ equipped with the Haagerup norm) is associative:

$$(V \otimes_h W) \otimes_h Z = V \otimes_h (W \otimes_h Z).$$

Furthermore, the norm is injective i.e. for subspaces $V_o \subseteq V$ and $W_o \subseteq W$ the restriction of the Haagerup norm from $V \otimes_h W$ to $V_o \otimes W_o$ is the Haagerup norm. Also the tensor product $T \otimes S$ of completely bounded operators T and S on V and W is completely bounded on $V \otimes_h W$. Moreover, $||T \otimes S||_{cb} \leq ||T||_{cb}||S||_{cb}$ (see[8]).

Theorem 3.5.4. *For any operator spaces V and W, $||.||_h$ is an operator* *space matrix norm on $V \otimes W$, and for any $u \in M_n(V \otimes W)$*

$$||u||_\vee \leq ||u||_h \leq ||u||_\wedge.$$

Proof. Let us suppose that $u \in M_n(V \otimes W)$, $u' \in M_n(V \otimes W)$, and $\epsilon > 0$. By definition, we may find $v \in M_{m,r}(V)$ and $w \in M_{r,m}(W)$ such that $u = v \odot w$, $||w|| = 1$, and $||v|| \leq ||u||_h + \epsilon$.

Similarly, we let $u' = v' \odot w'$, $v'' = v \oplus v'$ and $w'' = w \oplus w'$, $||w'|| = 1$ and $||v'|| \leq ||u'||_h + \epsilon$, then from the expression

$v'' \odot w'' = (v \oplus v') \odot (w \oplus w')$ we have

$$\begin{aligned} u \oplus u' &= (v \odot w) \oplus (v' \odot w') \\ &= (v \oplus v') \odot (w \oplus w') \\ &= v'' \odot w'' \end{aligned}$$

and thus

$$\begin{aligned} ||u \oplus u'||_h &\leq ||v''|| ||w''|| \\ &= \max\{||v||, ||v'|| \text{ since } ||w|| = ||w'|| = 1\} \\ &\leq \max\{||u||_h, ||u'||_h\} + \epsilon. \end{aligned}$$

Since ϵ is arbitrary, we obtain

$||u \oplus u'||_h \leq \max\{||u||_h, ||u'||_h\}$.

If $\alpha, \beta \in M_n(\mathbb{C})$, then

$$\begin{aligned} \alpha u \beta &= \alpha(v \odot w)\beta \\ &= (\alpha v) \odot (w\beta) \end{aligned}$$

and hence

$$\begin{aligned} ||\alpha u\beta||_h &\leq ||\alpha v||||w\beta|| \\ &\leq ||\alpha||||v||||w||||\beta|| \\ &\leq ||\alpha||(||u||_h + \epsilon)||\beta||. \end{aligned}$$

Since ϵ is arbitrary, we obtain
$||\alpha u\beta||_h \leq ||\alpha||||u||_h||\beta||$. So $||.||_h$ is an operator space matrix norm and consequently, $(V \otimes W, ||.||_h)$ is an operator space.

Next we prove the inequality. If $u \in M_n(V \otimes W)$, let us suppose that $\epsilon > 0$, and that
$u = v \odot w$,where
$v = [v_{i,k}] \in M_{n,r}(V)$ and $w = [w_{k,j}] \in M_{r,n}(W)$ satisfy
$||v||||w|| \leq ||u||_h + \epsilon$.
Let us suppose that $f \in M_p(V^*)$ and $g \in M_q(W^*)$ are contractions. Then we have

$$\begin{aligned} (f \otimes g)_n(v \odot w) &= \left[\sum f(v_{i,k}) \otimes g(w_{k,j})\right] \\ &= [f(v_{i,k}) \otimes I_q][I_p \otimes g(w_{k,j})], \end{aligned}$$

where we are using a product of matrices over $M_p(\mathbb{C}) \otimes M_q(\mathbb{C})$. Since matrix multiplication is contractive bilinear function on $M_p(\mathbb{C}) \otimes M_q(\mathbb{C})$,
$||(f \otimes g)_n(v \odot w)|| \leq ||f_{n,r}(v)||||g_{r,n}(w)|| \leq ||v||||w||$.
Now from
$||u||_\vee = \sup\{||(f \otimes g)_n(u) : f \in M_p(V^*), g \in M_q(W^*), ||f||, ||g|| \leq 1\}$, it

follows that

$||u||_\vee \leq ||v||||w|| \leq ||u||_h + \epsilon,$

from which we obtain

$$||u||_\vee \leq ||u||_h.$$

Recall that the Haagerup norm is defined by

$$||u||_h = \inf\{||v||||w|| : u = v \odot w\}.$$

Now given $u = \alpha(v \otimes w)\beta$ with $||\alpha||\,||v||\,||w||\,||\beta|| \leq ||u||_\wedge + \epsilon$, we have that

$$||u||_h \leq ||v||\,||w|| \leq ||u||_\wedge + \epsilon$$

and since $\epsilon > 0$ was chosen arbitrarily, it follows that

$$||u||_h \leq ||u||_\wedge.$$

which completes the proof. □

The above proof can be obtained from [7].

Remark 3.5.5. We observe from this theorem that the projective tensor product matrix norm is the largest among the operator space matrix norms.

Chapter 4

Conclusions and Recommendations

Calculating norms of matrices when the entries are not constants is not easy. In this thesis, we have approximated this norm when the elements of matrices are operators.To do this we had to first identify the space $M_{m,n}(B(\mathcal{H}))$ of $m \times n$ matrices with entries from $B(\mathcal{H})$ with the space $B(\mathcal{H}^n, \mathcal{H}^m)$ of bounded linear operators from $\mathcal{H}^n$ to $\mathcal{H}^m$. We observed that, the elements of the space $M_{m,n}(B(\mathcal{H}))$ are the bounded linear operators acting on the n-dimensional Hilbert space $\mathcal{H}^n$. The notion of completely bounded operators is a new area in Mathematics. It started its life in the early 1980's following Stinespring and Arveson's work. This later gave rise to operator spaces, a new branch in operator algebra. Progress in this new area of Mathematics has been rapid and it is difficult to say which results motivated others. Here, we have investigated the norm of completely bounded operators and have looked at certain examples of completely bounded operators. Among the results, we have obtained include:

- Showing that, for any $n \in \mathbb{N}$, $||\phi_n|| \leq ||\phi_{n+1}|| \leq ||\phi||_{cb}$ and that $||\phi_n|| \leq n||\phi||$ for any completely bounded operator ϕ.

- Giving four examples of completely bounded operators, something that is missing in the available literature.

- Giving a counterexample to show that not all bounded linear operators are completely bounded.

We hope that the results we have obtained are fundamental to the development of this area of Mathematics. Moreover, this thesis opens the way for further research in other aspects of completely bounded operators and operator spaces.

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