NORMS OF ELEMENTARY OPERATORS

BY

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ABSTRACT

The norm of an elementary operator has been investigated over long period by several mathematician under various special circumstances. Timoney working on algebra of bounded linear operators on Hilbert spaces, established the lower bound of norms of eleementary operators on Calkin algebra.

Similary, mathieu studied norm properties of elementary operators on Calkin algebra and established a result whose key basis is the Haagerup tensor norm. We joined results from these eminent mathematicians to establish norms of elementary operators, particularly determine the lower bounds of elementary operators.

Chapter 1

Introduction

1.1 Background Information

Here, we introduce essential concepts and relevant definitions we shall use in the sequel. We have also presented a review of literature. We conclude the chapter by a brief on the methodology we have used.

Definition 1.1.0: Field

A field is a set $\mathbb{K}$, containing at least two elements, together with two binary operations, called addition (denoted, $+$) and multiplication (denoted, $\times$) for which the following hold:

(a) $\mathbb{K}$ is an abelian group under addition,

(b) The set $\mathbb{K}^*$ of all nonzero elements in $\mathbb{K}$ is an abelian group under multiplication,

(c) (Distributivity), $\forall a, b, c, d \in \mathbb{K}:$

(i) $(a + b)c = ac = bc$

(ii) $c(a + b) = ca + cb$

Definition 1.1.1: Algebra

Let $\mathbb{K}$ be a field, and let A be a vector space over $\mathbb{K}$ equipped with an additional binary operation from $A \times A$ to A, denoted here by juxtaposition (i.e if x and y are elements of A, then xy is the product of x and y). Then A is an algebra over $\mathbb{K}$ if the following identities hold for any three elements x, y and z of A, and all elements ("scalars") a and b of $\mathbb{K}$:

- Left distributivity: $(x + y)z = xz + yz$
- Right distributivity: $x(y + z) = xy + xz$
- Compatibility with scalars: $(ax)(by) = (ab)(xy)$

An algebra over $\mathbb{K}$ is sometimes also called a $\mathbb{K}$-algebra, and $\mathbb{K}$ is called the base field of A. The binary operation is often referred to as multiplication in A.

Definition 1.1.2: Associative algebra

An associative algebra A over a field $\mathbb{K}$ is defined to be a vector space over $\mathbb{K}$ together with a $\mathbb{K}$-bilinear multiplication $A \times A \to A$ (where the image of(x,y) is written as xy) such that the associative law holds:

- $(xy)z = x(yz)$ $\forall$ x, y and $z \in A$.

If A contains an identity element, i.e an element 1 such that $1x = x1 = x$ for all x in A, then we call A an associative algebra with one or a unital (or unitary) associative algebra. Such an algebra is a ring, and contains all elements a of the field $\mathbb{K}$ by identification with a1.

The dimension of the associative algebra A over the field $\mathbb{K}$ is its dimension as a $\mathbb{K}$ -vector space.

Definition 1.1.3: An involution.

Let Ω be an algebra, a linear mapping $T : \Omega \longrightarrow \Omega$ defined by $x \rightarrow x^*$ is called an involution on Ω if it satisfies the following conditions:

$$\forall\, x, y \in \Omega, \lambda \in \mathbb{K}$$

- $(x+y)^* = x^* + y^*$
- $(\lambda x)^* = \overline{\lambda} x^*$
- $(xy)^* = y^* x^*$
- $x^{**} = x$

Definition 1.1.4: C^*-algebra

An algebra Ω is said to be a C^*-algebra if an involution $x \longrightarrow x^*$ is defined on it, which satisfy;

- $(x^*)^* = x \;\; \forall \;\; x \in \Omega$
- $(\lambda x)^* = \bar{\lambda} x^*, \;\; x \in \Omega, \; \lambda \in \mathbb{C}$
- $(xy)^* = y^* x^* \;\; \forall\, x, y \in \Omega$
- $\| xx^* \| = \| x \|^2 \;\; \forall\, x \in \Omega$

Definition 1.1.5: A linear Map

Let X and Y be vector spaces over the same field $\mathbb{K}$. A function $f : X \rightarrow Y$ is said to be a linear map or linear transformation if for any two vectors x and y in X and any scalar a in $\mathbb{K}$, the following two conditions are satisfied:

- f(x +y) = f(x) + f(y) additivity

- f(ax)= af(x) homogeneity of degree 1

Definition 1.1.6: A linear operator.

An operator is linear mapping of vector space X onto itself or to another vector space.

Definition 1.1.7: A linear functional.

A functional is a linear mapping of a vector space into a scalar $\mathbb{K}(\mathbb{C}, \mathbb{C})$.

Definition 1.1.8: Hilbert space.

A Hilbert space H, is a complete inner product space i.e a Banach space whose norm is generated by an inner product.

Definition 1.1.9:A bounded linear operator.

A linear operator $T : X \to Y$ is called bounded if and only if there exists a constant $M > 0$.

$$\| T(x) \| \leq M \| (x) \| \; \forall x \in X$$

We denote the set bounded linear operators on X and Y by B(X,Y) . Note also that; for a constant N, a bounded linear functional f on X satisfies the inequality,

$$| \; f(x) \; | \leq N \| (x) \| \; \forall x \in X, N > 0.$$

Definition 1.1.10: Norm of a bounded operator.

Let $T \in$ B(X,Y). Then the norm of T is defined as

$$\| T \| = sup\{\| Tx \|; x \in D(T), \| x \| = 1\} = sup\{\frac{\| Tx \|}{\| x \|}; x \in D(T),\} < \infty$$

That the supremum is finite follows from the fact that

$$\| T(x) \| \leq M \| (x) \| \; \forall x \in X, M > 0$$

Definition 1.1.11: Completely bounded norm.

Any operator $T : B(H) \to B(H)$ that induces a family of maps $T_n : M_n(B(H)) \to M_n(B(H)), n \geq 1$, defined by $T_n([x_{ij}]) = [T(x_{ij})]$ for any matrix $[x_{ij}] \in M_n(B(H))$ is said to be a completely bounded operator if $\sup_n \parallel T_n \parallel$ is finite.

Definition 1.1.12: Self Adjoint Transformation.

A bounded linear transformation $T \in B(H)$ is said to be self-adjoint if $T^* = T$. Thus T is Hermitian and $D(T) = H$ if and only if T is self adjoint.

Definition 1.1.13: Normal operator.

A bounded linear operator T on a Hilbert space H is said to be normal if it commute with its adjoint i.e $TT^* = T^*T$.

Definition 1.1.14: Unitary operator.

A unitary operator is a bounded linear operator T on a Hilbert space satisfying $T^*T = TT^* = I$, where I is the identity operator.

This property is equivalent to the following:

- T preserves inner product on the Hilbert space, so that for all vectors x and y in the Hilbert ,

$$\langle Tx, ty \rangle = \langle x, y \rangle.$$

- T is surjective isometry(distance preserving map) i.e

$$\|T(x-y)\| = \|(x-y)\|.$$

Definition 1.1.15: Compact operator.

An operator $T \in B(H)$ is said to be a compact operator if for every bounded sequence x_n in H the sequence (Tx_n) contains a convergent subsequent.

Definition 1.1.16: Finite operator.

An operator $T \in B(H)$ is finite rank operator if the dimension of the range of T is finite.

Definition 1.1.17: Unitary element.

Let Ω be an algebra. An element u in Ω is a unitary element if $u^*u = uu^* = 1$

Remark 1.1.18: if $u^* = 1$ then u is an isometry and if $uu^* = 1$, then u is a co-isometry. If $\varphi : A \to B$ is a homomorphism. Let X and Y be Banach spaces and $T \in B(X, Y)$ which is bijection, then there exists $T^{-1} \in B(Y, X)$.

Definition 1.1.19: Unitary element.

Let Ω be an algebra. An element x in Ω is a unitary element if $x^*x = xx^* = 1$

Definition 1.1.20: Orthonormal basis.

Let V be a vector space. A subset $(v_1, ..., v_n)$ of a vector space V, with the inner product $\langle , \rangle$ is called orthonormal if $\langle v_i, v_j \rangle = 0$ i.e the vectors are mutually perpendicular. An orthonormal set must be linearly independent and so it's a vector space basis for the space it spans. Such a basis is called an orthonormal basis.

Definition 1.1.21: Trace-class operator.

Let T be an operator on a Hilbert space H. We define it's trace-class norm to be $\| T \|_1 = \| \mid T \mid^{1/2} \|_2^2$. If E is an orthonormal basis of it, then

$$\| T \|_1 = \sum_{x \in E} \langle \mid T \mid (x), x \rangle$$

If $\| T \|_1 < +\infty$, we call T a trace-class operator.The trace of a trace-class operator T is given by $t_r(T) = \Sigma_{x \in E} \langle T(x), x \rangle$, where E is an orthonormal basis.

Definition 1.1.22: Homomorphism.

A homomorphism from an algebra Ω_1 to an algebra Ω_2 is a linear map φ :

$\Omega_1 \to \Omega_2; \varphi(ab) = \varphi(a)\varphi(b) \forall a, b \in \Omega_1$. It's kernel, ker φ is an ideal in Ω_1 and it's image $\varphi(\Omega_1)$ is a subalgebra of Ω_2.

Definition 1.1.23: A positive linear functional.

A positive linear functional is a functional on a Banach algebra Ω with an involution that satisfies the condition

$$f(xx^*) \geq 0 \forall x \in \Omega.$$

Definition 1.1.24: A state.

A state on an algebra Ω, is a continuous positive linear functional which satisfies the Schwartz inequality,

$$\mid f(x_* y) \mid_2 \leq f(x_* x) f(y_* y).$$

Definition 1.1.25: Inner product space.

An inner product space X is a complex linear space together with an inner product $\langle , \rangle : X \times X \to C$ such that;

- $\langle x, y \rangle = \overline{\langle y, x \rangle}$
- $\langle \lambda x + \mu y, z \rangle = \lambda \langle x, z \rangle + \mu \langle x, z \rangle : x, y, z \in X, \mu, \lambda \in \mathbb{K}$
- $\langle X, X \rangle \geq 0$, with $\langle x, x \rangle = 0 \Rightarrow x = 0$

Definition 1.1.26: Ideal.

Let Ω be an algebra. A left (respectively, right) ideal in Ω is a vector subspace I of Ω such that $a \in \Omega$ and $b \in I \Rightarrow ab \in I$ (respectively, $ba \in I$). An ideal in Ω is therefore a vector sub space that is simultaneously a left and a right ideal in Ω.

Definition 1.1.27: Essential Ideal.

Let I be a closed ideal in an algebra Ω. We say I is essential ideal in Ω if $aI = 0 \Rightarrow A = o \forall a \in \Omega$. (equivalently, $Ia = 0 \Rightarrow a = 0 \forall a \in \Omega$)

Definition 1.1.28: Free vector space.

Let $\mathbb{K}$ be a field.Given any non-empty set X,we may construct a vector space f_x over $\mathbb{K}$ with X as the basis, simply by taking f_x be the set of all formal finite linear combinations of elements of X

$$f_x = \{\sum_{i=1}^{k} r_i x_i : r_i \in \mathbb{K}, x_i \in X\}$$

Where the operations combine like terms using the rules; $rx_i + sx_i = (r+s)x_i$ and $r(sx_i) = (rs)x_i$. The vector space f_x is called **free vector space** of X.

Definition 1.1.29: Tensor product.

Let U and V be vector spaces over $\mathbb{K}$ and let T be a subspace of the free vector space $fu \times v$ generated by all vectors of the form

$$r(u,v) + s(u',v) = (ru + su', v)$$

$$r(u,v) + s(u,v') = (u, rv + sv')$$

$$\forall r, s \in F, u, u' \in U and v, v' \in V$$

The quotient space $fu \times v/T$ is called Tensor product of U and V and is denoted by $U \otimes V$.

Definition 1.1.30: Matrix numerical ranges.

For a tuple $(c_1, c_2........., c_n)$ of operators $c_i \in B(H)$, we denote by $W_m(c_1, c_2........., c_n)$ the "matrix numerical range"

$$W_m(c_1, c_2........., c_n) = \{(\langle c_j^* c_i \zeta, \zeta \rangle)_{i,j}^n = 1 : \zeta \in H, \| \zeta \| = 1\} \subset M_n$$

(where M_n is the positive semi definite $n \times n$ matrices.) Now, a subset of the closure of W_m which we call the 'extremal matrix numerical range' and denote by

$$W_{m,e}(c_1, c_2........, c_n) = \{\alpha \in \overline{W_m(c_1, c_2........, c_n)} : trace(\alpha) = \| \sum_{i=1}^{n} c_i^* c_i \|\}$$

Definition 1.1.31: Semigroups.

A semigroup is an algebraic structure consisting of a set closed under an associative binary operation. It is denoted as a pair $(X, *)$ where X is a set and binary function $* : X \times X \rightarrow X$ which is called the operation of the semigroup.The application of the operation is required to be associative i.e

$$(x * y) * z = x * (y * z) \forall x, y, z \in X$$

Examples of semigroups

(a) Positive integers with addition

(b) Any ideal of a ring, given multiplication. Thus any ring including integers, rational, real and complex numbers.

(c) Any subject of a semigroup closed under the semigroup operation.

(d) Any monoid, and therefore any group

Definition 1.1.32: SubSemigroups.

A subset Y of a semigroup X is called a subsemigroup if it is closed under the semigroup operation, that is Y^*Y is a subset of Y.

Definition 1.1.33: Positivity of a projection.

A projection P is to be positive if

$$\langle Px, x \rangle \geq 0 \, \forall \, x \in H$$

Definition 1.1.39: Multiplier algebra.

Let Ω be a non-unital C^*- algebra. Then there is a unique (up to isomorphism) C^*- algebra which contains Ω as an essential ideal and is maximal in the sense that any other algebra can be embedded in it. This C^*- algebra is called multiplier algebra and denoted by $M(\Omega)$.

Definition 1.1.40: Convex set.

Let X be a linear space. A subset M of the linear space X is convex if for all $x, y \in M$, for any positive real number t satisfying $0 < t < 1$ we have

$$tx + (1-t)y \in M.$$

Definition 1.1.41: Convex hull.

If M is a subset of a linear space X, then a convex hull M, represented by $Co(M)$ is the smallest convex subset of X containing M and it is the intersection of all the convex subsets of X that contain M.

If X is a linear topological space then the set $\overline{Co(M)}$ called the closed convex hull of M, is the intersection of all closed convex sets containing M.

Properties of $Co(M)$ and $\overline{Co(M)}$

Lemma

Let M and N be arbitrary sets in a linear space X, then

(a) $Co(\alpha M) = \alpha Co(M)$ and $Co(M+N) = Co(M) + Co(N)$

If X is a topological space, then

(b) $\overline{Co}(M) = \overline{Co(M)}$

(c) $\overline{Co}(\alpha M) = \alpha \overline{Co}(M)$

(d) If $\overline{Co}(M)$ is compact then $\overline{Co}(M+N) = \overline{Co}M + \overline{Co}N$

Remark 1.1.42:

The intersection of any convex subset of x also convex.

Definition 1.1.43: Irreducible.

A Representation (H,T),where $T : H \longrightarrow H$, of a C^*- algebra Ω is said to be irreducible if the algebra T (Ω) acts irreducibly on H.If two representation are unitarily equivalent. then irreducibility of one implies irreducibility of the other. Further, if H is a one dimensional Hilbert space,then the zero representation of any C^*- algebra on H is irreducible.

Definition 1.1.44: Diagonal matrix.

The matrix $D = [d_i j] \in M_n$ is called diagonal if $d_{ij} = 0$ whenever $j \neq i$. Conventionally, we denote such a matrix as $D = diag(d_{11}, ...d_{nn})$ or $D = diag d$, where d is the vector of diagonal entries of D.

Definition 1.1.45: Triangular matrix.

The matrix $T = [t_{ij}] \in M_n$ is said to be *upper triangular* if $t_{ij} = 0$ whenever $j < i$.

Analogously, T is said to be *lower triangular* if its transpose is upper triangular.

Definition 1.1.46: Unitary equivalent.

A matrix $B \in M_n$ is said to be unitary equivalent to $A \in M_n$ if there is a unitary matrix $U \in M_n$ such that $B = U^*AU$.

If U may be taken to be real, then B is said to be orthogonally equivalent to A.

Definition 1.1.47: Binary relation.

A binary relation or simply a relation from a set A to a set B is a subset $R \subset A \times B$.

Definition 1.1.48: Trace.

The trace of an $n \times n$ square matrix $A = (\alpha_{ij})$ is defined to be the sum of the elements on the main diagonal (the diagonal from the upper left to the lower right) of A i.e.,

$$tr(A) := \alpha_{11} + \alpha_{22} + ... + \alpha_{nn} = \sum_{i=1}^{n} \alpha_{ii}$$

where α_{ij} represents the entry on the ith row and jth column of A.

Definition 1.1.49: Rank.

The rank of a matrix A is defined as the order of the largest square array in A with a non zero determinant.

Definition 1.1.50: A Hilbert-Schmidt operator.

Hilbert-Schmidt Operator is a bounded operator T on a Hilbert space H with finite Hilbert-Schmidt norm, meaning that there exists an orthonormal basis $\{e_i : i \in I\}$ of H with the property

$$\sum_{i \in I} \| Te_i \|^{1/2} < \infty.$$

If this is for one orthonormal basis, it is true for any other orthonormal basis. Let A and B be two Hilbert-Schmidt operators. The *Hilbert-Schmidt inner product* can be defined as

$$\langle A, B \rangle_{HS} = tr(A^* B) = \sum_{i \in I} \langle Ae_i, Be_i \rangle$$

The induced norm is called the Hilbert-Schmidt norm:

$$\| A \|_{HS}^2 = \sum_{i \in I} \| Ae_i \|^2$$

Definition 1.1.51: Trace class operator.

Trace class operator is a compact operator for which a trace may be defined, such that the trace is finite and independent of the choice of the basis.

Properties of trace class operator

(a) A self adjoint operator **T** is trace class if and only if it's positive part T^+ and negative part T^- are both trace class.

(b) The trace is a linear functional over the space of trace class operators, i.e

$$tr(\alpha T + \beta T') = \alpha tr(T) + \beta tr(T') \quad \forall \alpha, \beta \in \mathbb{K}$$

(c) The bilinear map $\langle T, T' \rangle = tr(T^* T)$ is an inner product on the trace class.

Definition 1.1.52: Semiprime algebra.

An algebra Ω is to be prime if $a\Omega b = \{0\}, \Rightarrow a = 0$ or $b = 0$ and Ω is said to be semiprime if $a\Omega a = \{0\} \Rightarrow a = 0$.

Definition 1.1.53: Let Ω be non-unital C^*-algebra. Then there is a unique C^*-algebra which contains Ω as an essential ideal and is maximal in the sense that any other algebra can be embedded in it. This C^*-algebra is called multiplier algebra and denoted $M(\Omega)$.

Definition 1.1.54: If S is a subset of a Banach algebra Ω, the center/ centralizer of S is the set $Z(S) = \{x \in \Omega : xs = sx \ \forall \ s \in S\}$.

Definition 1.1.55: Completely bounded norm.

If a map $\alpha : B(H) \to B(H)$ induces family of maps $\alpha : M_n(B(H)) \to M_n(B(H))$, defined by $\alpha_n([x_{ij}]) = [\alpha(x_{ij})]$ for any $n \times n$ matrix $[x_{ij}] \in M_n(B(H))$, (where $n \geq 1$) and $\sup_n \parallel \alpha_n \parallel$ is finite, then α is said to be completely bounded norm of α denoted by $\|\alpha\|_{cb}$.

1.2 Statement of the problem

Let Ω be an algebra. The operator $T : \Omega \longrightarrow \Omega$ defined by

$$Tx = \sum_{i=1}^{k} a_i x b_i, a_i, b_i \in \Omega \qquad (1.2.1)$$

is called an elementary operator. We shall denote the class of elementary operators on Ω by $\mathcal{E}\ell(\Omega)$. The problem for computing $\|T\|$ has been considered over a long period by many mathematicians and there are some solutions known under various circumstances . We attempt to find $\| T \|$

when k=1,2 in (1.2.1)

1.3 Review of Related Literature.

Let Ω be an algebra. The operator $T : \Omega \rightarrow \Omega$ defined by

$$Tx = \sum_{i=1}^{k} a_i x b_i, a_i b_i \in \Omega, x \in \Omega$$

is called elementary operator. We shall denote the class of elementary operators on Ω by $\mathcal{E}\ell(\Omega)$. The problem for computing $\| T \|$ has been considered over a long period by many mathematicians and there are some solutions known under various circumstances. We attempt to find $\| T \|$: (i) when $k = 1, 2$ in (1.2.1). There are various results of norms of elementary operators known under special circumstances. One way the literature that relates to the problem has been viewed is to separate two strand of problems. One strand to concentrate on elementary operators with $k = 2$ in (1.2.1), and the other, which will be of much interest to us is dealing with the case when $k = 2$

In the case $k = 1$ in (1.2.1) we have

$$Tx = a_1 x b_1$$

It follows that

$$\| Tx \| = \| a_1 x b_1 \| \leq \| a_1 \| \| b_1 \|$$

i.e

$$\| T \| = \sup\{\| Tx \| : x \in \Omega, \| x \| = 1\} \leq \| a_1 \| \| b_1 \|$$

Therefore,

$$\| T \| \leq \| a_1 \| \| b_1 \|$$

Mathieu [15, 16], working in the opposite direction of (1.2.1) showed that there is a constant $C > 0$ such that

$$\| T \| \leq C \| a_1 \| \| b_1 \|$$

He even conjectured that $C = 1$.Timoney [25], Stacho and Zalar [22] working independently established that:

$$C = 1 \text{ and } C = 2(\sqrt{2} - 1)$$

respectively.

Theorem 1.3.1 (Timoney). If $a, b \in B(H)$ *(algebra of bounded linear operators on Hilbert space) and*

$$T_{a,b}(x) = axb + bxa$$

Then

$$\| T_{a,b} \| \geq \| a \| \| b \|$$

More generally, the same inequality holds if Ω is a prime C^*-algebra, a,b are in the multiplier algebra of Ω and

$$T_{a,b}(x) = axb + bxa$$

Remark: 1.3.2 A closed ideal I in a C^*-algebra is prime if whenever j_1 and j_2 are closed ideals of Ω such that $j_1, j_2 \in I$, we necessarily have $j_1 \in I$ or $j_2 \in I$.

We therefore say that an algebra Ω is a prime if the zero ideal of C^*-algebra if the zero of Ω is prime. Before we state Mathieu's theorem, we note that Calkin algebra is the quotient space $B(H)/K(H)$ where $B(H)$ is the space of bounded linear operators on Hilbert space and $K(H)$ is a set of compact operators on Hilbert spaces.
Mathieu [15], working on Calkin algebras, obtained the following theorem.

Theorem 1.3.3 (Mathieu). Let T be an elementary operator on Calkin algebra. Then,

$$\| T \| = \inf(\| \sum_{j=i}^{n} a_j a_j^* \|^{1/2} \| \sum_{j=i}^{n} b_j b_j^* \|^{1/2})$$

Where the infimum is taken over all representations of T
We note that Christensen and Sinclair [4] , Effros and Ruan [6] and Paulsen [18] showed that the elementary operator norm $\| T \|$ and the estimate in terms of the Haagerup tensor norm are related as follows

$$\| T \| \leq \| T \|_{cb} \leq \| \sum_{i=1}^{k} a_i \otimes b_i \|_h$$

where the subscript h denote the Haagerup norm. By $\| T \|_{cb}$, we imply the completely bounded norm of an elementary operator.

Theorem 1.3.4: Norm of an inner derivation A derivation on an algebra Ω, is a linear map from Ω to Ω satisfying

$$D(ab) = D(a)b + aD(b) \ \forall \, a, b \in \Omega$$

In particular, for a fixed $a \in \Omega$, the inner derivation $D(a, \Omega)$ is given by

$$D_a(b) = ab - ba \quad \forall \, b \in \Omega$$

For a fixed $a \in \Omega$, the norm of an inner derivation D_a is

$$\| D_a \| = \sup\{\| ab - ba \| : b \in \Omega, \| b \| = 1\} = \sup\{\| D_a(b) \| : b \in \Omega, \| b \| = 1\}$$

$$\leq 2 \inf\{\| a - z \| : z \in Z(M(\Omega))\}$$

With the infimum over z in the center $Z(M(\Omega))$ of $M(\Omega)$ with $M(\Omega)$ the multiplier algebra.

Alternatively, Mathieu and Ara [16] showed that (for general Ω)

$$\| D_a \| = 2 \inf\{\| a - z \| : z \in Z({}^cM(\Omega))$$

with ${}^cM(\Omega)$ the bounded central closure of $M(\Omega)$. For generalized (inner) derivations, $D(a, b)(x) = ax - ax$, there are results that are less comprehensive than for $D(a)$. In particular, Stampfli [23] established that

$$\| D_{a,b} \| = \inf \{\| a - \lambda \| + \| b - \lambda \| ; \lambda \in \mathbb{C}\}$$

when $\Omega = B(H)$.

1.4 Objective of Study

We consider an elementary operator

$$T_x = \sum_{i=1}^{k} a_i x b_i; \quad a_i, b_i \in \Omega$$

and investigate the operator norm $\| T \|$ for the case when k = 1,2.

1.5 Significance of study

The study will provide knowledge on the existing relation between norms of elementary operators and the norms of inner derivations. It's also hoped that, it will provide a wider avenue for research on the norms of elementary operators.

1.6 Methodology

For the research to succeed, knowledge of elementary operators on Calkin algebras, Banach spaces, norms of tensor product and Hilbert spaces was necessary. A good library, a lot of hardwork and patience was also required. Wide consultation with the supervisors,colleagues and other experts in the field was done. The internet facility came in handy.

Chapter 2

Elementary Operators on semi-prime algebras

2.1 Introduction

In this chapter we determine the norm properties of Haagerup estimate.
We also considered a set of bounded linear maps called double centralizers on a C^*- algebra and define a norm on them.
Further, we show that the set of commuting operator on Hilbert space yield convexity on matrix numerical range.
Finally, we investigate the concept of elementary operators on semigroups, semiprime algebras and polynomial algebras.

2.2 Haagerup norm

Let Ω be a prime C^*−algebra and and $T : \Omega \longrightarrow \Omega$ an elementary operator. Then T is representable as

$$Tx = \sum_{i=1}^{n} \lambda_i a_i^* x a_i$$

with $a_i \in \Omega$ linearly independent $\forall i = 1, 2, ..., n$ and $\lambda_i (1 \leq i \leq n)$ non-zero real numbers, see[14]. Since T is a hermitian preserving elementary operator, the existence of linearly independent operators $a_i \in \Omega$ that satisfy above equation is guaranteed by ([14], corollary 4.9). Since hermitian operators are self adjoint, an elementary operator defined from these operators is either self adjoint or not. If the elementary operator is self adjoint, then by ([25], Lemma 3.11 and Theorem 3.12),

$$\|T\| = \|T\|_{cb} = \left\{ \| \sum_{i=1}^{n} a_i a_i^* \| \, \| \sum_{i=1}^{n} b_i^* b_i \| \right\}^{1/2},$$

the Haagerup norm formula. The Haagerup norm of an element $T \in B(H) \otimes B(H)$ (algebraic tensor product) is defined as follows,

$$\|T\| = inf \left\{ \| \sum_{i=1}^{n} a_i a_i^* \| \, \| \sum_{i=1}^{n} b_i^* b_i \| \right\}^{1/2}$$

$$= inf \left\{ \| \sum_{i=1}^{n} a_i \otimes b_i \| \right\}$$

To show the Haagerup norm defines a norm;

(i) Clearly $\| T \| \geq 0$.

(ii) Also $\| T \| = 0 \Longleftrightarrow T = 0$.

(iii) To show that $\forall\alpha \in \mathbb{K}$

$$\| \alpha T \| = | \alpha | \| T \|,$$

we have,

$$\| \alpha T \| = inf \left\{ \| \sum_{i=1}^{n} (\alpha a_i)(\alpha a_i)^* \| \| \sum_{i=1}^{n} b_i^* b_i \| \right\}^{1/2}$$

$$= inf \left\{ \| \sum_{i=1}^{n} a_i a_i^* \| \| \sum_{i=1}^{n} (\alpha b_i)^* (\alpha b_i) \| \right\}^{1/2}$$

$$= inf \left\{ | \alpha |^2 \| \sum_{i=1}^{n} a_i a_i^* \| \| \sum_{i=1}^{n} b_i^* b_i \| \right\}^{1/2}$$

$$= |\alpha| \inf \left\{ \| \sum_{i=1}^{n} a_i a_i^* \| \| \sum_{i=1}^{n} b_i^* b_i \| \right\}^{1/2}$$

$$= |\alpha| \, \| T \|$$

(iv)Lastly ,we show that if $T, T' \in B(H) \otimes B(H)$ then

$$\| T + T' \| \leq \| T \| + \| T' \|$$

We have,$\| (T + T')(a_i \otimes b_i) \|$

$$\| T(a_i \otimes b_i) + T'(a_i \otimes b_i) \|$$

$$= \inf \left\{ \| \sum_{i=1}^{n} a_{i_1} \otimes b_{i_1} + \sum_{i=1}^{n} a_{i_2} \otimes b_{i_2} \| \right\}$$

$$\leq \inf \left\{ \| \sum_{i=1}^{n} a_{i_1} \otimes b_{i_1} \| \right\} + \inf \left\{ \| \sum_{i=1}^{n} a_{i_2} \otimes b_{i_2} \| \right\}$$

$$= \inf \left\{ \| \sum_{i=1}^{n} a_{i_1} a_{i_1}^* \| \| \sum_{i=1}^{n} b_{i_1}^* b_{i_1} \| \right\}^{1/2} + \inf \left\{ \| \sum_{i=1}^{n} a_{i_2} a_{i_2}^* \| \| \sum_{i=1}^{n} b_{i_2}^* b_{i_2} \| \right\}^{1/2}$$

$$= \|T(a_i \otimes b_i)\| + \|T'(a_i \otimes b_i)\|$$

Thus,

$$\|T + T'\| \leq\| T \| + \| T' \|$$

Therefore, Haagerup tensor norm defines a norm.

Definition 2.1.1: BILINEAR MAPS

let U, V and W be vector spaces over $\mathbb{K}$. A function $f : UXV \longrightarrow W$ is bilinear, if it's linear in both variables U and V separately.

To show it's linear in U, we let $u, u' \in U$ and $r; s \in \mathbb{K}$. Now by universal property of tensor product we know that,

$$f(u, v) = u \otimes v$$

Thus,

$$f(ru + su', v) = (ru + su') \otimes v$$

$$= (ru \otimes v) + (su' \otimes v)$$

$$= r(u \otimes v) + s(u' \otimes v)$$

$$= rf(u, v) + sf(u', v)$$

Hence, $f : UXV \longrightarrow W$ is linear in U.

To show linearity in V, let $v, v' \in V$ and $r, s \in \mathbb{K}$.

Therefore,

$$f(u, rv + sv') = u \otimes (rv + sv')$$

$$= (u \otimes rv) + (u \otimes sv')$$

$$= r(u \otimes v) + s(u \otimes v')$$

$$= rf(u, v) + sf(u', v)$$

Hence $f : UXV \longrightarrow W$ is linear in V.

Thus it is a bilinear map. The set of all bilinear map. The set of all bilinear functions from

$$UXV \longrightarrow W$$

is denoted by B(U,V;W).

Examples

(1) A real inner product $\langle , \rangle : UXV \longrightarrow \mathbb{R}$ is a bilinear form on U X V (2) If Ω is an algebra, the product map

$$\mu : \Omega X \Omega \longrightarrow \Omega$$

defined by

$$\mu(a,b) = ab \quad \forall a,b \in \Omega$$

is bilinear.

Definition 2.1.2: LINEARITY OF AN OPERATOR ON A TENSOR PRODUCT

From the universal property of tensor product, we know that to each bilinear function $f : U \times V \longrightarrow W$, there corresponds a unique linear function $\alpha : U \otimes V \longrightarrow W$, through which f can be factored (that is $f = \alpha ot$) (U,V,W are vector spaces). This establishes a map

$$\phi : B(U,V;W) \longrightarrow \ell(U \otimes V, W)$$

given by $\phi(f) = \alpha$, In other words, $\phi(f)$ is the linear map for which

$$\phi(f) : U \otimes V \longrightarrow W;$$

$$\phi(f) U \otimes V = f(U,V)$$

Thus

$$\| \phi(f)(U \otimes V) \| = \| f(U, V) \| .$$

Now to show that ϕ is linear. Let $f, g \in B(U, V; W)$, and $r, s \in K$ then

$$r\phi(f) + s\phi(g)(U \otimes V)$$

$$= r\phi f(U, V) + s\phi g(U, V)$$

$$= \phi(rf + sg)(U, V).$$

thus ϕ is linear.

Definition 2.1.3: DOUBLE CENTRALIZER

A double centralizer for a C^*- algebra Ω is a pair (L,R) of bounded linear maps on Ω, such that $\forall a, b \in \Omega, L(ab) = L(a)b, R(ab) = aR(b)$ and $R(a)b = aL(b)$. For example if $c \in \Omega$ and L_c, R_c are the map on Ω defined by $L_c(a) = ca$ and $R_c(a) = ca$ then (L_c, R_c) is a double centralizer on Ω. Now for all $c \in \Omega$ we have,

$$\| c \| = \sup \| cb \| : \| b \| \leq 1,$$

$$\| c \| = \sup \| bc \| : \| b \| \leq 1$$

and therefore $\| L_c \| = \| R_c \| = \| c \|$

Lemma 2.1.4

If (L,R) is a double centralizer on a C^*-algebra Ω then, $\|L\| = \|R\|$

Proof

Since $\|aL(b)\| = \|R(a)b\| \leq \| R \| \| a \| \| b \|$.

We have,

$$\| L(b) \| = sup\{\| aL(b) \| : \| a \| \leq 1\} \leq \| R \| \| b \|$$

and therefore,

$$\| L \| \leq \| R \| . \tag{2.2.1}$$

Also, $\| R(a)b \| = \| aL(b) \| \leq \| L \| \| a \| \| b \|$

$$\Longrightarrow \| R(a) \| = sup\{ \| R(a)b \| : \| b \| \leq 1\}$$

$$\leq \| L \| \| a \| \text{ i.e } \| R(a) \| \leq \| L \| \| a \|$$

Thus,

$$\| R \| \leq \| L \| . \tag{2.2.2}$$

From (2.1) and (2.2), we have $\| R \| = \| L \|$. If Ω is a C^*- algebra, we denote the set of it's double centralizers by $M(\Omega)$. We define the norm of the double centralizer (L,R) to be $\| L \| \| R \|$.
It is easy to check $M(\Omega)$ is a vector subspace of $B(\Omega) \otimes B(\Omega)$.
For, if we let $(u_1, v_1), (u_2, v_2) \in M(\Omega) \quad \forall u_1, u_2, v_1, v_2 \in \Omega$ then, $(u_1, v_1) + (u_2, v_2) = (u_1 + u_2, v_1 + v_2) \in M(\Omega)$. Also, let (r,u) and (s,u)$\in M(\Omega) \quad \forall r, s \in \mathbb{K}$ and $u, v \in \Omega$, then, (r,u)(s,v) = (rs,uv)$\in M(\Omega)$ which gives the desired results. If (L_1, R_1) and $(L_2, R_2) \in M(\Omega)$, we define their product to be $(L_1, R_1)(L_2, R_2) = (L_1L_2, R_1R_2)$. This product is also a double centralizer of Ω i.e $\forall a_1, a_2, b_1, b_2 \in \Omega$, we have,

$$\{(L_1, R_1)(a_1, b_1)\}\{(L_2, R_2)(a_2, b_2)\}$$

$$= \{L_1(a_1, b_1), R_1(a_1, b_1)\}\{L_2(a_2, b_2), R_2(a_2, b_2)\}$$

$$= L_1(a_1, b_1)L_2(a_2, b_2), R_2(a_2, b_2)R_1(a_1, b_1) \in M(\Omega).$$

If $L : \Omega \longrightarrow \Omega$, define $L^* : \Omega \longrightarrow \Omega$, by setting $L^*(a) = (L(a^*))^*$, then L^* is linear.

For if $\forall \alpha, \beta \in \mathbb{K}$ and $a, b \in \Omega$, we have $L^*(\alpha a + \beta b) = (L(\alpha a)^* + L(\beta b)^*)^* = (L(\alpha a)^* + L(\beta b)^*)^* = \alpha L^*(a) + \beta L^*(b)$, thus L^* is linear. The map $L \longrightarrow L^*$ is an isometric conjugate linear map from Banach algebra to itself such that $L^{**} = L$ and

$$(L_1 L_2)^* = L_2^* L_1^*.$$

If (L,R) is a double centralizer on Ω, so is $(L, R)^* = (R^*, L^*)$ we can easily verify that the map $(L, R) \longrightarrow (L, R)^*$ is an involution to $M(\Omega)$. For, $((L, R)^*)^* = (R^*, L^*)^* = (L, R)^* = (R^*, L^*)$

Theorem 2.1.5

If Ω is a C^*-algebra, then $M(\Omega)$ is a C^*-algebra under the multiplication, involution and norm defined above.

Proof

We only need to check that if $T = (L, R)$ is a double centralizer, then $\| T^*T \| = \| T \|^2$. If $\| a \| \leq 1$, then $\| L(a) \|^2 = \| (L(a))^* L(a) \|$

$$= \| L^*(a^*) L(a) \|$$

$$= \| L^*(a^*) L(a) \| = \| a^* R^* L(a) \| \leq \| R^* L \| = \| T^*T \| .$$

So,

$$\| T \|^2 = sup\{\| L(a) \|^2 : \| a \| \leq 1\} \leq \| T^*T \| < \| T \|^2$$

and therefore, $\| T^*T \| = \| T \|^2$. The algebra $\Omega \longrightarrow M(\Omega)$, $a \longrightarrow (L_a, R_a)$, is an isometric $^*-$ homomorphism, and therefore we can, and do identify Ω as a C^*- subalgebra of $M(\Omega)$

Theorem 2.1.6

Let I be a closed ideal in a C^*-algebra Ω. Then there is a unique $^*-$homomorphism

$\alpha : \Omega \longrightarrow M(I)$ extending the inclusion $I \longrightarrow M(I)$. Moreover, α is injective if I is essential Ω.

Proof

Since the inclusion map $I \longrightarrow M(I)$ admits a $*-$homomorphic extension $\alpha : \Omega \longrightarrow M(I)$. Suppose that $\psi : \Omega \longrightarrow M(I)$ is another such extension. If $a \in \Omega$ and $b \in I$, then $\alpha(a)b = \alpha(ab) = ab = \psi(ab) = \psi(ba)$. Hence, $(\alpha(a) - \psi(a))I = 0$, so $\alpha(a) = \psi(a)$. Since I is essential in M(I). Thus $\alpha = \psi$. Suppose now that I is essential in Ω and let $a \in ker(\Omega)$. Then $aI = L_a(I) = 0$. So a=0. Thus, α is injective. The results tell us that the multiplier algebra M(I) of I is the largest unital C^*-algebra containing I as an essential closed ideal.

Definition 2.1.7

For a Tuple $(C_1, C_2, ..., C_n)$ operators, $C_i \in B(H)$, we denote by $W_m(C_1, C_2, ..., C_n)$ the matrix range.

$$W_m(C_1, C_2, ..., C_n) = \{(\langle C_j^* C_i \xi, \xi \rangle)_{i,j=1}^n : \xi \in H, \parallel \xi \parallel = 1\} \subset M_n$$

(where M_n is the positive semidefinite n X n matrices.) Now, a subset of the of W_m which we call the ' extremal matrix numerical range' and denote by;

$$W_{m,e}(C_1, C_2, ..., C_n) = \left\{ \alpha \in \overline{W_m(C_1, C_2, ..., C_n)} : trace(\alpha) = \parallel \sum_{i=1}^{n} C_i^* C_i \parallel \right\}$$

Theorem 2.1.8

Let H be an Hilbert space. Consider a set of bounded linear operators$(T_i)_{i=1}^n \in B(H)$. Denote by $T_{i=1}^n = (T)_{i=1}^n \otimes i_n \in M_n(B(H)) = B(H^n)$ that is the block diagonal n X n matrix with $T_{i=1}^n$ in the diagonal blocks. Let T^n denote the corresponding n-tuple of $(T)_{i=1}^{n}$. Then,

$$W_m(T^n) = \left\{ \sum_{j=1}^{n} t_j \alpha_j : \alpha_j \in W_m(T), t_j \geq 0, \sum_{j=1}^{n} t_j = 1 \right\}$$

(the set of convex combinations of k elements of $w_m(T)$)

Proof. Consider $x_1, \ldots\ldots, x_n \in T$ which are unit vectors, then

$$\langle (t_{i=1}^{n})x, (t_{j=1}^{n}x \rangle = \sum_{i,j,k=1}^{n} tr\langle A_i x_k^{'}, A_j x_k^{'} \rangle,$$

where $tr\langle x_k, x_k \rangle = \|x_k\|^2$ and $x_k^{'}$ is the unit vector in the direction of x_k and $A \in B(H)$. Alternatively, if we denote by $x_i^* \otimes x_i$ the rank one operator on H given by $\theta : x_i^* \otimes x_i \longrightarrow \langle \theta, x_i \rangle x_i$. Let as take $R = \sum_{i=1}^{n} x_i \otimes x_i$, we can see that such R is a positive operator of trace $\sum_{i=1}^{n} \|x_i\|^2 = 1$. Every such R can be written in the form $\| \sum_{i=1}^{n} x^* \otimes x_i \|$. Moreover,

$$(\langle (T_{i=1}^{n})x, (T_{j=1}^{n})x \rangle)_{i,j=1}^{n} = (trace(A_{i=1}^{*} AR))_{i,j=1}^{n}$$

To show that $W_m(T^n)$ is convex, we need only show that,

$$W_m(T^{n+1}) = W_m(T^n)$$

And if n = dim H, then that is clearly true. For $k = n < dimH$. Now take

$$\pi = (trace(x_j^* x_i R_o))_{i,j=1}^{n} = W_n(T^{n+1})$$

where $R_o = \sum_{i=1}^{n+1} x_i \otimes x$, which is positive, and rank at most n+1. If the rank of R_o is $< k+1$ we are done and so we assume that the rank is n+1. We will work within the span of the x_i by taking P to be the orthogonal projection onto the span, temporarily restricting H to P(H) and considering $A^*_{i,j} = PA^*_jA_jP \in B(P(H))$ in place of $A^*_jA_i$. Note that $A^*_{i,j} = A_{j,i}$. Considering,

$$S_{n+1} = \{R \in B(PH) : R > 0, traceR = 1, trace(A_{i,j}, x) = \Theta_{i,j}\}$$

for $1 \leq i, j \leq k$ Note that this set is compact (a closed a subset of the trace one and positive definite matrices).The total number of real linear equations to be satisfied by $R \in S_{n+1}$ is $1 + k^2$ and we are working inside the hermitian elements of B(PH),a space of dimension $dim(PH)^2 = (K+1)^2 > 1 + K^2$. More precisely, we have

$$S^k_{n+1} \subset \{R = R^* \in B(P(H)), traceR = 1\}$$

$= \Pi_{n+1}$, an affine space of dimension $(k+1)^2 - 1$. S^k_{n+1} is the intersection of the convex set $\sum_{n+1}$ of positive element of $\prod_{n+1}$ with an affine subspace of $\prod_{n+1}$ of co-dimension k^2. $S_{n+1} \neq \phi$ because of R_O. Thus S_{n+1} must contain some point R which is not a relative interior point of $\sum_{n+1}$ of positive elements of $\prod_{n+1}$. Such an R must have rank $\leq n$ and so $\phi(trace(T^*_jT_i, x))^k_{i,j} \subset W_m(T^n)$

□

Remark 2.1.9

The argument above is a proof of a remnant of convexity for the joint (spatial) numerical range of the finite list of operators on B(H).The Toeplitz-Hausdorff theorem (see [1]), asserts that the numerical range of a single operator is convex. That is known to be false in general for the numerical range of two operators

$$\{(\langle T_1x, x\rangle, \langle T_2x, x\rangle) : x \in H, \parallel x \parallel = 1\}$$

though it is for two hermitian operators T_1, T_2. The argument above shows that the set of all convex combinations of n elements of the joint numerical range of k operators $T_1, T_2, ..., T_n \in B(H)$ is convex provided $(n+1)^2 > 1+d$ where d is the dimension of the real span of the real and imaginary parts of the T(or n=dim H).There is a case where the joint numerical range is known to be convex i.e for a commuting n-tuple of normal operators $(T_1, T_2......., T_n)$. It follows that if $T_j^* T_i$ are commuting operators then $W_m(T)$ is convex.

Notation. The cb norm of a linear map $T : \Omega \to \Omega$ is defined as $\| T \|_{cb} = \sup_{k\geq 1} \| T \|_k = \| T^k \|$ and

$$T^k M_k(\Omega) \to M_k(\Omega)$$

($M_k(\Omega)$ means that the $n \times n$ matrices with entries in Ω) is defined via

$$T^k(x_{i,j})_{i,j=1}^k = (T(x_{i,j}))_{i,j=1}^k$$

If $\Omega \subset B(H)$ then we can regard $M_k(\Omega) = \Omega \otimes M_k$ as a C^*- subalgebra of $B(H) \otimes M_k = B(H \otimes C^k) = B(H^k)$.

Theorem 2.1.10

Let Ω be an algebra. Let $\Omega = B(H)$ and $T \in \mathcal{E}\ell B(H)$. Then we have equality in

$$\| T \| \leq \| T \|_{cb} \leq \frac{1}{2} \left\{ \| \sum_{j=1}^{n} a_i a_j^* \| + \| \sum_{j=1}^{n} b_j^* b_j \| \right\} \quad \forall a_j, b_j \in B(H)$$

if and only if the intersection

$$W_{m,e}(a_1^*, ..., a_n^*) \bigcap W_{m,e}(b_1, ..., b_n) \neq \emptyset$$

Proof(see[25])

Consider first the case when H is finite dimensional and the intersection is non-empty. Thus there exist unit vectors x,y with

$$\langle a_j a_i^* x, x\rangle = \langle b_j^* b_i y, y\rangle$$

for $1 \leq i, j \leq n$ and

$$\sum_{i=1}^{n}\langle a_j a_i^* x, x\rangle = \sum_{i=1}^{n}\langle b_j^* b_i y, y\rangle = \| \sum_{i=1}^{n} a_i a_i^* \| = \| \sum_{i=1}^{n} b_i^* b_i \|$$

Now, $U(b_i y = a_i^* x)$ specifies unique unitary map from the span of $b_i y$ to the span of $a_i y$.We can then extend U to a unitary map on H and compute that,

$$\langle T(U)y, x\rangle = \sum_{i=1}^{n}\langle U b_i y, a_i^* x\rangle = \sum_{i=1}^{n}\langle a_i a_i^* y, y\rangle$$

$=\| \sum_{i=1}^{n} a_i a_i^* \| = \| \sum_{i=1}^{n} b_i^* b_i \|$. Thus we have $\| T \| \geq (1 \setminus 2)(\| \sum_{i=1}^{n} a_i a_i^* \| + \| \sum_{i=1}^{n} b_i^* b_i \|) \geq \| T \|_{cb} \geq \| T \|_1 = \| T \|$ forcing equality all round in this case. When H is infinite dimensional we have to modify the argument only slightly to take account of that fact that we can only find unit x and y so as to get arbitrarily close approximations $\langle a_j a_i^* x, x\rangle \cong \langle b_j^* b_i y, y\rangle$ for $1 \leq i, j \leq n$ and

$$\sum_{i=1}^{n}\langle a_i a_i^* x, x\rangle \cong \| \sum_{i=1}^{n} a_i a_i^* \| = \| \sum_{i=1}^{n} b_i^* b_i \|$$

We can then say that our $\|T\|$ will have norm approximately 1. For the converse, that is if $\| \sum_{i=1}^{n} a_i a_i^* \| \neq \| \sum_{i=1}^{n} b_i^* b_i \|$. A well known estimate due to Haagerup states that,

$$\| T \| \leq \| T \|_{cb} \leq \sqrt{\| \sum_{i=1}^{n} a_i a_i^* \| \| \sum_{i=1}^{n} b_i^* b_i \|} \tag{2.2.3}$$

where $\| T \|_{cb}$ is the completely bounded norm of T. The Haagerup estimate

(2.3) can be derived from the following matrix formulation

$$T_x = [a_1, a_2, ...a_n](x \otimes I_n)[b_1 b_2 ... b_n]$$

Taking $a_1, ..., a_n = (a_i)_{i=1}^n$ and $b_1, ..., b_n = (b_i)_{i=1}^n$, then we have $T_x = a_i(x \otimes I_n)b_i$ where $(x \otimes I_n)b_i$ is the block diagonal element of $M_N(\Omega) = \Omega \otimes M_n$ with x's along the diagonal. From $T_x = a_i(x \otimes I_n)b_i(x \in \Omega), T^k(X) = a_i^k(X \otimes i_n)b_i^k(X \in M_k(\Omega))$ where

$$a_i^k = [a_1 \otimes I_k, ..., a_n \otimes I_k]$$

and

$$b_i^k = [b_1 \otimes I_k, ..., b_n \otimes I_k]$$

We get the estimate (2.3) from

$$\| T^k \| \leq \| (a_i^n)^k \| \| (b_i^n)^k \| = \| a_i^n \| \| b_i^n \|$$

From (2.3)we get,

$$\| T \|_{cb} \leq 1/2\{\| \sum_{i=1}^n a_i a_i^* \| + \| \sum_{i=1}^n b_i^* b_i \|\} \qquad (2.2.4)$$

Therefore, for the converse, $\| \sum_{i=1}^n a_i a_i^* \| \neq \| \sum_{i=1}^n b_i^* b_i \|$, then we have strict inequality between the right hand sides of (2.3) and (2.4).So we may suppose equality and normalize

$$\| \sum_{i=1}^n a_i a_(i)^* \| = \| \sum_{i=1}^n b_i^* b_i \| = 1$$

We know that $\| T \| = \sup \| T(u) \|$ over u unitary (by the Russo-Dye theorem [20], or the more elementary fact that each element of the open unit ball of B(H) is an average of unitaries [22]). Now,

$$\| T(u) \| = \sup\{R\langle T(u)x, y\rangle\}$$

over unit vectors $x, y \in H$ and we note that, $R\langle T(u)x, y\rangle = \Sigma_{i=1}^{n} R\langle ub_i x, a_i^* y\rangle$.
Let $r_1 = ub_i x, r_2 = a_i^* y$
Now, $(\langle r_{2_i}, r_{1_i}\rangle) \in W_m(a_i^*)$. Clearly,

$$\mathbb{R}\langle T(u)x, y\rangle = \sum_{i=1}^{n} \mathbb{R}\langle r_{1_i}, r_{2_i}\rangle \leq \sum_{i=1}^{n} \| r_{1_i} \| \| r_{2_i} \|$$

$$\leq \sqrt{\sum_{i=1}^{n} \| r_{1_i} \|^2} \sqrt{\sum_{i=1}^{n} \| r_{2_i} \|^2} \leq 1$$

,

$$(\langle r_{1_i}, r_{1_j}\rangle)_{i,j=1}^{n} = (\langle r_{2_i}, r_{2_j}\rangle)_{i,j=1}^{n} \in W_{m,e}(a_i^*) \cap W_{m,e}(b_i^*) \neq \phi$$

Rank one operators on Hilbert space

We recall that $\| T \| \leq \| a_j \otimes b_j \|_h$ in term of the Haagerup norm on $B(H) \otimes B(H)$. Equality of this upper bound holds in case the operators $a_i a_j^*$ commute and operators $b_j^* b_i$ also commute as shown by theorem 2.1.5. For $x, y \in H$ we use the notation $x, y^* \in H$ for the rank one operator on H with $(x, y^*)(\theta) = \langle \theta, y\rangle x$ which is a linear operator.
To show it's linearity we let $\alpha_1, \alpha_2 \in K$ and $x_1, x_2 \in H, y_1^*, y_2^* \in H^*$ then

$$\begin{aligned}
(\alpha_1 x_1 \otimes \alpha_1 y_1^*) + (\alpha_2 x_2 \otimes \alpha_2 y_2^*)(\theta) &= (\alpha_1 x_1 \otimes \alpha_1 y_1^*)(\theta) + (\alpha_2 x_2 \otimes \alpha_2 y_2^*)(\theta) \\
&= \alpha_1 (x_1 \otimes y_1^*) + \alpha_2 (x_2 \otimes y_2^*)(\theta) \\
&= \alpha_1 \langle \theta, y_1^*\rangle x_2 + \alpha_2 \langle \theta, y_2^*\rangle x_2
\end{aligned}$$

Therefore, the operator

$$(x \otimes y^*)(\theta) = \langle \theta, y \rangle x$$

is linear.

Lemma 2.1.11:

Let H be a Hilbert space and B(H) an algebra of bounded linear operators on H. For $T \in El(B(H)), T(x) = \Sigma_{i=1}^{n} a_i x b_i \forall a_i, b_i \in B(H)$, then

$$\| T \| = \sup_{p_1, p_2} \| \sum_{i=1}^{n} P_1 a_i \otimes b_i P_2 \|_h$$

where $P_1, P_2 \in b(H)$ are rank one projections $(P_i^2 = P_i = P_i^*)$ We will show that $\| T \|$ above is a norm.

- It's clear that $\| T \| = 0$ imply that $P_1 a_i \otimes b_i P_2 = 0$
- It's also clear that $\| T \| = 0 \Leftrightarrow (P_1 a_i) \otimes b_i P_2 = 0$
- To show that $\| \lambda T \| = | \lambda | \| T \|$. We let $\lambda \in \mathbf{K}$,therefore,

$$\begin{aligned}
\| \lambda T \| &= \sup_{p_1, p_2} \| \sum_{i=1}^{n} (\lambda P_1 a_i) \otimes (\lambda b_i P_2) \|_h \\
&= \sup_{p_1, p_2} \| \sum_{i=1}^{n} \lambda (P_1 a_i) \otimes \lambda (b_i P_2) \|_h \\
&= \sup_{p_1, p_2} \| \lambda \sum_{i=1}^{n} (P_1 a_i) \otimes (b_i P_2) \|_h \\
&= | \lambda | \sup_{p_1, p_2} \| \sum_{i=1}^{n} (P_1 a_i) \otimes (b_i P_2) \|_h \\
&= | \lambda | \| T \|
\end{aligned}$$

Thus, $\| \lambda T \| = | \lambda | \| T \|$

- (iv) Finally, we show that for $T, T' \in B(H) \otimes B(H)$, we have

$$\| T + T' \| \leq \| T \| + \| T' \|$$

i.e triangle inequality. Let

$$\| T \| = \sup_{p_1,p_2} \| \sum_{i=1}^{n} (P_1 a_{i_1}) \otimes (b_{i_2} P_2) \|_h$$

and

$$\| T' \| = \sup_{p_1,p_2} \| \sum_{i=1}^{n} (P_1 a_{i_2}) \otimes (b_{i_2} P_2) \|_h$$

It follows that

$$\begin{aligned} \| (T + T')(P_1 a_i)(b_i P_2) \| &= \| T(P_1 a_i) \otimes (b_i P_2) \| + \| T'(P_1 a_i) \otimes (b_i P_2) \| \\ &= \sup_{p_1,p_2} \{ \| \sum_{i=1}^{n} (P_1 a_{i_1}) \otimes (b_{i_1} P_2) + (P_1 a_{i_2}) \otimes (b_{i_2} P_2) \|_h \} \\ &\leq \sup_{p_1,p_2} \| \sum_{i=1}^{n} (P_1 a_{i_1}) \otimes (b_{i_1} P_2) \|_h + \\ &\quad sup_{p_1,p_2} \| \sum_{i=1}^{n} (P_1 a_{i_2}) \otimes (b_{i_2} P_2) \|_h \\ &= \| T \| + \| T' \| \end{aligned}$$

Therefore, $\| T + T' \| \leq \| T \| + \| T' \|$. This implies $\| T \| = \sup_{p_1,p_2} \| \Sigma_{i=1}^{n} P_1 a_i \otimes b_i P_2 \|_h$ is a norm.

Notation 2.1.12

Let H be a Hilbert space. For $u, v \in H$ we use the notation $u \otimes v$ for the rank one operator on H with the property that $(u \otimes v^*)(\theta) = \langle \theta, v \rangle u$. This specifically is a linear operator

$$T : x \longrightarrow \langle (Txu, v) \rangle v \otimes u^*$$

i.e

$$Tp_1, p_2(x) = \sum_{i=1}^{n} P_1 a_i(x) b_i P_2.$$

Linearity of this operator can easily seen; Let $\alpha, \beta \in \mathbb{K}$ and $x, y, \in H$.
It follows that,

$$\begin{aligned} \{\langle T(\alpha x + \beta y)u, v \rangle\} v \otimes v^* &= \{\langle (\alpha Tx + \beta Ty)u, v \rangle\} v \otimes v^* \\ &= \{\langle (\alpha Tx)u + (\beta Ty)u, v \rangle\} v \otimes v^* \\ &= \langle (\alpha Tx)u, v \rangle v \otimes v^* + \langle (\beta Ty)u, v \rangle v \otimes v^* \end{aligned}$$

Which gives the desired results. For this operator $(P_1 a_i)(P_1 a_j)^*$ are commuting and so are $(b_i P_2)^*(b_j P_2)$. To prove commutativity,we note that $T_{p_1}(x) = P_1 a_j(x)$ and $T^*_{P_1} = (P_1 a_j)^*(x) \forall x \in H$. Thus

$$\| T_{p_1}(x) \|^2 = \langle T_{p_1}(x), T_{p_1}(x) \rangle = \langle T^*_{p_1} T_{p_1}(x), (x) \rangle.$$

Since $P_1 \in B(H)$ and $P_1 = P_1^* = P^2$ (rank one projection operator), we have,

$$
\begin{aligned}
\| T_{p_1}(x) \|^2 &= \langle T_{P_1} T^*_{p_1}(x), (x)\rangle \\
&= \langle T^*_{P_1}(X), T^*_{p_1}(x)\rangle \\
&= \| T^*_{p_1}(x) \|^2 \\
\Rightarrow \langle T_{p_1}(x), T_{p_1}(x)\rangle &= \langle T^*_{P_1}(x), T^*_{P_1}(x)\rangle \\
&= \langle T_{P_1} T^*_{p_1}(x), (x)\rangle \\
&= \langle T^*_{P_1} T_{p_1}(x), (x)\rangle
\end{aligned}
$$

$T^*_{p_1} T_{p_1} = T_{p_1} T^*_{p_1}$ i.e $(P_1 a_j)(P_1 a_j)^* = (P_1 a_j)(P_1 a_j)$.

2.3 Concept of elementary operators on semigroups

Let H be a semigroup. A double centralizer on H is an ordered pair $(T, T'$ of maps of H into itself satisfying $x(Tx) = (T')y \forall x, y \in H$. Let us denote by $\mathcal{E}\ell(H)$ the set of all ordered pairs M, M^* of linear maps of H into itself that satisfy the following identities

$$M(x(M^*y)z) = (Mx)y(Mz) \tag{2.3.1}$$

$$M^*(x(My)z) = (M^*x)y(M^*z) \forall x, y, z \in H \tag{2.3.2}$$

We now state some basic properties of $\mathcal{E}\ell(\Omega)(H)$

(a) Defining multiplication by $(M, M^*)(N, N^*) = (MoN, N^*oM)$ (here o denote composition maps)$\mathcal{E}\ell(H)$ becomes a semigroup with identity element. Moreover, defining $(M, M^*)^* = (M^*, M), \mathcal{E}\ell(H)$ becomes a semigroup with involution.

(b) The set of all pairs (a M b,b M a) where $a, b \in H$ is a subsemigroup of $\mathcal{E}\ell(H)$. Now let Ω_1 and Ω_2 be algebras over a field $\mathbb{K}$. By $\mathcal{E}\ell(\Omega_1\Omega_2)$ we denote the set of all ordered pairs

(M, M^*) where $M : \Omega_1 \to \Omega_2$ and $M^* : \Omega_1 \to \Omega_2$ which are linear maps such that (2.5) holds for all $x, z \in \Omega_2$ and (2.6) holds for all $x, z \in \Omega_2, y \in \Omega_1$ Linearity of M, M^* can be easily shown. For, if we let $\alpha, \beta \in \mathbb{K}$ and $x_1x_2, z_1, z_2 \in \Omega_1 y_1 y_2 \in \Omega_2$ we have,

$$\begin{aligned} M\{\alpha(x_1(M^*y_1)z_1) + \beta(x_2(M^*y_2)z_2)\} &= M\alpha(x_1z_1) + M\beta(x_2(M^*y_2)z_2) \\ &= \alpha\{(Mx_1)y_1(Mz_1)\} + \beta\{Mx_2)y_2(Mz_2)\} \\ &= M\{\alpha(x_1(M^*y_1)z_1) + \beta(x_2(M^*y_2)z_2)\} \end{aligned}$$

hence linearity of $M : \Omega_1 \to \Omega_2$. Further, for

$$x_1, x_2, z_1, z_2 \in \Omega_2, y_1, y_2 \in \Omega_1$$

and $\alpha, \beta \in \mathbb{K}$. We have,

$$\begin{aligned} M^*\{\alpha(x_1(My_1)z_1) + \beta(x_2(My_2)z_2)\} &= M^*\alpha(x_1(My_1)z_1) + M^*\beta(x_2(My_2)z_2) \\ &= \alpha((M^*x_1)y_1(M^*z_1)) + \beta((M^*x_2)y_2(M^*z_2)) \\ &= \alpha M^*(x_1(My_1)z_1) + \beta M^*(x_2(My_2)z_2) \end{aligned}$$

Hence linearity of $M^* : \Omega_2 \to \Omega_1$

Further, $\mathcal{E}\ell\Omega_1\Omega_2$ denote the set of all pairs $(\Sigma_{i=1}^n M_i, \Sigma_{i=1}^n M_i^*)$ where $M_i, M_i^* \mathcal{E}\ell(\Omega_1\Omega_2)$. The elements of $\mathcal{E}\ell(\Omega_1\Omega_2)$ called elementary operators of an algebra $\Omega_1$to an algebra Ω_2. Considering every pair of the form (aMb, bMa) with $a, b \in \Omega_1$ we have that the pairs,

$$\left(\sum_{i=1}^{n} a_i M b_i, \sum_{i=1}^{n} b_i M a_i\right) \forall a_i, b_i \in \Omega_1$$

belonging to $\mathcal{E}\ell(\Omega_1)$. We can make the following observations concerning these sets.

(i) Defining addition and scalar multiplication by,

$$\begin{aligned}
\left(\sum_{i=1}^{n} M_i, \sum_{i=1}^{n} M_i^*\right) + \left(\sum_{j=1}^{n} N_j, \sum_{j=1}^{n} N_j^*\right) &= \left(\sum_{i=1}^{n} M_i + \sum_{j=1}^{n} N_j, \sum_{i=1}^{n} M_i^* + \sum_{j=1}^{n} N_j^*\right) \\
&= \lambda\left(\sum_{i=1}^{n} M_i, \sum_{i=1}^{n} M_i^*\right), \forall \lambda \in \mathbb{K} \\
&= \left(\sum_{i=1}^{n} \lambda M_i, \sum_{i=1}^{n} \lambda M_i^*\right)
\end{aligned}$$

then $\mathcal{E}\ell(\Omega_1, \Omega_2)$ becomes a vector space.

(ii) Defining multiplication by

$$\left(\sum_{i=1}^{n} M_i, \sum_{i=1}^{n} M_i^*\right)\left(\sum_{j=1}^{n} N_j, \sum_{j=1}^{n} N_j^*\right) = \sum_{i,j=1}^{n} M_i N_j, \sum_{j=i}^{n} N_j^* M_j^*$$

then $\mathcal{E}\ell(\Omega_1)$ becomes an algebra without identity element.

2.4 Elementary operators on semiprime algebra

Lemma 2.2.1

Let Ω_1 and Ω_2 be any algebras. Let $v \in \Omega_1, (M, M^*), (N, N_*) \in \varepsilon\iota\Omega_1\Omega_2$ and define $P:\Omega_1 \to \Omega_2, P_*:\Omega_2 \to \Omega_1$ by

$$Px = M(x(N^*)v) - (Mx)w(Nv), P^*x = -M^*(w(Nv)x) + (N^*w)v(M^*x)$$

respectively. Then $(P,P) \in \mathcal{E}\ell(\Omega_1\Omega_2)$

Proof. For $x, z \in \Omega_2$ we have,

$$\begin{aligned}
P(x(P^*y)z) &= P(-xM^*(w(Nv)y)z) + x(N^*w)v(M^*y)z \\
&= M(-xM^*(w(Nv)y)z(N^*w)v) + M(x(N^*w)v(M^*y)z(N^*w)v) + \\
&\quad M(xM^*(w(Nv)y)z)w(Nv) - M(x(N^*w)v(M^*y)z)w(Nv) \\
&= -(Mx)w(Nv)yM(z(N^*w)v) + M(x(N^*w)v)yM(z(N^*w)v) + \\
&\quad (Mx)w(Nv)y(Mz)w(Nv) - M(x(N^*w)v)y(Mz)w(Nv) \\
&= M(x(N^*w)v) - (Mx)w(Nv))y(M(z(N^*w)v) - (Mz)w(Nv)) \\
&= (Px)y(Pz)
\end{aligned}$$

Similarly,we have

$$
\begin{aligned}
P^*(Py)z &= P^*xM(y(N^*wu)z - x(My)w(Nu)z \\
&= -M^*(xM(y(N^*w)u)z(Nu)x) - M^*(x(My)w(Nu)z(Nu)x) \\
&\quad +M^*(xM(y(N^*w)u)z)u(N^*x) + M^*(x(My)w(Nu)z)u(N^*x) \\
&= -M^*(xy(N^*w))uM(z(Nu)x) - M^*(x(My)w)uN^*(z(Nu)x) \\
&\quad +(M^*x)y(N^*w)u(Mz)u(N^*x) + M^*(x(My)w)u(N^*z)u(N^*x) \\
&= M^*(x(Ny)w) + (M^*x)y(M^*w)u - N^*(z(Nu)x) - (N^*z)u(N^*) \\
&= (P^*x)y(P^*z) \\
\Rightarrow (P^*, P) &\in \mathcal{E}\ell(\Omega_1, \Omega_2)
\end{aligned}
$$

□

The following theorem shows that the sum of elementary operators and that of their adjoints are equivalent on semiprime algebras (see [2]). Let Ω_1 and Ω_2 be semiprime algebras.If $(M_i M_i^*) \in \mathcal{E}\ell(\Omega_1, \Omega_2), i = 1, \ldots\ldots, n$ and $\Sigma_{i=1}^{n} M_i = 0$, then $\Sigma_{i=1}^{n} M_i^* = 0$

2.5 Elementary operators on polynomial algebras

We shall determine elementary operators on the algebra $\mathbb{K}[X]$ of polynomials coefficients in a field $\mathbb{K}$.

Theorem 2.2.3

A linear operator $T \in \mathcal{E}\ell(\mathbb{K}[X])$ if and only if there exist $a(X) \in \mathbb{K}[X]$ and $a, \beta \in \mathbb{K}, \alpha \neq 0$,such that (i)$T(fX) = a(X)f(\alpha X + \beta)$ (ii)$T^*(f(X)) = a(\alpha^{-1}\alpha^{-1}\beta)f$. Therefore,every operator$T \in \mathcal{E}\ell(\mathbb{K}[X])$ is of the form $T(f(X)) =$

$a_1(X)f(\alpha_1 X+\beta_1)+ \ldots\ldots + a_s(X)f(\alpha_s X+\beta_s)$ where $a_i(X) \in \mathbb{K}[X], \alpha_s, \beta_s \in \mathbb{K}, \alpha_s \neq 0$.

proof(see [3])

Let$T \in \mathcal{E}\ell(\mathbb{K}[X])$ and set a$(X) = T(1)$, b$(X) = T^*(1)$, c$(X) = T(X)$. If $a(X) = 0$, then $T^*(f(X))^2$

$$= T^*(f(X) * a(X) * f(X)) = 0 \quad \forall f(X) \in \mathbb{K}[X],$$

so that $T = T^* = 0$. Therefore,there is no loss of generality in assuming that $a(X) \neq 0$,and similarly, $b(X \neq) = 0$. Given any $f(x) \in \mathbb{K}[X]$, we have $T(f(X)T^*(1)1) = T(f(X))1T(1)$. That is,

$$T(b(X)f(X)) = a(X)T(f(X)) \tag{2.5.1}$$

Clearly (2.7) implies that

$$T(b(X)^{n-1}X^n) = a(X)^{n-1}T(X^n) \tag{2.5.2}$$

for any positive integer n. We can claim that $T(b(X)^{n-1}X^n)$. Indeed, by definition,this is true when n=1, and assuming that it is also true for n-1,we obtain

$$T(b(X)^{n-1}X^n) = T(X * b(X) * b(X)^{n-2}X^{n-1}) = T(X) * 1 * T(b(X)^{n-2}X^{n-1})$$

$$= c(X) * c(X)^{n-1} = c(X)^n \tag{2.5.3}$$

Comparing equation (1.14) and (1.15) we arrive at

$$c(X)^n T(X^n) \tag{2.5.4}$$

for every positive integer n.Inparticular, $a(X)^{n-1}$ divides $c(X)^n, n = 2, 3, \ldots.$
We claim that this yields that a(X) divides c(X).Writing $c(X) = \alpha P_1(X)^{k_1} \ldots\ldots P_r(X)^{k_r}$,
where $\alpha \in \mathbb{K}$, P_i are irreducible, and first using the that a(X) divides $c(X)^2$,
it follows that

$$a(X) = \beta P_1(X)^{S_1} \ldots\ldots P_r(X)^{s_r}$$

where $\beta \in F$ and $0 \leq s_i \leq 2k_i$. However, since $a(X)^{n-1}$ divides $c(X^n$ for every positive integer n,we have that $(n-1)s_1 \leq nk_i$, which clearly gives $s_i \leq k_i$. Therefore,$c(X) = a(X)d(X)$ for some $d(X) \in \mathbb{K}[X]$.Now (1.16)gives $a(X)^{n-1}T(X)^n = c(X)^n = a(X)^n d(X^n)$, so that $T(X^n) = a(X)d(X)^n$.But then, since T is linear,

$$T(f(X)) = a(X)f(d(X)) \tag{2.5.5}$$

for any $f(X) \in \mathbb{K}[X]$.By symmetry, there is $d^* \in \mathbb{K}[X]$ such that

$$T^*(f(X)) = b(X)f(d^*(X)) \tag{2.5.6}$$

for any $f(X) \in \mathbb{K}[X]$. Consider $T(T^*(X))$. On the one hand we have

$$T(T^*(X)) = T(1 * T^*(X).1) = T(1)XT(1) = A(X)^2X, \tag{2.5.7}$$

and on the hand, using (2.10,2.11and2.12),respectively we get

$$T(T^*) = T(b(X)d^*(X)) = a(X)T(d^*(X))$$

$$= a(X)^2 d^*(d(X)) \tag{2.5.8}$$

Comparing (2.14)and (2.15) we have $d^*(d(X)) = X$. By symmetry ,$d(d^*(X)) = X$.Thereforeby computation we have$d(X) = \alpha X + \beta$ and $d^*(X) = \alpha^{-1}\alpha^{-1}\beta$

for $\alpha, \beta \in \mathbb{K}, \alpha \neq 0$. Finally,from $b(X)^2 = T^*(1 * T(1) * 1) = T^*(a(X))$

$$= b(X)a(\alpha^{-1}X - \alpha^{-1}\beta).$$

We conclude that

$$b(X) = a(\alpha^{-1}X - \alpha^{-1}\beta).$$

Thus we have proved indeed that every $T \in \varepsilon\ell(F[X])$ is of the form

$$T(f(X)) = a(X)f(\alpha X + \beta),$$

with

$$T^*(f(X)) = a(\alpha^{-1}X - \alpha^{-1}\beta)f(\alpha^{-1}X - \alpha^{-1}\beta)$$

Chapter 3

Elementary operators on prime C^*-algebra

3.1 Introduction

In this chapter we establish the norm of elementary operator on prime C^* -algebra

Theorem 3.1.0: Let Ω prime C^* -algebra, therefore

$$\| M_{T,T'} \|=\| T \|\| T' \| \, \forall T, T' \in M(\Omega)$$

Proof. For $T, T('), X \in M(\Omega)$, we have that

$$M_{T,T'}X = TXT'.$$

Therefore,

$$\| M_{T,T'}X \|=\| TXT' \|\leq\| T \|\| X \|\| T' \|$$

Taking sup on both sides we have,

$$\| M_{T,T'} \|\leq\| T \|\| T' \|$$

Consequently,

$$\| M_{T,T'} \| \leq \| T \| \| T' \| . \qquad (3.1.1)$$

If $T, T' \in M(\Omega)$ are arbitrary, the inequality

$$\begin{aligned} \| T \|^2 \| T' \|^2 &= \| T^*T \| \| T'T'^* \| \\ &= \| M_{T^*T,T'T'^*} \| M_{T,T'} \| \\ &\leq \| M_{T^*T'^*} \| \| \| \\ &\leq \| T \| \| T' \| \| M_{T,T'} \| \end{aligned}$$

i.e. $\| T \|^2 \| T' \|^2 \leq \| T \| \| T' \| \| M_{T,T'} \|$. Dividing both sides by $\| T \| \| T' \|$, we have

$$\| M_{T,T'} \| \geq \| T \| \| T' \| \qquad (3.1.2)$$

Comparing equations(3.1)and(3.2), we have

$$\| M_{T,T'} \| = \| T \| \| T' \| \; \forall T, T' \in M(\Omega).$$

□

3.2 Elementary operators and the maximal numerical range

Definition 3.2.0: Let H be an Hilbert space and B(H) an algebra of bounded linear operators on Hilbert space. The Maximal numerical range, $W_o(T)$ of an operator $T \in B(H)$ is the set,

$$W_o(T) = \{\lambda : \langle Tx_n, x_n \rangle \to \lambda, \| x_n \| = 1 \quad and \quad \| Tx_n \| \to \| T \|\}$$

and the normalized maximal numerical range, $W_N(A)$ of the operator A to be the set $W_o(A/\parallel A\parallel) \quad for \quad A \neq 0$

Lemma 3.2.1

If $\parallel T\parallel=\parallel x\parallel=1 \quad and \quad \parallel Tx\parallel^2<(1-\epsilon) \quad then \quad \parallel (T^*T-1)x\parallel\leq 2\epsilon$

Proof.

$$\begin{aligned} 0 &\leq \parallel (T^*T-I)x\parallel^2 \\ &= \langle (T^*T-I)x,(T^*T-I)x\rangle\langle T^*Tx-x,T^*Tx-x\rangle \\ &= \langle T^*Tx,T^*Tx\rangle-\langle T^*Tx,x\rangle-\langle x,T^*Tx\rangle+\langle x,x\rangle \\ &= \langle T^*Tx,T^*Tx\rangle-2\langle Tx,Tx\rangle+\langle x,x\rangle \\ &= \parallel T^*Tx\parallel^2-2\parallel Tx\parallel^2+\parallel x\parallel^2\leq(1-\parallel Tx\parallel)^2\leq 2\epsilon \end{aligned}$$

□

Lemma 3.2.3

The set $W_o(T)$ is convex.

Proof(See [9])

Let $\lambda,\mu\in W_o(T)$.We assume without loss of generality that $\parallel T\parallel=1$. Assume also that,

$$\parallel x_n\parallel=\parallel y_n\parallel=1,\langle Tx_n,x_n\rangle\to\lambda$$

and

$$\langle Ty_n,y_n\rangle\to\mu.$$

Consider, $T_n = P_nTP_n$, Where P_n is the projection on H of x_n,y_n. Let ξ be a point on the line segment joining λ and μ. Then for each n, it is possible, by the Toeplitz-Hausdorff Theorem to choose $\alpha_n\beta_n$ such that

$\langle Tu_n, u_n\rangle = \langle T_n u_n, u_n\rangle \longrightarrow \xi$ and $\| u_n \|= 1$, where $u_n = \alpha_n x_n + \beta_n y_n$. Note that $|\langle x_n, y-n\rangle| \leq \theta < 1$, for n sufficiently large; that is, the angle between x_n and $y-n$ is bounded away from 0. Thus, there exist a constant M such that $|\alpha_n| \leq M$ and $|\beta_n| \leq M$ for large n,where$\| \alpha_n x_n + \beta_n y_n \|= 1$. By lemma3.2.1, $\| Tu_n \|^2 = \langle T^*Tu_n, u_n\rangle = \langle Tu_n, Tu_n\rangle =\| Tu_n \|^2 \| u \|^2 - 2m\epsilon_n$, where $\epsilon_n \longrightarrow 0$, and thus it follows that $\| Tu_n \| \longrightarrow 1$, since $\langle Tu_n, u_n\rangle \longrightarrow \xi$ Now, for $T, T' \in B(H)$, we define elementary operators as $M_{T,T'}(x) = TxT', (x \in B(H))$. We shall give necessary and sufficient conditions for any pair of operators T and T' to satisfy the equation $\| I + M_{T,T'} \|= 1 + \| T \| T' \|$ where I is the identity operator on H.

Theorem 3.2.4:

For $T, T' \in B(H)$ the following are equivalent

(a) $\| I + M_{T,T'} \|= 1 + \| T \|\| T' \|$

(b) $W_N(T^*) \cap W_N(T') \neq \phi$

Proof. See([12] for part of the proof.) $(a) \Longrightarrow (b)$, suppose that $\| I + M_{T,T'} \|= 1 + \| T \|\| T' \|$, then we can find two sequences $\{X_n \subseteq B(H)\}$ and $\{x_n \subseteq H\}$ with $\| X_n \|=\| x_n \|= 1$ for each n:

$$
\begin{aligned}
\lim_n \| X_n x_n + TX_nT'x_n \|^2 &= \lim_n \| (X_n + TX_nT')x_n \|^2 \\
&= \lim_n \langle X_n x_n + TX_nT'x_n, X_n x_n + TX_nT'x_n\rangle \\
&= \lim_n \langle X_n x_n, X_n x_n\rangle + \langle X_n x_n, TX_nT'x_n\rangle \\
&= +\langle TX_nT'x_n, X_n x_n\rangle + \langle TX_nT'x_n, TX_nT'x_n\rangle \\
&= \| X_n x_n, X_n \|^2 + 2\mathbb{R}\langle X_n x_n, TX_nT'x_n\rangle + \| TX_nT'x_n \|^2 \\
&= 1 + \| T \|\| T' \|
\end{aligned}
$$

Since

$$\| X_n x_n + T X_n T' x_n \| \leq \| X_n x_n \| + \| T X_n T' x_n \|$$

It follows that $\lim_n \| T X_n T' x_n \| = \| T \| \| T' \|$.
Consequently, we derive that $\lim_n \langle X_n x_n, T X_n T' x_n \rangle = \| T \| \| T' \|$. Thus

$$\lim_n \| T^* X_n x_n \| = \| T \| \quad and \quad \lim_n \| X_n T' x_n \| = \| T' \|$$

because

$$| \langle X_n x_n, T X_n T' x_n \rangle | \leq \| \lim_n \| T^* X_n x_n \| \| X_n T' x_n \|$$

For each $n \geq 1$ we have, $\| M_{T^*,-T'} \| \geq \| T^* X_n + X_n T' \| \geq \| T^* X_n x_n + X_n T' x_n \|$, since $\lim_n \| T^* X_n x_n + X_n T' x_n \| = \| T \| + \| T' \|$ and $\| M_{T^*,-T'} \| \leq \| T \| + \| T' \|$.

We conclude that $\| M_{T^*,-T'} \| = \| T \| + \| T' \|$. Thus it follows from Stampfli [23], that $W_N(T^*) \cap W_N(T') \neq \phi$ $(b) \to (a)$ (This solution is found in Stampfli [23]). Let $\mu \in W_N(T^*) \cap W_N(T')$. Then there exist two sequences x_n and y_n in H such that $\| x_n \| = \| y_n \| = 1$, $\lim_n \| T^* x_n \| = \| T \|$, $\lim_n \| T' y_n \| = \| T \|$, $\lim_n \langle T^* x_n, x_n \rangle = \mu \| T \|$ $\lim_n \langle T' y_n, y_n \rangle = \mu \| T' \|$.

Set $T^* x_n = \alpha_n x_n + \beta_n u_n$, where $\alpha_n, \beta_n \in \mathbb{K}$, $U_n \in H$ with $u_n = 1$ and $\langle x_n, u_n \rangle = 0$. We may choose u_n so that $\langle T^* x_n, u_n \rangle = \beta_n \geq 0 \; \forall \; n$.

Set also $T' y_n = \gamma_n y_n + \delta_n x_n$ where $\gamma_n, \delta_n \in \mathbb{K}$, $\| v_n \| = 1$, $\langle y_n, v_n \rangle = 0$ and $\langle T' y_n, v_n \rangle = \delta_n \geq 0$.

Define a sequence $\{X_n\}_n \subseteq B(H)$ by $X_n = \langle *, y_n \rangle x_n + \langle *, u_n \rangle u_n$. Then clearly $\| X_n \| = 1 \; \forall \; n$, we have $\langle X_n y_n, T X_n T' y_n \rangle = \langle T^* y_n, \gamma_n T y_n + \delta_n u_n \rangle = \alpha_n \gamma_n + \beta_n \delta_n$.
By the definition of the sequences x_n and y_n, we derive that $\lim_n |\alpha_n|^2 + |\beta_n|^2 = \| T \|^2$ and $\lim_n |\alpha_n| = |\mu| \; \| T \|$.

Thus $\lim_n \beta_n = \sqrt{1 - |\mu|^2} \; \| T \|$. In a similar way we obtain,

$\lim_n \delta_n = \sqrt{1-|\mu|^2} \parallel T' \parallel$. Hence,

$$\begin{aligned}
\lim_n \langle X_n y_n, T X_n y_n \rangle &= \lim_n \alpha_n \gamma_n + \beta_n \delta_n \\
&= |\mu|^2 \parallel T \parallel \parallel T' \parallel + (1-|\mu|^2) \parallel T \parallel \parallel T' \parallel \\
&= \parallel T \parallel \parallel T' \parallel
\end{aligned}$$

From this we conclude that $\lim_n \parallel T X_n T' y_n \parallel = \parallel T \parallel \parallel T' \parallel$. Now, we have for each $n \geq 1$,

$$\begin{aligned}
1 + \parallel T \parallel \parallel T' \parallel &\geq \parallel 1 + M_{T,T'} \parallel \\
&\geq \parallel X_n + T X_n T' \parallel \\
&= \parallel X_n y_n + T X_n T' y_n \parallel
\end{aligned}$$

Therefore,

$$\begin{aligned}
\parallel X_n y_n + T X_n T' y_n \parallel &= 1 + \parallel T \parallel \parallel T' \parallel \\
&\leq \parallel 1 + M_{T,T'} \parallel \\
&\leq 1 + \parallel T \parallel \parallel T' \parallel
\end{aligned}$$

Consequently,

$$\parallel 1 + M_{T,T'} \parallel = 1 + \parallel T \parallel \parallel T' \parallel .$$

□

Chapter 4

CONCLUSION

4.1 Concluding remarks.

The study of norms of elementary operators has emerged that,for general elementary operators, a full description of their properties is rather intricate since these are often interwoven with the structure of underlying algebra.
In our thesis, we used the theory of tensor norms of Hilbert spaces and tensor product of operators to work out the lower bound of elementary operators.
The study also investigated elementary operators on prime C^* -algebra. Under both circumstances, the value of the constant C i.e. the greatest lower bound of norms of elementary operators, and for any arbitrary k (where $k = 1, 2, \ldots, n$) is less than one.
There are still many results of elementary operators in terms of multiplication operators or tensor products of operators which other researchers can pursue.

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