ON A GENERALIZED q-NUMERICAL RANGE

by

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A thesis submitted in partial fulfilment of the requirements for

the award of the degree of

MASTER OF SCIENCE IN PURE MATHEMATICS

in the

Faculty of Science

MASENO UNIVERSITY

Maseno

2008

ABSTRACT

We consider numerical ranges of a bounded linear operator on complex Hilbert spaces. Many properties of the classical numerical range are known. We investigate the properties of the q-numerical range in relation to those of the classical numerical range. We also establish the relationship between the q-numerical range and the algebra q-numerical range. Furthermore, we extend the results of the classical numerical range and q-numerical range to the C-numerical range and investigate how the C-numerical range is an explicit generalization of both the classical numerical range and q-numerical range.

Chapter 1

INTRODUCTION

During the last decades, the study of numerical range $W(T)$ has attracted attention of many Mathematicians and several results have been obtained. These results are helpful in studying and understanding matrices and operators. Among the main results is the convexity of the numerical range as established in the classical Toeplitz-Hausdorff theorem [9, 17]. This set function also has various generalizations. In our thesis we considered finite dimensional linear maps on normed spaces, primarily we were concerned with a generalized q-numerical range of a bounded linear operator in complex Hilbert spaces.

The first chapter is composed of the basic results which are used in the subsequent chapters. Here we also present terminologies and symbols. Some generalizations of the numerical range are also mentioned.

In chapter two we investigate the properties of the q-numerical range. We show that some properties that hold for the classical numerical range also hold for q-numerical range, that is, non-emptiness, closedness, convexity among others. We also look at the relationship between the q-numerical range and the algebraic q-numerical range.

In chapter three, we extend some results of the classical numerical range and q-numerical range to the C-numerical range. Finally, we characterize the C-numerical range as a generalization of both the classical numerical range and q-numerical range.

1.1 Background Information

We first introduce some essential concepts involving definitions and other useful notions used in the sequel.

Definition 1.1.1: An operator.

Is a mapping of a vector space X onto itself or to another vector space.

Definition 1.1.2: A linear operator.

Let X and Y be linear spaces. Then a function $T : X \to Y$ is called a linear operator if and only if for all $x_1, x_2 \in X$ and all scalars $\lambda, \mu \in \mathbf{K}$ we have

$$T(\lambda x_1 + \mu x_2) = \lambda T(x_1) + \mu T(x_2).$$

Remark 1.1.3:

Operators on normed spaces, which are linear and continuous, are of special interest in this study. The continuity is understood to be the metric continuity given by the norm. Thus $T : X \to Y$ is continuous at $x_o \in X$ if for every $\epsilon > 0$ there exists $\delta(x_o, \epsilon)$ such that

$$||T(x) - T(x_o)|| < \epsilon \text{ whenever } ||x - x_o|| < \delta.$$

Definition 1.1.4: A bounded linear operator.

A linear operator $T : X \to Y$ is called bounded if and only if there exists a constant $M > 0$ such that,

$$||T(x)|| \leq M||(x)|| \quad \forall x \in X.$$

Definition 1.1.5: Norm of a bounded operator.

Let $T \in B(X, Y)$. Then the norm of T is defined as

$$||T|| = \{sup||Tx||; x \in \mathbf{D}(T), ||x|| < 1\} = \{sup||Tx||; x \in \mathbf{D}(T), x \neq 0\} < \infty.$$

That the supremum is finite follows from the fact that

$$||T(x)|| \leq M||(x)|| \quad \forall x \in X, \ M \geq O.$$

Definition 1.1.6: Inverse operator (T^{-1}) .

Let X and Y be vector spaces, $T : \mathbf{D}(T) \subset X \to \mathbf{R}(T) \subseteq Y$ a linear operator, then the inverse operator (T^{-1}) of T is the mapping

$$T^{-1} : \mathbf{R}(T) \to \mathbf{D}(T).$$

Theorem 1.1.7: Banach Inverse Theorem.

Let X and Y be Banach spaces and $T \in B(X, Y)$ which is bijection. Then there exists $T^{-1} \in B(Y, X)$.

Proof. [(10), pp 88-101].

Definition 1.1.8: A functional.

A functional is a mapping of a vector space into a scalar $\mathbf{K}(\mathbf{C},\mathbf{R})$.

Definition 1.1.9: A Linear functional.

f is a linear functional on X if $f : X \to \mathbf{C}$ is a linear operator, i.e. a linear functional is a complex-valued linear operator.

Definition 1.1.10: A bounded Linear functional.

A linear functional f is called bounded if and only if there exists a constant $N > 0$ such that,

$$|f(x)| \leq N||(x)|| \quad \forall x \in X.$$

Definition 1.1.11: Norm of a bounded Linear functional.

Let $f : X \to \mathbf{R}\, or\ \mathbf{C}$ be a bounded linear operator on X. Then the norm of f is defined as

$$||f|| = sup\left\{\frac{|f(x)|}{||x||} x \neq 0\right\}$$

Definition 1.1.12: The Dual space.

Let X be a vector space and X^* the set of all linear functionals on X. X^* is called the dual space of X.

Remark 1.1.13:

The dual space X^* of a normed space X is a Banach space whether or not X is a Banach space.

Definition 1.1.14: An algebra.

An algebra A over X is a linear space together with an internal multiplication of elements of A, such that
$\forall x, y, z \in A$;

(i) $x(yz) = (xy)z$

(ii) $x(y+z) = xy + xz : (y+z)x = yx + zx$

(iii) $\lambda(xy) = (\lambda x)y = x(\lambda y),\ \lambda \in \mathbf{K}$

Definition 1.1.15: A normed algebra.

A normed linear space $(A, ||.||)$ over $\mathbf{K}$ is said tobe a normal algebra if A is an algebra and

$$||xy|| \leq ||x||||y||,\ \forall x, y \in A$$

Definition 1.1.16: An Involution.

Let A be an algebra, a mapping $A \to A$ defined by $x \to x^*$ is called *an involution* on A if it satisfies the following conditions: $\forall x, y \in A,\ \lambda \in \mathbf{K}$

(i) $(x+y)^* = x^* + y^*$

(ii) $(\lambda x)^* = \bar{\lambda} x^*$

(iii) $(xy)^* = y^* x^*$

(iv) $x^{**} = x$

Definition 1.1.17: A positive linear functional.

A positive linear functional is a linear functional on a Banach algebra A with an involution that satisfies the condition

$$f(xx^*) \geq 0 \quad \forall x \in A.$$

Definition 1.1.18: A state.

A state on an algebra A, is a continous positive linear functional that satisfies the Schwartz inequality,

$$|f(x^*y)|^2 \leq f(x^*x)f(y^*y).$$

Definition 1.1.19: Inner product space.

An inner product space X is a complex linear space together with an inner product $\langle , \rangle : X \times X \to \mathbf{C}$ such that;

(i) $\langle x, y\rangle = \overline{\langle y, x\rangle}$

(ii) $\langle \lambda x + \mu y, z\rangle = \lambda\langle x, z\rangle + \mu\langle x, z\rangle : x, y, z \in X, \lambda \in \mathbf{K}$

(iii) $\langle x, x\rangle \geq 0$, with $\langle x, x\rangle = 0 \Rightarrow x = 0$.

Definition 1.1.20: Hilbert space.

A Hilbert space is a complete inner product space i.e a Banach space whose norm is generated by an inner product.

Definition 1.1.21: Hermitian operator.

Let H be a Hilbert space and $\mathbf{D}$ a linear manifold of H. The mapping $T : \mathbf{D} \to H$ is said to be Hermitian if

$$\langle Tx, y\rangle = \langle x, Ty\rangle \quad \forall\ x, y \in \mathbf{D}.$$

Definition 1.1.22: Adjoint.

If $T \in B(\mathcal{Y}, \mathcal{K})$, where $\mathcal{Y}$, $\mathcal{K}$ are Hilbert spaces, then the unique linear operator $T^* \in B(\mathcal{K}, \mathcal{Y})$ satisfying

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \quad \forall\ x \in \mathcal{Y}\ and\ y \in \mathcal{K}$$

is called the *(Hilbert space) Adjoint* of T.

Definition 1.1.23: Self-adjoint operator.

A bounded operator $T \in B(H)$ is said to be self-adjoint if $T^* = T$.
Thus T is Hermitian and $\mathbf{D}(T) = H$ if and only if T is self-adjoint.

Proposition 1.1.24:

Let $T \in B(H)$ where H is a complex Hilbert space. Then the following statements are equivalent.

(i) T is self-adjoint, i.e $T = T^*$.

(ii) $\langle Tx, x \rangle$ is a real number, for all x $\in H$.

Proof. [(10), pp 400].

Definition 1.1.25: Normal operator.

A bounded linear operator T on a Hilbert space H is said to be normal if it commutes with its adjoint i.e $TT^* = T^*T$.

Definition 1.1.26: Unitary operator.

A unitary operator is a bounded linear operator U on a Hilbert space satisfying: $U^*U = UU^* = I$, where I is the identity operator.
This property is equivalent to the following:

(i) U preserves inner product on the Hilbert space, so that for all vectors x and y in the Hilbert space H,

$$\langle Ux, Uy \rangle = \langle x, y \rangle.$$

(ii) U is a surjective isometry (distance preserving map) i.e

$$\|U(x-y)\| = \|(x-y)\|.$$

Definition 1.1.27: Compact operator.

If H is a Hilbert space, then an operator $T \in B(H)$ is finite rank operator if the dimension of the range of T is finite and compact operator if for every bounded sequence (x_n) in H the sequence (Tx_n) contains a convergent subsequence.

Definition 1.1.28: Orthogonal compliment.

Let X be a vector space, Y a closed subspace of X, the orthogonal compliment of the subspace Y denoted by $Y^{\perp}$ is defined as

$$Y^{\perp} = \{x \in X : x \perp Y\}.$$

Definition 1.1.29: Projection operator.

The concept of a projection operator P or briefly, projection P is defined on a Hilbert space H where H is represented as the direct sum of a closed subspace Y and its orthogonal compliment $Y^{\perp}$ thus

$$H = Y \bigoplus Y^{\perp}$$

$$x = y + z, \qquad (y \in Y, z \in Y^{\perp}) \tag{1.1}$$

Since the sum is direct, y is unique for any given $x \in H$. Hence (1.1) defines a linear operator

$$P : H \to H$$

$$x \mapsto y = Px$$

P is called an orthogonal projection or projection on H onto Y. Hence a linear operator $P : H \to H$ is a projection on H if there is a closed subspace Y of H such that Y is the range of P and $Y^{\perp}$ is the null space of P and $P/_{Y}$ is the identity operator on Y.

Note that in (1.1) we can now write

$$x = y + z = Px + (1 - P)x.$$

This shows that the projection on H onto $Y^{\perp}$ is $I - P$.

There is another characterization of a projection on H, which is sometimes used as a definition:

Theorem 1.1.30: (Projection).

A bounded linear operator $P : H \to H$ on a Hilbert space H is a projection if and only if P is self-adjoint ($P^* = P$) and idempotent ($P^2 = P$)

Proof. [(10), pp 408].

Definition 1.1.31: (A positivity).

A projection P is said to be positive if

$$\langle Px, x\rangle \geq 0 \quad \forall x \in H.$$

Theorem 1.1.32: (Positivity, norm).

For any projection P on a Hilbert space H.

(i) $\langle Px, x\rangle = ||Px||^2$.

(ii) $P \geq 0$.

(iii) $\|P\| \leq 1; \quad \|P\| = 1 \quad \text{if} \quad P(H) \neq \{0\}$.

Proof. [(10), pp 410-411].

Theorem 1.1.33: (The product of projections).
In connection with products (composites)of projections on a Hilbert space H,the following two statements hold.

(a) $P = P_1P_2$ is a projection on H if and only if the projections P_1 and P_2 commute, that is, $P_1P_2 = P_2P_1$. Then P projects H onto $Y = Y_1 \cap Y_2$, where $Y_j = P_j(H)$.

(b) Two closed subspaces Y and V of H are orthogonal if and only if the corresponding projections satisfy $P_YP_V = 0$.

Proof. [(15), pp 414].

Theorem 1.1.34: (Sum of projections).
Let P_1 and P_2 be projections on a Hilbert space H. Then

(a) The sum $P = P_1 + P_2$ is a projection on H if and only if $Y_1 = P_1(H)$ and $Y_2 = P_2(H)$ are orthogonal.

(b) If $P = P_1 + P_2$ is a projection, P projects H onto $Y = Y_1 \bigoplus Y_2$.

Proof. [(10), pp 417].

Corollary 1.1.35: (Sum of finite number of projections).
If H is a Hilbert space, $(M_i)_{i=1}^{n}$ are closed subspaces of H and $(P_i)_{i=1}^{n}$ are

projections onto closed subspaces, then

$$\sum_{i=1}^{n} P_i = I. \tag{1.2}$$

if and only if $(M_i)_{i=1}^{n}$ are pairwise orthogonal and span H, that is, if and only if for all $x \in H$ has a unique representation

$$x = x_1 + \cdots + x_n \textit{ where } x_i \in M_i \quad \forall i = 1, 2, \ldots, n.$$

This shows that the sum of a finite number of projections onto pairwise orthogonal subspaces is itself a projection.

Definition 1.1.36:

Let H be a Hilbert space and $T : H \to H$ be a linear operator.

(i) The subset

$$\gamma(T) = \{\lambda \in \mathbf{C} : \lambda I - T \quad \text{is not invertible}\}$$

is called the **spectrum** of T.

(ii) The compliment $\mathbf{C} - \gamma(T)$ is called the **resolvent** set of T.

(iii) The number

$$r(T) = sup\{|\lambda| : \lambda \in \gamma(T)\}$$

is called the **spectral radius** of T.

(iv) A number $\lambda \in \mathbf{C}$ is called the **eigenvalue** of T if there is a non-zero $x \in H$ such that $Tx = \lambda x$; the vector x is then called an **eigenvector** for T corresponding to the eigenvalue λ.

Convex sets:

Definition 1.1.37: Convex set.

Let X be a linear space. A subset M of the linear space X is convex if for all $x, y \in M$, and for any positive real number t satisfying $0 < t < 1$ we have

$$tx + (1-t)y \in M.$$

Definition 1.1.38: Convex hull.

If M is a subset of a linear space X, then a convex hull M, represented by $conv(M)$ is the smallest convex subset of X containing M and it is the intersection of all the convex subsets of X that contain M.

Remark 1.1.39:

The intersection of any convex subsets of X is also convex.

Definition 1.1.40: Span of M. Let M be a non-void subset of a linear space $(X, \mathbf{K})$. The set of all linear combinations of elements of M is called the space spanned by M and is represented by $[M]$. That is

$$[M] = \{\alpha_1 x_1 + \cdots + \alpha_n x_n\} : n \in \mathrm{N},\ x_i \in M \ and\ \alpha_i \in \mathbf{K} \quad i = 1, \ldots, n.$$

Special types of matrices:

Definition 1.1.41: Diagonal matrix.

The matrix $D = [d_{ij}] \in M_n$ is called *diagonal* if $d_{ij} = 0$ whenever $j \neq i$. Conventionally, we denote such a matrix as $D = diag(d_{11}, \cdots d_{nn})$ or $D =$diag d, where d is the vector of diagonal entries of D.

Remark 1.1.45:

Neither the unitarily matrix U nor the triangular matrix T of the above theorem is unique.

Definition 1.1.46: Unitarily diagonalizable matrix.

If $A \in M_n$ is unitarily equivalent to a diagonal matrix, then A is said to be unitarily diagonalizable.

Theorem 1.1.47: (The spectral theorem for normal matrices).

If $A = [a_{ij}] \in M_n$, has eigenvalues $\lambda_1, \ldots, \lambda_n$, then the following statements are equivalent. $\forall i, j = 1, 2, \ldots, n$.

(a) A is normal.

(b) A is unitarily diagonalizable.

(c) $\sum_{i,j=1}^{n} |a_{ij}|^2 = \sum_{i=1}^{n} |\lambda_i|^2$.

(d) There is an orthonormal set of n eigenvectors of A.

Proof. [(13), pp 101].

Theorem 1.1.48: (The spectral theorem for Hermitian matrices).

Let $A \in M_n$ be given. Then A is Hermitian if and only if there is a unitary matrix $U \in M_n$ and a real diagonal matrix $\Lambda \in M_n$ such that $A = U^* \Lambda U$. Moreover, A is real and Hermitian if and only if there is a real orthogonal matrix $P \in M_n$ and a real diagonal matrix $\Lambda \in M_n$ such that $A = P^* \Lambda P$.

Proof. [(13), pp 172-173].

Definition 1.1.49: Trace.

Let $A = (\alpha_{jk}) \forall j, k = 1, 2, \ldots, n$, be an n-rowed square matrix. Then the sum of its eigenvalues equals to the trace of A, that is, the sum of the elements of the principal diagonal:

$$\sum_{i=1}^{n} \lambda_i = trace\, A = \alpha_{11} + \cdots + \alpha_{nn}.$$

Definition 1.1.50: Rank.

The rank of a matrix A is defined as the order of the largest square array in A with a nonzero determinant.

1.2 Numerical Ranges

1.2.1 The Classical Numerical Range.

Let H be a complex Hilbert space, $T : H \to H$ a bounded linear operator and $B(H)$ the set of bounded linear operators on H. For any $T \in B(H)$, the set $W(T)$ given by

$$W(T) = \{\langle Tx, x\rangle : x \in H, \|x\| = 1\} \quad (1.3)$$

is the *classical numerical range* of the operator T.

In the case of a matrix $A \in C^{n\times n}$ which represents a linear operator from $\mathbf{C}^n$ to $\mathbf{C}^n$, the numerical range can be thought of as the image of the surface of the Euclidian unit ball in $\mathbf{C}^n$ (a compact set) under the continous transformation $x \mapsto f(Ax)$ i.e

$$W(A) = \{f(Ax) : x \in \mathbf{C}^n,\ f \in (\mathbf{C}^n)^*, \quad f(x) = \|x\| = 1 = \|f\|\} \quad (1.4)$$

f is defined based on the consequences of the extension form of the Hahn-Banach Theorem i.e

Theorem 1.2.1: (Extension form of the Hahn-Banach Theorem).

Let X be a real vector space, M a subspace of X and let p be a seminorm on X. Suppose f is a complex-valued linear functional on M such that

$$|f(x)| \leq p(x)\, \forall x \in M$$

Then $\exists$ a bounded linear functional F on X for which

(i) $|F(x)| \leq f(x) \quad \forall x \in X$.

(ii) $F(x) = f(x) \quad \forall x \in M$

In other words, $\exists$ an extension F of f having the same property of f
Proof. [(11), pp 136].

Corollary 1.2.2 : Let w be a non-zero vector in a normed space X. Then there exists a continous linear functional F, defined on the entire space X, such that $||F|| = 1$ and $F(w) = ||w||$.
Proof. [(15), pp 150].

Corollary 1.2.3: If X is a normed space such that $F(w) = 0 \quad \forall F \in X^*$, then $w = 0$.
Proof. [(15), pp 151].

Corollary 1.2.4: Let X be a normed space and M be its closed subspace. Further, assume that $w \in X - M$. Then there exists $F \in X^*$ such that $||F|| = 1$ and $F(w) = ||w|| = 1$.
Proof. [(15), pp 151].

1.2.1. Properties of the classical numerical range $W(T)$:
For any $T \in B(H)$;

(i) $W(T)$ is non-empty.

(ii) $W(U^*TU) = W(T)$, U is unitary operator on H.

(iii) $W(T)$ lies in the closed disc of radius $||T||$ centered at the origin.

(iv) $W(T)$ contains all the eigenvalues of T i.e $\lambda \in W(T)$.

(v) $W(T^*) = \{\bar{\lambda} : \lambda \in W(T)\}$.

(vi) $W(I) = \{1\}$, I is the identity of $B(H)$.

(vii) $W(\alpha T + \beta I) = \alpha W(T) + \beta \quad \alpha, \beta, \in \mathbf{K}$.

(viii) $W(T)$ is a convex set (The Toeplitz-Hausdorff Theorem).

Proof. [14].

1.2.2 Other forms of Numerical ranges.

The q-Numerical Range.

For any $T \in B(H)$, the q-numerical range is defined

$$W_q(T) = \{\langle Tx, y\rangle : x, y \in H, \langle x, x\rangle = \langle y, y\rangle = 1, \langle x, y\rangle = q\} \tag{1.5}$$

for $q \in \mathbf{C}$ satisfying $|q| \leq 1$.

The Maximal Numerical Range and the δ-Numerical Range.

These types of numerical range were introduced by Stampfli [16]. For any $T \in B(H)$, the maximal numerical range is defined as

$$W_o(T) = \{\lambda : \langle Tx_n, x_n\rangle \to \lambda,\ \|x_n\| = 1, \quad \text{and } \|Tx_n\| \to \|T\|\}. \tag{1.6}$$

Whereas the δ-numerical range is given by

$$W_\delta(T) = \text{closure}\,\{\langle Tx, x\rangle : x \in H, \|x\| = 1, \|Tx\| \geq \delta\}\,. \tag{1.7}$$

However, by considering a matrix $\mathbf{A} \in \mathbf{C}^{n\times n}$, Chi-Kwong Li [4] defines these numerical ranges as

$$W_o(T) = \{x^*Ax :\ x \in C^n, x^*x = 1,\ \|Ax\| = \|A\|\|x\|\}\,. \tag{1.8}$$

and δ-numerical range by

$$W_\delta(T) = \{x^*Ax :\ x \in C^n, x^*x = 1,\ \|Ax\| \geq \delta\}\,. \tag{1.9}$$

The Algebra Numerical Range and the Spatial Numerical Range.

Let X be a complex normed algebra with unit. Denote by X^* the set of all bounded linear functionals on X. The algebra numerical range of an element $a \in X$ is defined by

$$V(a) = \{f(a) : f \in X^*,\ f(I) = 1 = \|f\|\}\,. \tag{1.10}$$

It is well-known that $V(a)$, [3] is a compact convex subset of the complex plane.

Let X be a complex Banach space and $B(X)$ the Banach algebra of bounded linear operators on X. For $T \in B(X)$, the spatial numerical range is given by

$$V_{sp}(T) = \{f(Tx) : x \in X, f \in X^*,\ \|f\| = \|x\| = f(x) = 1\}\,. \tag{1.11}$$

The algebra and spatial numerical range of an operator are closely connected. We have

$$V(a) = \overline{conv}\, V_{sp}(T) \tag{1.12}$$

where $\overline{conv}$ denotes the closed convex hull, [3].

The k-Numerical Range, c-Numerical Range and C-Numerical Range.

For $k \in \{1, \ldots, n\}$, Halmos [7] introduced the k-numerical range of $T \in \mathbf{C}^{n \times n}$ defined by

$$W_k(T) = \left\{ \sum_{i=1}^{k} x_i^* T x_i : x_1, \ldots x_k \text{ are orthonormal vectors in } \mathbf{C}^n \right\}. \tag{1.13}$$

$W_k(T)$ is convex, [2].

For a real vector $c = (c_1, \ldots, c_n)$, Westwick [18] introduced the c-numerical range of T defined by

$$W_c(T) = \left\{ \sum_{i=1}^{n} c_i x_i^* T x_i : x_1, \ldots x_n \text{ are orthonormal vectors in } \mathbf{C}^n \right\}. \tag{1.14}$$

$W_c(T)$ is convex (i.e Westwick's convexity theorem), [18].

Replacing c in $W_c(T)$ above with $C = diag(c_1, \cdots, c_n)$ where $C \in \mathbf{C}^{n \times n}$, the c-numerical range is C-numerical range which can also be denoted by

$$W_C(T) := \{tr(CUTU^*) : U \in unitary)\}. \tag{1.15}$$

This set was introduced by Goldberg and Straus [5]. The convexity of $W_C(T)$ does not always hold for $C \in \mathbf{C}^{n\times n}$.

Remark: For any $T \in B(H)$ the k-numerical range, c-numerical range and C-numerical range can also be expressed respectively as:

$$W_k(T) = \left\{ \sum_{i=1}^{k} \langle Tx_i, x_i \rangle : x_1, \ldots x_k \text{ are orthonormal vectors in } H \right\} \tag{1.16}$$

where $k \in \{1, \ldots, n\}$,

$$W_c(T) = \left\{ \sum_{i=1}^{n} c_i \langle Tx_i, x_i \rangle : x_1, \ldots x_n \text{ are orthonormal vectors in } H \right\} \tag{1.17}$$

where $c = (c_1, \ldots, c_n)$ is a real vector, and

$$W_C(T) = \left\{ \sum_{i=1}^{n} c_i \langle Tx_i, x_i \rangle : x_1, \ldots x_n \text{ are orthonormal vectors in } H \right\} \tag{1.18}$$

where $C = diag(c_1, \cdots, c_n)$ is a complex vector. This set can also be expressed as

$$W_C(T) := \{tr(CU^*TU) : U \in B(H) \text{ is unitary}\}. \tag{1.19}$$

1.3 Review of Related Literature.

Toeplitz [17] in 1918 proved that $W(T)$ has a convex outer boundary and Hausdorff [9] in 1919 showed that the intersection of every line with $W(T)$ is connected or empty. It is from the conjecture of these two that gave rise to the result of the classical Toeplitz-Hausdorff Theorem [(8), 1967]. It is remarkable for it states that the image of the unit sphere in $\mathbf{C}^n$ (a hollow object) is a compact set in $\mathbf{C}$ under the quadratic form $x \mapsto \langle Tx, x\rangle$. Since then various generalization have been considered ranging from finite dimensional to infinite dimensional linear maps on normed spaces. Sadia [14] showed that $W(T)$ is identical to $V(T)$. Stampfli [16] introduced the maximal numerical range $W_o(T)$ and the δ-numerical range $W_\delta(T)$ of the bounded linear operator T. He [16] proved the convexity of $W_o(T)$ on a Hilbert space. Whereas, the convexity of $W_\delta(T)$ on a Hilbert space was proved by Agure [1]. However, the convexity of both sets of the numerical ranges on a general Banach space is still open.

Agure [1], considered $V_\delta(T)$ for $T \in B(H)$, where $B(H)$ is a unital algebra and called it the algebra δ-numerical range, i.e

$$V_\delta(T) = \text{closure}\left\{f(T) : f(I) = \|f\| = 1 \quad \text{and} \quad f(T^*T) \geq \delta^2\right\}. \quad (1.20)$$

Having proved that both sets $W_\delta(T)$ and $V_\delta(T)$ are convex, he [1] showed that the two sets are actually identical. i.e

$$W_\delta(T) = V_\delta(T) \quad \forall T \in B(H).$$

Halmos [7] in 1964 introduced the k-numerical range $W_k(T)$ of $T \in B(H)$ and its convexity was proved by Berger [2]. Then Westwick [18] considered the c-numerical range $W_c(T)$ of $T \in B(H)$. While Goldberg and Straus [5] in 1977 introduced and studied the C-numerical range $W_C(T)$ of $T \in B(H)$. Westwick [18] proved that $W_C(T)$ is always convex for $C \in \mathbf{R}^n$ (i.e for C a Hermitian) but left open for a complex C.

Therefore the purpose of our study is to further extend the properties of $W(T)$ and $W_q(T)$, in particular the convexity property to $W_C(T)$ for a complex C and determine how $W_C(T)$ is a generalization of both the classical numerical range, and the q-numerical range of a Hilbert space operator T.

1.4 Statement of The Problem.

Let H be a Hilbert space, $T : H \to H$ a bounded linear operator and $B(H)$ the set of bounded linear operators on H. $\forall T \in B(H)$, we investigate the properties of $W_q(T)$. In particular, we confirm that $W_q(T)$ is convex. We also establish that the other properties of $W(T)$ in section (1.2.1) are also true for $W_q(T)$. Furthermore, we investigate the relationship between $W_q(T)$ and the algebra q-numerical range. Finally, we extend properties of $W(T)$ and $W_q(T)$ to $W_C(T)$ and investigate how $W_C(T)$ is a generalization of both $W(T)$ and $W_q(T)$.

1.5 Objectives of The study.

The main purpose of this study was to:

- Investigate the properties of $W_q(T)$. In particular, to confirm whether $W_q(T)$ is convex and establish whether the other properties of $W(T)$ are also true for $W_q(T)$.
- Investigate whether $W_q(T)$ is related to the algebra q-numerical range $V_q(T)$.
- Extend the properties of $W(T)$ and $W_q(T)$ to $W_C(T)$, and finally show how the $W_C(T)$ is a generalization of both $W(T)$ and $W_q(T)$ $\forall T \in B(H)$.

Chapter 2

THE q-NUMERICAL RANGE

Introduction.

In this chapter we investigate the properties of the q-numerical range. We show that some properties that hold for $W(T)$ in section (1.2.1) also hold for $W_q(T)$, for example non-emptiness, closedness, convexity among others. We also look at the relationship between the q-numerical range $W_q(T)$ and the algebra q-numerical range $V_q(T)$.

2.1 Properties of q-numerical range.

For any $T \in B(H)$, we recall the definition of q-numerical range as

$$W_q(T) = \{\langle Tx, y\rangle : x, y \in H, \langle x, x\rangle = \langle y, y\rangle = 1, \langle x, y\rangle = q\}$$

for $q \in \mathbf{C}$ satisfying $|q| \leq 1$.

(i) $W_q(T) \neq \emptyset$.

(ii) $W_q(T)$ is unitary invariant i.e $W_q(U^*TU) = W_q(T)$ for all unitary operator $U \in B(H)$.

(iii) $W_q(T)$ lies in the closed disc of radius $||T||$ centered at the origin i.e for $\lambda \in \mathbf{C}$, $\{|\lambda| \leq ||T||\}$.

(iv) $W_q(T)$ contains all the eigenvalues of T i.e $\lambda \in W_q(T)$.

(v) $W_q(T^*) = \{\overline{\lambda} : \lambda \in W_q(T)\}$.

(vi) $W_q(I) = \{q\}$, I is the identity operator $\in B(H)$.

(vii) $W_q(\alpha I + \beta T) = \alpha\{q\} + \beta W_q(T)$, I is identity operator $\in B(H)$ and $\alpha, \beta \in \mathbf{K}$.

(viii) $W_q(T)$ is a convex set.

Proof.

(i). To prove that $W_q(T) \neq \emptyset$.

Since T is an operator on $H \neq 0$.

Take $x, y \in H : ||x|| = ||y|| = 1,\ \langle x, y\rangle = q$. Then

$$W_q(T) = \{\langle Tx, y\rangle : x, y \in H, \langle x, x\rangle = \langle y, y\rangle = 1, \langle x, y\rangle = q\} \neq \emptyset.$$

(ii). To prove that $W_q(U^*TU) = W_q(T)$ for all unitary operator $U \in B(H)$.

We need to show that

$$W_q(U^*TU) \subseteq W_q(T) \tag{2.1}$$

and that

$$W_q(T) \subseteq W_q(U^*TU). \tag{2.2}$$

To prove (2.1).

Let $\lambda \in W_q(U^*TU)$ for some $U\ T \in B(H)$, where U is unitary.

Then $\exists\ x, y \in H$ with $\|x\| = \|y\| = 1,\ \langle x, y\rangle = q$ such that

$$\langle (U^*TU)x, y\rangle = \lambda.$$

Thus

$$\lambda = \langle (U^*TU)x, y\rangle = \langle TUx, Uy\rangle$$

Taking $Ux = x_1, Uy = y_1$ with $\|x_1\| = \|y_1\| = 1$,

we see that

$$\langle x_1, y_1\rangle = \langle Ux, Uy\rangle = \langle (U^*U)x, y\rangle = \langle x, y\rangle = q$$

and that

$$\lambda = \langle TUx, Uy\rangle = \langle Tx_1, y_1\rangle$$

$$\Rightarrow \lambda \in W_q(T).$$

$$\text{Thus} \quad W_q(U^*TU) \subseteq W_q(T).$$

Next to show that $W_q(T) \subseteq W_q(U^*TU)$.

Let $\lambda \in W_q(T)$. Then,$\exists\ x, y \in H$ with $\|x_1\| = \|y_1\| = 1,\ \langle x_1, y_1\rangle =$

q

such that

$$\langle Tx, y\rangle = \lambda.$$

Then since U is unitary, we have

$$\langle Tx, y\rangle = \langle TUU^*x, UU^*y\rangle$$

Taking $U^*x = x_1, U^*y = y_1$ with $||x_1|| = ||y_1|| = 1$,
we see that

$$\langle x_1, y_1\rangle = \langle Ux, Uy\rangle = \langle (U^*U)x, y\rangle\langle x, y\rangle = q$$

and that

$$\langle TUU^*x, UU^*y\rangle = \langle TUx_1, Uy_1\rangle$$

$$= \langle (U^*TU)x_1, y_1\rangle$$

$$\Rightarrow \lambda \in W_q(U^*TU).$$

$$\text{Thus} \quad W_q(T) \subseteq W_q(U^*TU)$$

(iii). To show that $W_q(T)$ lies in the closed disc of radius $||T||$ centered at the origin i.e $\{|\lambda| \leq ||T||\}$.
Take $\lambda \in W_q(T)$, then $\exists\ x, y \in H, \langle x, x\rangle = \langle y, y\rangle = 1, \langle x, y\rangle = q$ such that

$$\lambda = \langle Tx, y\rangle$$

$$\Rightarrow |\lambda| = |\langle Tx, y\rangle| \leq ||Tx||||y|| \leq ||T||||x||||y||.$$

Since $||x|| = ||y|| = 1$, we get

$$|\lambda| \leq ||T||.$$

Thus the result i.e $\omega_q(T) = sup\{|\lambda| : \lambda \in W_q(T)\}$.

(iv). To show that $W_q(T)$ contains all the eigenvalues of T i.e $\lambda \in W_q(T)$.

Let $T : H \to H$ (H, a finite dimensional Hilbert space):

$Tx = \lambda x$ with $||x|| = ||y|| = 1, \langle x, y\rangle = q \quad \forall x, y \in H$.

Then,

$$\langle Tx, y\rangle = \langle \lambda x, y\rangle = \lambda q$$

$$\Rightarrow |\lambda q| = |\lambda||q| \leq |\lambda| \leq ||T||$$

$$\Rightarrow \lambda \in W_q(T).$$

(v). To show that $W_q(T^*) = \{\overline{\lambda}.: \lambda \in W_q(T)\}$

$$W_q(T^*) = \{\langle T^*x, y\rangle : \ x, y \in H, \langle x, x\rangle = \langle y, y\rangle = 1, \langle x, y\rangle = q\}.$$

$$= \{\langle x, Ty\rangle : \ x, y \in H, \langle x, x\rangle = \langle y, y\rangle = 1, \langle x, y\rangle = q\}$$

$$= \left\{\overline{\langle Ty, x\rangle} : \ x, y \in H, \langle x, x\rangle = \langle y, y\rangle = 1, \langle x, y\rangle = q\right\}$$

$$= \left\{\overline{\lambda} : \lambda \in W_q(T)\right\}.$$

(vi). To show that $W_q(I) = \{q\}$, I is the identity operator $\in B(H)$.

$$W_q(I) = \{\langle Ix, y\rangle : \langle x, x\rangle = \langle y, y\rangle = 1, \langle x, y\rangle = q\}$$

$$= \{\langle x, y\rangle : x, y \in H, \langle x, y\rangle = q\} = \{q\}.$$

(vii). To show that

$$W_q(\alpha I + \beta T) = \alpha\{q\} + \beta W_q(T)$$

where I is identity operator $\in B(H)$ and $\alpha,\ \beta \in \mathbf{K}$.

$$W_q(\alpha I + \beta T) = \{\langle(\alpha I + \beta T)x, y\rangle : x, y \in H, \langle x, x\rangle = \langle y, y\rangle = 1, \langle x, y\rangle = q\}$$

$$= \{\langle \alpha I x + \beta T x, y\rangle : x, y \in H, \langle x, x\rangle = \langle y, y\rangle = 1, \langle x, y\rangle = q\}$$

$$= \{\langle \alpha x, y\rangle + \langle \beta T)x, y\rangle : x, y \in H, \langle x, x\rangle = \langle y, y\rangle = 1, \langle x, y\rangle = q\}$$

$$= \{\alpha\langle x, y\rangle : x, y \in H, \langle x, x\rangle = \langle y, y\rangle = 1, \langle x, y\rangle = q\} \quad +$$

$$\{\beta\langle Tx, y\rangle : x, y \in H, \langle x, x\rangle = \langle y, y\rangle = 1, \langle x, y\rangle = q\}$$

$$= \alpha\{q\} + \beta W_q(T).$$

(viii). To show that $W_q(T)$ is a convex set.

We would like to point out that this proof follows from the argument of Halmos [8]. Here need to show that the intersection of every line with $W_q(T)$ is connected or empty.

Suppose T is a bounded linear operator on the Hilbert space H.

Suppose λ and μ are distinct points of $W_q(T)$. We desire to show that the line segment $[\lambda, \mu]$ lies entirely in $W_q(T)$.

We have unit vectors x and y with

$$\lambda = \langle Tx, y\rangle \quad \text{and} \quad \mu = \langle Ty, x\rangle. \tag{2.3}$$

These vectors are linearly independent (else $\lambda = \mu$). Hence

$$\langle Tx, y\rangle \neq \langle Ty, x\rangle.$$

If $\{x, y\}$ are linearly dependent, then since they are unit vectors, one could be written as a multiple of the other. Since, moreover,the factor should have to have absolute values of 1, it could follow that

$$\langle Tx, y\rangle = \langle Ty, x\rangle, \text{ a contradiction.}$$

Two preliminary reductions simplifies the proof:

- If $\lambda = \mu$, this is trivial.

  Now let $\lambda \neq \mu$. Then

$$\exists \quad \alpha, \beta \in \mathbf{C} : \alpha\lambda + \beta = 1 \quad \text{and} \quad \alpha + \mu\beta = 0. \tag{2.4}$$

It suffices to prove that $[0, 1]$ is induced in

$$W_q(\alpha T + \beta I) = \alpha W_q(T) + \beta q. \tag{2.5}$$

The reason is that if

$$\alpha\langle Tx^1, y^1\rangle + \beta q = t \; for \, x^1, \, y^1 \, \in H,$$

then

$$\alpha\langle Tx^1, y^1\rangle + \beta q = t(\alpha\lambda + \beta q) + (1 - t)(\alpha\mu + \beta q)$$

$$= \alpha t\lambda + (1-t)\alpha\mu + \beta q$$

$$= \alpha\left[t\lambda + (1-t)\mu\right] + \beta q.$$

There is no loss of generality in assuming that $\lambda = 1$ and $\mu = 0$. Then $(\alpha + \beta) = 1$.

•• Write $T = T_1 + iT_2$ with T_1 and T_2 Hermitian.

Since $\langle Tx, y\rangle (= 1)$ and $\langle Ty, x\rangle (= 0)$ are real, it follows that the imaginary parts i.e $\langle T_2x, y\rangle$ and $\langle T_2y, x\rangle$ vanish. Replacing x and y by λx and λy respectively where $|\lambda| = 1$, we have that $\langle Tx, y\rangle$ and $\langle Ty, x\rangle$ remains the same i.e

$$\langle Tx, y\rangle = \langle T\lambda x, \lambda y\rangle = |\lambda|^2\langle Tx, y\rangle = \langle Tx, y\rangle.$$

Analogously $\langle Ty, x\rangle$ remains $\langle Ty, x\rangle$ and $\langle T_2x, y\rangle$ becomes

$$\langle T_2\lambda x, \lambda y\rangle = |\lambda|^2\langle T_2x, y\rangle = \langle T_2x, y\rangle.$$

There is no loss of generality in assuming that $\langle T_2x, y\rangle$ is purely imaginary.

We may then proceed by putting

$$h(t) = tx + (1-t)y \qquad 0 \leq t \leq 1 \tag{2.6}$$

and assert that $h(t) \neq 0$.

Since

$$\langle T_2h(t), h(t)\rangle = \langle T_2(tx + (1-t)y), tx + (1-t)y\rangle$$

$$= \langle T_2 tx + T_2(1-t)y, tx + (1-t)y\rangle + \langle T_2(1-t)y, tx\rangle +$$

$$\langle T_2(1-t)y, (1-t)y\rangle$$

$$= t^2\langle T_2x, x\rangle + t(1-t)\left[\langle T_2x, y\rangle + \langle T_2x, y\rangle^*\right] + (1-t)^2\langle T_2y, y\rangle$$

$$= t^2\langle T_2x, x\rangle + 2t(1-t)\mathbf{Re}\langle T_2x, y\rangle + (1-t)^2\langle T_2y, y\rangle.$$

Since $\langle T_2x, y\rangle$ (and hence $\langle T_2x, x\rangle, \langle T_2y, y\rangle$) is purely imaginary, we see that

$$\langle T_2h(t), h(t)\rangle = 0 \ \forall t \ (0 \leq t \leq 1).$$

It follows then that

$$\langle Th(t), h(t)\rangle = t^2\langle Tx, x\rangle + 2t(1-t)\mathbf{Re}\langle Tx, y\rangle + (1-t)^2\langle Ty, y\rangle \quad (2.7)$$

is real and lies entirely in $W_q(T)$ $\forall t$ $(0 \leq t \leq 1)$.

i.e when $t = 0$, then

$$\langle Th(t), h(t)\rangle = \langle Ty, y\rangle$$

and when $t = 1$, then

$$\langle Th(t), h(t)\rangle = \langle Tx, x\rangle.$$

The function

$$t \longrightarrow \frac{\langle Th(t), h(t)\rangle}{||h(t)||^2} \quad (2.8)$$

is continous on the closed interval $[0, 1]$. Hence the range of the function contains every number in the unit interval $[0, 1]$.

2.2 The relationship between the q-Numerical Range and the Algebra q-Numerical Range.

Let H be a complex Banach space and $B(H)$ the set of bounded linear operators on H. For $\forall\, T \in B(H)$, $q \in \mathbf{C}\; : |q| \leq 1$, the algebra q-numerical range is defined by

$$V_q(T) = closure\{f(T) : f \in B(H)^*, f(I) = \langle x, y\rangle \;, \|f\| \leq 1 \;, f(T^*T) = \langle Tx, Ty\rangle\}\,. \tag{2.9}$$

$V_q(T)$ is a convex set. To show this, let $\lambda_1, \lambda_2 \in V_q(T)$. We shall show that

$$\alpha\lambda_1 + (1-\alpha)\lambda_2 \in V_q(T) \qquad (0 \leq \alpha \leq 1). \tag{2.10}$$

Now $\lambda_1, \lambda_2 \in V_q(T)$ implies that $\exists$ functionals $f_1, f_2 \in B(H)^*$ such that

$$f_1(T) = \lambda_1,\;\; f_2(T) = \lambda_2 \quad f_1 = \langle x, y\rangle\; f_2 = \langle x, y\rangle$$

$$\|f_1\| \leq 1,\; f_2(I) \leq = 1, \quad \text{and}\; f_1(T^*T) = \langle Tx, Ty\rangle,\;\; f_2(T^*T) = \langle Tx, Ty\rangle.$$

Define f in $B(H)^*$ by

$$f(T) = \alpha f_1(T) + (1-\alpha) f_2(T). \tag{2.11}$$

We demonstrate that f is a bounded linear functional such that

$$f(I) = \langle x, y\rangle \;, \|f\| \leq 1 \quad \text{and}\; f(T^*T) = \langle Tx, Ty\rangle$$

Thus $\forall\ \beta_1, \beta_2 \in \mathbf{C},\ 0 \leq \alpha \leq 1,$

$$f\left(\beta_1 T_1 + \beta_2 T_2\right) = \alpha f_1\left(\beta_1 T_1 + \beta_2 T_2\right) + (1-\alpha) f_2\left(\beta_1 T_1 + \beta_2 T_2\right)$$

$$= \alpha f_1(\beta_1 T_1) + \alpha f_1(\beta_2 T_2) + (1-\alpha) f_2(\beta_1 T_1) + (1-\alpha)\beta_2 f_2(T_2)$$

$$= \alpha f_1 \beta_1(T_1) + \alpha f_1 \beta_2(T_2) + (1-\alpha) f_2 \beta_1(T_1) + (1-\alpha)\beta_2 f_2(T_2)$$

$$= \beta_1 \left\{\alpha f_1(T_1) + (1-\alpha) f_2(T_1)\right\} + \beta_2 \left\{\alpha f_1(T_2) + (1-\alpha) f_2(T_2)\right\}$$

$$= \beta_1 f(T_1) + \beta_2 f(T_2) \quad \Rightarrow \ f \text{ is linear.}$$

Now letting

$$f(T) = \langle Tx, y\rangle\ \forall x, y \in H\ :\ \langle x, x\rangle = \langle y, y\rangle = 1,\ \langle x, y\rangle = q$$

it is clear that

$$f(I) = \langle x, y\rangle.$$

From the definition (2.11), we have

$$|f(T)| = |\alpha f_1(T) + (1-\alpha) f_2(T)|$$

$$\leq \alpha |f_1(T)| + (1-\alpha)|f_2(T)|$$

$$\leq \alpha ||f_1||||T|| + (1-\alpha)||f_2||||T||$$

$$\leq (\alpha + 1 - \alpha)||T||\ (since\ ||f_1|| \leq 1, ||f_2|| \leq 1)$$

$$\therefore\ |f(T)| \leq ||T||. \tag{2.12}$$

Taking sups. both sides of (2.12) : $||T|| \leq 1$, we obtain

$$||f|| \leq 1.$$

Also from definition (2.11), we see that

$$f(T^*T) = \alpha f_1(T^*T) + (1-\alpha) f_2(T^*T)$$

$$= \alpha f_1(T^*T) + f_2(T^*T) - \alpha f_2(T^*T)$$

$$= f_2(T^*T) = \langle Tx, Ty \rangle$$

$$\therefore\ f(T^*T) = \langle Tx, Ty \rangle.$$

It follows that f as defined above is a bounded linear functional satisfying

$$f(I) = \langle x, y \rangle, \ \ ||f|| \leq 1 \quad \text{and } f(T^*T) = \langle Tx, Ty \rangle.$$

Hence $f(T) \in V_q(T)$, implying that $V_q(T)$ is convex.
It is clear that f is not a state on $B(H)$.

Now to establish the relationship between $W_q(T)$ and $V_q(T)$, we first seek to show that

$$W_q(T) \subseteq V_q(T). \tag{2.13}$$

To establish this,
Let $\lambda \in W_q(T)$, then $\exists\ x, y \in H,\ ||x|| = ||y|| = 1, \langle x, y \rangle = q :$

$$\langle Tx, y \rangle = \lambda.$$

Define f in $B(H)^\star$ by

$$f(T) = \langle Tx, y\rangle, \quad f(T^*T) = \langle Tx, Ty\rangle \; \forall T \in B(H). \tag{2.14}$$

We show that f is a linear functional in $B(H)^\star$ satisfying

$$f(I) = \langle x, y\rangle, \; \|f\| \leq 1.$$

To show that f is linear,

we let $\alpha, \beta \in \mathbf{C}, \quad T_1, T_2 \in B(H)$, then

$$f\left(\alpha T_1 + \beta T_2\right) = \langle(\alpha T_1 + \beta T_2)x, y\rangle$$

$$= \langle \alpha T_1 x, y\rangle + \langle \beta T_2 x, y\rangle$$

$$= \alpha\langle T_1 x, y\rangle + \beta\langle T_2 x, y\rangle$$

$$= \alpha f(T_1) + \beta f(T_2) \quad \Rightarrow f \text{ is linear.}$$

From (2.14), it is clear that

$$f(I) = \langle x, y\rangle.$$

Also we have

$$|f(T)| = |\langle Tx, y\rangle| \leq \|T\|\|x\|\|y\| = \|T\|$$

$$\Rightarrow |f(T)| \leq \|T\| \quad (since \; \|x\| = \|y\| = 1) \tag{2.15}$$

Taking sups. both sides of (2.15): $||T|| = 1$,

$$\Rightarrow ||f|| \leq 1$$

So that

$$\lambda = \langle Tx, y \rangle = f(T) \in V_q(T).$$

$$\Rightarrow W_q(T) \subseteq V_q(T).$$

However, the reverse inclusion i.e $V_q(T) \subseteq W_q(T)$, does not hold.

Chapter 3

EXTENSION OF $W(T)$ AND $W_q(T)$ TO $W_C(T)$

Introduction.

It is not surprising that some results on $W(T)$ and $W_q(T)$ can be extended to the C-numerical range which is defined by

$$W_C(T) = \left\{ \sum_{i=1}^{n} c_i \langle Tx_i, x_i \rangle : x_i, \ldots, x_n \text{ orthonormal vectors in H} \right\} \quad (3.1)$$

for $\forall\, T \in B(H)$, with C, a k-tuple of nonzero (in general, complex) numbers $c_1, \ldots, c_n$. From these extended results, one can interprete the set C-numerical range as a generalization of both $W(T)$ and $W_q(T)$. This is explicitly established in this chapter.

3.1 Properties of C-Numerical Range.

For any $T \in B(H)$, C, a K-tuple of non-zero (in general, complex) numbers $c_1, \cdots, c_n$, and $\{x_i : x_1, \ldots, x_n\}$ an orthonormal subset of H;

(i) $W_C(T) \neq \emptyset$.

(ii) $W_C(T)$ is unitary invariant i.e $W_C(U^*TU) = W_C(T)$ for all unitary operators $U \in B(H)$.

(iii) $W_C(T)$ lies in the closed disc of radius $k||T||$ centered at the origin i.e $\{|\lambda| \leq k||T||\}$.
(Where we denote $k = \sum_{i=1}^{n} |c_i| = 1$).

(iv) $W_C(T)$ contains all the eigenvalues of T i.e $\lambda \in W_C(T)$.

(v) $W_C(T^*) = \{\overline{\lambda} : \lambda \in W_C(T)\}$.

(vi) $W_C(I) = \{\sum_{i=1}^{n} c_i\}$, I is the identity operator $\in B(H)$.

(vii) $W_C(\alpha I + \beta T) = \alpha \sum_{i=1}^{n} c_i + \beta W_C(T)$, I is identity operator $\in B(H)$ and $\alpha,\ \beta, \in \mathbf{K}$.

(viii) $W_C(T)$ is a convex set.

Proof.

(i). To prove that $W_C(T) \neq \emptyset$.
Since T is an operator on $H \neq 0$, also $\{x_i\}_i^n$ being an orthonormal subset of H, and C is a non-zero k-tuple complex numbers, it follows that

$$W_C(T) = \left\{ \sum_{i=1}^{n} c_i \langle Tx_i, x_i \rangle : x_i, \ldots, x_n \text{ orthonormal vectors in H} \right\} \neq \emptyset.$$

(ii). To prove that $W_C(U^*TU) = W_C(T)$ for all unitary operator $U \in B(H)$.

We need to show that

$$W_C(U^*TU) \subseteq W_C(T) \tag{3.2}$$

and that

$$W_C(T) \subseteq W_C(U^*TU). \tag{3.3}$$

To prove (3.2)

Let $\lambda \in W_C(U^*TU)$ for some unitary $U \in B(H)$,

then $\exists \ \{x_i\}_i^n$ an orthonormal subset of H such that

$$\sum_{i=1}^{n} c_i \langle U^*TUx_i, x_i\rangle = \sum_{i=1}^{n} c_i \langle TUx_i, Ux_i\rangle = \lambda.$$

Taking $Ux_i = y_i, \quad with \ \|y_i\| = 1$, we have

$$= \sum_{i=1}^{n} c_i \langle TUx_i, Ux_i\rangle = \sum_{i=1}^{n} c_i \langle Ty_i, y_i\rangle$$

$$\Rightarrow \ \lambda \in W_C(T).$$

Thus

$$W_C(U^*TU) \subseteq W_C(T).$$

Next to show that

$$W_C(T) \subseteq W_C(U^*TU).$$

Let $\lambda \in W_C(T)$, then $\exists\ \{x_i\}_i^n$ an orthonormal subset of H such that

$$= \sum_{i=1}^{n} c_i \langle Tx_i, x_i \rangle = \lambda.$$

Since U is unitary, we have

$$\sum_{i=1}^{n} c_i \langle Tx_i, x_i \rangle = \sum_{i=1}^{n} c_i \langle TUU^* x_i, UU^* x_i \rangle.$$

Taking $U^* x_i = y_i \quad with\ \|y_i\| = 1$, we obtain

$$\sum_{i=1}^{n} c_i \langle TUU^* x_i, UU^* x_i \rangle = \sum_{i=1}^{k} c_i \langle U^* TU y_i, y_i \rangle$$

$$\Rightarrow \lambda \in W_C(U^* TU).$$

Thus

$$W_C(T) \subseteq W_C(U^* TU).$$

(iii). To show that $W_C(T)$ lies in the closed disc of radius $k\|T\|$ centered at the origin i.e $\{|\lambda| \leq k\|T\|\}$.

(Where we denote $k = \sum_{i=1}^{n} |c_i| = 1$).

Let $\lambda \in W_C(T)$, then there exists $\{x_i\}_i^n$ an orthonormal subset of H such that

$$\lambda = \sum_{i=1}^{n} c_i \langle Tx_i, xi \rangle$$

$$\Rightarrow |\lambda| = |\sum_{i=1}^{n} c_i \langle Tx_i, xi \rangle| \leq \|Tx_i\| \|x_i\| \sum_{i=1}^{n} |c_i|$$

$$\leq \|T\| \|x_i\| \|x_i\| \sum_{i=1}^{n} |c_i|$$

$$(But\ \|x_i\| = 1, \quad \sum_{i=1}^{n} |c_i| = 1)$$

$$\therefore\ |\lambda| \leq \|T\|.$$

Thus the result i.e $\omega_q(T) = sup\{|\lambda| : \lambda \in W_C(T)\}$.

(iv). To show that $W_C(T)$ contains all the eigenvalues of T.
Let $T : H \to H$ (for H a finite dimensional Hilbert space):

$$Tx_i = \lambda_i x_i.$$

Take $\lambda \in W_C(T)$ such that

$$\lambda = \sum_{i=1}^{n} c_i\lambda_i, \ 0 \leq c_i \leq 1, \ \sum_{i=1}^{n} c_i = 1$$

and for $\{x_i\}_i^n$ an orthonormal subset of H, then

$$\sum_{i=1}^{n} c_i\langle Tx_i, x_i\rangle = \sum_{i=1}^{n} c_i\langle \lambda_i x_i, x_i\rangle = \lambda$$

$$= \sum_{i=1}^{n} c_i\lambda_i\langle x_i, x_i\rangle$$

$$= \sum_{i=1}^{n} c_i\lambda_i = \lambda.$$

$\Rightarrow$ All eigenvalues λ_i are a linear combination in the set $W_C(T)$.

(v). To show that $W_C(T^*) = \{\overline{\lambda} : \lambda \in W_C(T)\}$.
We have

$$W_C(T^*) = \left\{\sum_{i=1}^{n} c_i\langle T^* x_i, x_i\rangle : \|x_i\| = 1\right\}$$

$$= \left\{ \sum_{i=1}^{n} c_i \langle x_i, Tx_i \rangle : \|x_i\| = 1 \right\}$$

$$= \left\{ \sum_{i=1}^{n} c_i \overline{\langle Tx_i, x_i \rangle} : \|x_i\| = 1 \right\}$$

$$= \left\{ \overline{\lambda} : \lambda \in W_C(T) \right\}.$$

(vi). To show that $W_C(I) = \{\sum_{i=1}^{n} c_i\}$, I is the identity operator $\in B(H)$.

We have

$$W_C(I) = \left\{ \sum_{i=1}^{n} c_i \langle Ix_i, x_i \rangle : \|x_i\| = 1 \right\}$$

$$= \left\{ \sum_{i=1}^{n} c_i \langle x_i, x_i \rangle : \|x_i\| = 1 \right\}$$

$$= \left\{ \sum_{i=1}^{n} c_i \right\}.$$

(vii). To show that $W_C(\alpha I + \beta T) = \alpha \sum_{i=1}^{n} c_i + \beta W_C(T)$, I is identity operator $\in B(H)$.

$\forall\, T \in B(H)$,

$$W_C(\alpha I + \beta T) = \left\{ \sum_{i=1}^{n} c_i \langle (\alpha I + \beta T) x_i, x_i \rangle : \|x_i\| = 1 \right\}$$

$$= \left\{ \langle \sum_{i=1}^{n} c_i \alpha I x_i + \sum_{i=1}^{n} c_i \beta T x_i, x_i \rangle : \|x_i\| = 1 \right\}$$

$$= \alpha \left\{ \sum_{i=1}^{n} c_i \langle Ix_i, x_i \rangle : \|x_i\| = 1 \right\} + \beta \left\{ \sum_{i=1}^{n} c_i \langle Tx_i, x_i \rangle : \|x_i\| = 1 \right\}$$

$$= \alpha \sum_{i=1}^{n} c_i + \beta W_C(T).$$

(viii). To show that $W_C(T)$ is a convex set.

The convexity of $W_C(T)$ (for T normal) holds if and only if the eigenvalues of T are collinear on the complex plane.

To show this, let us take arbitrary $t,\ s \in W_C(T)$ *and* $0 \leq \alpha \leq 1$. Then we have

$$t = \left\{ \sum_{i=1}^{n} c_i \langle Tx_i, x_i \rangle : (x_i)_{i=1}^{n} \text{ are othornomal vectors in H} \right\},$$

$$s = \left\{ \sum_{i=1}^{n} c_i \langle Ty_i, y_i \rangle : (y_i)_{i=1}^{n} \text{ are othornomal vectors in H} \right\}.$$

Denote by K the n-dimensional subspace of H spanned by vectors $e_1, \ldots, e_n$. Let $P \in B(H)$ be the orthogonal projection from H to K. Then we have

$$t \in W_{P^*CP/K}(T) \quad and\ s \in W_{P^*CP/K}(T)$$

Here C is taken as an ordered vector so that

$$\mathbb{H}_{\mathbb{C}} = \mathbb{U}_{\mathbb{C}}\ [6],$$

where

$\mathbb{H}_{\mathbb{C}}$: a class of Hermitian matrices depending on C and

$\mathbb{U}_{\mathbb{C}}$: a class of matrices, $\mathbb{U}_{\mathbb{C}} = conv\,\{UCU^* :\ U \in unitary\}$.

From the spectral theorem for Hermitian matrices [(13), pp 172-173], we have

$$P^*CP/K = C \in\ Hermitian.$$

Thus $W_{P^*CP/K}(T)$ is equivalent to the c-Numerical Range that was introduced by Westwick [18] and whose convexity holds [18,19].
Whence we have

$$\alpha t + (1 - \alpha)s \in W_{P^*CP/K}(T) \subseteq W_C(T).$$

Conversely, if $W_C(T)$ is a convex complex plane, then there exists eigenvalues of T which are collinear.

3.2 The relationship between the C-Numerical Range and the Algebra C-Numerical Range.

Let H be a complex Banach space and $B(H)$ the set of bounded linear operators on H. For $\forall T \in B(H)$, the algebra C-numerical range is defined by

$$V_C(T) = closure\left\{f(t) : f \in B(H)^*,\ f(I) = \|f\| = 1,\ f(T^*T) = \sum_{i=1}^{n} c_i\|Tx_i\|^2\right\}.$$

$V_C(T)$ is convex. To show this, let $\lambda_1, \lambda_2 \in V_C(T)$. We seek to show that

$$\alpha\lambda_1 + (1-\alpha)\lambda_2 \in V_C(T) \quad (0 \leq \alpha \leq 1).$$

(Here we consider $0 \leq c_i \leq 1$, $\sum_{i=1}^{n} c_i = 1$).

Now $\lambda_1, \lambda_2 \in V_C(T)$ implies that $\exists$ functionals $f_1, f_2 \in B(H)^*$ such that

$$f_1(T) = \lambda_1,\ f_2(T) = \lambda_2, \quad f_1(I) = 1 = \|f_1\|,\ f_2(I) = 1 = \|f_2\|$$

and

$$f_1(T^*T) = \sum_{i=1}^{n} c_i\|Tx_i\|^2 \geq 0 \quad f_2(T^*T) = \sum_{i=1}^{n} c_i\|Tx_i\|^2 \geq 0$$

Define f in $B(H)^*$ by

$$f(T) = \alpha f_1(T) + (1-\alpha)f_2(T). \tag{3.4}$$

We demonstrate that f is a positive linear functional satisfying the condition

$$f(I) = \|f\| = 1.$$

$\forall\ \beta_1, \beta_2 \in \mathbf{C},\ 0 \leq \alpha \leq 1,$

$$f(\beta_1 T_1 + \beta_2 T_2) = \alpha f_1 (\beta_1 T_1 + \beta_2 T_2) + (1-\alpha) f_2 (\beta_1 T_1 + \beta_2 T_2)$$

$$= \alpha f_1(\beta_1 T_1) + \alpha f_1(\beta_2 T_2) + (1-\alpha) f_2(\beta_1 T_1) + (1-\alpha)\beta_2 f_2(T_2)$$

$$= \alpha f_1 \beta_1(T_1) + \alpha f_1 \beta_2(T_2) + (1-\alpha) f_2 \beta_1(T_1) + (1-\alpha)\beta_2 f_2(T_2)$$

$$= \beta_1 \{\alpha f_1(T_1) + (1-\alpha) f_2(T_1)\} + \beta_2 \{\alpha f_1(T_2) + (1-\alpha) f_2(T_2)\}$$

$$= \beta_1 f(T_1) + \beta_2 f(T_2)$$

$$\Rightarrow \quad f \quad \text{is linear.}$$

It is clear from the definition that f is positive. That is,

$$f(T^*T) = \alpha f_1(T^*T) + (1-\alpha) f_2(T^*T)$$

$$= \alpha f_1(T^*T) + f_2(T^*T) - \alpha f_2(T^*T)$$

$$= f_2(T^*T) = \sum_{i=1}^{n} c_i \|Tx_i\|^2 \geq 0.$$

Now $f(I) = \alpha f_1(I) + (1-\alpha) f_2(I)$

$$= \alpha + 1 - \alpha = 1 \qquad (\text{since } f_1(I) = f_2(I) = 1)$$

Thus

$$f(I) = 1. \tag{3.5}$$

Next we show that

$$||f|| = 1.$$

From the definition of f, we have

$$f(T) = \alpha f_1(T) + (1-\alpha)f_2(T) \qquad (0 \leq \alpha \leq 1)$$

$$\Rightarrow |f(T)| = |\alpha f_1(T) + (1-\alpha)f_2(T)|$$

$$\leq \alpha||f_1||||(T)|| + (1-\alpha)||f_2||||(T)||$$

$$= (\alpha + 1 - \alpha)\,(||T||) \qquad (\text{since } ||f_1|| = ||f_2|| = 1)$$

$$\Rightarrow |f(T)| \leq ||T||.$$

(Taking sups. both sides : $||T|| = 1$), we have

$$||f|| \leq 1. \tag{3.6}$$

Also from (3.5)

$$1 = |f(I)| \leq ||f||||I|| = ||f||$$

$$\Rightarrow ||f|| \geq 1. \tag{3.7}$$

From (3.6) and (3.7) we have

$$||f|| = 1.$$

It thus follows that f as defined above is a positive, linear functional satisfying

$$||f|| = 1 = f(I).$$

Hence

$$f(T) \subseteq V_C(T) \;\Rightarrow\; V_C(T) \text{ is convex.}$$

We then show that $W_C(T)$ is identical to $V_C(T)$ $\forall T \in B(H)$ i.e

$$W_C(T) = V_C(T).$$

First we show that

$$W_C(T) \subseteq V_C(T). \tag{3.8}$$

Let $\lambda \in W_C(T)$, then $\exists$ $(x_i)_{i=1}^n$ othornomal vectors in H and $0 \leq c_i \leq 1$, $\sum_{i=1}^n c_i = 1$:

$$\sum_{i=1}^n c_i \langle Tx_i, x_i \rangle = \lambda.$$

Define f in $B(H)^*$ by

$$f(T) = \sum_{i=1}^n c_i \langle Tx_i, x_i \rangle, \quad f(T^*T) = \sum_{i=1}^n c_i \|Tx_i\|^2, \ \forall T \in B(H) \tag{3.9}$$

We show that f is a state in $B(H)^*$.

To show that f is linear,

we let $\alpha, \beta \in \mathbf{C}$, $T_1, T_2 \in B(H)$, then

$$f\left(\alpha T_1 + \beta T_2\right) = \sum_{i=1}^n c_i \langle (\alpha T_1 + \beta T_2) x_i, x_i \rangle$$

$$= \sum_{i=1}^n c_i \langle \alpha T_1 x_i, x_i \rangle + \sum_{i=1}^n c_i \langle \beta T_2 x_i, x_i \rangle$$

$$= \alpha \sum_{i=1}^{n} c_i \langle T_1 x_i, x_i \rangle + \beta \sum_{i=1}^{n} c_i \langle T_2 x_i, x_i \rangle$$

$$= \alpha f(T_1) + \beta f(T_2)$$

$$\Rightarrow f \text{ is linear.}$$

From (3.9) it follows that f is positive.

Next we show that

$$f(I) = \|f\| = 1, \; I \text{ an identity operator } \in B(H).$$

From (3.9), we have

$$|f(T)| = |\sum_{i=1}^{n} c_i \langle T x_i, x_i \rangle|$$

$$\leq \sum_{i=1}^{n} c_i \|T\| \|x_i\|^2 \leq \sum_{i=1}^{n} c_i \|T\| \quad for \; \|x_i\| = 1$$

$$\Rightarrow \; |f(T)| \leq \|T\| \quad (since \sum_{i=1}^{n} c_i = 1, \; 0 \leq c_i \leq 1). \tag{3.10}$$

Taking sups. on both sides of (3.10), we obtain

$$\|f\| \leq 1. \tag{3.11}$$

We also have

$$f(I) = \sum_{i=1}^{n} c_i \langle I x_i, x_i \rangle = 1$$

$$\Rightarrow \; 1 = |f(I)| \leq \|f\| \|I\| = \|f\|.$$

So that

$$1 \leq \|f\|. \tag{3.12}$$

From (3.11) and (3.12), we have

$$\|f\| = 1.$$

Thus $\lambda = f(T)$ as defined in (3.8) is contained in $V_C(T)$

$$\Rightarrow W_C(T) \subseteq V_C(T).$$

To show the reverse inclusion i.e

$$V_C(T) \subseteq W_C(T).$$

We assume that $\lambda \in V_C(T)$ and $\lambda \notin W_C(T)$ and derive a contradiction. Since $\lambda \in V_C(T)$, it follows that $\exists$ a state f in $B(H)$ such that

$$f(T) = \lambda \quad \text{and} \quad f(T^*T) = \sum_{i=1}^{n} c_i \|Tx_i\|^2.$$

Since $W_C(T)$ is convex, by rotating T, we may assume that

$$\mathbf{Re} W_C(T) \leq \mathbf{Re}\lambda - \alpha, \quad \alpha > 0. \tag{3.13}$$

$$Let\ G = \left\{ (x_i)_{i=1}^{n} \text{ othonormal vectors in H and } \mathbf{Re} \sum_{i=1}^{n} c_i \langle Tx_i, x_i \rangle \geq \mathbf{Re}\lambda - \frac{\alpha}{2},\ \alpha > 0 \right\}. \tag{3.14}$$

The set G is non-empty because if it is not, then for all $(x_i)_{i=1}^{n}$ othonormal vectors in H, we shall have

$$\mathbf{Re}\sum_{i=1}^{n} c_i\langle Tx_i, x_i\rangle < \mathbf{Re}\lambda - \frac{\alpha}{2}, \quad \alpha > 0. \tag{3.15}$$

But since f is a *weak**-limit of convex combination of vector states, $\forall \epsilon > 0, \ \exists\ N = N(\epsilon) : \forall m > N,$

$$|f_m(T) - f(T)| < \epsilon.$$

Also we can find $M = M(\epsilon)$ such that for all $m > M$, the following inequality will hold.

$$|f_n(T^*T) - f(T^*T)| < \epsilon.$$

$$\text{Take} \quad \epsilon < \frac{\alpha}{2} \tag{3.16}$$

and $n > max(N, M)$

Since

$$f_m(T) = \sum_{i=1}^{n} c_i\langle Tx_i, x_i\rangle \quad 0 \leq c_i \leq 1, \ and \ \sum_{i=1}^{n} c_i = 1,$$

we have

$$\mathbf{Re}f_m(T) = \mathbf{Re}\sum_{i=1}^{n} c_i\langle Tx_i, x_i\rangle < \mathbf{Re}\lambda - \frac{\alpha}{2}$$

$$\Rightarrow \mathbf{Re}\ f_m(T) < \mathbf{Re}\lambda - \frac{\alpha}{2}. \tag{3.17}$$

But

$$f_m(T) > f(T) - \epsilon$$

so that

$$\mathbf{Re}f_m(T) > \mathbf{Re}f(T) - \epsilon.$$

Thus

$$\mathbf{Re} f_m(T) > \mathbf{Re}\lambda - \epsilon \quad (since \quad f(T) = \lambda). \tag{3.18}$$

From (3.18) and (3.19) we have

$$\mathbf{Re}\lambda - \epsilon < \mathbf{Re}\lambda - \frac{\alpha}{2}$$

$$\Rightarrow \epsilon > \frac{\alpha}{2} \tag{3.19}$$

which is a contradiction.

Therefore $\lambda \notin W_C(T)$ implies that $\lambda \notin V_C(T)$ and hence $\lambda \in V_C(T)$ implies that $\lambda \in W_C(T)$. Thus

$$V_C(T) \subseteq W_C(T). \tag{3.20}$$

(3.8) and (3.20)

$$\Rightarrow \; V_C(T) = W_C(T).$$

3.3 C-Numerical Range as a generalization of both the Classical Numerical Range and q-Numerical Range.

3.3.1 C-Numerical Range as a generalization of the Classical Numerical Range.

For $T \in B(H)$, with C, a k-tuple of nonzero (in general, complex) numbers $c_1, \ldots, c_n$, we recall the C-numerical range as the set

$$W_C(T) = \left\{ \sum_{i=1}^{n} c_i \langle Tx_i, x_i \rangle : x_i, \ldots, x_n \text{ orthonormal vectors in H} \right\}$$

If C consists of just one number, say $c_1 = 1$, then it reduces to the classical numerical range,

$$W(T) = \{\langle Tx, x\rangle : \ x \in H, \|x\| = 1\}.$$

3.3.2 C-Numerical Range as a generalization of the q-Numerical Range.

We assume that the Hilbert space H has a finite dimension n. Let p be an orthogonal projection in a subspace $K(H)$ of H and H_n designate the range of p. One significant property in this section is presence of an orthonormal basis. (An orthonormal basis for H is by definition an orthogonal system (x_λ) such that x_λ are basic vectors in the sense that

$\langle x_\lambda, x_\lambda \rangle$ are orthogonal projections in $K(H)$ of rank 1. The orthogonal dimension of H (i.e the cardinal number of any of its orthonormal bases) will be designated by $dim_{K(H)}$. Furthermore, the inner product on H will be denoted by

$$\langle \cdot, \cdot \rangle = tr\left(\langle \cdot, \cdot \rangle\right).$$

We shall fix a unit vector ε in H and denote by e the orthogonal projection $e_{\varepsilon,\varepsilon}$ to the one-dimensional subspace spanned by ε. More precisely, for a fixed orthogonal projection e in $K(H)$ of rank 1, H is given as the set of all ex, $x \in H$. Also, for all $x, y \in H$ we have that

$$\langle x, y \rangle = \langle x, y \rangle e.$$

We now choose an orthonormal basis $\{\varepsilon_1, \ldots \varepsilon_n\}$ for H_n which is to be held fixed for the rest of this section. We shall denote

$$e_1 = e_{\varepsilon i, \varepsilon i} \qquad for\ i = 1, \ldots, n.$$

Evidently,

$$p = e_1 + \ldots + e_n$$

Definition 3.3.2.1:

For any complex number q with $|q| \leq 1$ and $T \in B(H)$, we shall define the set

$$pW_q^n(T) = \{tr\langle Tx, y \rangle : \ x, y \in H, \ \langle x, x \rangle = \langle y, y \rangle = p, \ \langle x, y \rangle = qp\} \tag{3.21}$$

where $p \in K(H)$ is the rank n projection.

Remark 3.3.2.2:

Suppose that vectors $x, y \in H$ satisfy

$$\langle x, x\rangle = \langle y, y\rangle = p, \ \langle x, y\rangle = qp \tag{3.22}$$

where $q \in \mathbf{C}, \ |q| < 1$.

Let us put

$$x_i = e_{\varepsilon,\varepsilon i}x, \ y_i = e_{\varepsilon,\varepsilon i}y \quad for \ i = 1, \ldots, n.$$

We shall show that

$$\{x_1 \ldots, x_n, \ y_1, \ldots, y_n\}$$

is linearly independent set of a Hilbert space H. Namely, for $i, j = 1, \ldots, n$, we have

$$\langle x_i, x_j\rangle = e_{\varepsilon,\varepsilon i}\langle x, x\rangle e_{\varepsilon j,\varepsilon} = e_{\varepsilon,\varepsilon i}pe_{\varepsilon j,\varepsilon} = \delta_{i,j}e$$

and analogously

$$\langle y_i, y_j\rangle = \delta_{i,j}e.$$

Also, the condition

$$\langle x, y\rangle = qp$$

implies that

$$\langle x_i y_i\rangle = e_{\varepsilon,\varepsilon i}\langle x, y\rangle e_{\varepsilon j,\varepsilon} = e_{\varepsilon,\varepsilon i}qpe_{\varepsilon j,\varepsilon} = q\delta_{i,j}e.$$

From this we deduce that $x_i, y_i \in H$ and it holds that

$$\langle x_i, x_j\rangle = \langle y_i, y_j\rangle = \delta_{i,j}, \ \langle x_i, y_j\rangle = q\delta_{i,j} \quad for \ i, j = 1, \ldots, n. \tag{3.23}$$

Where $\delta_{i,j}$ is the Kronecker delta defined by

$$\forall\ i,j = 1,\ldots,n,\ \langle x_i, x_j\rangle = \delta_{i,j} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Let us now suppose that

$$\sum_{i=1}^{n} \alpha_i x_i + \sum_{i=1}^{n} \beta_i y_i = 0 \text{ for some } \alpha_i, \beta_i \in \mathbf{C}. \tag{3.24}$$

Multiplying this equality on its right-hand side by x_i and then by y_i (and by using 3.23) we get,

$$\sum_{i=1}^{n} \alpha_i x_i . x_i + \sum_{i=1}^{n} \beta_i y_i . x_i = 0$$

$$\Rightarrow \sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \beta_i \bar{q} = 0$$

$$= \alpha_i + \beta_i \bar{q} = 0 \quad (Taking \sum_{i=1}^{n} \alpha_i = \alpha_i \ and \sum_{i=1}^{n} \beta_i = \beta_i)$$

and

$$\sum_{i=1}^{n} \alpha_i x_i . y_i + \sum_{i=1}^{n} \beta_i y_i . y_i = 0$$

$$\Rightarrow \sum_{i=1}^{n} \alpha_i q + \sum_{i=1}^{n} \beta_i = 0$$

$$= \alpha_i q + \beta_i = 0 \quad (Taking \sum_{i=1}^{n} \alpha_i = \alpha_i \ and \sum_{i=1}^{n} \beta_i = \beta_i)$$

from which it follows that

$$\alpha_i(1 - |q|^2) = 0,\ i = 1,\ldots,n.$$

Hence,

$$\alpha_i = 0, \text{ and thus } \beta_i = 0, \ i = 1, \ldots, n.$$

Therefore,

$$\{x_i \ldots x_n, \ y_1 \ldots, y_n\}$$

is a linearly independent set in H.

Remark 3.3.2.3:

Observe that when $n = 1$, the condition

$$\langle x, x \rangle = \langle y, y \rangle = e_1, \ \langle x, y \rangle = qe_1$$

is equivalent to the fact that x and y are unit vectors of Hilbert space H such that

$$\langle x, y \rangle = q.$$

Moreover,

$$tr\langle Tx, y \rangle = \langle Tx, y \rangle$$

So the set $pW_q^1(T)$ is the q-numerical range $W_q(T)$ $\forall T \in B(H)$.

Remark 3.3.2.4:

The condition

$$\langle x, x \rangle = \langle y, y \rangle = p$$

obviously implies

$$\langle x - px, x - px \rangle = \langle y - py, y - py \rangle = 0 \ i.e \ x = px \ and \ y = py.$$

So we have

$$\langle Tx, y\rangle = p\langle Tx, y\rangle p.$$

Furthermore, for every $\eta \perp H_n$ we get

$$\langle Tx, y\rangle\eta = p\langle Tx, y\rangle p\eta = 0.$$

Therefore, $\langle Tx, y\rangle$ can be regarded as an operator acting on the n-dimensional space H_n.

Remark 3.3.2.5:

The definition of the set $pW_q^n(T)$ does not depend on the choice of the rank n projection $p \in K(H)$. Hence the set $pW_q^n(T)$ can be written as $W_q^n(T)$, and when n=1, then we have $W_q^1(T)$, which is the q-numerical range $W_q(T)$ of the operator $T \in B(H)$.

Lemma 3.3.2.6:

For any $q \in \mathbf{C}$ with $|q| \leq 1$, $p \in K(H)$ the rank n projection, then there exist $x, y \in H$ such that

$$\langle x, x\rangle = \langle y, y\rangle = p, \ \langle x, y\rangle = qp$$

and by the definition of $pW_q^n(T)$, it implies that $pW_q^n(T) \neq \emptyset$.

Proof:

Let $\{u_1, \ldots, u_n\}$ be an orthonormal set of the Hilbert space H. We define

$$v_i = \bar{q}u_i + \sqrt{1-|q|^2}u_{n+i} \quad for \ i = 1, \ldots, n.$$

Then we get

$$\langle v_i, v_j\rangle = \left\langle \bar{q}u_i + \sqrt{1-|q|^2}u_{n+i}, \bar{q}u_j + \sqrt{1-|q|^2}un + j\right\rangle$$

$$= \bar{q}q\langle u_i, u_j\rangle + \bar{q}\sqrt{1-|q|^2}\langle u_i, u_{n+j}\rangle + \sqrt{1-|q|^2}q\langle u_{n+i}, u_j\rangle + (1-|q|^2)\langle u_{n+i}, u_{n+j}\rangle$$

$$= |q|^2\delta_{i,j} + (1-|q|^2)\delta_{i,j} = \delta_{i,j} \quad i = 1, \ldots, n.$$

Also we have

$$\langle u_i, v_j\rangle = \left\langle u_i, \bar{q}u_j + \sqrt{1-|q|^2}u_{n+j}\right\rangle$$

$$= q\langle u_i, u_j\rangle + \sqrt{1-|q|^2}\langle u_i, u_{n+j}\rangle = q\delta_{i,j} \quad for\ i = 1, \ldots, n.$$

Thus we obtain

$$\langle x_i, x_j\rangle = e_{\varepsilon i,\varepsilon}\langle u_i, u_j\rangle e_{\varepsilon,\varepsilon j} = e_{\varepsilon i,\varepsilon}\langle u_i, u_j\rangle e_{\varepsilon,\varepsilon j} = \delta_{i,j}e_i$$

and analogously

$$\langle y_i, y_j\rangle = \delta_{i,j}e_i \quad for\ i, j = 1 \ldots, n.$$

Moreover,

$$\langle x_i, y_j\rangle = e_{\varepsilon i,\varepsilon}\langle u_i, v_j\rangle e_{\varepsilon,\varepsilon j} = e_{\varepsilon i,\varepsilon}\langle u_i, v_j\rangle ee_{\varepsilon,\varepsilon j} = q\delta_{i,j}e_i\ for\ i, j = 1 \ldots, n.$$

Then

$$x = x_1 + \ldots + x_n = x_i$$

and

$$y = y_1 + \ldots + y_n = y_i$$

are desired vectors, implying that the set $pW_q^n(T)$ is non-empty for all $T \in B(H)$.

We now prove our main result which is due to Rajna [12].

Theorem 3.3.2.7:
Let T be an operator in $B(H)$. Then

$$W_q(T) = \{tr(CU^*TU) : \ U \text{ is unitary}\}$$

where C, a k-tuple of complex numbers is regarded as a linear operator acting on an orthonormal basis $\{z_1, \ldots z_n\}$ of an n-dimensional subspace of H such that

$$Cz_i = qz_i \quad i = 1 \ldots, n$$

$$Cz_{n+i} = \sqrt{1-|q|^2}z_i \quad i = 1 \ldots, n.$$

Proof.
Given a unitary operator

$$U : \ H \rightarrow H,$$

we define

$$x_i = e_{\varepsilon i,\varepsilon} Uz_i$$

$$y_i = \overline{q}e_{\varepsilon i,\varepsilon}Uz_i + \sqrt{1-|q|^2}e_{\varepsilon i,\varepsilon}Uz_{n+i} \quad for \ i = 1, \ldots, n.$$

Since $\{Uz_1, \ldots, Uz_n\}$ is an orthonormal set in H, arguing as in the proof of lemma 3.3.2.6, we obtain that

$$\langle x_i, x_j\rangle = \langle y_i, y_j\rangle = \delta_{i,j}e_i, \ \langle x_i, y_j\rangle = q\delta_{i,j}e_i \quad \text{for all } i, j = 1, \ldots, n.$$

Let us put

$$x = x_1 + \ldots + x_n = x_i$$

and

$$y = y_i + \ldots + y_n = y_i.$$

Then we have

$$\langle x, x \rangle = \langle y, y \rangle = p, \ \langle x, y \rangle = qp.$$

Also it holds that

$$e_{\varepsilon,\varepsilon i}x = e_{\varepsilon,\varepsilon i}\,(x_i + \ldots + x_n) = e_{\varepsilon,\varepsilon i}x_i = e_{\varepsilon,\varepsilon i}e_{\varepsilon i,\varepsilon}Uz_i = eUz_i = Uz_i. \quad (3.25)$$

$$e_{\varepsilon,\varepsilon i}y = e_{\varepsilon,\varepsilon i}\,(y_1 + \ldots + y_n) = e_{\varepsilon,\varepsilon i}y_i = e_{\varepsilon,\varepsilon i}\left(\bar{q}e_{\varepsilon i,\varepsilon}Uz_i + \sqrt{1-|q|^2}e_{\varepsilon i,\varepsilon}Uz_{n+i}\right)$$

$$= \bar{q}eUz_i + \sqrt{1-|q|^2}eUz_{n+i} = \bar{q}Uz_i + \sqrt{1-|q|^2}UZ_{n+i} \; for \; i = 1, \ldots, n. \quad (3.26)$$

In particular,

$$e_{\varepsilon,\varepsilon i}x, \quad e_{\varepsilon,\varepsilon i}y \in H,$$

so we get

$$\langle Te_{\varepsilon,\varepsilon i}x, e_{\varepsilon,\varepsilon i}y \rangle = \langle Te_{\varepsilon,\varepsilon i}x, e_{\varepsilon,\varepsilon i}y \rangle \, e.$$

On the other hand, we have

$$\langle Te_{\varepsilon,\varepsilon i}x, \ e_{\varepsilon,\varepsilon i}y \rangle = e_{\varepsilon,\varepsilon i}\langle Tx, y \rangle e_{\varepsilon i,\varepsilon} = e_{\varepsilon,\varepsilon i}e_{\varepsilon i,\varepsilon i}\langle Tx, y \rangle e_{\varepsilon i,\varepsilon i}e_{\varepsilon i,\varepsilon}$$

$$= e_{\varepsilon,\varepsilon i}\langle Te_i x, e_i y \rangle e_{\varepsilon i,\varepsilon} = e_{\varepsilon,\varepsilon i}\,\langle Te_i x, e_i y \rangle \, e_i e_{\varepsilon i,\varepsilon}$$

$$= \langle Te_i x, e_i y \rangle \, e.$$

Therefore

$$\langle Te_{\varepsilon,\varepsilon i}x, e_{\varepsilon,\varepsilon i}y\rangle = \langle Te_i x, e_i y\rangle \ \ for\ i = 1, \ldots, n. \tag{3.27}$$

Notice that

$$C^* z_i = \bar{q} z_i + \sqrt{1-|q|^2} z_{n+i} \qquad i = 1, \ldots, n$$
$$C^* z_i = 0 \qquad i = n+1, \ldots, 2n.$$

So by (3.25), (3.26) *and* (3.27) we obtain that

$$\begin{aligned}
tr(CU^*TU) &= \sum_{i=1}^{n} \langle U^*TU z_i, C^* z_i\rangle = \sum_{i=1}^{n} \langle TU z_i, UC^* z_i\rangle \\
&= \sum_{i=1}^{n} \left\langle TU z_i, U\left(\bar{q} z_i + \sqrt{1-|q|^2} z_{n+i}\right)\right\rangle \\
&= q\sum_{i=1}^{n} \langle TU z_i, U z_i\rangle + \sum_{i=1}^{n} \langle Te_{\varepsilon,\varepsilon i}x, e_{\varepsilon,\varepsilon i}y - \bar{q} e_{\varepsilon,\varepsilon i}x\rangle \\
&= \sum_{i=1}^{n} \langle Te_{\varepsilon,\varepsilon i}x, e_{\varepsilon,\varepsilon i}y\rangle = \sum_{i=1}^{n} \langle Te_i x, e_i y\rangle \\
&= tr\left(\sum_{i=1}^{n} \langle Te_i x, e_i y\rangle\, e_i\right) = tr\left(\sum_{i=1}^{n} \langle Te_i x, e_i y\rangle\right) \\
&= tr\left(\sum_{i=1}^{n} e_i \langle Tx, y\rangle e_i\right) = tr\left(p\langle Tx, y\rangle p\right) \\
&= tr\langle Tx, y\rangle = \langle Tx, y\rangle
\end{aligned}$$

which reduces to the q-numerical range (by remarks 3.3.2.3 and 3.3.2.5). Conversely, let $x, y \in H$ satisfy

$$\langle x, x\rangle = \langle y, y\rangle = p \ and\ \langle x, y\rangle = qp.$$

Suppose first that $|q| < 1$. Define a linear operator

$$U : Y_n \to H$$

by its action on the basis $\{z_1 \dots z_n\}$:

$$Uz_i = e_{\varepsilon,\varepsilon i}x,$$

$$Uz_{n+i} = \frac{1}{\sqrt{1-|q|^2}}(e_{\varepsilon,\varepsilon i}y - \bar{q}e_{\varepsilon,\varepsilon i}x) \quad for\ i = 1, \dots, n.$$

It is easy to check that the operator U is a well-defined isometry. That is, for $i, j = 1, \dots, n$, we have

$$\langle Uz_i, Uz_j \rangle = e_{\varepsilon,\varepsilon i}\langle x, x\rangle e_{\varepsilon j,\varepsilon} = e_{\varepsilon,\varepsilon i}pe_{\varepsilon j\varepsilon} = \delta_{i,j}e$$

which implies

$$Uz_i \in H$$

and

$$\langle Uz_i, Uz_j \rangle = \delta_{i,j}.$$

Also, for $i, j = 1, \dots, n$ it holds that

$$\langle Uz_i, Uz_{n+j} \rangle = \frac{1}{1-|q|^2}\langle e_{\varepsilon,\varepsilon i}y - \bar{q}e_{\varepsilon,\varepsilon i}x, e_{\varepsilon,\varepsilon j}y - \bar{q}e_{\varepsilon,\varepsilon j}x\rangle$$

$$= \frac{1}{1-|q|^2}\left(e_{\varepsilon,\varepsilon i}\langle y, y\rangle e_{\varepsilon j,\varepsilon} - qe_{\varepsilon,\varepsilon i}\langle y, x\rangle e_{\varepsilon j,\varepsilon} - \bar{q}e_{\varepsilon,\varepsilon i}\langle x, y\rangle e_{\varepsilon j,\varepsilon} + |q|^2 e_{\varepsilon,\varepsilon i}\langle x, x\rangle e_{\varepsilon j,\varepsilon}\right)$$

$$= \frac{1}{1-|q|^2}\left(e_{\varepsilon,\varepsilon i}pe_{\varepsilon j,\varepsilon} - q\bar{q}e_{\varepsilon,\varepsilon i}pe_{\varepsilon j,\varepsilon} - \bar{q}qe_{\varepsilon,\varepsilon i}pe_{\varepsilon j,\varepsilon} + |q|^2 e_{\varepsilon,\varepsilon i}pe_{\varepsilon j,\varepsilon}\right)$$

$$= \frac{1}{1-|q|^2}\delta_{i,j}\left(e - |q|^2 e - |q|^2 e + |q|^2 e\right) = \delta_{i,j} e.$$

So

$$U z_{n+i} \in H$$

and

$$(U z_{n+i}, U z_{n+j}) = \delta_{i,j}.$$

Furthermore

$$\langle U z_i, U z_{n+j} \rangle = \frac{1}{1-|q|^2} e_{\varepsilon,\varepsilon i} \langle x, y - \bar{q} x \rangle e_{\varepsilon j,\varepsilon}$$

$$= \frac{1}{1-|q|^2} e_{\varepsilon,\varepsilon i} \left(\langle x, y \rangle - q \langle x, x \rangle\right) e_{\varepsilon j,\varepsilon}$$

$$= \frac{1}{1-|q|^2} e_{\varepsilon,\varepsilon i} \left(qp - qp\right) e_{\varepsilon j,\varepsilon} = 0 \quad \text{for } i, j = 1, \ldots, n.$$

Hence

$$\langle U z_i, U z_{n+j} \rangle = 0 \quad \text{for } i, j = 1, \ldots, n.$$

Therefore, U is an isometry and can be extended to a unitary operator

$$U : H \to H.$$

Finally, since (3.25) (3.26) *and* (3.27) are also valid, the same calculation as before shows that

$$tr(CU^* TU) = tr\langle Tx, y \rangle = \langle Tx, y \rangle .$$

Now, suppose that $|q| = 1$, then we have

$$C^* z_i = \bar{q} z_i \qquad for\ i = 1, \ldots, n.$$

$$C^* z_i = 0 \quad for\ i = 1 + n, \ldots, 2n.$$

Define a linear operator

$$U : Y_n \to H$$

on the orthonormal basis $\{z_1, \ldots z_n\}$ by putting

$$U z_i = e_{\varepsilon,\varepsilon i} x \quad \text{for } i = 1, \ldots, n.$$

It is clear that U is a well-defined isometry and can be extended to a unitary operator

$$U : H \to H.$$

Let us put

$$x_i = e_i x, \ y_i = e_i y \quad \text{for } i = 1, \ldots, n.$$

Thus we have

$$\langle x_i, x_i \rangle = e_i \langle x, x \rangle e_i = e_i p e_i = e_i$$

and analogously

$$\langle y_i, y_i \rangle = e_i \quad \text{for } i = 1, \ldots, n.$$

Moreover,

$$\langle x_i, y_i \rangle = e_i \langle x, y \rangle e_i = e_i q p e_i = q e_i \quad \text{for } i = 1, \ldots, n.$$

So we deduce that x_i, y_i are unit vectors of the Hilbert space H such that

$$\langle x_i, y_i \rangle = q, \quad \text{for } i = 1, \ldots, n.$$

Also, since

$$|\langle x_i, y_i \rangle| = |q| = 1 = |\langle x_i, x_i \rangle|^{\frac{1}{2}}.|\langle y_i, y_i \rangle|^{\frac{1}{2}}$$

it follows that

$$y_i = \alpha_i x_i \quad \alpha_i \in \mathbf{C} \qquad \text{for } i = 1, \ldots, n.$$

But

$$q = \langle x_i, y_i \rangle = (x_i, \alpha_i x_i) = \bar{\alpha}_i \langle x_i, x_i \rangle = \bar{\alpha}_i$$

from which it follows that

$$y = py = y_1 + \ldots + y_n = \bar{q}(x_i + \ldots x_n) = \bar{q}px = \bar{q}x.$$

Thus we get

$$tr(CU^*TU) = \sum_i^n \langle U^*TU z_i, C^* z_i \rangle = \sum_i^n \langle TU z_i, U(\bar{q} z_i) \rangle$$

$$= \sum_i^n \langle Te_{\varepsilon,\varepsilon i}x, \bar{q}e_{\varepsilon,\varepsilon i}x \rangle = \sum_i^n \langle Te_{\varepsilon,\varepsilon i}x, e_{\varepsilon,\varepsilon i}y \rangle$$

$$= \sum_{i=1}^n \langle Te_i x, e_i y \rangle = tr\left(\sum_{i=1}^n \langle Te_i x, e_i y \rangle e_i\right)$$

$$= tr\left(\sum_i^n \langle Te_i x, e_i y \rangle\right) = tr\left(\sum_i^n e_i \langle Tx, y \rangle e_i\right) = tr\,(p\langle Tx, y \rangle p)$$

$$= tr\langle Tx, y \rangle = \langle Tx, y \rangle$$

and by remarks (3.3.2.3), and (3.3.2.5) this set reduces to the q-numerical range $W_q(T)$ of an operator $T \in B(H)$.

Corollary 3.3.2.8: Let H be a Hilbert space of finite dimension. Denote by $\{e_i\}$ a fixed orthonormal basis of H. For $q \in \mathbf{C}$, $|q| \leq 1$, and C, a k-tuple of complex numbers representing a matrix with respect to the basis $\{e_1, \ldots, e_n\}$. Then the set

$$W_C(T) = tr(CU^*TU) \quad for\ T \in B(H)$$

can also be defined as:

$$W_C(T) = \left\{ \sum_{i=1}^{n} \langle Tx_i, y_i \rangle : x_i, y_i \in H\ \langle x_i, x_i \rangle = \langle y_i, y_i \rangle = 1,\ \langle x_i, y_j \rangle = q\delta_{i,j}\ i, j = 1, \ldots, n \right\}$$

Proof:

Given $t \in W_C(T)$, there is a unitary $U \in B(H)$ such that

$$t = tr(CU^*TU) = \sum_{i}^{n} \langle U^*TUe_i, C^*e_i \rangle = \sum_{i}^{n} \langle TUe_i, UC^*e_i \rangle.$$

Let us put

$$x_i = Ue_i,$$

$$y_i = UC^*e_i, \quad i, j = 1, \ldots, n.$$

Then we have

$$t = \sum_{i}^{n} \langle Tx_i, y_i \rangle \ :\ \langle x_i, x_j \rangle = \langle y_i, y_j \rangle = \delta_{i,j}\ and\ \langle x_i, y_i \rangle = q\delta_{i,j}.$$

Conversely, suppose that

$$t = \sum_{i}^{n} \langle Tx_i, y_i \rangle \ : \ \langle x_i, x_j \rangle = \langle y_i, y_j \rangle = \delta_{i,j} \ and \ \langle x_i, y_i \rangle = q\delta_{i,j}.$$

If $|q| < 1$, define

$$Ue_i = x_i, \ Ue_{n+1} = \frac{1}{\sqrt{1-|q|^2}(y_i - \bar{q}x_i)}, \ i = 1, \ldots, n.$$

In the case $|q| = 1$, let us put

$$Ue_i = x_i, \ i = 1, \ldots, n.$$

Then U is a well-defined isometry on the subspace of H and can be extended to a unitary operator $U \in B(H)$. Thereby,

$$t = tr(CU^*TU).$$

We observe that when $n = 1$, then

$$W_C(T) = \langle Tx, y \rangle$$

which is the q-numerical range of an operator $T \in B(H)$ and hence $W_C(T)$ is a generalization of the q-numerical range.
This concludes our assertion that $W_C(T)$ is a generalization of both $W(T)$ and $W_C(T)$.

Chapter 4

CONCLUSION

4.1 Concluding remarks.

The study of numerical range and its generalization has been a motivation to many mathematicians. In our study, we considered numerical ranges in a complex Hilbert space. We investigated the properties q-numerical range,whereby, we established that the properties for the classical numerical range in section (1.2.1) are also true for the q-numerical range. We also established that the q-numerical range is contained in the algebra q-numerical range but the reverse inclusion does not hold. Furthermore, we showed that the results of the classical numerical range and q-numerical range can be extended to the C-numerical range and that the C-numerical range is an explicit generalization of both the classical numerical range and q-numerical range.

It is our hope that the study will activate interest in posterity among students. For future research , efforts can be directed towards generalization of the numerical ranges on other Banach spaces apart from the Hilbert space.

References

[1] **Agure, J.O. (1996).** *"On the convexity of Stampfli's numerical range"*, Bull. Aust. Math. Society vol.53 , pp 33-37.

[2] **Berger, C.A. 1963.** *Normal dilations*, PhD. Thesis, Cornel University.

[3] **Bonsall, F.F. and Duncan, J. (1971).** *Numerical ranges of operators on normed spaces and elements of normed Algebras*, London Math. Soc. lecture notes 2. (Cambridge University Press).

[4] **Chi-Kwong, L. (2005).** *Lecture notes on Numerical Range.* AMS, Draft.

[5] **Goldberg, M. and Straus, E.G. (1977).** *Elementary inclusion relations for generalized numerical ranges*, Linear Algebra Appl.18, pp 1-24.

[6] **Goldberg, M. and Straus, E.G. (1977).** *On Characterization and Integrals of Generalized Numerical Ranges*, Pacific Journal of Maths. vol.69, No.1.

[7] **Halmos P.R. (1964).** *Numerical ranges and Normal dilations*, Acta Szeged, vol.25, pp 1-5.

[8] **Halmos P.R. (1967).** *A Hilbert space problem book*, Van Nostrand, Princeton, NJ.

[9] **Hausdorff, F. (1919)**. *Der Wertvorrat einer Bilinearform*, Math Z vol.3 , pp 314-316.

[10] **Kreyzig, E. (1978)**. *Introductory Functional Analysis with Applications*, John Wiley and sons.

[11] **Maddox, I.J, (1988)**. *Elements of Functional Analysis*, Universal Book Stall, New Delhi, 2^{nd} Ed.

[12] **Rajna, R. (2005)**. *A generalized q-numerical range*, Mathematical Communications vol.10, pp 31-45.

[13] **Roger, A.H. and Charles, R.J. (1985)**. *Matrix Analysis*, Cambridge University.

[14] **Sadia, H.M. (2005)**. *'On convexity of numerical ranges'*, Msc. Thesis (Maseno university).

[15] **Siddiqi, a h (1992)**. *Functional Analysis with Applications*, Tata McGraw Hill.

[16] **Stampfli, J.G. (1970)**. *The norm of a derivation*, Pacific Journal of Mathematics vol.33(3), pp 737-747.

[17] **Toeplitz, O. (1918)**. *Das algebraische Analogon Z U einem satz von fejer*, Math Z. vol.2, pp 469-472.

[18] **Westwick, R. (1975)**. *'A theorem on numerical range'*, Linear and Multilinear Algebra, pp 311-315.

[19] **Yeung, A.Y. and Tsing, N.K. (1983)**. *An extension of the Hausdorff-Toeplitz Theorem on the Numerical Range*, AMS vol.89, No.2.