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On the Solution of n-Dimensional Regular Cauchy Problem of Euler-Poisson-Darboux Equation (EPD)

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Abstract

A general exact solution to the n-dimensional regular Cauchy problem of Euler-Poisson-Darboux (EPD) equation has been studied. Firstly, the general exact solution for the one dimensional regular Cauchy problem of EPD has been worked out. The EPD which is a second order Partial Differential Equation (PDE) is converted into an Ordinary Differential Equation (ODE) by method of separation of variables. On solving the ODE, the first complementary function (cf) is obtained directly. The second cf is obtained when the first derivative is eliminated from the ODE and then the ODE solved. When the expression for eliminating the first derivative is solved, a third term is obtained. The general solution for the one dimensional regular Cauchy EPD is therefore the product of the three terms. The procedure has been repeated for the two dimensional and n-dimensional cases. The general solutions for these cases are products of four terms and n+2 terms respectively. Finally, the general exact solution for n-dimensional regular Cauchy wave equation when $k = 0$, has also been obtained.

Mathematics Subject Classification: 35Q05

Keywords: Method of separation of variables, complementary function, General exact solution

1 Introduction

The n-dimensional regular Cauchy problem of Euler-Poisson-Darboux equation is given by

$$\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} + ... + \frac{\partial^2 U}{\partial x_n^2} = \frac{\partial^2 U}{\partial t^2} + \frac{k}{t}\frac{\partial U}{\partial t} \tag{1}$$

$$U(x_1, ..., x_n, t) = f(x_1, ...x_n) \tag{2}$$

$$\frac{\partial U}{\partial t}(x_1, ..., x_n, t) = 0 \tag{3}$$

where $x_1, x_2, ...x_n$ are points in $\mathbb{R}^n$, k is a real parameter, t is a time parameter, f stands for function, U is diplacement of a wave perpendicular to its direction of travel and $\mathbb{R}^n$ is Euclidean space. Miles and Young [1] considered the generalised Euler-Poisson-Darboux equation of the form

$$L(u) = \frac{\partial^2 U}{\partial t^2} + \frac{k}{t}\frac{\partial U}{\partial t} - \nabla U - cU = 0 \tag{4}$$

$$U(x, 0) = P(x), U_t(x, 0) = 0 \tag{5}$$

where k and c are real parameters, $k \neq -1, -3, -5, ...$, $P(x)$ is assumed to be a polyharmonic function of order p and ∇ is the n dimensional Laplace operator. Equations (4) and (5) had a solution of the form

$$U(x, t) = \sum_{n=0}^{p-1} (\nabla P) U_n(t)$$

where $U_n(t)$ are determined by a system of Bessels equations. Dernek [2] solved the initial boundary value problem given by

$$\nabla U = U_{tt} + \frac{k}{t} U_t + g(x, t)\, (t > 0)$$

$$U(0, t) = U(a, t) = 0$$

$$U(x, 0) = f(x)$$

$$U_t(x, 0) = 0$$

where $k < 1, k \neq -1, -2, -3, ...$ f and g are real analytic functions. Finite Integral Transformation technique was used to find the solution. In our earlier paper, Kweyu and Manyonge [3], we found a general analytical solution to the one dimensional regular Cauchy problem of the Euler-Poisson Darboux equation. The methods used were Similarity Transformation and elimination of the first derivative. In the present article, the general exact solution for one dimensional , two dimensional and n-dimensional EPD's have been worked out. The techniques used are method of separation of variables and elimination of first derivative [4].

2 General exact solutions for regular Cauchy EPD for one-dimensional and two dimensional cases

2.1 One dimensional EPD

The one dimensional regular Cauchy EPD is given by

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 U}{\partial t^2} + \frac{k}{t}\frac{\partial U}{\partial t} \tag{6}$$

$$U(x,t) = f(x) \tag{7}$$

$$\frac{\partial U}{\partial t}(x,t) = 0 \tag{8}$$

Using the method of separation of variables , let $U(x,t) = X(x)T(t)$ to be a non trivial solution to equation (6) then

i. $U_{xx} = X''T$

ii. $U_{tt} = XT''$

iii. $U_t = XT'$

Putting (i) to (iii) in equation (6) gives

$$X''T = XT'' + \frac{k}{t}XT'$$

Dividing through by XT, gives

$$\frac{X''}{X} = \frac{T''}{T} + \frac{k}{t}\frac{T'}{T}$$

Let

$$\frac{X''}{X} = \frac{T''}{T} + \frac{k}{t}\frac{T'}{T} = A \tag{9}$$

$$X'' - XA = 0$$

The Auxilliary Quadratic Equation (AQE) becomes

$$m^2 - A = 0$$

$$m = \pm\sqrt{A}$$

so that

$$X = c_1 e^{\sqrt{A}x} + c_2 e^{-\sqrt{A}x} \tag{10}$$

where A, c_1 and c_2 are arbitrary constants. Equation (10) gives the first complementary function. Again from equation (9)

$$\frac{T''}{T} + \frac{k}{t}\frac{T'}{T} = A$$

which is solved by [4]

i. method of elimination of first derivative.

ii. finding second complementary function (cf).

iii. finding a term not part of complementary functions.

$$T'' + \frac{k}{t}T' - AT = 0$$

Let $\sigma = \frac{k}{t}, \varsigma = -A$ and $\tau = 0$ since

$$\varsigma_1 = \varsigma - \frac{1}{2}\frac{d\sigma}{dt} - \frac{\sigma^2}{4} = \frac{1}{2}\frac{k}{t^2} - \frac{k^2}{4t^2} - A$$

$$\tau_1 = \tau e^{\frac{1}{2}\int \sigma dt} = 0$$

Let $T = \xi\eta$ where ξ is the second cf and η, term which is not part of cf.

$$\frac{d^2\xi}{dt^2} + \left(\frac{1}{2}\frac{k}{t^2} - \frac{k^2}{4t^2} - A\right)\xi = 0$$

The AQE is

$$m^2 + \left(\frac{1}{2}\frac{k}{t^2} - \frac{k^2}{4t^2} - A\right) = 0$$

$$m = \pm i\sqrt{\frac{2k - k^2 - 4At^2}{4t^2}}$$

$$\xi = c_3 cos\sqrt{\frac{2k - k^2 - 4At^2}{4t^2}}t + c_4 sin\sqrt{\frac{2k - k^2 - 4At^2}{4t^2}}t$$

where c_3 and c_4 are arbitrary constants

$$\xi = c_3 cos\frac{\sqrt{2k - k^2 - 4At^2}}{2} + c_4 sin\frac{\sqrt{2k - k^2 - 4At^2}}{2} \tag{11}$$

$$\eta = e^{-\frac{k}{2}\int \frac{1}{t}dt} = e^{-\frac{k}{2}Int} = \frac{1}{t^{\frac{k}{2}}}$$

$$T = \xi\eta = \left(c_3 cos\frac{\sqrt{2k - k^2 - 4At^2}}{2} + c_4 sin\frac{\sqrt{2k - k^2 - 4At^2}}{2}\right)\frac{1}{t^{\frac{k}{2}}} \tag{12}$$

The general exact solution for one dimensional EPD is the product of equation (10) and (12) i.e

$$U(x,t) = \left(c_1 e^{\sqrt{A}x} + c_2 e^{-\sqrt{A}x}\right)\left(c_3 cos\frac{\sqrt{2k - k^2 - 4At^2}}{2} + c_4 sin\frac{\sqrt{2k - k^2 - 4At^2}}{2}\right)\frac{1}{t^{\frac{k}{2}}} \tag{13}$$

2.2 Two dimensional EPD

When we apply the same methods to the two dimensional EPD, the general exact solution for the two dimensional EPD is given by

$$U(x,y,t) = \left(c_5 e^{\sqrt{C}x} + c_6 e^{-\sqrt{C}x}\right)\left(c_7 cos\sqrt{D}y + c_8 sin\sqrt{D}y\right)$$
$$\left(c_9 cos\frac{\sqrt{2k-k^2-4Bt^2}}{2} + c_{10} sin\frac{\sqrt{2k-k^2-4Bt^2}}{2}\right)\frac{1}{t^{\frac{k}{2}}} \quad (14)$$

where $B, C, D, c_5, c_6, c_7, c_8, c_9$ and c_{10} are arbitrary constants.

3 General exact solution for n- dimensional EPD

Let $U(x_1, x_2, x_3, ...x_n, t) = X_1(x_1)X_2(x_2)X_3(x_3)...X_n(x_n)T(t)$ to be a non trivial solution to the n-dimensional EPD equation, then

i $U_{x_1x_1} = (X_1''X_2X_3...X_n)T$

ii $U_{x_2x_2} = (X_1X_2''X_3...X_n)T$

iii $U_{x_3x_3} = (X_1X_2X_3''...X_n)T$

iv .

v .

vi .

vii $U_{x_nx_n} = (X_1X_2X_3...X_n'')T$

Putting (i) to (vii) in equation (1) gives

$$(X_1''X_2X_3...X_n)T + (X_1X_2''X_3...X_n)T + (X_1X_2X_3''...X_n)T + ...(X_1X_2X_3...X_n'')T$$
$$= (X_1X_2X_3...X_n)T'' + \frac{k}{t}(X_1X_2X_3...X_n)T'$$

Dividing through by $(X_1X_2X_3...X_n)T$, we obtain

$$\frac{X_1''}{X_1} + \frac{X_2''}{X_2} + \frac{X_3''}{X_3} + ... + \frac{X_n''}{X_n} = \frac{T''}{T} + \frac{k}{t}\frac{T'}{T}$$

Let

$$\frac{X_1''}{X_1} + \frac{X_2''}{X_2} + \frac{X_3''}{X_3} + ... + \frac{X_n''}{X_n} = \frac{T''}{T} + \frac{k}{t}\frac{T'}{T} = \alpha$$

Let

$$\frac{X_1''}{X_1} + \frac{X_2''}{X_2} + \frac{X_3''}{X_3} + ... = \beta = \alpha - \frac{X_n''}{X_n}$$

$$X_n'' + (\beta - \alpha)X_n = 0$$

Let

$$\beta - \alpha = \gamma$$

$$X_n'' + \gamma X_n = 0$$

$$X_n = c_{n-1}\cos\sqrt{\gamma}x_n + c_n \sin\sqrt{\gamma}x_n$$

$\alpha, \beta, \gamma, c_{n-1}$ and c_n are arbitrary constants. From

$$\frac{X_1''}{X_1} + \frac{X_2''}{X_2} + \frac{X_3''}{X_3} = \beta$$

Let

$$\frac{X_1''}{X_1} + \frac{X_2''}{X_2} = \beta - \frac{X_3''}{X_3} = \delta$$

$$X_3'' + (\delta - \beta)X_3 = 0$$

Let $\delta - \beta = \varepsilon$

$$X_3 = c_{11}\cos\sqrt{\varepsilon}x_3 + c_{12}\sin\sqrt{\varepsilon}x_3$$

$\delta, \varepsilon, c_{11}$ and c_{12} are arbitrary constants. From

$$\frac{X_1''}{X_1} + \frac{X_2''}{X_2} = \delta$$

Let

$$\frac{X_1''}{X_1} = \delta - \frac{X_2''}{X_2} = \zeta$$

$$X_2'' + (\zeta - \delta)X_2 = 0$$

Let $\zeta - \delta = \eta$

$$X_2 = c_{13}\cos\sqrt{\eta}x_2 + c_{14}\sin\sqrt{\eta}x_2$$

ζ, η, c_{13} and c_{14} are arbitrary constants. From

$$\frac{X_1''}{X_1} = \zeta$$

$$X_1 = c_{15}e^{\sqrt{\zeta}x_1} + c_{16}e^{-\sqrt{\zeta}x_1}$$

From

$$\frac{T''}{T} + \frac{k}{t}\frac{T'}{T} = \alpha$$

$$T = \left(c_{17}cos\frac{\sqrt{2k - k^2 - 4\alpha t^2}}{2} + c_{18}sin\frac{\sqrt{2k - k^2 - 4\alpha t^2}}{2}\right)\frac{1}{t^{\frac{k}{2}}}$$

The general exact solution for n-dimensional EPD is therefore given by

$$U(x_1, x_2, x_3, ...x_n, t) = \left(c_{15}e^{\sqrt{\zeta}x_1} + c_{16}e^{-\sqrt{\zeta}x_1}\right)\left(c_{13}\cos\sqrt{\eta}x_2 + c_{14}\sin\sqrt{\eta}x_2\right)$$
$$\left(c_{11}\cos\sqrt{\varepsilon}x_3 + c_{12}\sin\sqrt{\varepsilon}x_3\right) ... \left(c_{n-1}\cos\sqrt{\gamma}x_n + c_n\sin\sqrt{\gamma}x_n\right)$$
$$\left(c_{17}cos\frac{\sqrt{2k - k^2 - 4\alpha t^2}}{2} + c_{18}sin\frac{\sqrt{2k - k^2 - 4\alpha t^2}}{2}\right)\frac{1}{t^{\frac{k}{2}}} \quad (15)$$

The subscripts of arbitrary constants of equation (15) are now written in terms of n so that it becomes a generalized equation, it now takes the form

$$U(x_1, x_2, x_3, ...x_n, t) = \left(c_{n-10}cos\frac{\sqrt{2k - k^2 - 4\alpha t^2}}{2} + c_{n-9}sin\frac{\sqrt{2k - k^2 - 4\alpha t^2}}{2}\right)$$
$$\left(c_{n-8}e^{\sqrt{\zeta}x_1} + c_{n-7}e^{-\sqrt{\zeta}x_1}\right)\left(c_{n-6}\cos\sqrt{\eta}x_2 + c_{n-5}\sin\sqrt{\eta}x_2\right)$$
$$\left(c_{n-4}\cos\sqrt{\varepsilon}x_3 + c_{n-3}\sin\sqrt{\varepsilon}x_3\right) ... \left(c_{n-1}\cos\sqrt{\gamma}x_n + c_n\sin\sqrt{\gamma}x_n\right)\frac{1}{t^{\frac{k}{2}}} \quad (16)$$

Results and Conclusion

i. The general exact solution to the n-dimensional regular Cauchy problem of the EPD is given by equation (16).

ii. When we put $k = 0$ in equation (16), we obtain the general exact solution to the n-dimensional wave equation. The equation takes the form

$$U(x_1, x_2, x_3, ...x_n, t) = \left(c_{n-10}cosht\alpha^{\frac{1}{2}} + ic_{n-9}sinht\alpha^{\frac{1}{2}}\right)$$
$$\left(c_{n-8}e^{\sqrt{\zeta}x_1} + c_{n-7}e^{-\sqrt{\zeta}x_1}\right)\left(c_{n-6}\cos\sqrt{\eta}x_2 + c_{n-5}\sin\sqrt{\eta}x_2\right)$$
$$\left(c_{n-4}\cos\sqrt{\varepsilon}x_3 + c_{n-3}\sin\sqrt{\varepsilon}x_3\right) ... \left(c_{n-1}\cos\sqrt{\gamma}x_n + c_n\sin\sqrt{\gamma}x_n\right) \quad (17)$$

since [4] $\sin it\alpha^{\frac{1}{2}} = i\sinh t\alpha^{\frac{1}{2}}$ and $\cos it\alpha^{\frac{1}{2}} = \cosh t\alpha^{\frac{1}{2}}$.

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