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Numerical Solution of Non-Linear Boundary Value Problems of Ordinary Differential Equations Using the Shooting Technique

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Abstract

Ordinary Differential Equations (ODEs) of the Initial Value Problem (IVP) or Boundary Value Problem (BVP) type can model phenomena in wide range of fields including science, engineering, economics, social science, biology, business, health care among others. Often, systems described by differential equations are so complex that purely analytical solutions of the equations are not tractable. Therefore techniques for solving differential equations based on numerical approximations take centre stage. In this paper we review the shooting method technique as a method of solution to both linear and non-linear BVPs.

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1 Non Linear BVP of Ordinary Differential Equations

1.1 Introduction

Virtually all systems that undergo change can be described by differential equations. Differential equations model phenomena in wide range of fields including science, engineering, economics, social science, biology, business, health care among others. Often, systems described by differential equations are so complex that purely analytical solutions of the equations are not tractable. Therefore techniques for solving differential equations based on numerical approximations take centre stage.

1.2 Initial and Boundary Value Problems

A differential equation defines a relationship between unknown function and one or more of its derivatives. The derivatives are of the dependent variable with respect to independent variable(s). If the independent variable is single, the differential equation is called an ordinary differential equation (ODE), otherwise it is a partial differential equation (PDE). An Initial Value Problem of an ODE usually abbreviated as IVP is an ordinary differential equation whose solution is specified at only one given point in the domain of the equation. This condition is often called an initial condition. An ordinary differential equation whose solution in an interval domain say [a,b]where $a, b \in \mathbb{R}$ is specified at more than one point is called a Boundary Value Problem (BVP). The conditions are then called boundary conditions. Some specific examples of ODEs are: in engineering, the dissolution of a contaminant into groundwater is governed by a first order ordinary differential equation. In science, the rate of cooling of a beverage is proportional to the difference in temperature between the beverage and the surrounding air. This is also governed by a first order ODE.

An analytical solution of a differential equation (partial or ordinary) is also called 'closed form solution'. At best, there are only a few differential equations that can be solved analytically in a closed form. There exist many different methods in the literature for the analytical solutions of both IVPs and BVPs[1]. These include methods for first order ordinary differential equations such as linear equations solved by use of an integrating factor, exact equations, homogeneous equations, ODEs in which variables are separable and Bernoulli type equations. For second and higher order ODEs, techniques such as use of complementary functions and particular integrals and variation of parameters method are available. There are only fairly few kinds of equations for which the solution is in terms of standard elementary mathematical functions such as cosine, sine logarithms, exponentials etc. Some simple second order linear differential equations can be solved using various special functions such Legendre, Bessel's etc. Beyond second order, the kinds of functions needed to solve even fairly simple linear differential equations become extremely complicated. Solutions of most practical problems involving differential equations require the use of numerical methods. Numerical solution of IVPs of ODEs are classified into two major groups namely, One-step methods and multi-step methods. The one-step methods include among others, Taylor's methods, Euler's method, Heun's method and Runge-Kutta methods. Linear multi-step methods include, implicit Euler's method, Trapezium rule method, Adams-Bashforth method, Adams-Moulton method and predictor-corrector methods. For BVPs of ODEs there exist some methods such as the shooting method and finite difference method for both linear and non-linear BVPs. The shooting method is the subject of this paper.

2 The Shooting Method

This method transforms a boundary value ordinary differential equation into a system of first order equations solved as IVPs. The solution of the IVP obtained is evaluated at the second boundary point and its value then is compared with the actual boundary value given. An iterative approach is employed to vary the assumed initial condition until the specified boundary conditions of the solution are satisfied. As an illustration of the concept, consider a second order BVP of ODE of the form

$$y'' = f(x, y, y'), \quad a \le x \le b, \quad y(a) = \alpha, \quad y(b) = \beta \tag{1}$$

We reduce this to a system of first order equations by letting

$$\frac{dy}{dx} = z, \quad y(a) = \alpha \tag{2}$$

$$\frac{dz}{dx} = f(x, y, z), \quad z(a) = y'(a) = s_i \tag{3}$$

Equations (2) and (3) constitute a system of two initial value problems. The s_i are real numbers. Notice that for each i, we are making a guess of the slope of the solution at the first boundary $x = a$ i.e. $y'(a)$ which produces a solution of the IVP. Since the solution of the IVP depends on the guessed value of s_i, let $y(x, s_i)$ be the solution of the IVP. We shall generate a sequence $\{s_i\}$ such that

$$\lim_{i\to\infty} y(x, s_i) = y(x) \tag{4}$$

Now notice that the solution of the IVP at the second boundary point depends on both b and s. Hence for any choice or guess of the slope value s_i, $y(b, s_i) - \beta$

is the error made. Suppose s is the value of the slope which makes the error zero, then

$$y(b, s) - \beta = 0 \tag{5}$$

is the equation to solve which is a root finding problem. There are numerical methods such the Bisection method, Secant method, Newton-Raphson method, Fixed point methods, linear interpolation which can be used to find roots of non linear equations of the form $f(s) = 0$[2]. Here we describe the Secant method as shown in figure A below. Consider a function $f(x)$ in the form shown below

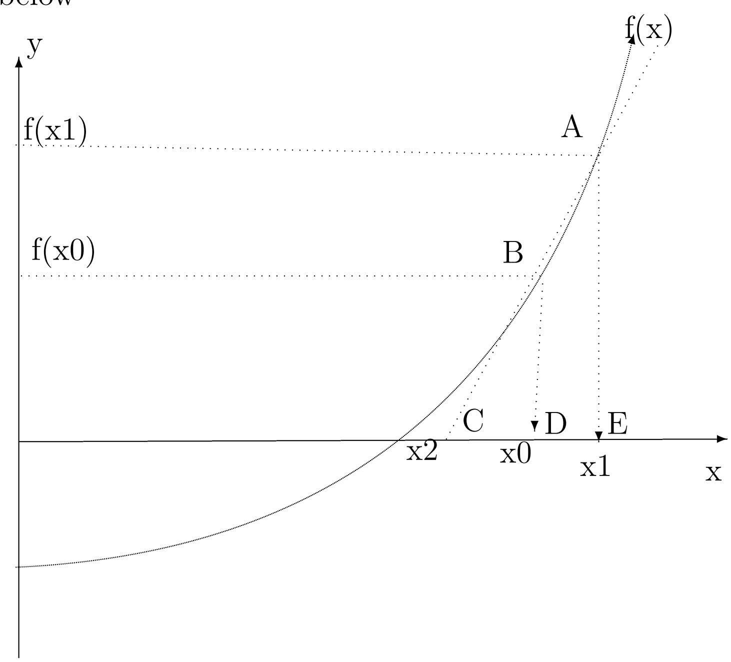

Figure A: Derivation of the Secant method

Take the initial guesses to the solution asx_0 and x_1. Draw a straight line(secant line, tangent line) between $f(x_0)$ and $f(x_1)$ passing through the $x-$ axis at x_2. EAC and DBC are similar triangles. Hence EA/CE=DB/CD or

$$\frac{f(x_1)}{x_1 - x_2} = \frac{f(x_0)}{x_0 - x_2}$$

on rearranging we find

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

or

$$x_i = x_{i-1} - \frac{f(x_{i-1})(x_{i-1} - x_{i-2})}{f(x_{i-1}) - f(x_{i-2})}$$

This will iteratively give you the solution of $f(x) = 0$. So for a given s_0 and s_1

$$s_i = s_{i-1} - \frac{f(s_{i-1})(s_{i-1} - s_{i-2})}{f(s_{i-1}) - f(s_{i-2})}$$

from which we can approximate the value of s that makes $f(s) = 0$. The secant method to approximate the solution of $y(b, s) - \beta = 0$ will be

$$s_i = s_{i-1} + \frac{[s_{i-2} - s_{i-1}]}{y(b, s_{i-2}) - y(b, s_{i-1})}[\beta - y(b, s_{i-1})] \qquad (6)$$

Equation (6) provides you with an opportunity to make a mathematical guess of the slope of the IVP. Note that we need two initial choices s_0 and s_1 to compute s_2 in order to continue the iteration. The beauty of the shooting method is that it applies to both linear and non linear odes. For linear odes, the accuracy is achieved in the first iteration in the Secant method. A few iterations are needed for non linear odes. Let us consider an an example.

Example:

Consider the the non-linear BVP defined by the differential equation. We solve using the shooting method.

$$\frac{d^2y}{dx^2} + \sin(y) = \cos(5x); \quad -1 \leq x \leq 1, \quad y(-1) = 0, y(1) = 0$$

Solution:

First, reduce the ODE to a system of first order equations by letting

$$\frac{dy}{dx} = z = f(x, y, z), \quad \text{then} \frac{dz}{dx} = \cos(5x) - \sin(y) = g(x, y, z)$$

Hence the system of equations becomes

$$\frac{dy}{dx} = z = f(x, y, z)$$

$$\frac{dz}{dx} = \cos(5x) - \sin(y)$$

with $y(-1) = 0$ and $z(-1) = ?(\Omega)$(unknown). Here is the description of the method. Guess the first slope call it $dydx[1]$ and find the solution of the IVP. Make a second guess of the slope call it $dydx[2]$ and find the solution of the IVP. The third guess of the slope is now obtained by some iterative protocol in this case we use the secant method. The obtained slope is used to find solution of he IVP. At this point we can now compare the obtained boundary point and the actual boundary value. If the absolute error say between them is greater than a given tolerance, we work out another slope $dydx[4]$ and find solution of IVP. This process goes on until tolerance is achieved. We can now output the final solution of the IVP which is then the solution of the BVP. The resulting IVP in each case is solved numerically using the fourth order Runge-Kutta method. Here is a mathematica code for the method.

```
tol = .001;
f[x_, y_, z_] := z;
g[x_, y_, z_] := Cos[5*x] - Sin[y];
n = 8;(*number of time steps which you can vary*)
(*initialize the arrays x, y and z*)
x = Table[j, j, 0, n];
y = Table[j, j, 0, n];
z = Table[j, j, 0, n];
(*initial conditions*)
system[Ω_] := {xI = -1;x[[1]] = xI;zI = 0; y[[1]] = yI;zI = Ω_;z[[1]] = zI;yF = 0; xF = 1.0;step = (xF - xI)/n;h = N[step];(*step size*) For[i = 1, i< n + 1, i++,
{ k1 = f[x[[i]], y[[i]], z[[i]]];
l1 = g[x[[i]], y[[i]], z[[i]]];
k2 = f[x[[i]] + h*.5, y[[i]] + k1*h*.5, z[[i]] + l1*h*.5];
l2 = g[x[[i]] + h*.5, y[[i]] + k1*h*.5, z[[i]] + l1*h*.5];
k3 = f[x[[i]] + h*.5, y[[i]] + k2*h*.5, z[[i]] + l2*h*.5];
l3 = g[x[[i]] + h*.5, y[[i]] + k2*h*.5, z[[i]] + l2*h*.5];
k4 = f[x[[i]] + h, y[[i]] + k3, z[[i]] + l3];
l4 = g[x[[i]] + h, y[[i]] + k3, z[[i]] + l3];
x[[i + 1]] = x[[i]] + h;
y[[i + 1]] = y[[i]] + (k1 + 2*k2 + 2*k3 + k4)*h/6;
z[[i + 1]] = z[[i]] + (l1 + 2*l2 + 2*l3 + l4)*h/6]};
(*incorporation of the shooting technique*) (*test and find where the root lies*)
```

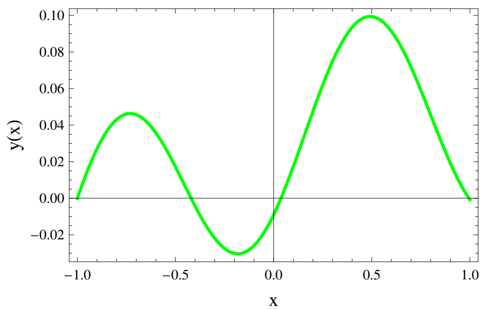

Figure 1: The shooting method

W1 = 1; Ω1 = N[W1]; system[Ω1]; Z1 = y[[n + 1]] - yF; W2 = 2; Ω2 = N[W2]; system[Ω2]; Z2 = y[[n + 1]] - yF;W3 = 3; Ω3 = N[W3]; system[Ω3]; Z3 = y[[n + 1]] - yF;W4 = 4; Ω4 = N[W4]; system[Ω4]; Z4 = y[[n + 1]] - yF;yEnd = W1, Z1, W2, Z2, W3, Z3, W4, Z4; ListLinePlot[yEnd];
(*apply the secant method*)
Q1 = 0; Ω1 = N[Q1]; system[Ω1]; R1 = y[[n + 1]]; Q2 = .5; Ω2 = N[Q2]; system[Ω2]; R2 = y[[n + 1]]; dydx[1] = Q1; dydx[2] = Q2;dydx[3] = dydx[2] + ((dydx[1] - dydx[2]))/(R1 - R2)*(yF - R2); Q3 = dydx[3]; Ω3 = N[Q3]; system[Ω3]; R3 = y[[n + 1]];
(*test for convergence*)
k = 3; R[2] = R2; R[3] = R3;
While[Abs[yF - R[k]] > tol, dydx[k + 1] = dydx[k] + ((dydx[k - 1] - dydx[k])/(R[k - 1] - R[k]))*(yF - R[k]); Q[k + 1] = dydx[k + 1]; Ω = N[Q[k + 1]]; system[Ω]; R[k + 1] = y[[n + 1]]; k = k + 1]; Q[k] = dydx[k]; Ω= N[Q[k]]; system[Ω]; R[k] = y[[n + 1]];
k;(*number of iterations*)
dydx[k];(*required slope*) R[k];(*yEnd*)
y = Table[x[[j]], y[[j]], j, 1, n + 1];
z = Table[x[[j]], z[[j]], j, 1, n + 1];
plot1 = ListPlot[y, Frame → True, Joined → True, PlotStyle → Green, Thick,FrameLabel → Style[”x”, 12], Style[”y(x)”, 12]]

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