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BIJECTIONS FOR CLASSES OF LABELLED TREES

ALBERT OLOO NYARIARO AND ISAAC OWINO OKOTH*

ABSTRACT. Trees are acyclic connected graphs. Plane trees, d-ary trees, binary trees, noncrossing trees and their generalizations, which are families of trees, have been enumerated by many authors using various statistics. These trees are known to be enumerated by Catalan or Catalan-like formulas (Fuss-Catalan numbers). One of the most common approaches to the enumeration of these trees is by means of generating functions. Another method that can be used to enumerate them is by constructing bijections between sets of the same cardinality. The bijective method is preferred to other methods by many combinatorialists. So, in this paper, we construct bijections relating k-plane trees, k-noncrossing increasing trees, k-noncrossing trees, k-binary trees and weakly labelled k-trees.

1. Introduction and preliminary results

Counting of trees, which are simple combinatorial structures consisting of vertices and edges such that the resulting graph is connected and has no loops, multiple edges and cycles, can be traced back to the work of Cayley [2]. Various tree families have been considered in the literature. These include plane trees, binary trees, d-ary trees, noncrossing trees among others. See [3, 4, 11, 20] among many other papers and books. These trees have been generalized by

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*Corresponding author.

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assigning colours or integers to the vertices of the tree. In this section, we present definitions of various tree families, their enumerative results and our results which are proved in the subsequent sections.

Definition 1.1. *A* plane tree *is a rooted tree such that all the children of its internal vertices are ordered. It is also referred to as an* ordered tree*.*

Plane trees are counted by Catalan numbers [3]. In the last decade, research has been conducted on the generalization of plane trees. Of interest is the generalization of Gu, Prodinger and Wagner obtained in [9], which we now define:

Definition 1.2. *If vertices of a plane tree are labelled with integers in the set* $\{1, 2, \dots, k\}$ *such that all edges* (i, j) *satisfy the condition* $i + j \leq k + 1$ *then we obtain a* k-plane tree*.*

In Figure 1, we have 2-plane trees on 8 vertices whose roots are labelled by 2.

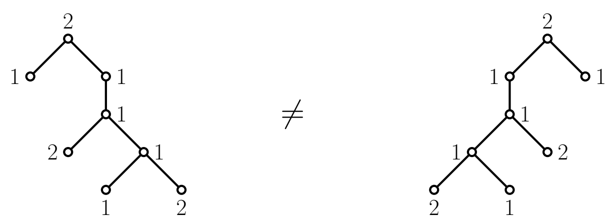

Figure 1. Two different 2-plane trees on 8 vertices with root labelled by 2.

The authors of [9] used both generating functions and bijections to enumerate k-plane trees. They proved that the number of k-plane trees with root labelled by ℓ on $n > 1$ vertices is given by

$$\frac{k+1-\ell}{kn-\ell+1}\binom{(k+1)n-\ell-1}{n-1}. \tag{1.1}$$

So, summing over all ℓ in Equation (1.1), we find that there are

$$\frac{k}{n}\binom{(k+1)(n-1)}{n-1}$$

k-plane trees on $n > 1$ vertices.

Moreover, we obtain a special case by setting $\ell = k$ in Equation (1.1): The number of k-plane trees whose roots are of label k on n vertices is given by

$$\frac{1}{n-1}\binom{(k+1)(n-1)}{n-2}.$$

The case of $k = 2$ was earlier considered by Gu and Prodinger in [8]. It should be noted that 1-plane trees are plane trees.

Another tree family that we are interested in is noncrossing trees:

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Definition 1.3. *A* noncrossing tree *is a tree drawn in the plane with its vertices on the boundary of a circle and the edges are line segments which do not cross inside the circle.*

Definition 1.4. [5] *In a noncrossing tree, a* butterfly *is an ordered pair of trees that have a common root. The subtrees that share a root with v but have lower labels (resp. higher labels) than v are called the* right *(resp.* left*)* wing *of a butterfly rooted at v.*

Panholzer and Prodinger in [17] introduced a representation for noncrossing trees as plane trees. Let u be the parent of a non-root vertex v. If u is greater than v then the vertex that corresponds to u is represented by r. Otherwise, it is represented by l. Vertex 1 is taken as the root. A subtree whose root is represented by l (resp. r) is said to be a *left* (resp. *right*) *subtree*. The right (resp. left) wing of a butterfly rooted at v consists of the right (resp. left) subtrees that share a root with v. In Figure 2, we show a noncrossing tree and its (l, r)-representation.

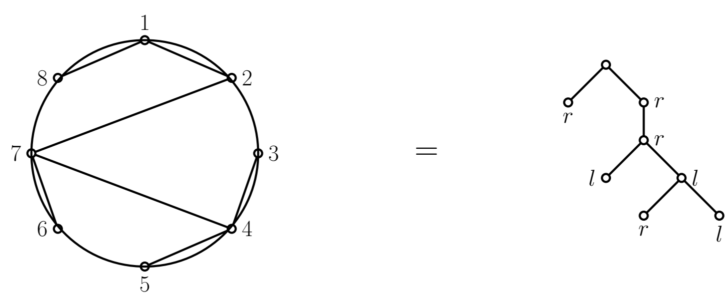

FIGURE 2. A noncrossing tree on 8 vertices with its (l, r)-representation.

Definition 1.5. *Consider a path from the root to a given vertex in a noncrossing tree. An edge (i, j) in the path is called an* ascent *if $i < j$. Otherwise, the edge is a* descent*.*

Note that in the (l, r)-representation of noncrossing trees, if we consider paths from the root, then an edge whose terminal vertex is represented by r is an ascent. Otherwise, it is a descent.

Noncrossing trees were generalized by Pang and Lv [18] as follows:

Definition 1.6. *A k-*noncrossing tree *is a noncrossing tree whose vertices are labelled with integers in the set $\{1, 2, \ldots, k\}$ such that if an edge (i, j) is an ascent in the path from the root then $i + j \leq k + 1$.*

For any k-noncrossing tree with vertices from the set $\{1, 2, \ldots, n\}$, we shall always call an element of this set as *vertex number* and an element of $\{1, 2, \ldots, k\}$ as the *label* of the vertex. The authors of [18] showed that the number of k-noncrossing trees on $n > 1$ vertices is given by

$$\frac{1}{n-1}\binom{(2k+1)(n-1)}{n-2}. \tag{1.2}$$

These trees have been enumerated by root degree [13], number of forests [13] and prescribed labels of each kind [15].

The case of $k = 2$ was initially considered by Yan and Liu [21], in which they obtained the number of these trees according to the label of the root and the number of vertices. Of course if $k = 1$, we get noncrossing trees.

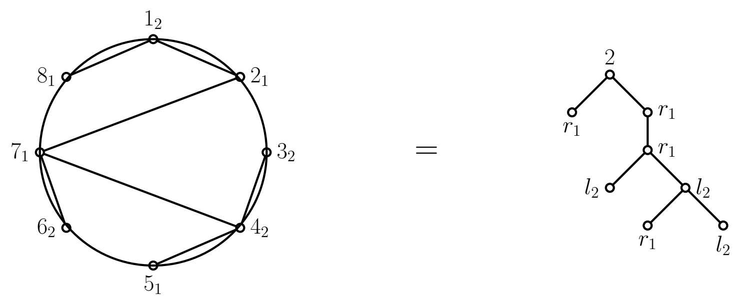

FIGURE 3. A 2-noncrossing tree on 8 vertices whose root is labelled by 2. On the right is its (l, r)-representation. The subscripts are labels of the vertices.

Noncrossing trees whose vertex numbers increase along paths from the root were first studied by Asinowski and Mansour [1] in 2008. They are commonly called *noncrossing increasing trees* and they are one of the combinatorial structures counted by the famous Catalan numbers.

Definition 1.7. [12] *A k-*noncrossing increasing tree *is a k-noncrossing tree whose vertex numbers increase as one moves away from the root.*

All vertices, except the root, in noncrossing/ k-noncrossing increasing trees are represented by r in the (l, r)-representation of these trees. The k-noncrossing increasing trees on n vertices with root labelled by k are enumerated by Fuss-Catalan numbers,

$$\frac{1}{n-1}\binom{(k+1)(n-1)}{n-2}, \tag{1.3}$$

as showed in [12], with the special case $k = 2$, obtained earlier in [14].

From Equations (1.2) and (1.3), we get the following result, which we prove in Section 2 and also present its consequences.

Theorem 1.8. *There is a bijection between the set of $2k$-noncrossing increasing trees on n vertices with root labelled by $2k$ and the set of k-noncrossing trees on n vertices with root labelled by k.*

In 2010, Pang and Lv [18] defined a *k-labelled tree* as a plane tree whose vertices, except for the root, receive labels from $\{1, 2, \ldots, k\}$. Given a vertex u, all the vertices at a lower level which are connected to u, are said to be *children* of u and such children are called *brothers*.

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In a k-labelled tree, for any two brothers, a brother that appears on the right is the younger brother to the one on the left. Or we say that the one on the left is the elder brother to the one on the right. We define a *weakly labelled k-tree* as a k-labelled tree such that the children of the root are all labelled by k and each vertex, except for the root, receives a label not less than any of its elder brothers. The authors of [18] proved the following result:

Theorem 1.9. [18] *The number of weakly labelled k-trees on $n+1$ vertices is given by*

$$\frac{1}{n}\binom{(k+1)n}{n-1}. \tag{1.4}$$

Panholzer and Prodinger [16] introduced and studied d-ary trees in which the vertices are labelled using the integers in the set $\{1, 2, \ldots, k\}$ and having the property that the label of the child in the rightmost path must be greater than or equal to the label of the parent. They showed that these trees with root labelled by k on n vertices are enumerated by the formula

$$\frac{1}{k(d-1)n+1}\binom{(k(d-1)+1)n}{n}. \tag{1.5}$$

Besides proving the formula from a generating function approach, the authors also constructed a bijection between these trees and unlabelled $(k(d-1)+1)$-ary trees. Setting $d = 2$, we obtain a k-binary tree, which we now define.

Definition 1.10. [16] *A k-*binary tree *is a binary tree whose vertices are labelled with integers in the set $\{1, 2, \ldots, k\}$ such that for any right child of label j whose parent is of label i, we have $i \leq j$.*

In Figure 4, we have a 5-binary tree on 20 vertices whose root is labelled by 5.

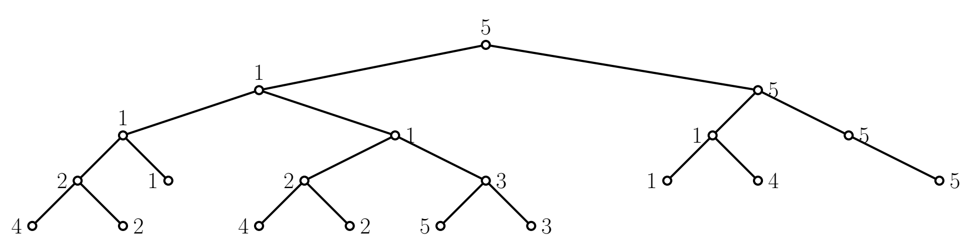

FIGURE 4. 5-binary tree on 20 vertices with root of label 5.

Setting $d = 2$ in Equation (1.5), we get Equation (1.4) and thus the following result, which we prove in Section 3.

Theorem 1.11. *There is a bijection between the set of k-binary trees on n vertices whose roots are of label k and the set of weakly labelled k-trees on $n+1$ vertices.*

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From Theorems 1.9 and 1.11 we have:

Corollary 1.12. *The number of k-binary trees on n vertices whose roots are of label k is given by*

$$\frac{1}{n}\binom{(k+1)n}{n-1}. \tag{1.6}$$

Note that Equation (1.6) is just a special case of Equation (1.5), i.e., we set $d = 2$ in Equation (1.5). When $k = 1$ in Equation (1.6), we obtain the Catalan number that counts binary trees with n vertices and $k = 2$ results in a formula for 2-binary trees obtained by Gu, Li and Mansour [7]. A bijection between the set of 2-binary trees and the set of ternary trees was also obtained by the authors of [7]. A more straightforward and simpler bijection was obtained by Prodinger [19], which was later on generalized by the authors of [16].

Consider a binary tree T. We call a path in T, a *right path* (resp. *left path*) if it starts at a left child (resp. right child) or the root, and all other vertices in the path are right (resp. left) children. A right path (resp. left path) that starts at the root is called a *rightmost path* (resp. *leftmost path*) of T. The *length* of a path is the number of edges in the path. A *right planted binary tree* (resp. *left planted binary tree*) is a binary tree with leftmost (resp. rightmost) path of length 0. A *planted binary tree* is a binary tree with either leftmost or rightmost path of length 0.

Gu, Li and Mansour in [7] proved the following result:

Theorem 1.13. [7] *There is a bijection between the set of 2-binary trees on n vertices whose roots are of label 2 and the set of noncrossing trees on $n+1$ vertices.*

In Section 4, we give a new and simpler bijection to prove Theorem 1.13. We then obtain several corollaries. We conclude the paper in Section 5.

2. Proof and Consequences of Theorem 1.8

Proof of Theorem 1.8. Consider a $2k$-noncrossing increasing tree T on n vertices with root labelled by $2k$. We obtain the corresponding k-noncrossing tree on n vertices with root labelled by k, by the following steps:

(i) Obtain the (l, r)-representation of the noncrossing increasing tree T that was introduced by Panholzer and Prodinger [17]. All the vertices, except the root, are represented by r, i.e., all the edges are ascents.

(ii) Relabel the root of the tree by k and change the representation of the $1, 2, \ldots, k$ vertices, except the children of the root to l.

(iii) Relabel vertices labelled by $k+1, k+2, \ldots, 2k$ by $1, 2, \ldots, k$ respectively. Note that these vertices are represented by r. Change the representation of their children back to r.

(iv) Perform the following procedure as one traverses the tree by Breadth First Search (BFS). Let v be a vertex , other than a child of the root, represented by r with children $v_1, v_2, \ldots, v_t$ represented by r. Let T_{v_i} be the subtree consisting of v, all children of v lying between v_i and v_{i+1} that are represented by l, and all the descendants.The subtree T_{v_t} consists of v, all the children of v to the right of v_t that are represented by l, and all their descendants. For all $i = 1, 2, \ldots, t$, remove all branches of T_{v_i} from its root v and attach them in the same order to v_i to form a left subtree rooted at v_i. The subtree T_{v_i} is a left wing of a butterfly rooted at v_i. The resultant tree is a k-noncrossing tree.

We now check if the tree obtained is a valid k-noncrossing tree with root labelled by k. If we consider a path from the root, an edge is an ascent if the terminal vertex is represented by r. There are two cases where the final vertex is represented by r.

(i) The final vertex is a child of the root. Here, since all children of root are labelled by 1 then the ascent rule holds.

(ii) The final vertex is a not a child of the root. This is the case where the initial vertex or terminal vertex was initially labelled by $k+m$ and was relabelled by m. The ascent condition of $i + j \leq 2k + 1$ in $2k$-noncrossing increasing trees becomes $i + j \leq k + 1$ in k-noncrossing trees. Thus the ascent rule holds.

The trees produced are thus valid k-noncrossing trees on n vertices with roots labelled by k. Figure 5 is an illustration of how to obtain a k-noncrossing tree from a $2k$-noncrossing increasing tree.

We now describe the reverse procedure: Consider a k-noncrossing tree T' on n vertices with root labelled by k. We obtain the corresponding $2k$-noncrossing increasing tree on n vertices with root labelled by $2k$, by the following steps:

(i) Relabel the root by $2k$.

(ii) We perform the following procedure as we traverse T' by BFS. Let v be a vertex, other than the children of the root, represented by r with children $u_1, u_2, \ldots, u_s$ and $v_1, v_2, \ldots, v_t$ represented by l and r respectively. Note that the sets $\{u_1, u_2, \ldots, u_s\}$ and $\{v_1, v_2, \ldots, v_t\}$ can be empty. Let the parent of v be x. Relabel v by $k+m$ if it was labelled by m for $m = 1, 2, \ldots, k$. Now, attach the subtree rooted at v with children $u_1, u_2, \ldots, u_s$ as a subtree rooted at x such that $u_1, u_2, \ldots, u_s$ appear on the right of v and the other younger siblings of v appearing on the right of u_s. Change the representation of vertices $v_1, v_2, \ldots, v_t$ to l.

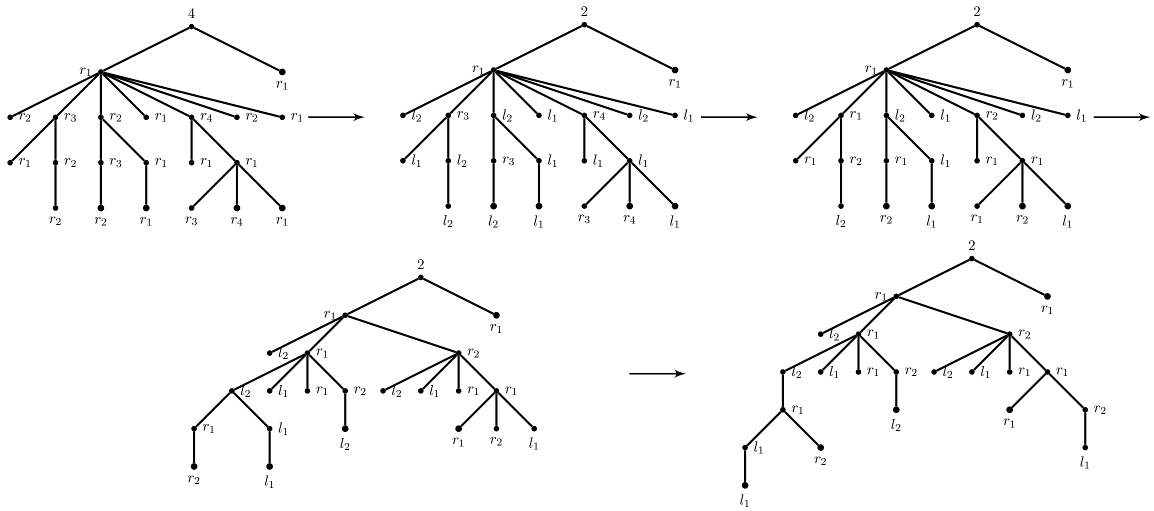

FIGURE 5. Obtaining a 2-noncrossing tree on 22 vertices whose root is labelled by 2 from a 4-noncrossing increasing tree on 22 vertices whose root is labelled by 4.

(iii) Once all the vertices have been traversed, change the representation of all the vertices to r. The tree obtained is a $2k$-noncrossing increasing tree with root labelled by $2k$.

We check for the validity of the tree: All the children of the root are labelled by 1 and the root is labelled by $2k$. So the ascent condition for $2k$-noncrossing increasing trees is satisfied. As we traverse the k-noncrossing tree:

(i) A vertex labelled by m and represented by r is relabelled by $k+m$ in the resultant $2k$-noncrossing increasing tree. Then the ascent condition of $i+j \leq k+1$ in k-noncrossing tree becomes $i+j \leq 2k+1$ in the $2k$-noncrossing increasing tree. So the ascent rule holds.

(ii) A vertex labelled by m and represented by l retains its label in the resultant $2k$-noncrossing increasing tree. Thus the maximum possible sum of the labels of the end-points of an edge is $2k$. So the ascent condition for $2k$-noncrossing increasing tree is satisfied. The ascent condition holds.

The tree obtained is thus a valid $2k$-noncrossing increasing tree on n vertices with root labelled by $2k$. A depiction of this procedure is given in Figure 6. □

The following result was obtained by Okoth in [12].

Corollary 2.1. [12] *There is a bijection between the set of k-plane trees on n vertices whose root is labelled by h and the set of k-noncrossing increasing trees on n vertices whose root is labelled by h.*

From Theorem 1.8 and Corollary 2.1, we get the following corollary:

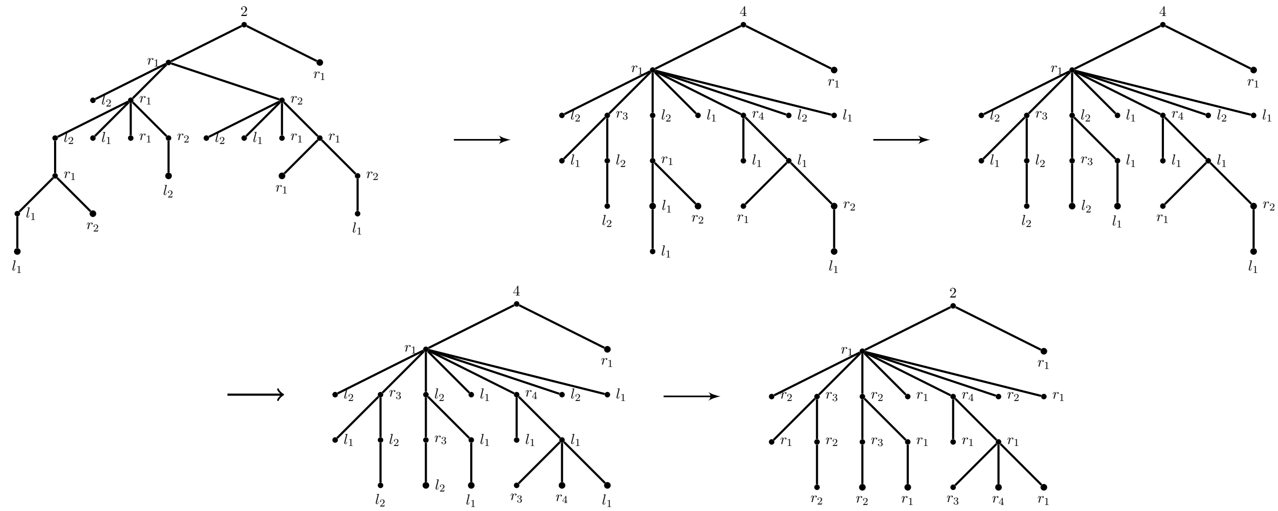

FIGURE 6. Obtaining a 4-noncrossing increasing tree on 22 vertices whose root is labelled by 4 from a 2-noncrossing tree on 22 vertices whose root is labelled by 2.

Corollary 2.2. *There is a bijection between the set of $2k$-plane trees with root labelled by $2k$ on n vertices and the set of k-noncrossing trees with root labelled by k on n vertices.*

3. Proof of Theorem 1.11

Let T be a weakly labelled k-tree on $n+1$ vertices where all the children of the root are of label k. Let the non-root vertices be $v_1, v_2, \ldots, v_n$. We build the corresponding k-binary tree B on the non-root vertices of T as follows:

(i) For each non-root vertex v_i of T, the left child of v_i in B is the eldest child of v_i in T. If v_i is a leaf in T, then there is no left child of v_i in B.

(ii) For each non-root vertex v_i of T, the right child of v_i in B is the next younger sibling of v_i in T. If v_i has no younger sibling in T, then there is no right child of v_i in B.

The tree established is a k-binary tree on n vertices whose root is of label k.

To prove bijection, we need to provide the reverse procedure: Let B be a k-binary tree on n vertices whose root is of label k. Let these vertices be $v_1, v_2, \ldots, v_n$ where v_1 is the root. We build the corresponding weakly labelled k-tree T on $n+1$ vertices, starting at a new vertex (root) as follows:

(i) Draw an edge from the root to a vertex which will be the eldest child of the root in T. This vertex corresponds to v_1 in the k-binary tree. Now, we build the tree recursively.

(ii) The left child of v_i in B is the eldest child of v_i in T. If v_i has no left child in B, then v_i is a leaf.

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(iii) The right child of v_i in B is the next younger sibling of v_i in T. If v_i has no right child in B, then v_i has no younger sibling in T.

For an example of the bijection, see Figure 7.

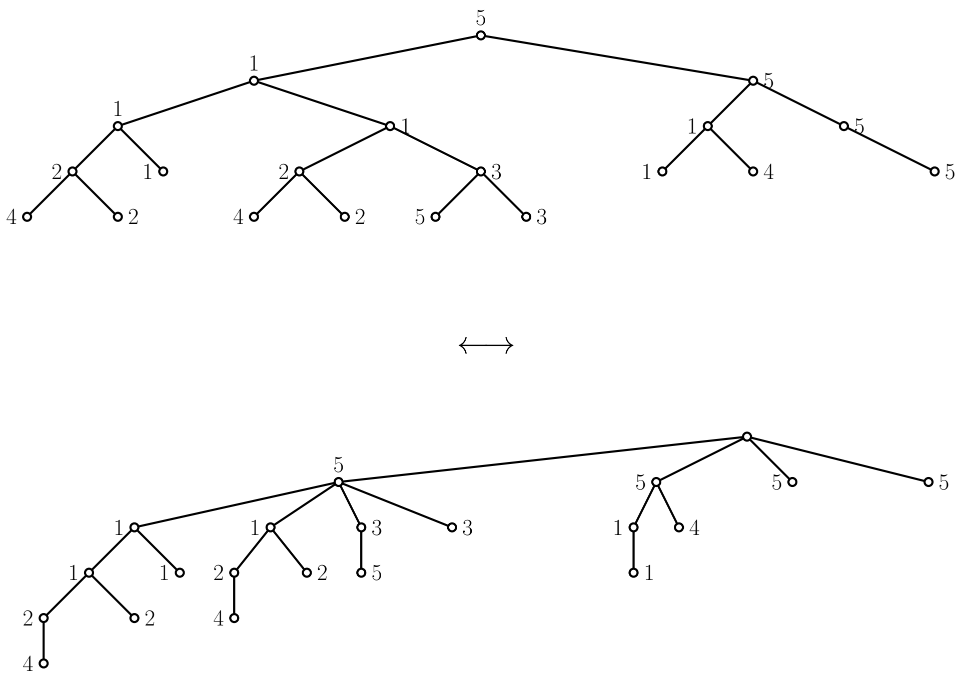

FIGURE 7. Bijection between 5-binary tree on 20 vertices whose root is of label 5 and weakly labelled 5-tree on 21 vertices.

When $k = 1$, this bijection is equivalent to what is known as natural correspondence [10] or rotation correspondence [6].

4. Proof and Consequences of Theorem 1.13

Proof of Theorem 1.13. Consider a 2-binary tree T on n vertices with root labelled by 2. We obtain a noncrossing tree T' on $n + 1$ vertices by the following steps:

(i) Consider the rightmost path $v_1, v_2, \ldots, v_m$ in T with the root of T being v_1. Place vertices $v'_0, v'_1, \ldots, v'_m$ clockwise on the circle and create edges from v'_0 to v'_i where $i = 1, 2, \ldots, m$. Here, v'_0 is the root of the noncrossing tree T'.

(ii) Let L_i be the left subtree rooted at v_i and let w_i be the root of L_i.
Let $w_i, x_1, x_2, \ldots, x_s, y_1, y_2, \ldots, y_t$ be the rightmost path starting at the root w_i of L_i such that all vertices on the paths $x_1, x_2, \ldots, x_s$ and $y_1, y_2, \ldots, y_t$ are labelled by 1 and 2 respectively. Note that if vertex w_i is labelled by 2 then the path $x_1, x_2, \ldots, x_s$ is empty. Also if all the vertices on the rightmost path starting at w_i are all labelled by 1 then the path $y_1, y_2, \ldots, y_t$ is empty. There are two cases to consider:

(a) If w_i is labelled by 1, then place vertices $w'_i, x'_1, x'_2, \dots, x'_s$ clockwise on the circle after vertex v'_i and create edges from v'_i to w'_i and from v'_i to x'_j where $j = 1, 2, \dots, s$. Also place vertices $y'_1, y'_2, \dots, y'_t$ anti-clockwise on the circle before vertex v'_i and create edges from v'_i to y'_j where $j = 1, 2, \dots, t$.

(b) If w_i is labelled by 2, then place vertices $w'_i, y'_1, y'_2, \dots, y'_t$ anti-clockwise on the circle before vertex v'_i and create edges from v'_i to w'_i and from v'_i to y'_j where $j = 1, 2, \dots, t$.

(iii) We repeat steps (ii) until all the vertices of T are mapped onto T'.

We now describe the reverse procedure:

(i) Consider a noncrossing tree on $n + 1$ vertices whose root r has degree m. In clockwise direction, let the vertices that are incident to r be $v_1, v_2, \dots, v_m$. These vertices correspond to the rightmost path $v'_1, v'_2, \dots, v'_m$ in the 2-binary tree such that v'_1 is the root of the binary tree and all these vertices are labelled by 2.

(ii) Let $w_p, w_{p-1}, \dots, w_1 v_i x_1, x_2, \dots, x_q$ be a sequence of vertices in the noncrossing tree such that all the vertices $w_1, w_2, \dots, w_p, x_1, x_2, \dots, x_q$ are incident to v_i. Moreover, $w_{i+1} < w_i$ and $x_j < x_{j+1}$ for all $1 \le i < p - 1$ and $1 \le j < q - 1$, and $w_1 < v_i < x_1$. Note that the vertex sets $W = \{w_p, w_{p-1}, \dots, w_1\}$ and $X = \{x_1, x_2, \dots, x_q\}$ can be empty. If $X = \emptyset$, then create a left edge from v'_i to a new vertex w'_1 in the binary tree such that the new vertex is of label 2. Now create a right path starting at w'_1 and having $w'_1, w'_2, \dots, w'_p$ (can be empty) in this order as the vertices, all labelled by 2. If $X \neq \emptyset$, then create a left edge from v'_i to a new vertex x'_1 in the binary tree such that the new vertex is of label 1. Now create a right path starting at x'_1 and having $x'_1, x'_2, \dots, x'_q$ in this order as the vertices, all of label 1, followed by $w'_1, w'_2, \dots, w'_p$ (can be empty) in this order as vertices, all labelled by 2.

(iii) Repeat steps (ii) until all the vertices in the noncrossing tree, except the root, have their corresponding vertices in the 2-binary tree.

□

Consider the 2-binary tree on 11 vertices whose root is labelled by 2 given in Figure 8. In Figure 9, we illustrate the bijection between this tree and a noncrossing tree. Vertices of the 2-binary tree which have been considered in the noncrossing tree are filled up.

Theorem 4.1. [11] *For all $n > d$, the number of noncrossing trees on n vertices such that the degree of the root is $d \geq 1$ is given by*

$$\frac{2d}{3n-d-3}\binom{3n-d-3}{n-d-1}.$$

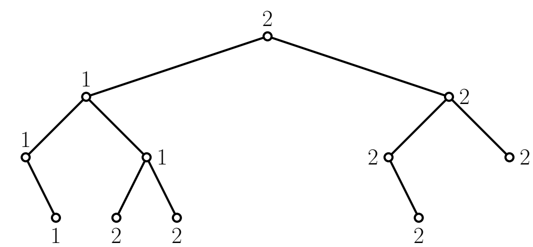

FIGURE 8. 2-binary tree on 11 vertices with root of label 2.

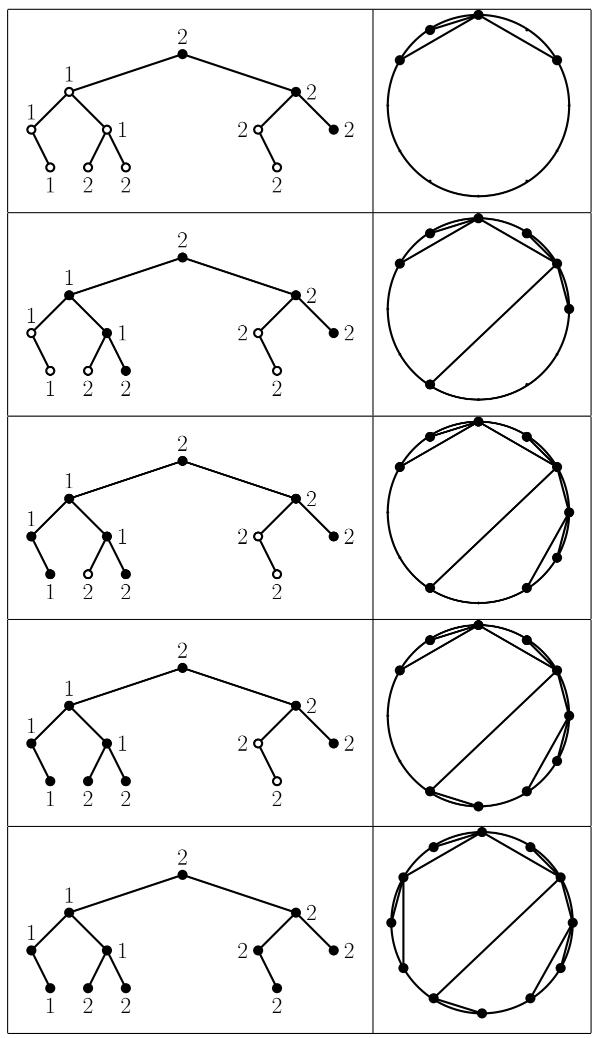

FIGURE 9. Bijection between 2-binary trees with root of label 2 and noncrossing trees.

Based on the proof of Theorem 1.13, we obtain the following corollary:

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Corollary 4.2. *The number of 2-binary trees on n vertices with roots of label 2 such that the rightmost path is of length $d-1$ is given by*

$$\frac{2d}{3n-d}\binom{3n-d}{n-d}. \tag{4.1}$$

By setting $d=1$ in Equation (4.1), we obtain the following:

Corollary 4.3. *There are*

$$\frac{1}{n}\binom{3n-2}{n-1} \tag{4.2}$$

left planted 2-binary trees on n vertices whose roots are labelled by 2.

Corollary 4.4. *The number of right planted 2-binary trees on n vertices whose roots are labelled by 2 is*

$$\frac{1}{2n-1}\binom{3n-3}{n-1}. \tag{4.3}$$

Proof. Attach a 2-binary tree on n vertices whose root is labelled by 2 to a new vertex labelled by 2, as a right subtree of the root. □

From Equations (4.2) and (4.3), we obtain:

Corollary 4.5. *The total number of planted 2-binary trees on n vertices with root of label 2 is*

$$\frac{2}{n}\binom{3n-3}{n-1}.$$

The following result is proved by Flajolet and Noy in [5]:

Theorem 4.6. [5] *The number of noncrossing trees on n vertices and having ℓ leaves is equal to*

$$\frac{1}{n-1}\binom{n-1}{\ell}\sum_{i=0}^{\ell-1}\binom{n-1}{i}\binom{n-\ell-1}{\ell-1-i}2^{n-2\ell+i}.$$

Definition 4.7. *Let T be a binary tree. A vertex of T which is either a leaf or an internal vertex without a left child is called a* leafy vertex.

From Theorem 4.6 and the proof of Theorem 1.13, we obtain Corollary 4.8.

Corollary 4.8. *The number of 2-binary trees on n vertices whose root is of label 2 such that there are ℓ leafy vertices is given by*

$$\frac{1}{n-2}\binom{n-2}{\ell}\sum_{i=0}^{\ell-1}\binom{n-2}{i}\binom{n-\ell-2}{\ell-1-i}2^{n-2\ell+i-1}.$$

5. Conclusion

In this paper, we have constructed various bijections concerning k-plane trees, k-binary trees, weakly labelled k-trees, k-noncrossing increasing trees and k-noncrossing trees. In Section 2, we obtained a bijection between the set of $2k$-noncrossing increasing trees on n vertices and the set of k-noncrossing trees on n vertices. As a corollary, we showed that a bijection between the set of $2k$-plane trees with root labelled by $2k$ and the set of k-noncrossing trees with root labelled by k is obtained via $2k$-noncrossing increasing trees with root labelled by $2k$. It would be interesting to obtain a direct bijection between the set of $2k$-plane trees with root labelled by $2k$ and the set of k-noncrossing trees with root labelled by k. We also constructed a bijection between the set of k-binary trees with root labelled by k and the set of weakly labelled k-trees. This was achieved in Section 3. However, bijections between the set of these weakly labelled k-trees and k-plane trees as well as k-noncrossing trees are yet to be established. Lastly, in Section 4, we obtained a new and simpler bijection between the set of 2-binary trees on n vertices whose root is labelled by 2 and the set of noncrossing trees on $n+1$ vertices. It will be interesting to construct a bijection between the set of $2k$-binary trees whose root is labelled by $2k$ and the set of k-noncrossing trees whose root is labelled by k.

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Albert Oloo Nyariaro

Department of Pure and Applied Mathematics, Maseno University, P. O. Box 333, Maseno, Kenya.

Email: anyariaro@maseno.ac.ke

Isaac Owino Okoth

Department of Pure and Applied Mathematics, Maseno University, P. O. Box 333, Maseno, Kenya.

Email: ookoth@maseno.ac.ke

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